Department of Computer Science and Engineering (CSE)
BRAC University

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CSE250 - Circuits and Electronics

AC Fundamentals, AC Circuits, and AC Power

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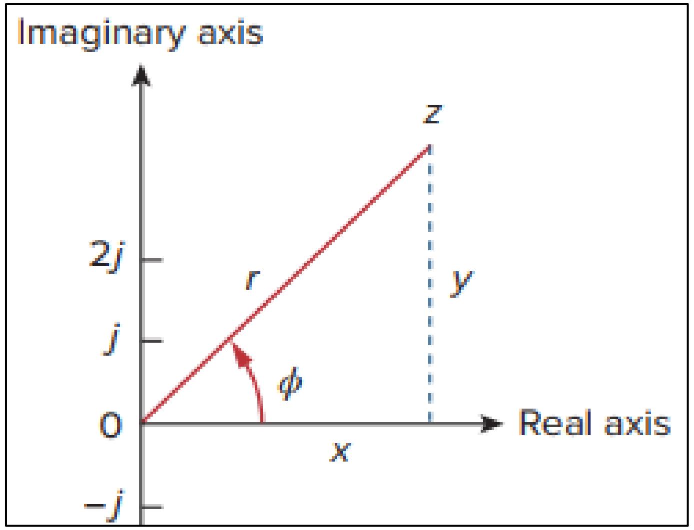
Complex Number

- A complex number z can be written as,
- $z = x + jy$ Rectangular form
- $z = r\angle\varphi$ Polar form $j = \sqrt{-1} = -\frac{1}{j}$
- $z = re^{j\varphi}$ Exponential form
- The relationship between the rectangular and the polar form can be written from the figure as,
- $r = \sqrt{x^2 + y^2}$
- $\phi = \tan^{-1}\frac{y}{x}$;
- $x = r\cos\theta$
- $y = r\sin\theta$



- The complex exponential can be expanded by a Taylor's series expansion as,
- $e^{j\varphi} = 1 + j\varphi + \frac{(j\varphi)^2}{2!} + \frac{(j\varphi)^3}{3!} + \frac{(j\varphi)^4}{4!} + \cdots$
- Separating the real and imaginary parts,

$$\Rightarrow \quad e^{j\varphi} = \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots\right) + j(x - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots)$$

- The real and imaginary parts are the Taylor's series expansion of a cosine and a sine respectively.

$$\Rightarrow \quad e^{j\varphi} = cos\varphi + jsin\varphi$$

$$\Rightarrow \quad re^{j\varphi} = rcos\varphi + jrsin\varphi$$

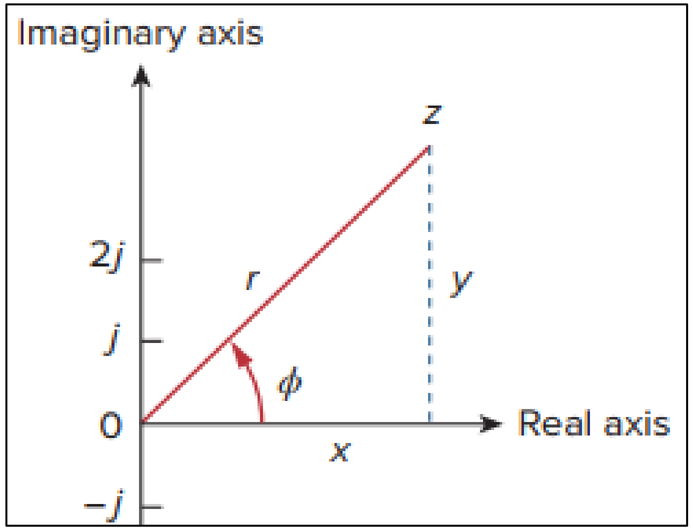
$$\text{So,} \quad z = re^{j\varphi} = x + jy$$

Operations of Complex Numbers

- Rectangular form $z = x + jy$
- Polar form $z = r\angle\varphi$ where, $j = \sqrt{-1} = -\frac{1}{j}$
- Exponential form $z = re^{j\varphi}$

- Let, $z_1 = x_1 + jy_1 = r_1\angle\varphi_1 = r_1e^{j\varphi_1}$ and $z_1 = x_2 + jy_2 = r_2\angle\varphi_2 = r_2e^{j\varphi_2}$

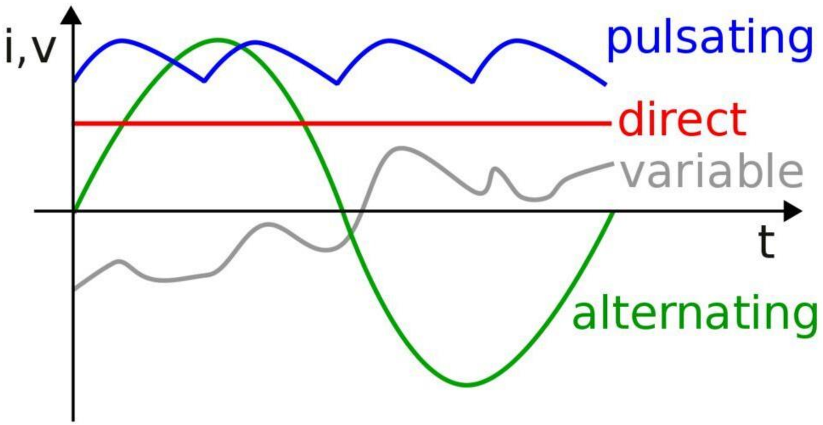
- Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
- Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
- Multiplication: $z_1z_2 = r_1r_2\angle(\varphi_1 + \varphi_2)$
- Division: ${}^{z_1}/_{z_2} = \frac{r_1}{r_2}\angle(\varphi_1 - \varphi_2)$
- Reciprocal: $\frac{1}{z} = \frac{1}{r}\angle - \varphi$
- Square root: $\sqrt{z} = \sqrt{r}\angle\frac{\varphi}{2}$

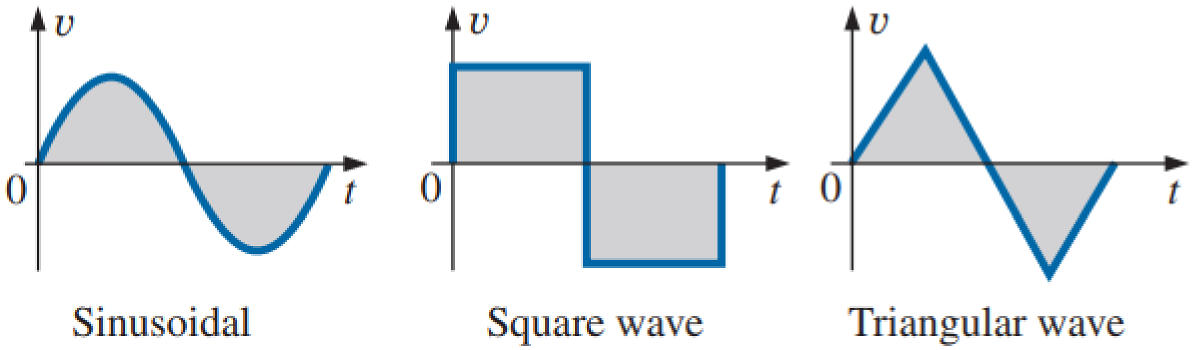


- Complex Conjugate: $z^* = x - jy = r\angle{-\varphi} = re^{-j\varphi}$

Alternating Current or Voltage

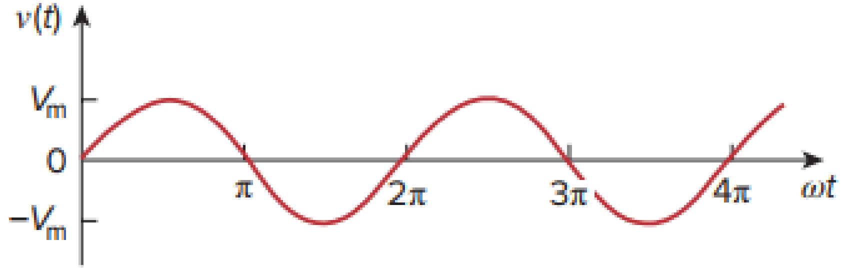
- *Alternating current (AC)* is a flow of electric charge that periodically reverses its direction, in contrast to direct current (DC) which only flows in a single direction. It starts, say, from zero, grows to a maximum, decreases to zero, reverses, reaches a maximum in the opposite direction, returns again to the original value, and repeats this cycle indefinitely.
- The interval of time between the attainment of a definite value on two successive cycles is called the *period*, the number of cycles or periods per second is the *frequency*, and the maximum value in either direction is the *amplitude* of the alternating current.

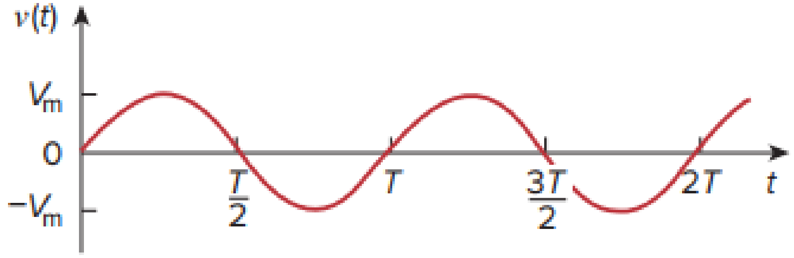




Alternating waveforms.

Sinusoid





- Among different types of ac waveforms, the pattern of particular interest is the sinusoidal because it is the voltage generated by utilities throughout the world and supplied to homes, factories, laboratories, and so on.
- A *sinusoid* is a signal that has the form of the sine or cosine function. A sinusoidal current is usually referred to as *alternating current (ac)*. Circuits driven by sinusoidal current sources or voltage sources are called *ac circuits*.
- The basic mathematical format for the sinusoidal waveform is $A_m sin\omega t$ or $A_m cos\omega t$
- For electrical quantities such as current and voltage, the general format is,

$$v(t) = V_m sin(\omega t) \; or \; V_m cos(\omega t) \qquad and \qquad i(t) = I_m sin(\omega t) \; or \; I_m cos(\omega t)$$

where, V_m, I_m = amplitudes of the sinusoids; ω = the angular frequency in radians/s; ωt = the argument of the sinusoid in deg or rad; time period, $T = \frac{2\pi}{\omega} = \frac{1}{f}$, f = frequency

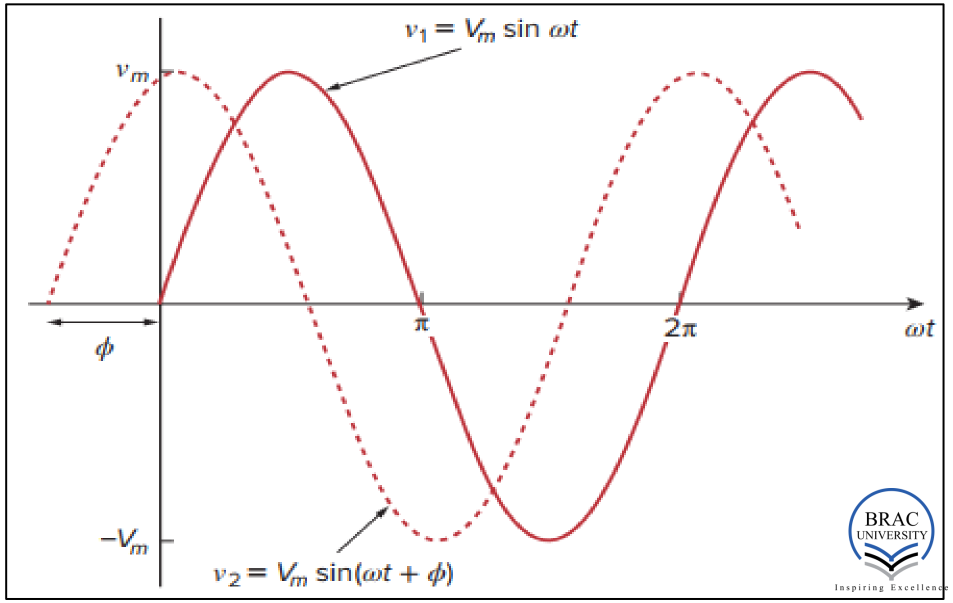
Leading and Lagging Sinusoids

- The equation for a sinusoid in more general form,

$$v(t) = V_m \, sin(\omega t + \varphi)$$

  where $(\omega t + \varphi)$ is the argument and φ is the initial phase. Both argument and phase can be in radians or degrees.

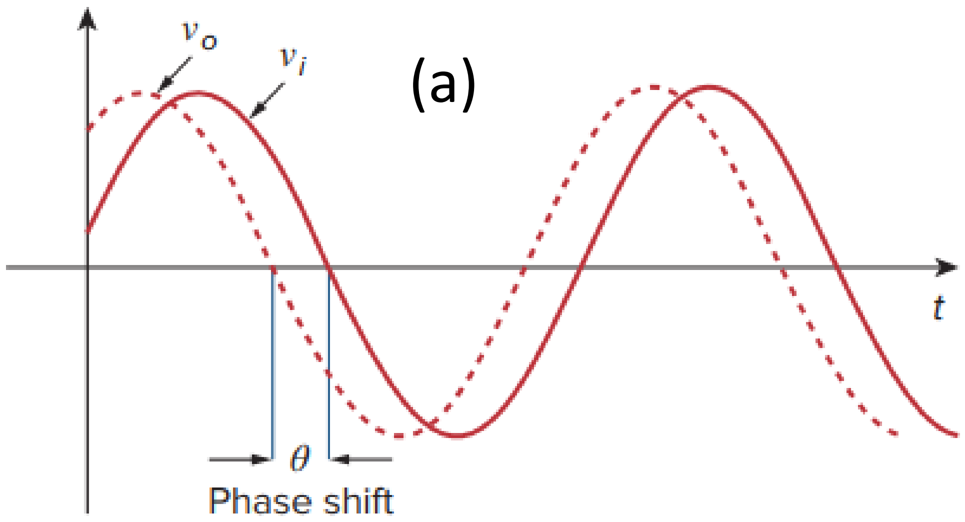
- Let's examine two sinusoids $v_1(t) = V_m sin\omega t$ and $v_2(t) = V_m sin(\omega t + \varphi)$

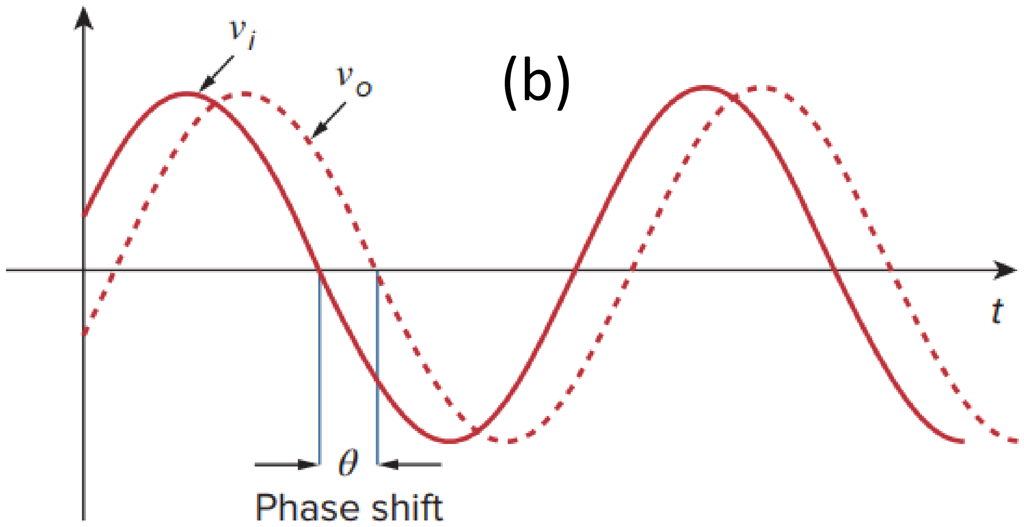


- The starting point of v_2 occurs first in time. Therefore, we say that v_2 leads v_1 by φ or that v_1 lags v_2 by φ.
- If $\varphi \neq 0$, v_1 and v_2 are out of phase.
- If $\varphi = 0$, v_1 and v_2 are in phase

Problem 1

- Determine for each of the plots, which one is leading/lagging.



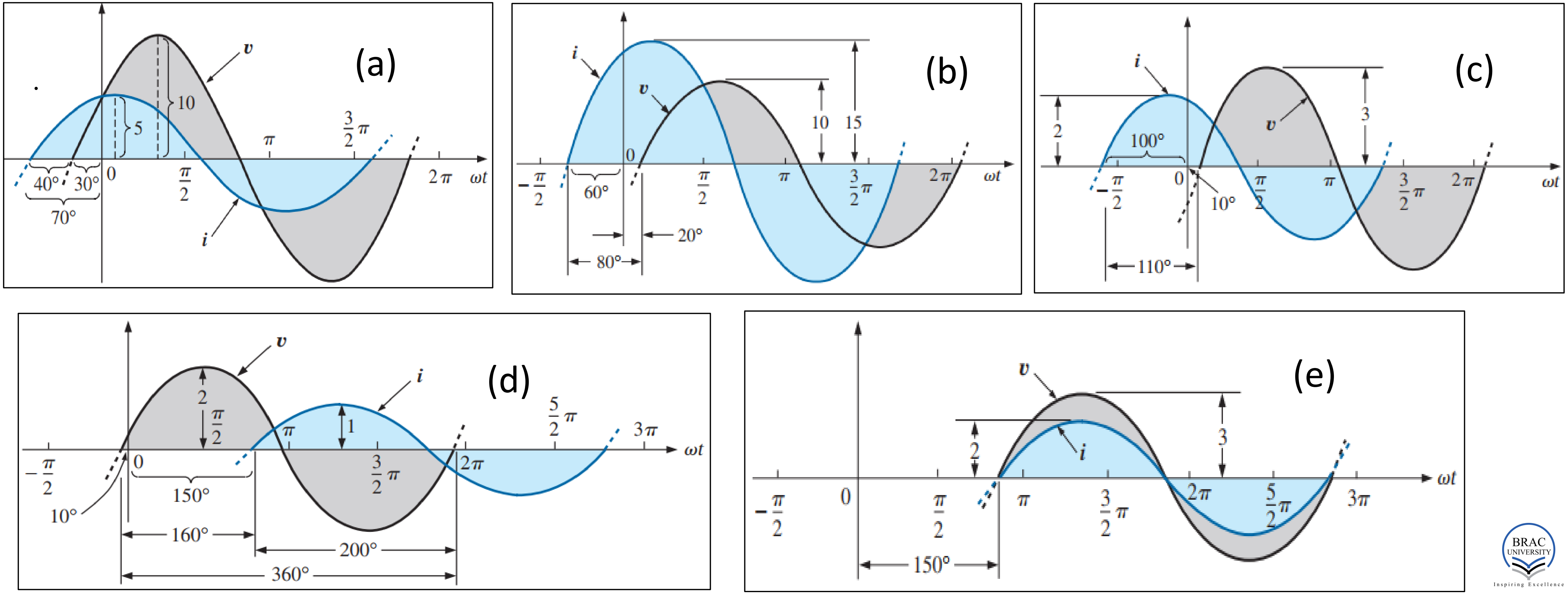


Ans: (a) v_o **leading**; (b) v_i **leading**

Problem 2

Ans: (a) i leads v by 40°; (b) i leads v by 80°; (c) i leads v by 110°; (d) v leads i by 160°; (e) v and i are in phase

- Determine for each of the plots, which one (v or i) is leading/lagging and by how much.



Sine-Cosine Conversion

- A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes.

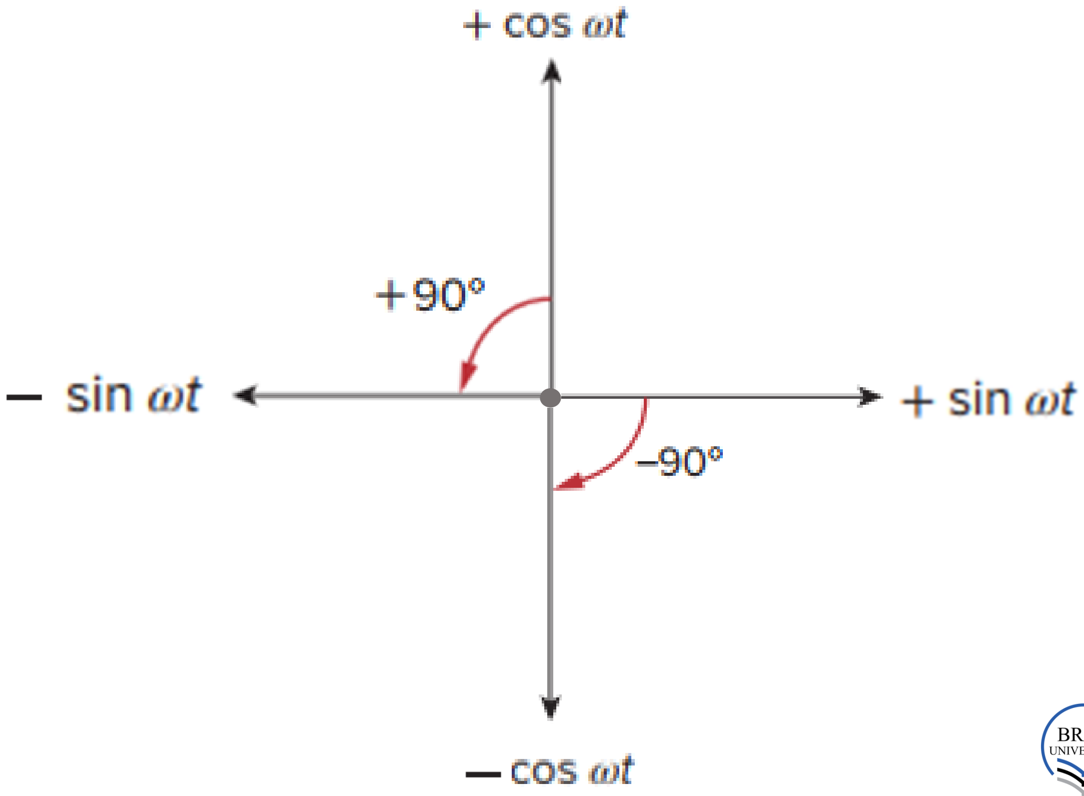
- The following trigonometric identities can be used to convert from sine to cosine or vice versa.

- $\sin(\omega t \pm 180°) = -\,sin\omega\mathrm{t}$

- $\cos(\omega t \pm 180°) = -cos\omega\mathrm{t}$

- $\sin(\omega t \pm 90°) = \pm\,cos\omega\mathrm{t}$

- $\cos(\omega t \pm 90°) = \mp\,sin\omega\mathrm{t}$



Problem 3

- Calculate amplitude, initial phase $(0 \le \varphi \le 180°)$, angular frequency, period, and frequency for the following sinusoids. What are the values of V_S and I_s at $t = 20\,ms$? *[Both sin and cosine forms can be employed]*

  I. $V_s = 45\cos(5\pi t + 36°)\ (V)$

  II. $I_s = 15\cos(25\pi t + 25°)\ (A)$

  III. $I_s = -20\cos(314t - 30°)\ (A)$

  IV. $V_s = -4\sin(628t + 55°)\ (V)$

<u>Ans</u>: $I.\ 45\,V;\ 36°\ or\ 126°;\ 5\pi\,rad/s;\ 0.4\,s;\ 2.5\,Hz;\ 26.45\,V$

$II.\ 15\,A;\ 25°\ or\ 115°;\ 25\pi\,rad/s;\ 80\,ms;\ 12.5\,Hz;\ -6.34\,A$

$III.\ 20\,A;\ 150°\ or\ 60°;\ 314\,rad/s;\ 20\,ms;\ 50\,Hz;\ -17.29\,A$

$IV.\ 4\,V;\ -130°\ or\ 140°;\ 628\,rad/s;\ 10\,ms;\ 100\,Hz;\ -3.26\,V$

Problem 4

- For the following pairs of sinusoids, determine which one leads and by how much $(0 \leq \theta \leq 180°)$.

  I. $v(t) = 10\cos(4t - 60°)$ & $i(t) = 4\sin(4t + 50°)$

  II. $v_1(t) = 4\cos(377t + 10°)$ & $v_2(t) = -20\cos 377t$

  III. $v_1(t) = 45\sin(\omega t + 30°)\,V$ & $v_2(t) = 50\cos(\omega t - 30°)$

  IV. $i_1(t) = -4\sin(377t + 55°)$ & $i_2(t) = 5\cos(377t - 65°)$

  V. $x(t) = (13\cos 2t + 5\sin 2t)$ & $y(t) = 15\cos(2t - 11.8°)$

Hint: convert both the sinusoids into sine or cosine form → convert them into phasors → add them in frequency domain → convert them back in the time domain and compare

Ans: I. i leads v by $20°$

II. v_2 leads v_1 by $170°$

III. v_2 leads v_1 by $30°$

IV. i_1 leads i_2 by $155°$

V. y leads x by $9.24°$

Phasor

- A *phasor* is a complex number that represents the amplitude and phase of a sinusoid. It is a useful notion for solving ac circuits excited with sinusoids.
- Let $z = x \pm jy = r\angle\varphi = r(cos\varphi \pm jsin\varphi) = re^{\pm j\varphi} = r(cos\varphi \pm jsin\varphi)$
- We can write, $cos\varphi = Re\{e^{\pm j\varphi}\}$ and $sin\varphi = \pm\, Im\{e^{\pm j\varphi}\}$
- Given a sinusoid, $v(t) = V_m cos(\omega t + \varphi) = Re\left\{V_m e^{j(\omega t+\varphi)}\right\} = Re\{V_m e^{j\omega t} e^{j\varphi}\} = Re\left\{\boldsymbol{V} e^{j\omega t}\right\}$
- where $\boxed{\boldsymbol{V} = V_m e^{j\varphi} = V_m\angle\varphi}$ is the phasor representation of $v(t)$
- $v(t) = \underbrace{V_m cos(\omega t + \varphi)}_{\text{Time domain representation (cosine)}} \Leftrightarrow \underbrace{\boldsymbol{V} = V_m\angle\theta}_{\text{Phasor domain representation}} \Leftrightarrow \underbrace{V_m sin(\omega t + \varphi)}_{\text{Time domain representation (sine)}}$

** *from now on, phasors will be represented in bold letters*

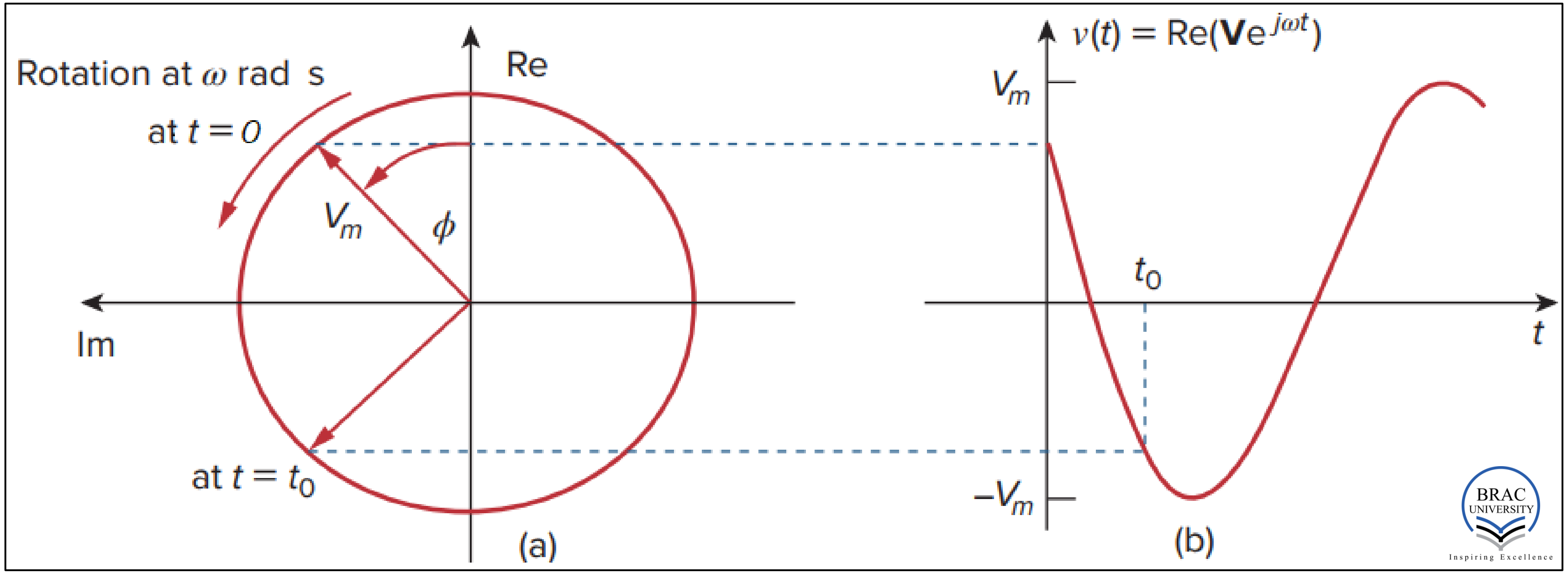
Time-domain vs Phasor-domain

$$\underbrace{v(t) = V_m cos(\omega t + \varphi)}_{\text{Time domain representation (cosine)}} \Leftrightarrow \underbrace{\boldsymbol{V} = V_m e^{j\varphi} = V_m \angle \varphi}_{\text{Phasor domain representation}}$$

- $v(t)$ is the instantaneous or time domain representation, while $\boldsymbol{V}$ is the frequency or phasor domain representation.
- $v(t)$ is time dependent, while $\boldsymbol{V}$ is not.
- $v(t)$ is always real with no complex term, while $\boldsymbol{V}$ is generally complex.
- Phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Phasor: graphical representation

- A sinusoidal voltage of the form $V_m cos(\omega t + \varphi)$ or $V_m sin(\omega t + \varphi)$ can be expressed as $\boldsymbol{V}e^{j\omega t}$, where $\boldsymbol{V} = V_m e^{j\varphi} = V_m \angle\varphi$ is the phasor representation of $v(t)$.
- Consider the plot of the *sinor* $\boldsymbol{V}e^{j\omega t} = V_m e^{j(\omega t+\varphi)}$ on the complex plane in figure (a). As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the CCW direction.
- We may regard $v(t)$ as the projection of the sinor $\boldsymbol{V}e^{j\omega t}$ on the real axis, as shown in (b). The value of the sinor at time $t = 0$ is the phasor $\boldsymbol{V}$ of the sinusoid $v(t)$. The sinor may be regarded as a rotating phasor.
- Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important to keep in mind the frequency ω of the phasor.



Problem 5

- Transform the following sinusoids to phasors. Specify if they are sin or cosine phasors. *[Answers are given in such a way that the phase angle is between* $0°$ & $180°$)

I. $-14\sin(5t - 22°)$

II. $-8\cos(16t + 15°)$

III. $-20\cos(314t - 30°)$

IV. $-4\sin(628t + 55°)$

Ans: $I.\ \mathbf{14\angle 158°}$ (sin)

$II.\ \mathbf{8\angle{-165}°}$ (cos)

$III.\ \mathbf{20\angle 150°}$ (cos)

$IV.\ \mathbf{4\angle{-125}°}$ (sin)

Problem 6

- Find the sinusoids (in sin or cosine form as specified) corresponding to these phasors:

   I. $-25\angle 40°$ to sine

   II. $j(12 - j5)$ to cosine

   III. $-10\angle -30°$ to sine

   IV. $20\angle 45°$ to cosine

   V. $If\ v_1(t) = -10\sin(\omega t - 30°)\,V\ and$
   $v_2(t) = 20\cos(\omega t + 45°)\,V,$
   find $v(t) = v_1(t) + v_2(t)$

<u>Ans</u>: I. $25\sin(\omega t - 140°)$
II. $13\sin(\omega t - 67.38°)$
III. $10\sin(\omega t + 120°)$
IV. $29.77\sin(\omega t + 140°)$
Or, $29.77\cos(\omega t + 50°)$

Phasor relationship for a Resistor

- If current through a resistor R is,

$$i(t) = I_m cos(\omega t + \varphi),$$

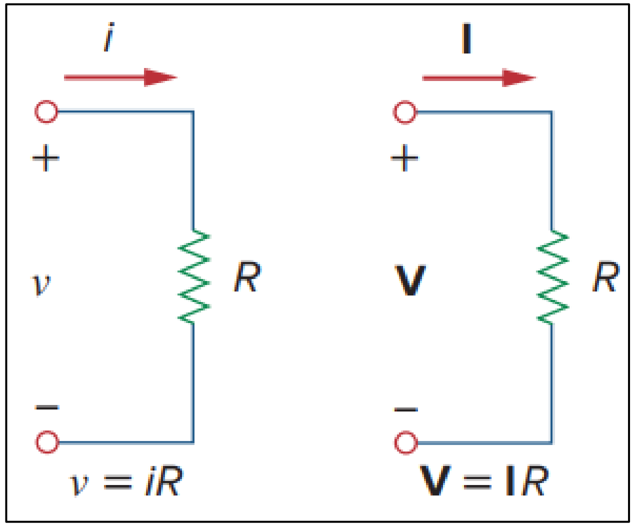
the voltage across it is given by Ohm's law as,

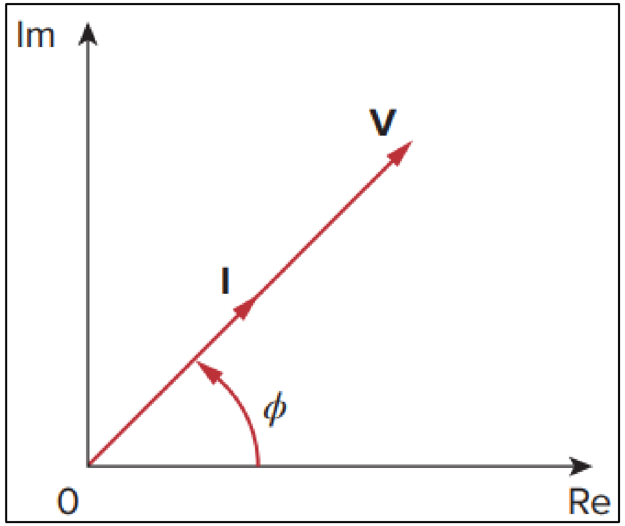
$$v(t) = Ri(t) = RI_m cos(\omega t + \varphi).$$

In phasor form, $\boldsymbol{V} = RI_m\angle\varphi = \boldsymbol{RI}$

- Hence, the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain.

- The angles of the voltage and current of a resistor are identical, hence, current and voltage of a resistor are always in phase.





Phasor relationship for an Inductor

- For an inductor L, assume the current through it is

$$i(t) = I_m cos(\omega t + \varphi)$$

- The voltage across the inductor is,

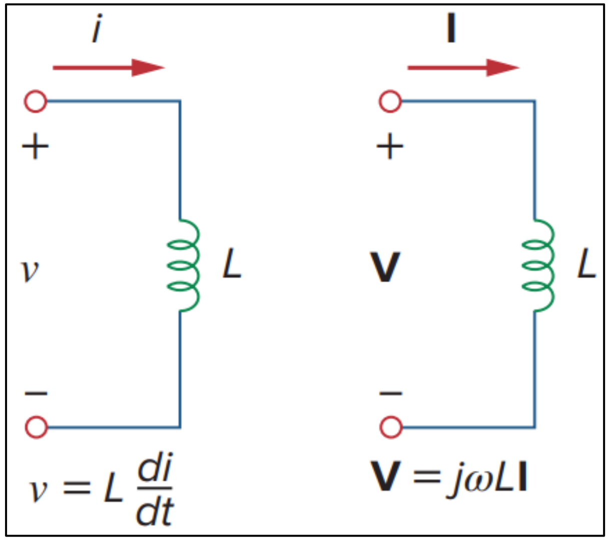
$$v(t) = L\frac{di}{dt} = -\,\omega L I_m sin(\omega t + \varphi) = \omega L I_m cos(\omega t + \varphi + 90°)$$

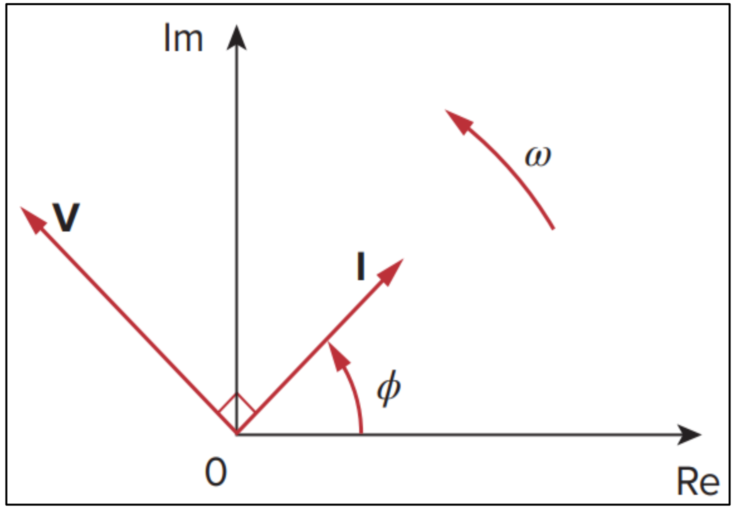
- In phasor form, $\boldsymbol{V} = \omega L I_m e^{j(\varphi+90°)} = \omega L I_m e^{j\varphi} e^{j90}$

$$= \omega L I_m e^{j\varphi}(cos90° + jsin90°) = j\omega L I_m e^{j\varphi} = j\omega L I_m \angle\varphi$$

$$\Rightarrow \boldsymbol{V} = j\omega L\boldsymbol{I}$$

- The voltage has a magnitude of $\omega L I_m$ and a phase of φ + 90°. Voltage and current of an inductor are 90° out of phase. Specifically, the current lags the voltage by 90°.





Phasor relationship for a Capacitor

- For a capacitor C, assume the voltage across it is

$$v(t) = V_m cos(\omega t + \varphi)$$

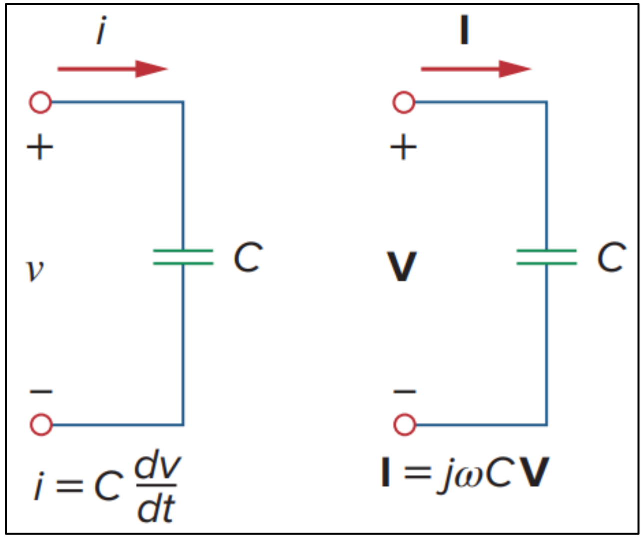
- The current through the capacitor is,

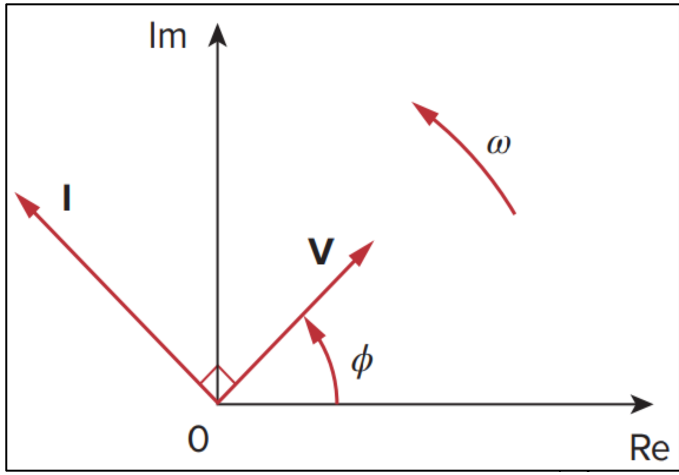
$$i(t) = C\frac{dv}{dt} = -\,\omega C V_m sin(\omega t + \varphi) = \omega L C cos(\omega t + \varphi + 90°)$$

- In phasor form, $\boldsymbol{I} = \omega C V_m e^{j(\varphi+90)} = \omega C V_m e^{j\varphi} e^{j90°}$
  $= \omega C V_m e^{j\varphi}(cos90° + jsin90°) = j\omega C V_m e^{j\varphi} = j\omega C V_m \angle\varphi$

$$\Rightarrow \boldsymbol{I} = j\omega C\boldsymbol{V} \qquad \Rightarrow \boldsymbol{V} = \frac{1}{j\omega C}\boldsymbol{I}$$

- The current has a magnitude of $\omega C V_m$ and a phase of $\phi + 90°$. Voltage and current of a capacitor are 90° out of phase. Specifically, the current leads the voltage by 90°.





Impedance

- In the preceding section, we obtained the voltage-current relations for the three passive elements as,

$$\boldsymbol{V} = \boldsymbol{RI} \qquad \boldsymbol{V} = j\omega L\boldsymbol{I} \qquad \boldsymbol{V} = \frac{1}{j\omega C}\boldsymbol{I}$$

$$\frac{\boldsymbol{V}}{\boldsymbol{I}} = \boldsymbol{R} \qquad \frac{\boldsymbol{V}}{\boldsymbol{I}} = j\omega L \qquad \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{1}{j\omega C}$$

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

- From these three expressions, we obtain Ohm's law in phasor form for any type of element as, $\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} \Rightarrow \boldsymbol{V} = \boldsymbol{ZI}$

where $\boldsymbol{Z}$ is a frequency-dependent quantity known as impedance, measured in $ohms$.

- The *impedance* $\boldsymbol{Z}$ of a circuit is the ratio of the phasor voltage $\boldsymbol{V}$ to the phasor current $\boldsymbol{I}$, measured in ohms (Ω).

Impedances of passive elements

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \frac{1}{j\omega C}$

Impedance: frequency dependency

- *The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current.*

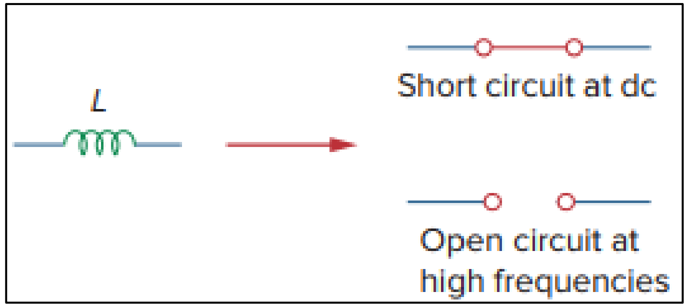
- Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

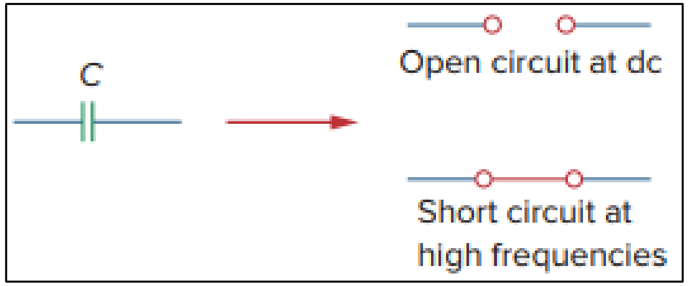
- When $\omega = 0\,(for\ dc\ sources)$ $\boldsymbol{Z_L} = j\omega L = 0$ *and* $\boldsymbol{Z_C} = \dfrac{\mathbf{1}}{\boldsymbol{j\omega C}} \to \infty$, confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit.

- When $\omega \to \infty\,(for\ high\ frequencies)$ $\boldsymbol{Z_L} \to \infty$ *and* $\boldsymbol{Z_C} = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit.

Impedances of passive elements.

Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = j\omega L$
C	$\mathbf{Z} = \dfrac{1}{j\omega\, C}$





Impedance: resistance & reactance

- As a complex quantity, the impedance may be expressed in rectangular form as,

$$\boldsymbol{Z} = R \pm jX$$

where, $R = Re\{\boldsymbol{Z}\}$ (Ω) is the *resistance* and $X = Im\{Z\}$ (Ω) is the *reactance*

- The reactance, X, is just a magnitude, a positive value, but when used as a vector, a j is associated with inductance and a $-j$ is associated with capacitance.

- Impedance $\boldsymbol{Z} = R + jX$ is said to be inductive or lagging since current lags voltage, while impedance $\boldsymbol{Z} = R - jX$ is capacitive or leading because current leads voltage.

- In polar form, $\boldsymbol{Z} = R \pm jX = |\boldsymbol{Z}|\angle \pm \varphi$ where, $|\boldsymbol{Z}| = \sqrt{R^2 + X^2}$; $\theta = \tan^{-1}\frac{\pm X}{R}$

$$R = |\boldsymbol{Z}|cos\theta; \quad X = |\boldsymbol{Z}|sin\theta$$

Admittance

- It is sometimes convenient to work with the reciprocal of impedance, known as admittance.

- The *admittance* $\boldsymbol{Y}$ is the reciprocal of impedance, measured in siemens (S).

$$\boldsymbol{Y} = \frac{\boldsymbol{I}}{\boldsymbol{V}} = \frac{1}{\boldsymbol{Z}}$$

- As a complex quantity, we may write $\boldsymbol{Y}$ as,

$$\boldsymbol{Y} = G + jB$$

where, $G = Re\{\boldsymbol{Y}\}$ is called the conductance and $B = Im\{\boldsymbol{Y}\}$ is called the susceptance.

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

- $G + jB = \frac{1}{R+jX} = \frac{(R-jX)}{(R+jX)(R-jX)} = \frac{(R-jX)}{R^2+X^2};$ $\Rightarrow$ $\mathrm{G} = \frac{R}{R^2+X^2};$ $\mathrm{B} = -\frac{X}{R^2+X^2}$

KVL and KCL in phasor domain

- ***KVL***

- Let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then, $v_1 + v_2 + \ldots + v_n = 0$.
- In the sinusoidal steady state, each voltage may be written in cosine form, so that

$\Rightarrow$ $V_{m_1}\cos(\omega t + \theta_1) + V_{m_2}\cos(\omega t + \theta_2) + \cdots + V_{m_n}\cos(\omega t + \theta_n) = 0$

$\Rightarrow$ $Re\{V_{m_1}e^{j\theta_1}e^{j\omega t}\} + Re\{V_{m_2}e^{j\theta_2}e^{j\omega t}\} + \cdots + Re\{V_{m_n}e^{j\theta_n}e^{j\omega t}\} = 0$

$\Rightarrow$ Let $\boldsymbol{V_k} = V_{mk}e^{\theta_k}$, then $Re\{(\boldsymbol{V_1} + \boldsymbol{V_2} + \cdots + \boldsymbol{V_n})e^{j\omega t}\} = 0$

$\Rightarrow$ Because $e^{j\omega t} \neq 0$, $\boldsymbol{V_1} + \boldsymbol{V_2} + \cdots + \boldsymbol{V_n} = \boldsymbol{0}$, indicating that KVL holds for phasors

- ***KCL***

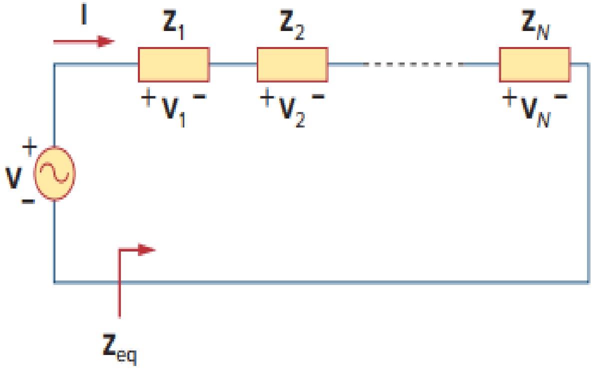
$\Rightarrow$ By following a similar procedure, we can show that KVL holds for phasors.

$\Rightarrow$ $i_1 + i_2 + \cdots + i_n = 0 \quad \leftrightarrow \quad \boldsymbol{I_1} + \boldsymbol{I_2} + \cdots + \boldsymbol{I_n} = \boldsymbol{0}$

Series and Parallel Impedances

- ***Series:*** Consider the N series-connected impedances. The same current $\boldsymbol{I}$ flows through the impedances. Applying KVL around the loop gives,

$$\boldsymbol{V_1 + V_2 + \cdots + V_N = I(Z_1 + Z_2 + \cdots + Z_N)}$$

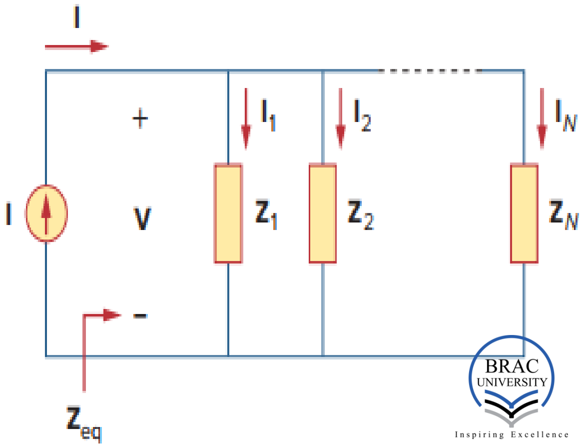


$\Rightarrow$ The equivalent impedance at the input terminals is,

$$\boldsymbol{Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + \cdots + Z_N}$$

- ***Parallel:*** In the same manner , we can obtain the equivalent impedance or admittance of the N parallel-connected impedances. The voltage across each impedance is the same. Applying KCL at the top node,

$$\boldsymbol{I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right)}$$



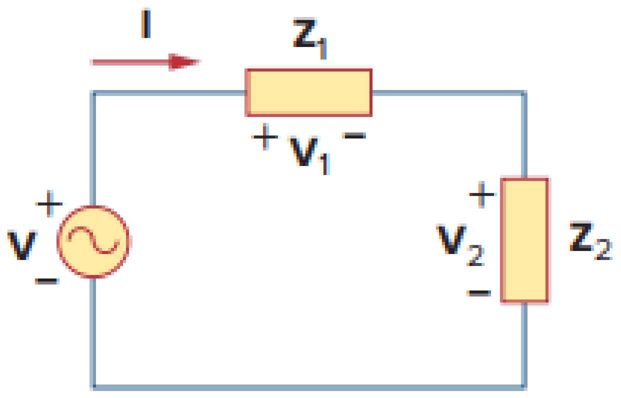
$\Rightarrow$ The equivalent impedance at the input terminals is,

$$\boldsymbol{\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}}$$

Voltage Division Rule

- For ac circuits in phasor domain, the voltage division rule applies exactly as they do in dc circuits.

$$I = \frac{V}{Z_1 + Z_2}$$



- Because $V_1 = IZ_1$ *and* $V_2 = IZ_2$

$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \qquad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

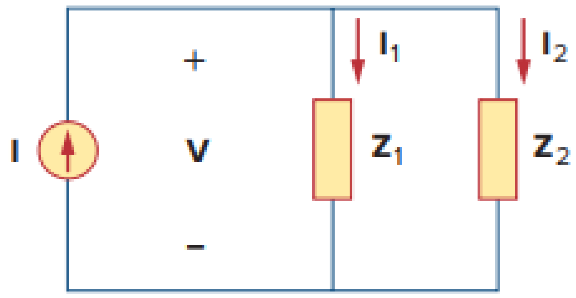
- In general, for any number of impedances connected in series to a supply voltage, the voltage across any particular impedance Z_x is,

$$\boxed{V_x = \frac{Z_x}{Z_1 + Z_2 + Z_3 + \dots\dots + Z_N} \times V}$$

Current Division Rule

- For ac circuits in phasor domain, the current division rule applies exactly as they do in dc circuits.

$$\boldsymbol{Z_{eq}} = \frac{\boldsymbol{Z_1 Z_2}}{\boldsymbol{Z_1} + \boldsymbol{Z_2}}$$



- Because $\boldsymbol{V} = \boldsymbol{IZ_{eq}} = \boldsymbol{Z_1 I_1} = \boldsymbol{Z_2 I_2}$

$$\boldsymbol{I_1} = \frac{\boldsymbol{Z_2}}{\boldsymbol{Z_1} + \boldsymbol{Z_2}} \times \boldsymbol{I} = \frac{(\boldsymbol{Z_1})^{-1}}{(\boldsymbol{Z_1})^{-1} + (\boldsymbol{Z_2})^{-1}} \times \boldsymbol{I}$$

$$\boldsymbol{I_2} = \frac{\boldsymbol{Z_1}}{\boldsymbol{Z_1} + \boldsymbol{Z_2}} \times \boldsymbol{I} = \frac{(\boldsymbol{Z_2})^{-1}}{(\boldsymbol{Z_1})^{-1} + (\boldsymbol{Z_2})^{-1}} \times \boldsymbol{I}$$

- In general, for any number of impedances connected in parallel to a supply current, the current through any particular impedance $\boldsymbol{Z_x}$ is,

$$\boxed{\boldsymbol{I_x} = \frac{(\boldsymbol{Z_x})^{-1}}{(\boldsymbol{Z_1})^{-1} + (\boldsymbol{Z_2})^{-1} + (\boldsymbol{Z_3})^{-1} + \ \dots\dots \ + (\boldsymbol{Z_x})^{-1}} \times \boldsymbol{I}}$$

Nodal Analysis: frequency domain equivalent circuit

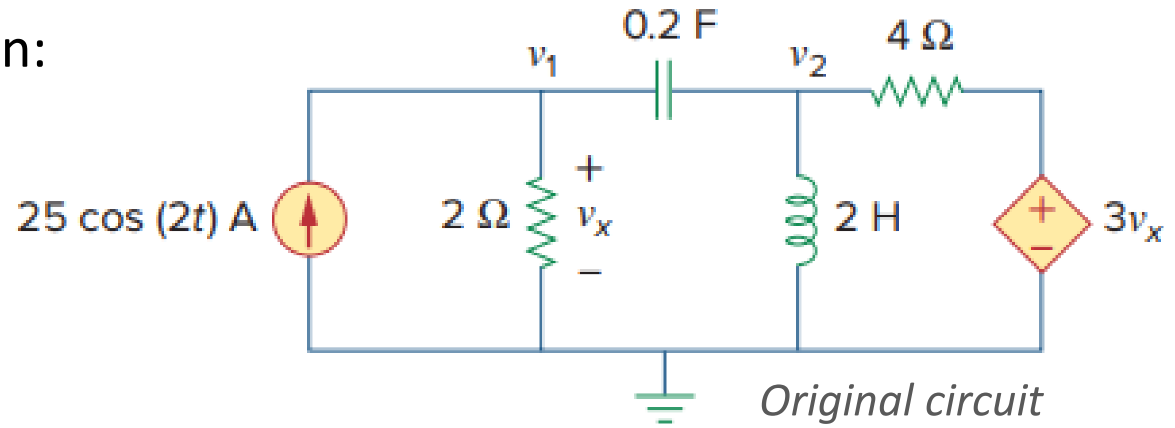
The basis of nodal analysis is KCL. Since KCL is valid for phasors, we can analyse ac circuits by nodal analysis. The following examples illustrate this.

We first convert the circuit to the frequency domain:

$$25\cos(2t) \Rightarrow 25\angle 0^\circ \qquad \omega = 2\, rad/s$$

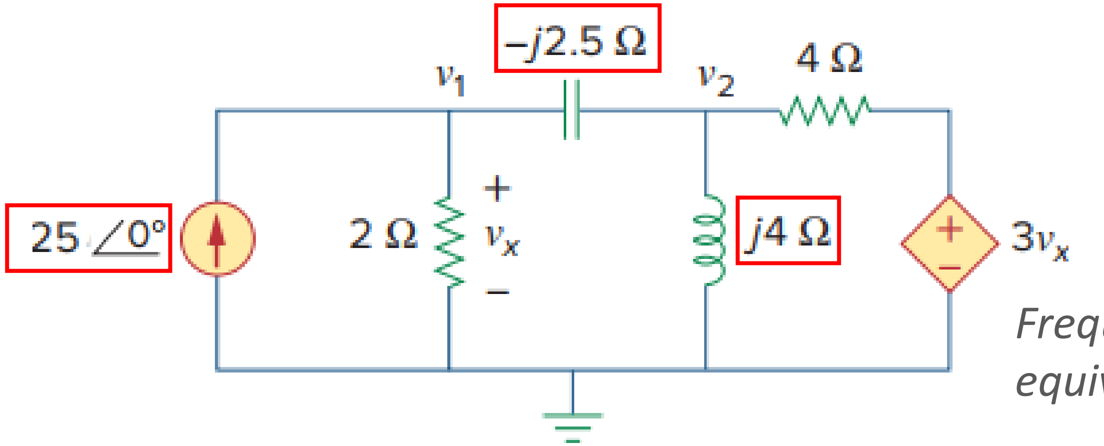
$$2\text{ H} \Rightarrow j\omega L = j \times 2 \times 2 = j4 \quad (\Omega)$$

$$0.2\text{ F} \Rightarrow \frac{1}{j\omega C} = \frac{-j}{2 \times 0.2} = -j2.5 \quad (\Omega)$$



Original circuit

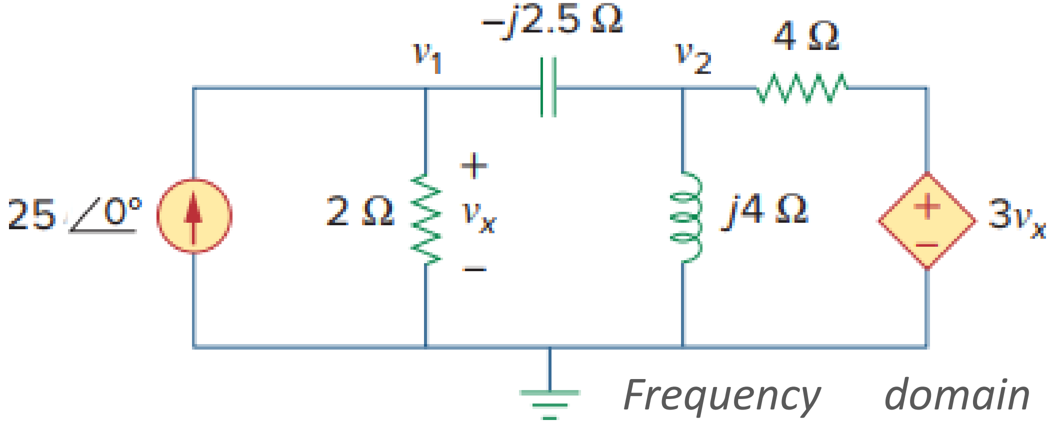
The frequency domain equivalent circuit is as shown below



Frequency domain equivalent circuit

Example 1

Nodal Analysis: forming node equations



Frequency domain equivalent circuit

Applying KCL at node 1,

$$\frac{\boldsymbol{V}_1}{2} + \frac{\boldsymbol{V}_1 - \boldsymbol{V}_2}{-j2.5} = 25\angle 0°$$

$$\Rightarrow (0.5 + j0.4)\boldsymbol{V}_1 - (j0.4)\boldsymbol{V}_2 = 25 \;\; ----(i)$$

Applying KCL at node 2,

$$\frac{\boldsymbol{V}_2}{j4} + \frac{\boldsymbol{V}_2 - \boldsymbol{V}_1}{-j2.5} + \frac{\boldsymbol{V}_2 - 3\boldsymbol{V}_x}{4} = 0$$

$\Rightarrow -(0.75 + j0.4)\boldsymbol{V}_1 + (0.25 + j0.15)\boldsymbol{V}_2 = 0 \;\; ----(ii)$ [where $\boldsymbol{V}_x = \boldsymbol{V}_1$]

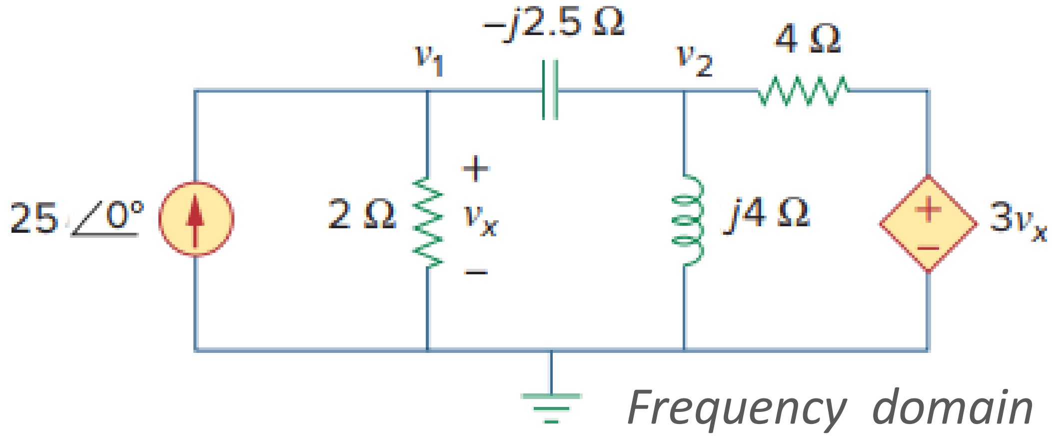
$\Rightarrow$ The two simultaneous equation can be solve using any one of the methods (by addition/subtraction or by substitution/elimination or by cross multiplication.

$\Rightarrow$ Another method is the Cramer's rule, which is shown in the next slide.

Solving complex equations using Cramer's rule

Equations (i) and (ii) can be put in matrix form as,

$$\begin{bmatrix} 0.5 + j0.4 & -j0.4 \\ -0.75 - j0.4 & 0.25 + j0.15 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$$



Frequency domain equivalent circuit

We obtain the determinants as,

$$\Delta = \begin{vmatrix} 0.5 + j0.4 & -j0.4 \\ -0.75 - j0.4 & 0.25 + j0.15 \end{vmatrix} = 0.225 - j0.125$$

$$\Delta_1 = \begin{vmatrix} 25 & -j0.4 \\ 0 & 0.25 + j0.15 \end{vmatrix} = 6.25 + j3.75 \qquad \Delta_2 = \begin{vmatrix} 0.5 + j0.4 & 25 \\ 0.75 + j0.4 & 0 \end{vmatrix} = 18.75 + j10$$

$$\boldsymbol{V}_1 = \frac{\Delta_1}{\Delta} = \frac{6.25 + j3.75}{0.225 - j0.125} = 14.15 + j24.53 = 28.31\angle 60.01°$$

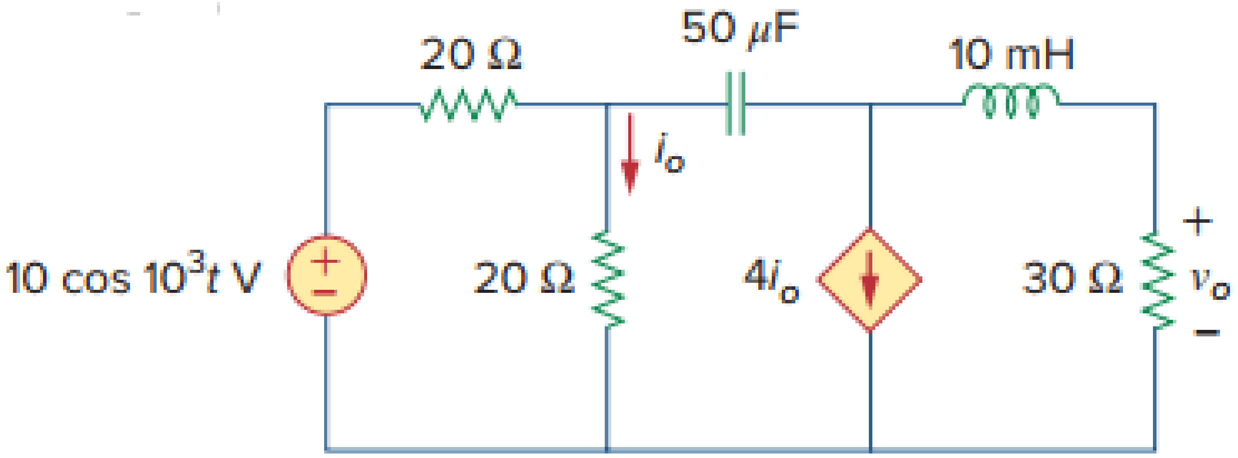
$$\boldsymbol{V}_2 = \frac{\Delta_2}{\Delta} = \frac{18.75 + j10}{0.225 - j0.125} = 44.81 + j69.34 = 82.56\angle 57.12°$$

In time domain,

$$v_1(t) = 28.31\cos(2t + 60.01°)$$
$$v_2(t) = 82.56\cos(2t + 57.12°)$$

Problem 7

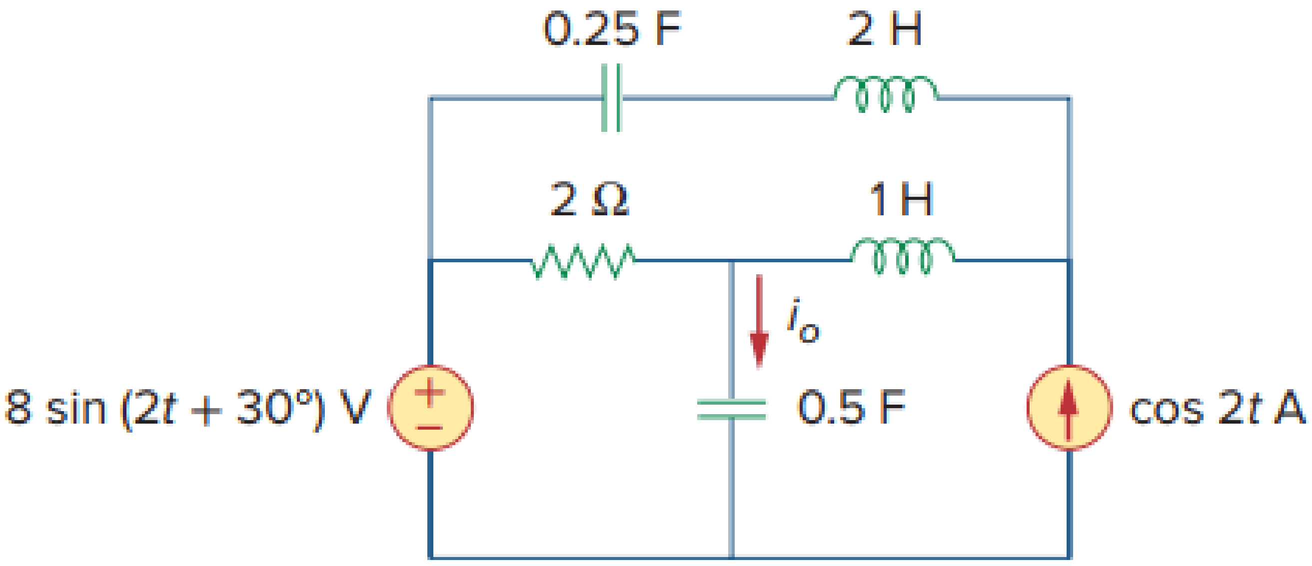
- Use Nodal Analysis to find $v_o(t)$ in the following circuit.



Ans: $\mathbf{6.154\ cos(10^3 t + 70.26°)\ V}$

Problem 8

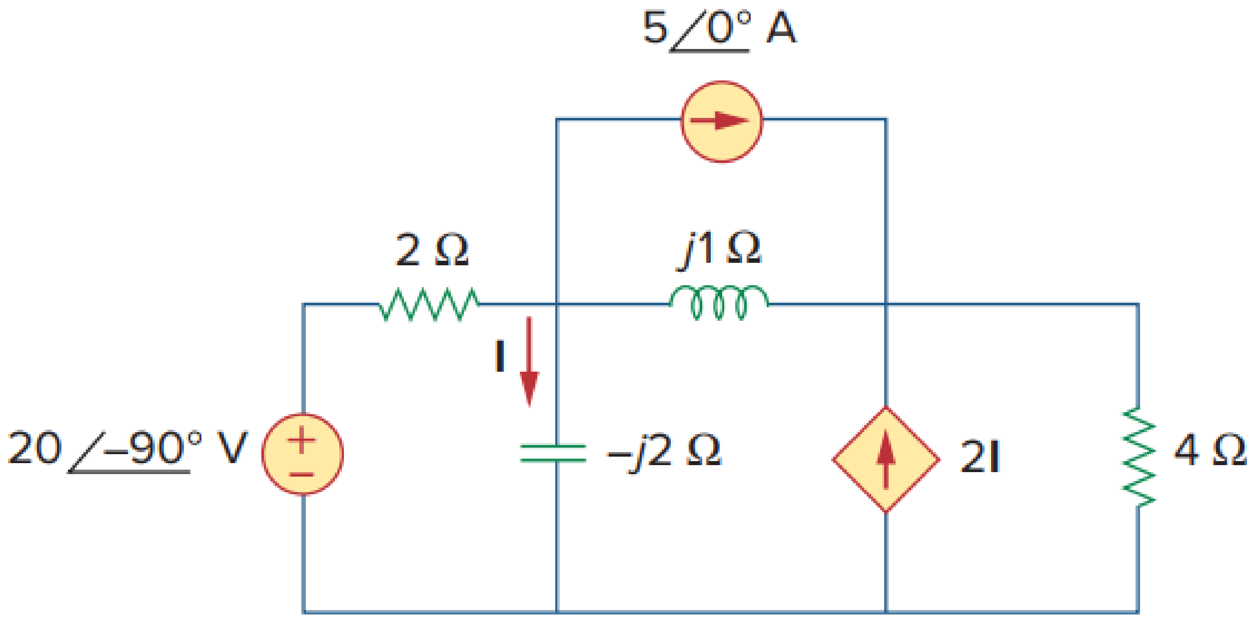
- Use Nodal Analysis to find i_o in the circuit.



Ans: $\mathbf{199.5\angle 86.89^\circ}\ \boldsymbol{mA}$

Problem 9

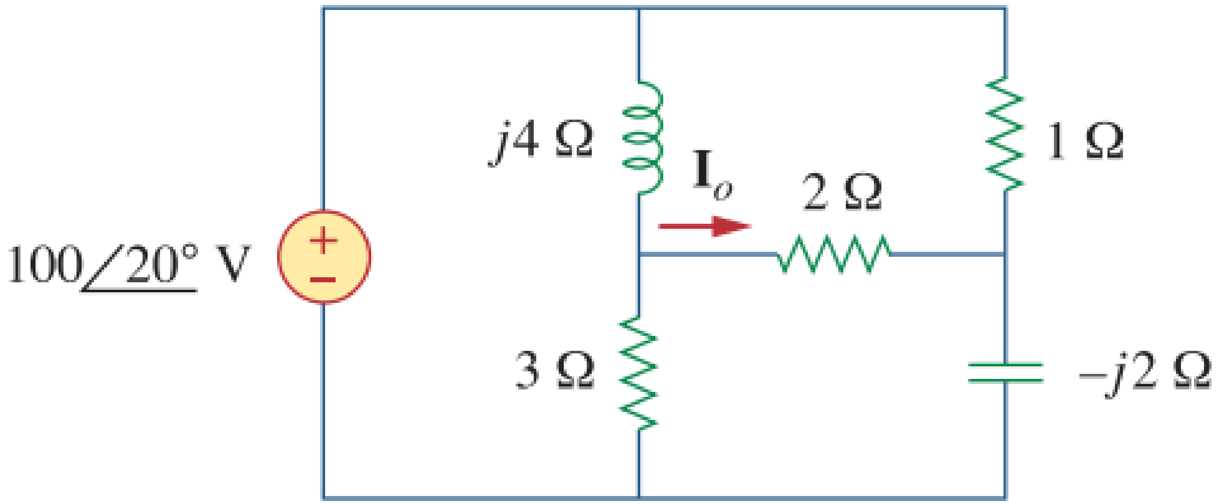
- Solve for the current $\boldsymbol{I}$ using Nodal Analysis.



Ans: $\mathbf{7.906\angle 43.49^\circ}\ \boldsymbol{A}$

Problem 10

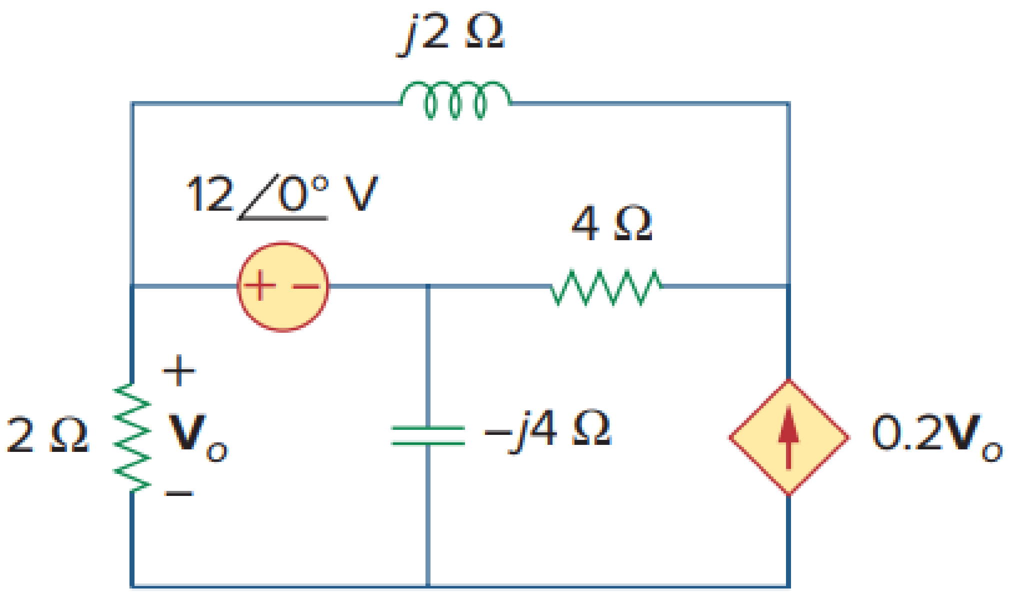
- Solve for the current $\boldsymbol{I}$ using Nodal Analysis.



Ans: $\mathbf{9.25\angle{-162.12}^{\circ}}\boldsymbol{A}$

Problem 11

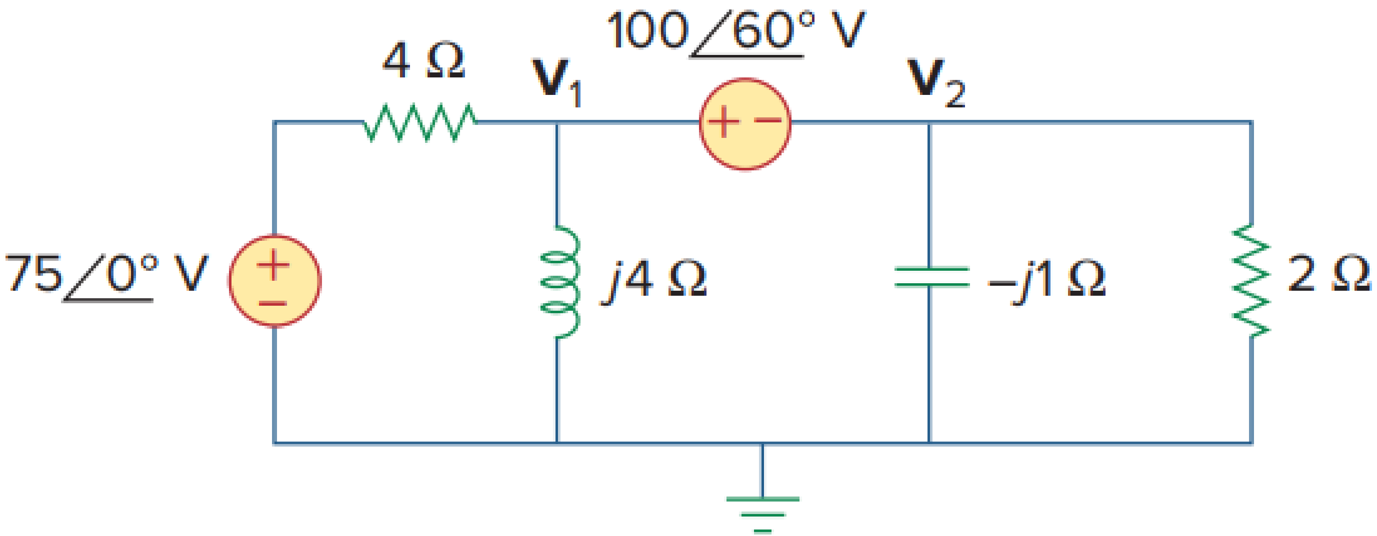
- Use Nodal Analysis to find $v_o(t)$ in the circuit.



Ans: $\mathbf{7.682}\cos(\mathbf{50.19}°)\ \boldsymbol{V}$

Problem 12

- Evaluate $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ using Nodal Analysis.



Ans: $\boldsymbol{V_1} = \mathbf{96.8\angle{-69.66}^\circ}\ \boldsymbol{V};\ \boldsymbol{V_2} = \mathbf{16.88\angle 165.72^\circ}\ \boldsymbol{V}$

Mesh Analysis: frequency domain equivalent circuit

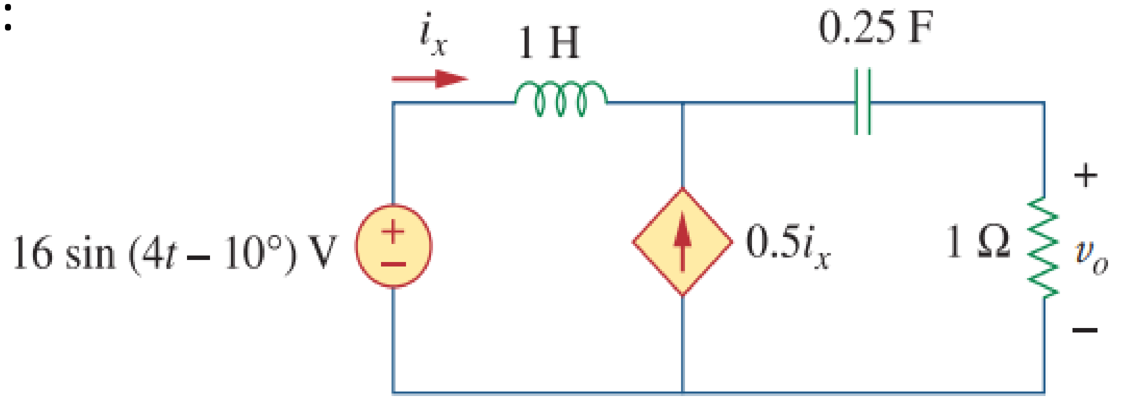
The basis of mesh analysis is KVL. Since KVL is valid for phasors, we can analyse ac circuits by mesh analysis. The following examples illustrate this.

We first convert the circuit to the frequency domain:

$16\sin(4t - 10°) \Rightarrow 16\angle - 10°$ $\quad\omega$ = 4 rad/s

$$1\text{ H} \Rightarrow j\omega L = j \times 4 \times 1 = j4 \quad (\Omega)$$

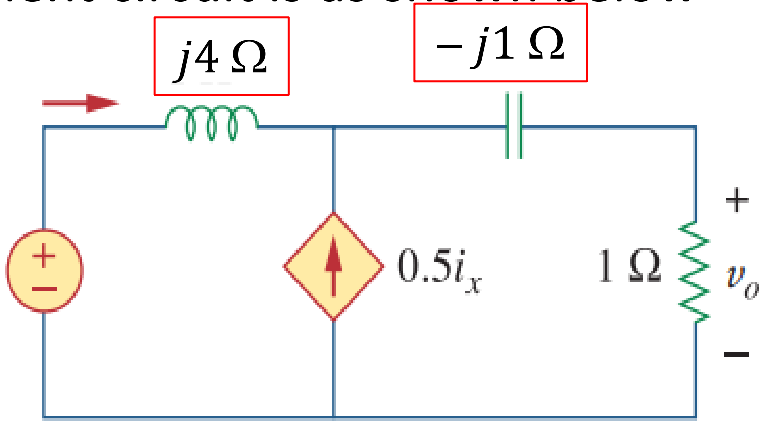
$$0.25\text{ F} \Rightarrow \frac{1}{j\omega C} = \frac{-j}{4 \times 0.25} = -j1 \quad (\Omega)$$



Original circuit

The frequency domain equivalent circuit is as shown below

Keep in mind that this is a **sin** phasor



Frequency domain equivalent circuit

Example 2

Mesh Analysis: forming mesh equations

Applying KVL to the Supermesh,

$$-16\angle-10° + j4\boldsymbol{I}_1 + (-j1)\boldsymbol{I}_2 + 1\boldsymbol{I}_2 = 0$$

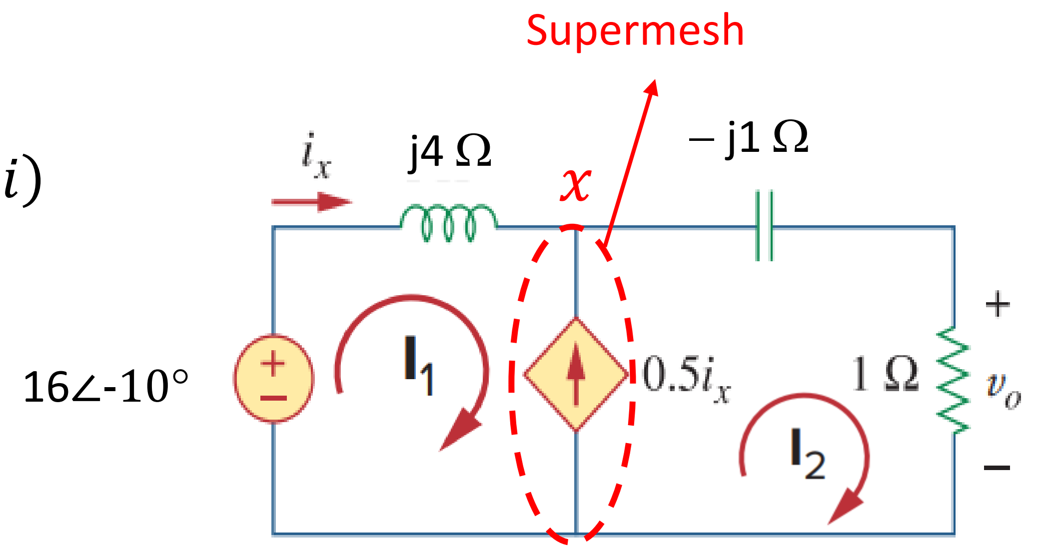
$$\Rightarrow \quad j4\boldsymbol{I}_1 + (1-j1)\boldsymbol{I}_2 = 16\angle-10° -----(i)$$

KCL at node x,

$$\boldsymbol{I}_2 - \boldsymbol{I}_1 = 0.5\boldsymbol{i}_x$$

$$\Rightarrow \quad \boldsymbol{I}_2 - \boldsymbol{I}_1 = 0.5\boldsymbol{I}_1$$

$$\Rightarrow \quad 1.5\boldsymbol{I}_1 - \boldsymbol{I}_2 = 0 ------(ii)$$



Frequency domain equivalent circuit

$\Rightarrow$ The two simultaneous equation can be solve using any one of the methods (by addition/subtraction or by substitution/elimination or by cross multiplication or by Cramer's rule.

Mesh Analysis: solving complex equations

$$j4\boldsymbol{I}_1 + (1 - j1)\boldsymbol{I}_2 = 16\angle - 10° - - - - - -(i)$$

$$1.5\boldsymbol{I}_1 - \boldsymbol{I}_2 = 0 - - - - - - -(ii)$$

Substituting $\boldsymbol{I}_2 = 1.5\boldsymbol{I}_1$ from (ii) into (i),

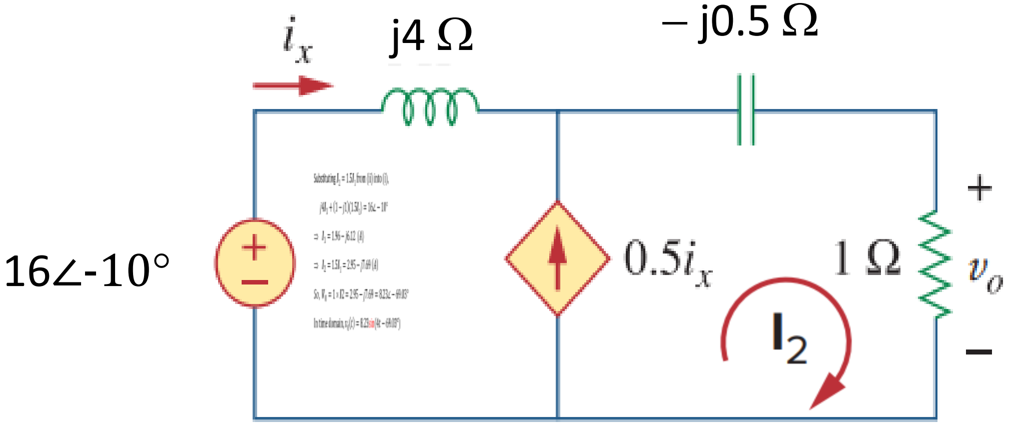
$$j4\boldsymbol{I}_1 + (1 - j1)(1.5\boldsymbol{I}_1) = 16\angle - 10°$$

$$\Rightarrow \quad \boldsymbol{I}_1 = 1.96 - j6.12 \ \ (A)$$

$$\Rightarrow \quad \boldsymbol{I}_2 = 1.5\boldsymbol{I}_1 = 2.95 - j7.69\ (A)$$

$$So,\ \boldsymbol{V}_0 = 1 \times \boldsymbol{I}2 = 2.95 - j7.69 = 8.23\angle - 69.03°$$

In time domain, $v_0(t) = 8.23\sin(4t - 69.03°)$

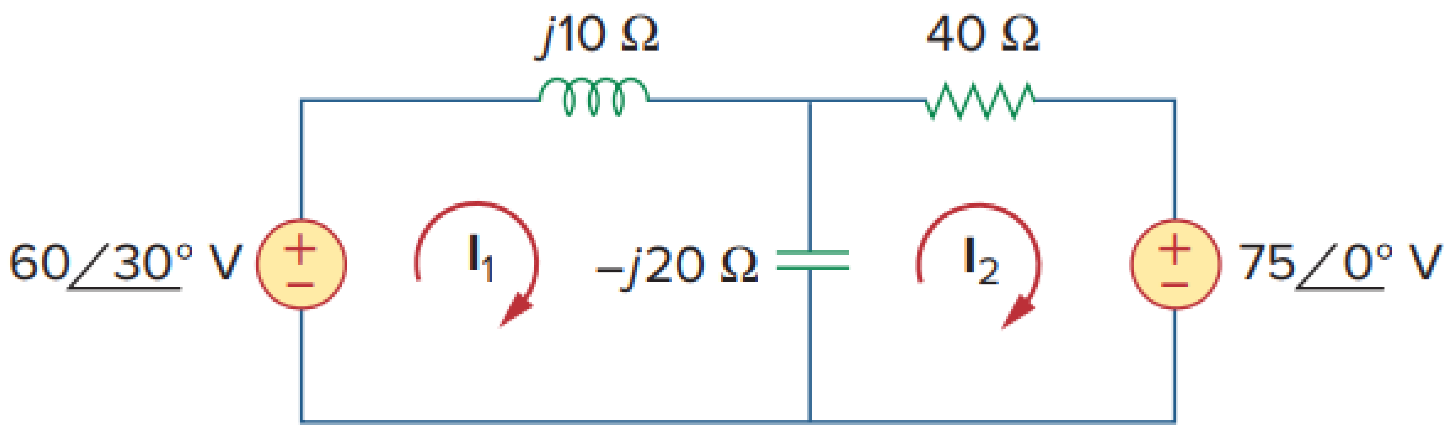


Frequency domain equivalent circuit

Problem 13

- Using Mesh Analysis, find $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$

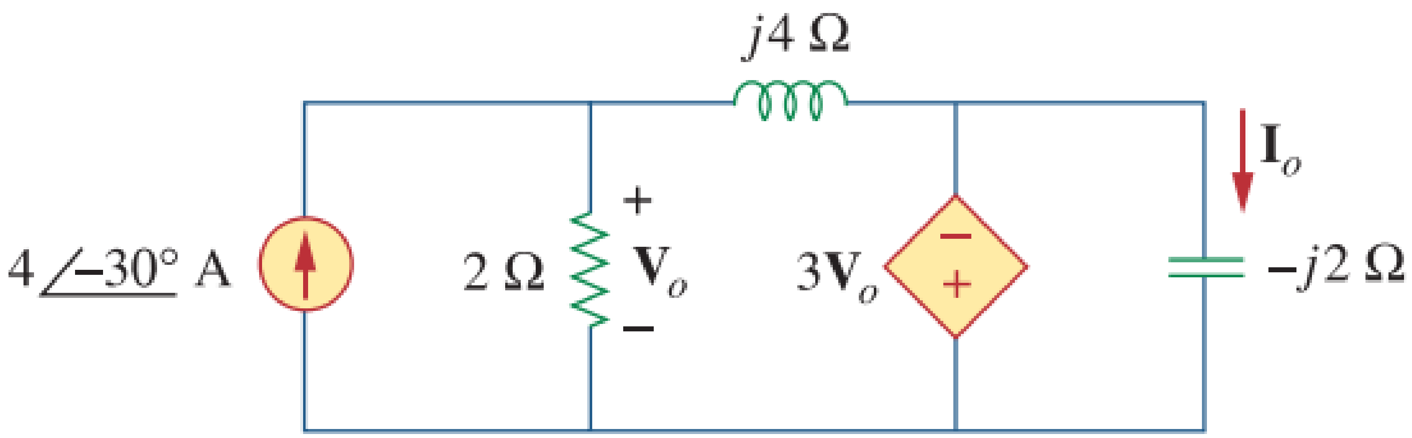
.



Ans: $\boldsymbol{I_1} = \mathbf{4.698\angle 95.24^\circ}\ \boldsymbol{A};\ \boldsymbol{I_2} = \mathbf{0.9928\angle 37.71^\circ}\ \boldsymbol{A}$

Problem 14

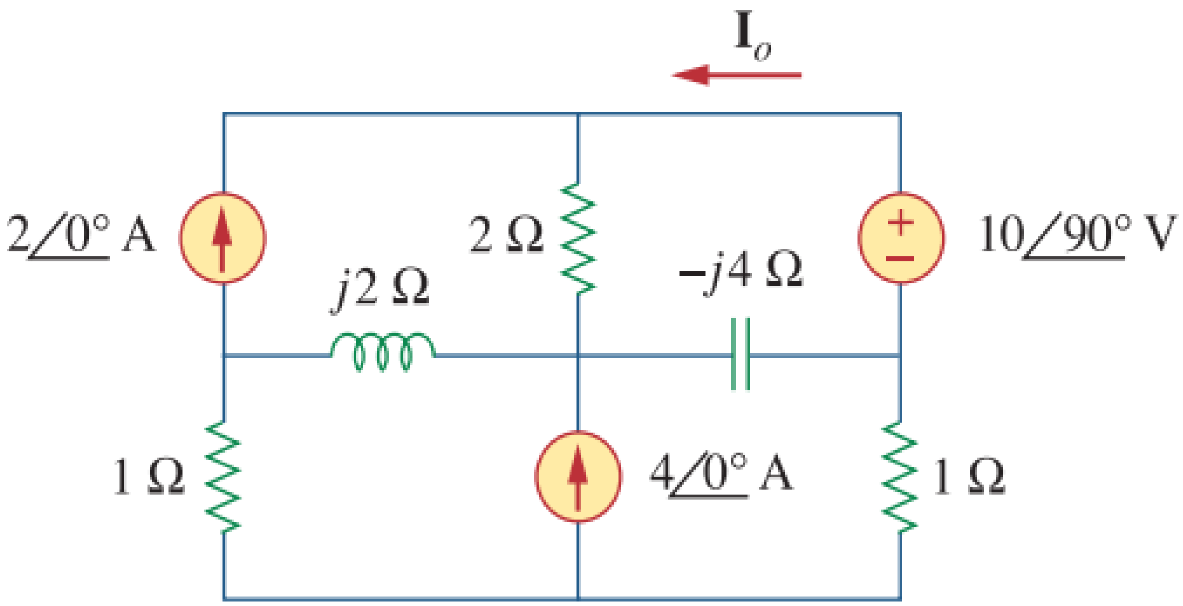
- Determine $\boldsymbol{V}_0$ and $\boldsymbol{I}_0$ using Mesh Analysis. What is the phasor current flowing through the dependent voltage source?



Ans: $\boldsymbol{I_0} = \mathbf{8.485\angle 15^\circ}\ \boldsymbol{A}$; $\boldsymbol{V_0} = \mathbf{5.657\angle{-75}^\circ}\ \boldsymbol{V}$

Problem 15

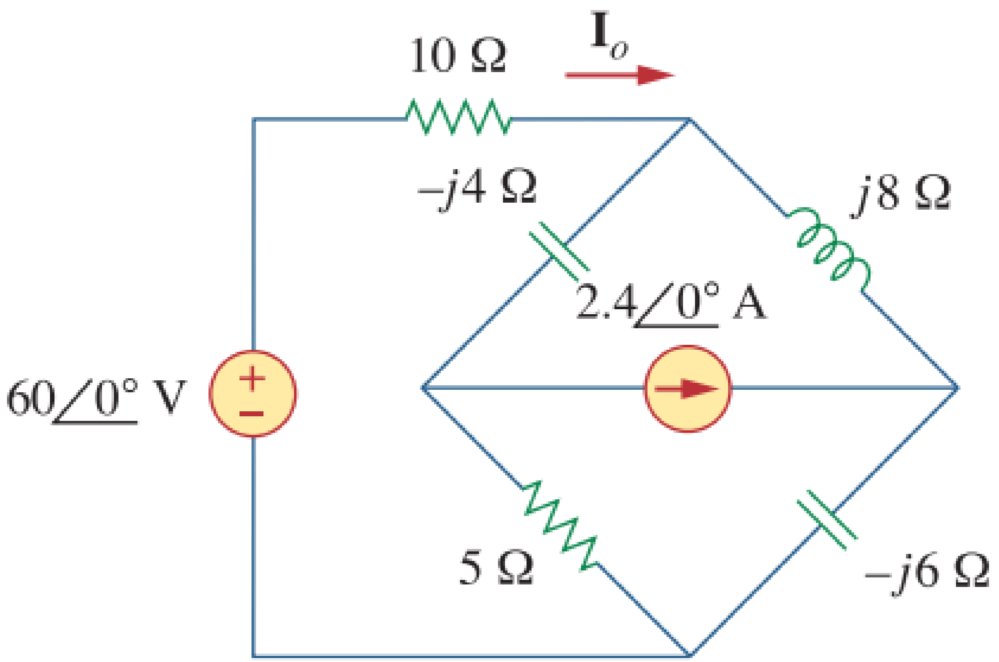
- Solve for the current $\boldsymbol{I}_0$ using Mesh Analysis.



Ans: $\boldsymbol{I}_0 = \mathbf{3.35\angle 174.3^\circ}\ \boldsymbol{A}$

Problem 16

- Solve for the current I_0 using Mesh Analysis.



Ans: $I_0 = 6.089\angle 5.94°\ A$

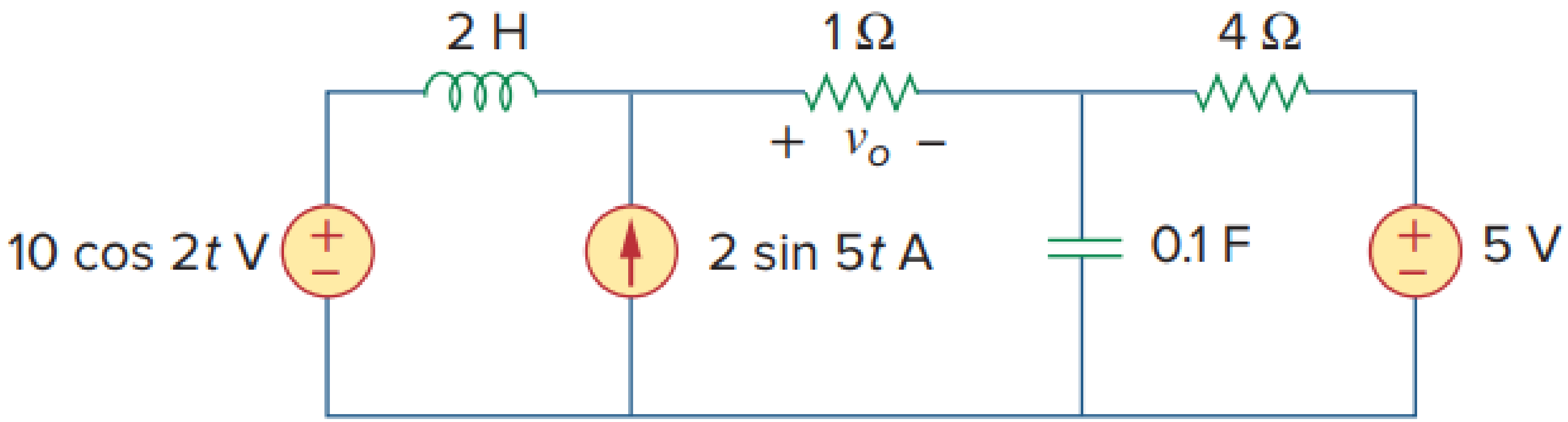
Superposition Principle (AC)

- Since ac circuits are linear, the superposition theorem applies to ac circuits the same way it applies to dc circuits.
- *The theorem becomes important if the circuit has sources operating at different frequencies*.
- In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each source of different frequency.
- The total response must be obtained by adding the individual responses.
- In which domain should we add the responses from individual independent sources?

$\Rightarrow$ *Because the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency ω. It would therefore not make sense to add responses at different frequencies in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.*

Superposition Principle: an example

- Find $\boldsymbol{v_0}$ using the Superposition Theorem.



Example 3

⇒ Because the circuit operates at three different frequencies (ω = 0 for the dc voltage source), one way to obtain a solution is to use superposition.

⇒ So let, $v_0 = v' + v'' + v'''$; where v' is due to the $10\cos 2t\ V$ voltage source, and v'' is due to the $2\sin 5t\ A$ current source, and v''' is due to the $5\ V$ dc voltage source

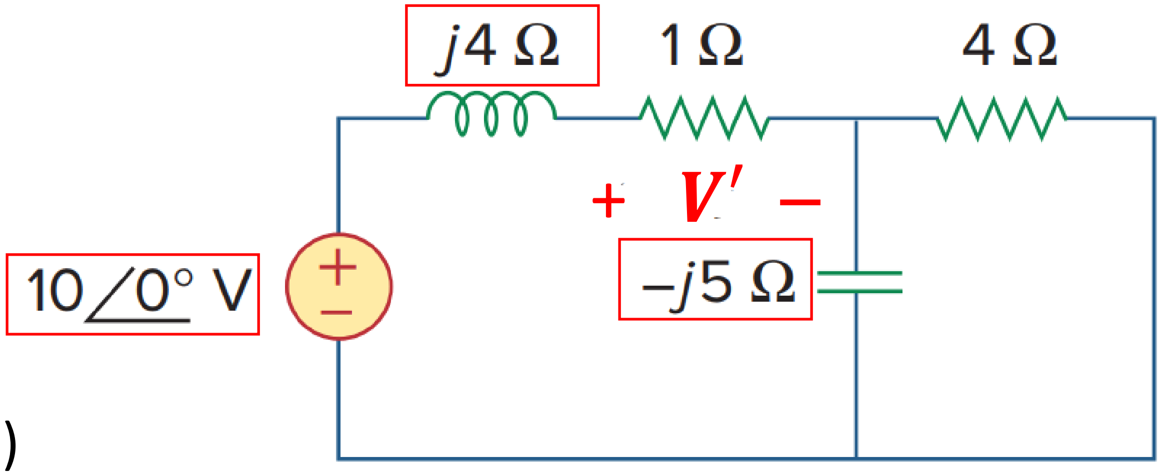
Only the 10cos(2t) V source is active

Both the $5\,V$ source and the $2\sin 5t$ current source are set to zero. Transforming the circuit to the frequency domain:

$$10\cos(2t) \Rightarrow\ 10\angle 0^\circ \qquad \omega\ =\ 2\,rad/s$$

$$2\text{ H} \Rightarrow j\omega L = j \times 2 \times 2 = j4 \quad (\Omega)$$

$$0.1\text{ F} \Rightarrow \frac{1}{j\omega C} = \frac{-j}{2 \times 0.1} = -j5 \quad (\Omega)$$



$$\boldsymbol{Z_1} = (-j5)||(4) = \frac{(-j5)\times 4}{4-j5} = 2.439 - j1.951\ (\Omega)$$

By voltage division, $\boldsymbol{V}' = \dfrac{1}{j4+1+(-j5)} \times 10\angle 0^\circ$

$$= 2.498\angle - 30.79^\circ$$

In time domain, $v''(t) = 2.498\cos(2t - 30.79^\circ)\ (V)$

Only the 2sin(5t) A source is active

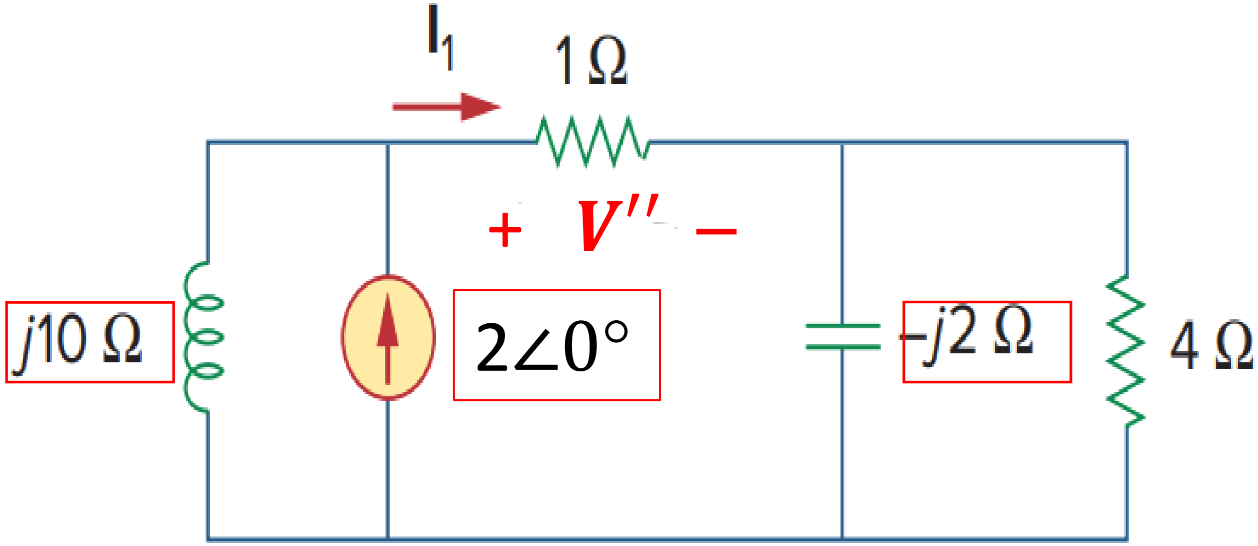
Both the $5\ V$ source and the $10\cos 2t$ current source are set to zero. Transforming the circuit to the frequency domain:

$$2\sin(5t) \Rightarrow 2\angle 0° \qquad \omega = 2\, rad/s$$

$$2\text{ H} \Rightarrow j\omega L = j \times 5 \times 2 = j10 \quad (\Omega)$$

$$0.1\text{ F} \Rightarrow \frac{1}{j\omega C} = \frac{-j}{5 \times 0.1} = -j2 \quad (\Omega)$$

$$\boldsymbol{Z_1} = (-j2)||(4) = \frac{(-j2)\times 4}{4-j2} = 0.8 - j1.6$$



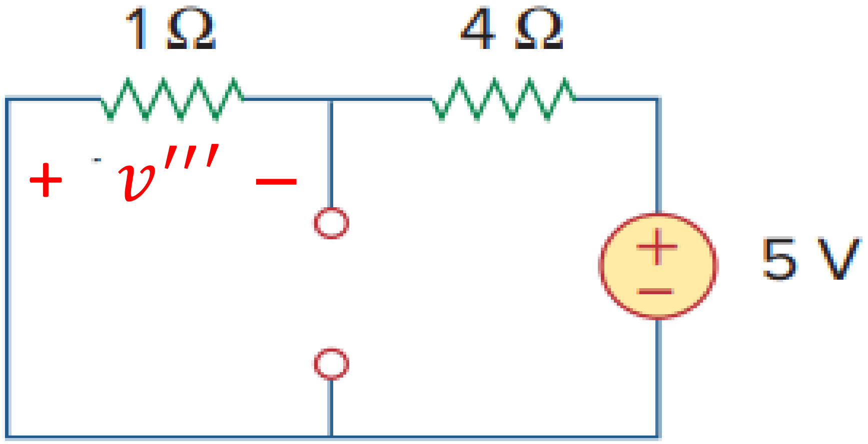
By current division, $\mathbf{I}_1 = \dfrac{j10}{j10+1+\boldsymbol{Z}_1} \times 2\angle 0° = 2.276 + j0.488\ (A)$

$$\mathbf{V}''' = \boldsymbol{I}_1 \times 1 = 2.328\angle 12.10°$$

In time domain, $v'''(t) = 2.328\sin(5t + 12.10°)\ (V)$

Only the 5 V dc source is active

As in dc, the capacitor and inductor have been replaced with open and short circuits, respectively.



Because $\omega = 0\ (dc)$,

$$j\omega L = 0, \qquad \frac{1}{j\omega C} \rightarrow \infty$$

v''' can be found by voltage division as,

$$v''' = \frac{1}{1+4} \times (-5)$$

$$\Rightarrow \ v''' = -1\ V$$

According to the superposition principle,

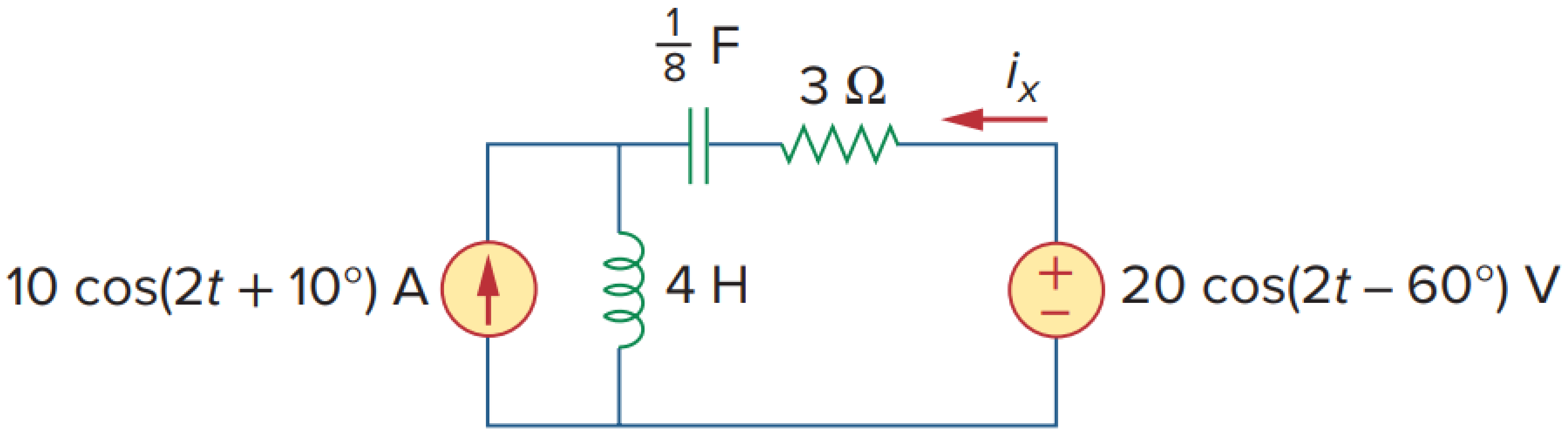
$$v_0(t) = v'(t) + v''(t) + v'''(t)$$

$$= 2.498\cos(2t - 30.79°) + 2.328\sin(5t + 12.10°) + (-1)$$

$$= \boxed{-1 + 2.498\cos(2t - 30.79°) + 2.328\sin(5t + 12.10°)\ V}$$

Problem 17

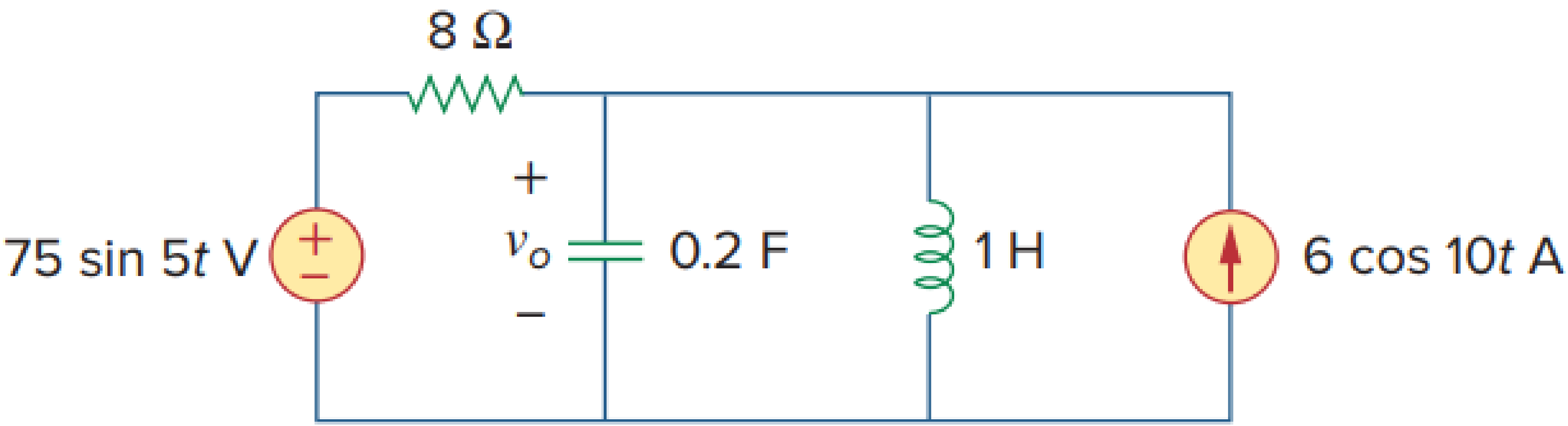
- Using the Superposition Principle, find $i_x(t)$ in the following circuit.



Ans: $\mathbf{19.804}\cos(\mathbf{2}\boldsymbol{t} - \mathbf{129.17}°)\ \boldsymbol{A}$

Problem 18

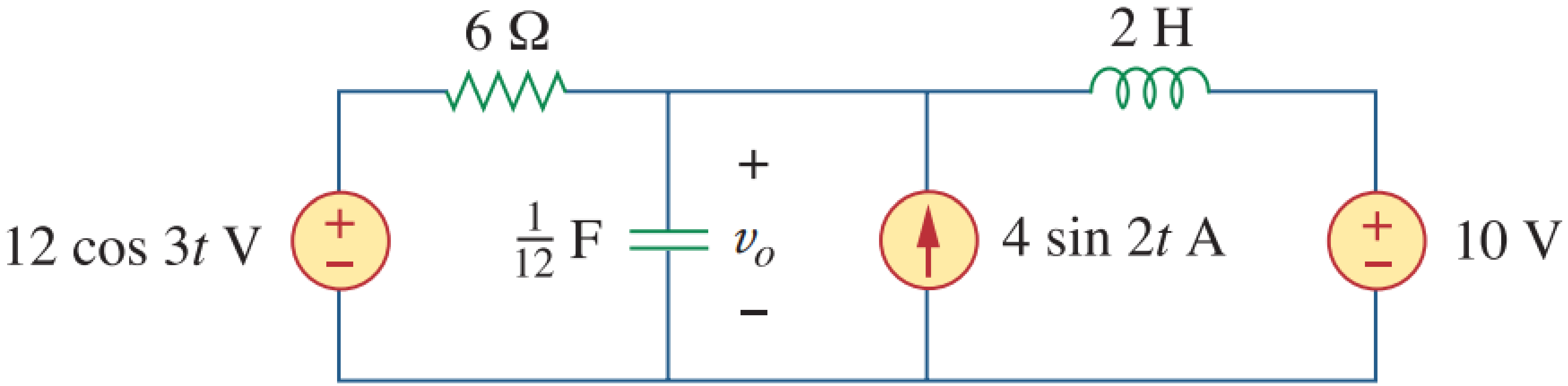
- Calculate $v_0(t)$ in the following circuit using the Superposition Principle.



Ans: $\mathbf{11.577}\,\sin(\mathbf{5}\boldsymbol{t}-\mathbf{81.12°})\;+\;\mathbf{3.154}\,\sin(\mathbf{10}\boldsymbol{t}-\mathbf{86.24°})\;\boldsymbol{V}$

Problem 19

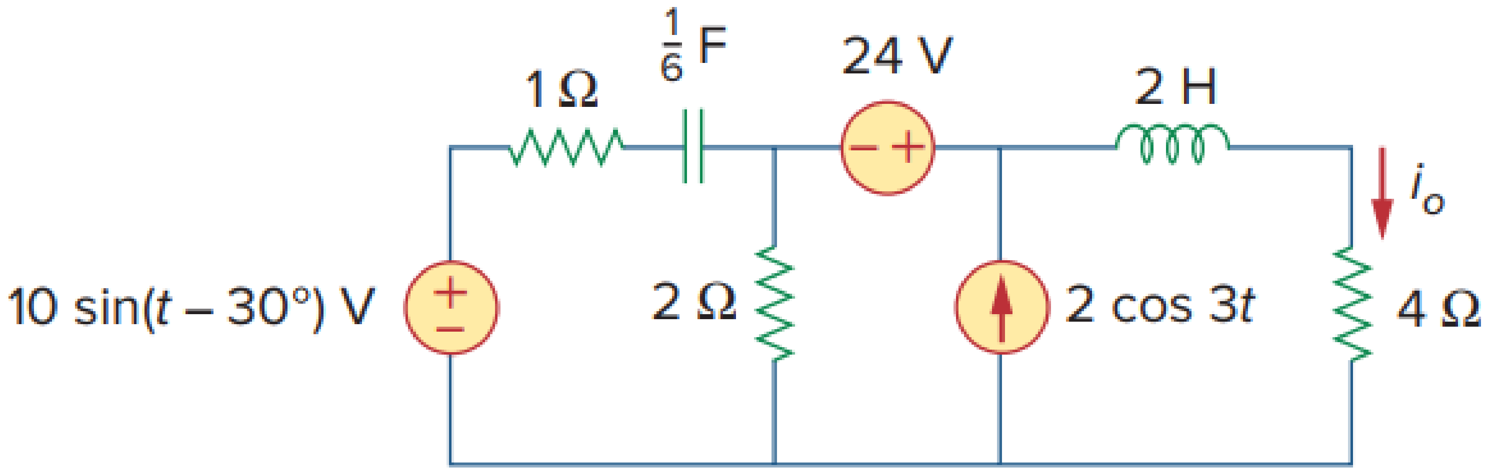
- Solve for the voltage $v_0(t)$.



Ans: $\mathbf{10} + \mathbf{21.47}\sin(\mathbf{2}\boldsymbol{t} + \mathbf{26.57}°) + \mathbf{10.73}\cos(\mathbf{3}\boldsymbol{t} - \mathbf{26.57}°)\ \boldsymbol{V}$

Problem 20

- Determine $i_0(t)$.

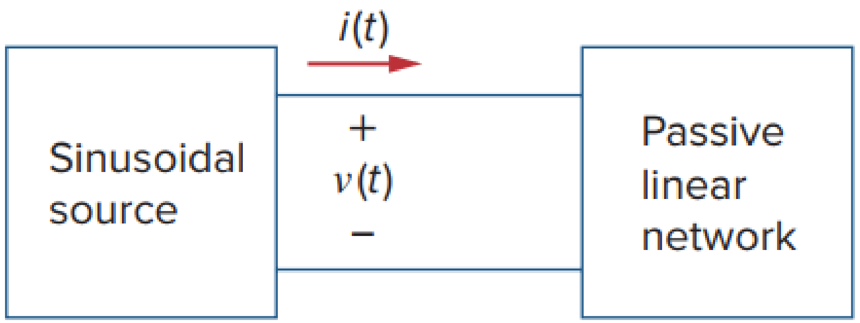


Ans: $\mathbf{4} + \mathbf{0.504}\sin(\boldsymbol{t} + \mathbf{19.1}^\circ) + \mathbf{0.3352}\cos(\mathbf{3}\boldsymbol{t} - \mathbf{76.43}^\circ)\ \boldsymbol{V}$

Instantaneous Power

- The *instantaneous power* (in *watts*) is the power at any instant of time rule. The instantaneous power $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$p(t) = v(t)i(t)$$



- For an arbitrary combination of circuit elements under sinusoidal excitation, if the voltage and current at the terminals are $v(t) = V_m\cos(\omega t + \theta_v)$ and $i(t) = I_m\cos(\omega t + \theta_i)$ respectively, the instantaneous power is,

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

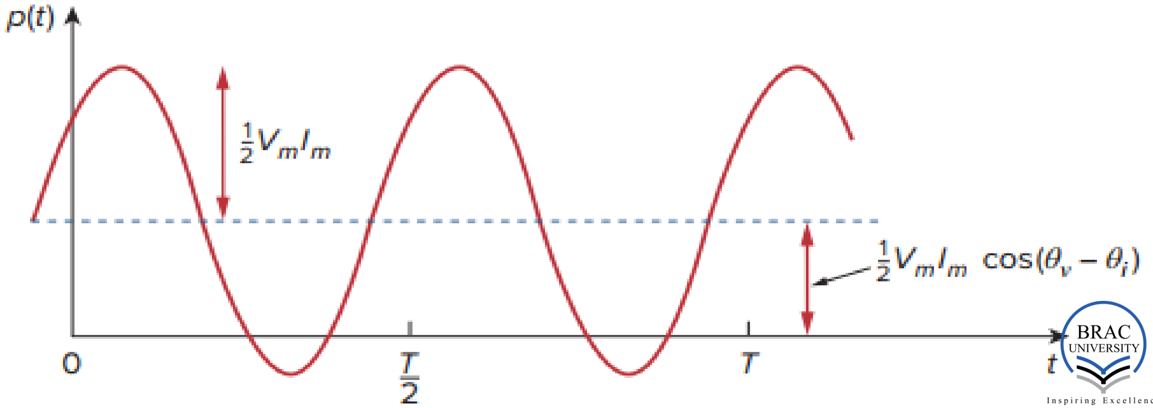
$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_I) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

Instantaneous Power: graphically

$$p(t) = \underbrace{\frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}_{\textit{Time independent}} + \underbrace{\frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)}_{\textit{Time dependent and periodic}}$$

- $p(t)$ is periodic as can be seen from $p(t)$ vs t plot. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

- The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure.



Average Power

- The *average power*, in *watts*, is the average of the instantaneous power over one period.

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T v(t)i(t)dt = \frac{1}{2}V_m I_m\left[\frac{1}{T}\int_0^T \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt\right]$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}[t]_0^T + \frac{1}{2}V_m I_m \frac{1}{T}\left[\frac{\sin(2\omega t + \theta_v + \theta_i)}{2\omega}\right]_0^T$$

$$\boxed{P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

- In phasor form for $\boldsymbol{V} = V_m\angle\theta_v$ and $\boldsymbol{I} = I_m\angle\theta_i$, $P = \frac{1}{2}Re\{\boldsymbol{V}\boldsymbol{I}^*\} = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$
- For a *purely reactive circuit*, $\theta_v = \theta_i$, and $P = \frac{1}{2}V_m I_m \cos(0°) = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2\,R$
- For a *purely reactive circuit*, $\theta_v - \theta_i = \pm 90°$, and $P = \frac{1}{2}V_m I_m \cos(90°) = 0$
- *A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.*

Problem 21

I. Calculate the instantaneous power and average power absorbed by a passive linear network. The voltage across the network and the current fed to the network are,

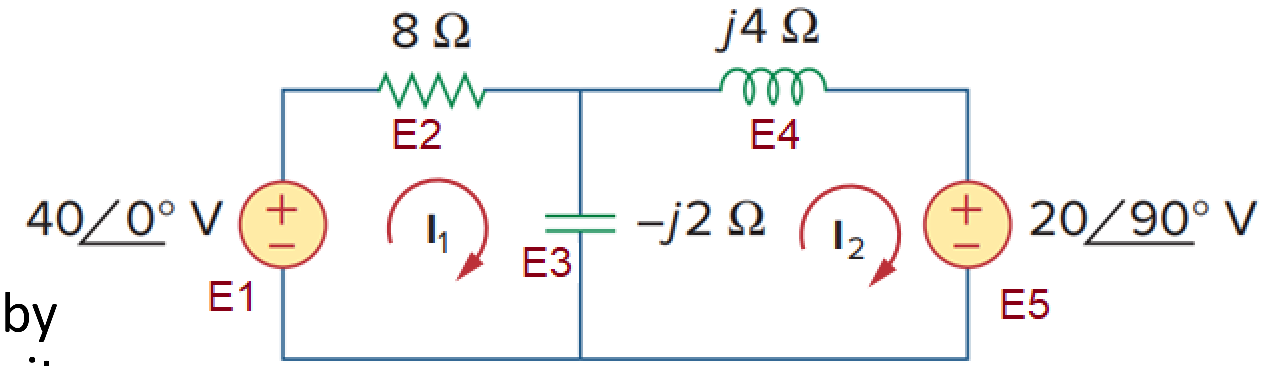
$$v(t) = 330\, cos(10t + 20°)\, V$$

$$i(t) = 33 \sin(10\text{t} + 60°)\text{ A}$$

II. A current $\boldsymbol{I} = 33\angle 30°\, A$ flows through an impedance $\boldsymbol{Z} = 40\angle -22°\, \Omega$. Find the average power delivered to the impedance.

Ans:
$I.\ \ \mathbf{3.5} + \mathbf{5.445} \cos(\mathbf{20}\boldsymbol{t} - \mathbf{10}°);\ \mathbf{3.5}\ \boldsymbol{kW}$
$II.\ \ \mathbf{20.19}\ \boldsymbol{kW}$

Example 4



- Calculate the average power absorbed by each of the five elements in the circuit below.

Let's use Mesh Analysis.

KVL at loop 1,

$$-40\angle 0° + 8\boldsymbol{I_1} + (-j2)(\boldsymbol{I_1} - \boldsymbol{I_2}) = \boldsymbol{0}$$

$$\Rightarrow \ (8 - j2)\boldsymbol{I_1} + j2\boldsymbol{I_2} = \ 40\angle 0°$$

KVL at loop 2,

$$(-j2)(\boldsymbol{I_2} - \boldsymbol{I_1}) + j4\boldsymbol{I_2} + 20\angle 90° = 0$$

$$\Rightarrow \ \ j2\boldsymbol{I_1} + j2\boldsymbol{I_2} = -20\angle 90°$$

Solving yields, $\boldsymbol{I_1} = 5\angle -53.13°$;

$$\boldsymbol{I_2} = 13.6\angle -162.897°$$

For element 1 (E1), supplying power (-ve) is

$$P_{E1} = \ -V_m I_m \cos(\theta_v - \theta_i)$$

$$P_{E1} = -40 \ \times 5 \times \cos(0° - (-53.13°))$$

$$= -120\ (W)$$

For a resistor we may use $P = \frac{1}{2} I_m^2 R$. So,

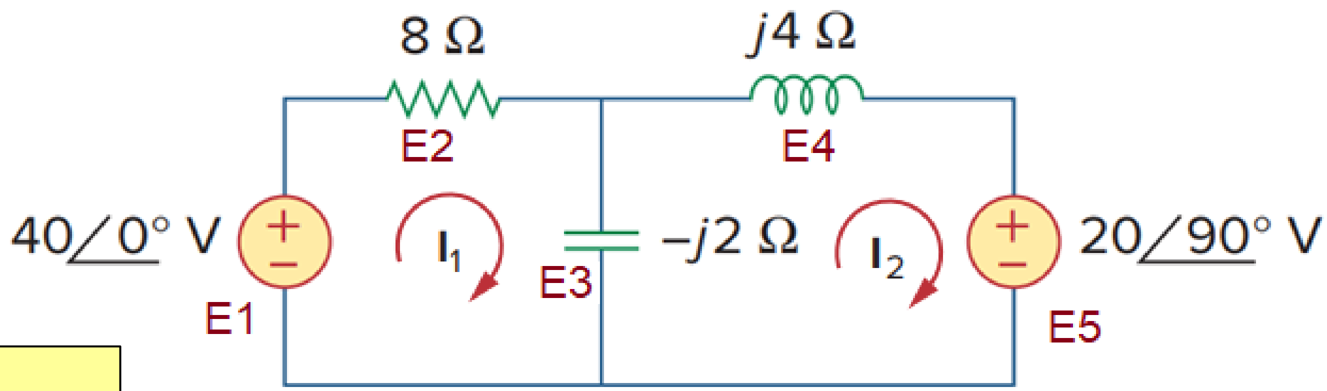
$$P_{E2} = \frac{1}{2} \times 5^2 \times 8 = 100\ (W)$$

Current through the $-j2$ impedance is,

$$(\boldsymbol{I_1} - \boldsymbol{I_2}) = 16\angle 0°$$

Example 4 (contd … 2)

- Calculate the average power absorbed by each of the five elements in the circuit below.



Voltage drop across the $-j2$ impedance is,

$$(\boldsymbol{I_1} - \boldsymbol{I_2}) \times (-j2) = 32\angle -90^\circ$$

So, the absorbing power of element E3 is,

$$P_{E3} = 32 \times 68 \times \cos(-90^\circ - 0^\circ)$$

$$= 0^\circ \text{ (as expected)}$$

Similarly, P_{E4} will be equal to zero as capacitors and inductors don't absorb real power.

The absorbing power of the $20\angle -90^\circ$ source is,

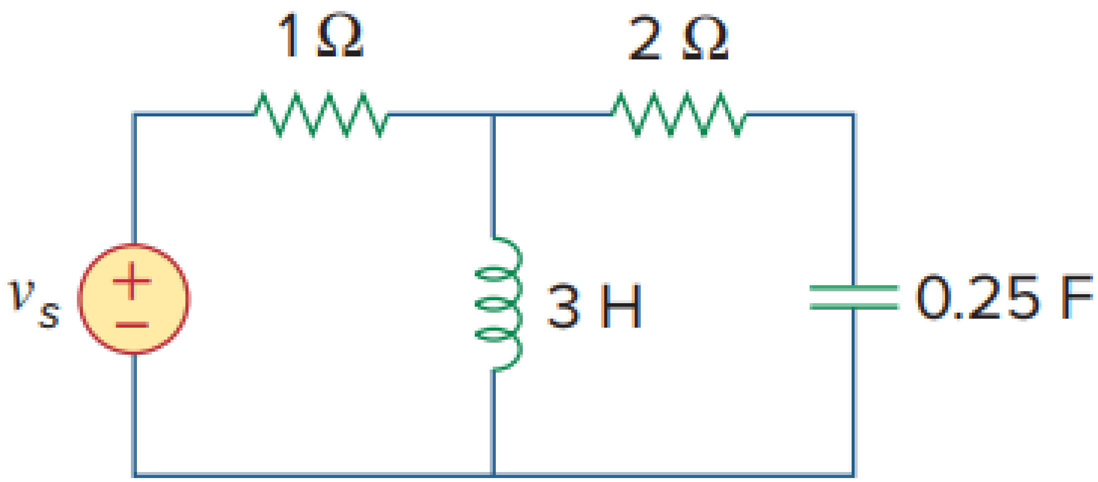
$$P_{E5} = 20 \times 13.6 \times \cos(90^\circ - (-162.897^\circ))$$

$$= -80\ (W)$$

It turns out it is also supplying power.

Problem 22

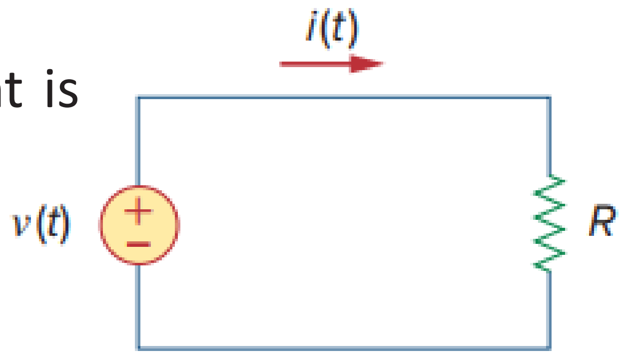
- Assume that $v_s = 8\cos(2t - 40°)\ V$ in the circuit below. Find the average power of each of the elements.



Ans: $\boldsymbol{P_{1\Omega} = 1.4159\ W, P_{2\Omega} = 5.097\ W,}$
$\boldsymbol{P_{3H} = P_{0.25F} = 0\ W}$

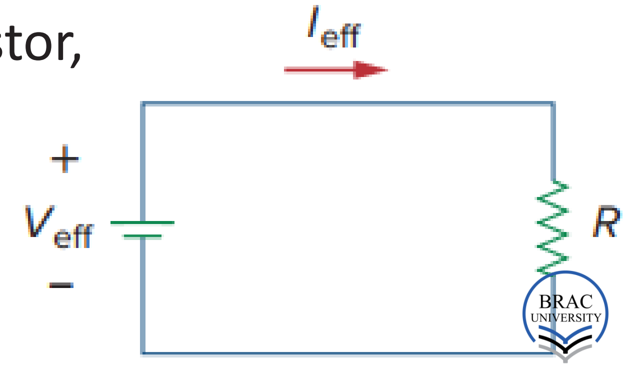
Effective or RMS Value

- The idea of effective value arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.
- The *effective value* of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.
- For a sinusoidal voltage $v(t)$ connected to a resistor, if the current is $i(t)$ the average power absorbed by the resistor is,



$$P = \frac{1}{T}\int_{0}^{T} i^2 R\, dt = R\frac{1}{T}\int_{0}^{T} i^2\, dt$$

- If a dc current I_{eff} transfers the same amount of power to the resistor, that is, $P = I_{eff}^2 R$, by comparing,



$$I_{eff} = \sqrt{\frac{1}{T}\int_{0}^{T} i^2\, dt}$$

RMS Value of Sinusoids

- For any periodic function $x(t)$ with period T, the effective value is, $X_{eff} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$
- This indicates that the effective value is the (square) root of the mean (or average) of the square of the periodic signal. Thus, the effective value is often known as the *root-mean-square value*, or *rms* value for short;
- The rms of a sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta_v)$ connected to a resistor, if the current is $i(t)$ the average power absorbed by the resistor is,

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 cos^2(\omega t + \theta_v)dt} = \frac{V_m}{\sqrt{2}}\sqrt{\frac{1}{T}\int_0^T (1 + 2cos\,(\omega t + \theta_v)dt} = \frac{V_m}{\sqrt{2}}$$

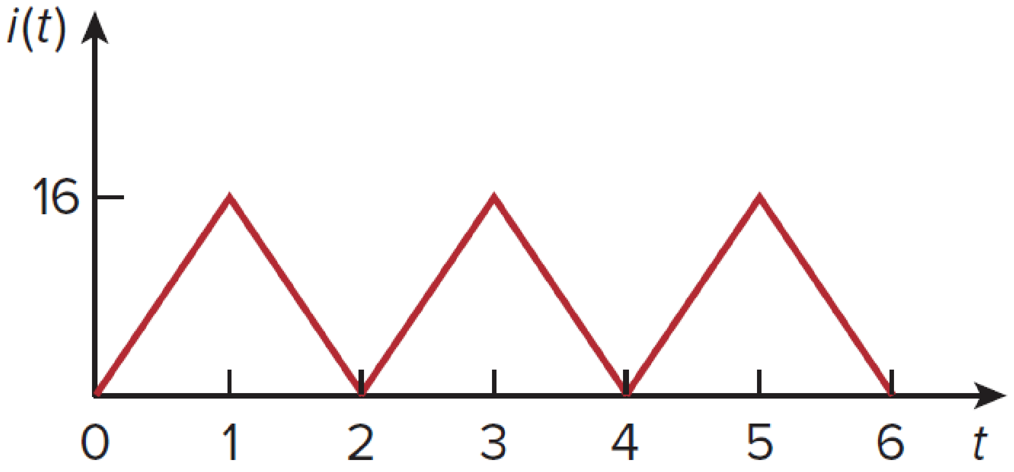
- The average power can be written in terms of V_{rms} and I_{rms} as,

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_I) = V_{rms} I_{rms} \cos(\theta_v - \theta_I)$$

- For a resistor, $P = \frac{V_{rms}^2}{R}$ or $P = I_{rms}^2 R$

Example 5

- Find the rms value of the current waveform shown below. If the current flows through a $9\,\Omega$ resistor, calculate the average power absorbed by the resistor.



Here, $T = 2\ s$.

The equation of the line segment between (0, 0) and (1, 16) is,

$$i(t) = 16t \quad [y = mx]$$

Similarly for the line segment between (1, 16) and (2,0),

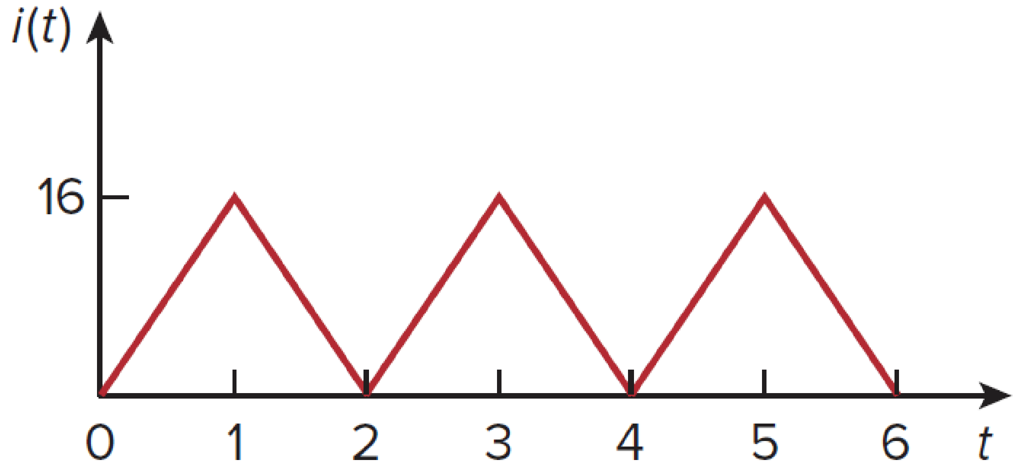
$$i(t) = -16t$$

The rms value of the current can be found as,

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

Example 5 (contd … 2)

- Find the rms value of the current waveform shown below. If the current flows through a $9\ \Omega$ resistor, calculate the average power absorbed by the resistor.



$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

$$= \sqrt{\frac{1}{2}\left[\int_0^1 (16t)^2\,dt + \int_1^2 (-16t)^2 dt\right]}$$

$$= 18.475\ A$$

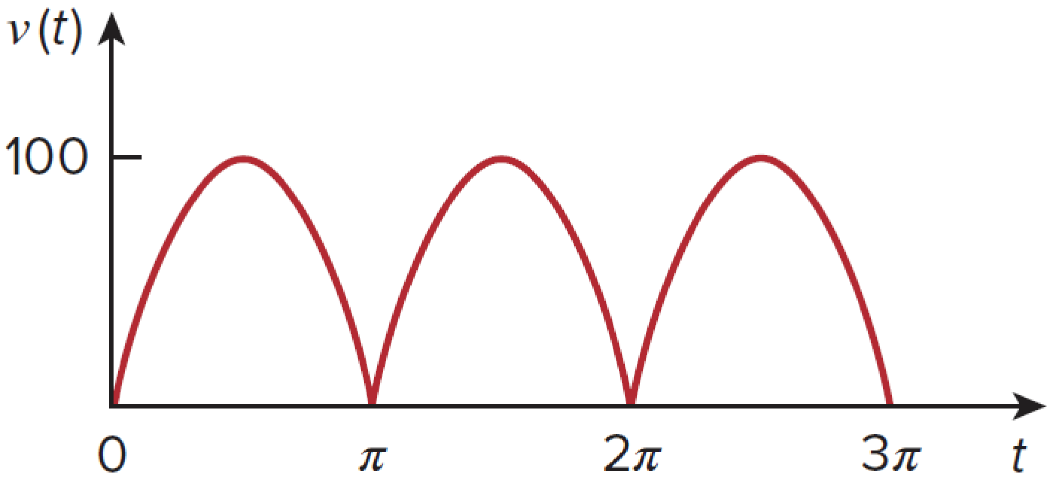
Average power absorbed by the resistor

$$= I_{rms}^2 R = (18.475)^2 \times 9$$

$$= 3071.93\ (W)$$

Problem 23

- Find the rms value of the full-wave rectified sine wave shown below. Calculate the average power dissipated in a 6 Ω resistor.



Ans: $\mathbf{70.71\ V;\ 833.3\ W}$

Problem 24

- Calculate the effective value of the current waveform shown and the average power delivered to a 12 Ω resistor when the current runs through the resistor.

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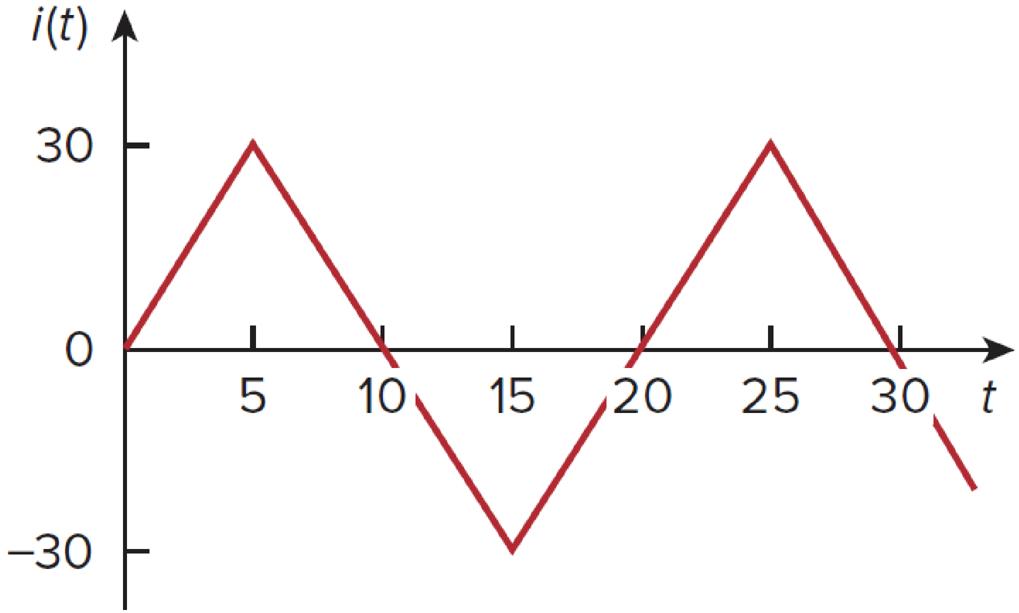
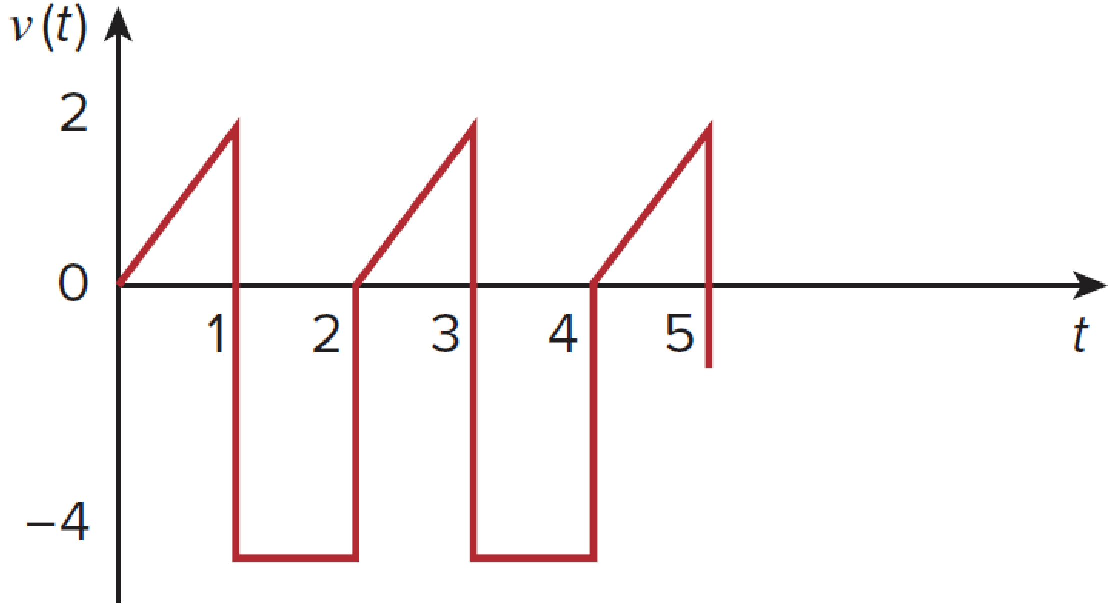


Ans: $\mathbf{17.321\ A, 3.6\ kW}$

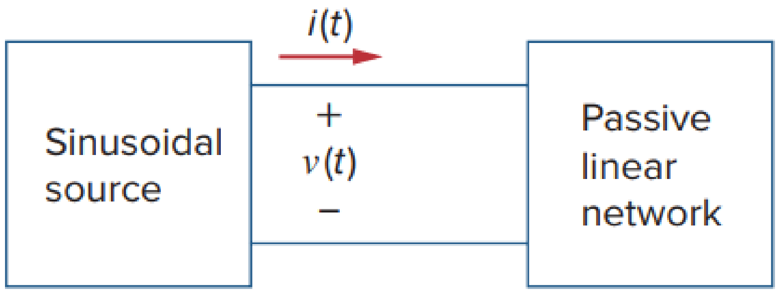
Problem 25

- Find the rms value of the signal shown below.



Ans: $\mathbf{2.944}\ \boldsymbol{V}$

Apparent Power and PF



- We saw that if the voltage and current at the terminals of a circuit are $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$ or, in phasor form, $\boldsymbol{V} = V_m \angle \theta_v$ and $\boldsymbol{I} = I_m \angle \theta_i$, the average power is,

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

- Where $S = V_{rms} I_{rms}$ is known as the apparent power. The *apparent power* (in VA) is the product of the rms values of voltage and current.

- The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. However, because of the term $\cos(\theta_v - \theta_I)$ the power is less than the apparent power for loads other than purely resistive.

- The term $\cos(\theta_v - \theta_I)$ is called the *Power Factor* of the load.

Power Factor

- The *power factor* is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

$$Pf = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- The angle $\theta_v - \theta_I$ is called the *power factor angle*.
- The power factor angle is equal to the angle of the load impedance if $\boldsymbol{V}$ is the voltage across the load and $\boldsymbol{I}$ is the current through it. This is evident from the fact that,

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_I} = \frac{V_m}{I_m} \angle(\theta_v - \theta_i) = \frac{V_{rms}/\sqrt{2}}{I_{rms}/\sqrt{2}} \angle(\theta_v - \theta_I) = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i)$$

- For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and $pf = 1$. This implies that the apparent power is equal to the average power.
- For a purely reactive load, $\theta_v - \theta_I = \pm 90°$ and $pf = 0$. In this case the average power is zero. In between these two extreme cases, pf is said to be leading or lagging. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load.

Example 6

- Obtain the power factor and the apparent power of a load whose impedance is $Z = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 155.56\cos(377t + 10°)\ V$.

The load impedance Z can be written in phasor form as,

$$\boldsymbol{Z} = 72.11\angle 33.69°\ (\Omega)$$

Cosine of the angle of impedance is the power factor. So,

$$pf = \cos(33.69) = 0.8321$$

The power factor is lagging (means current lags voltage) as can be seen from the reactance or angle of the impedance. The reactance $(+j40)$ is inductive or the angle is positive $(33.69°)$.

The load current can be found by using Ohm's law as, $\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}} = \frac{155.56\angle 10°}{72.11\angle 33.69°} = 2.16\angle -23.69°$

Apparent power,

$$S = V_{rms} I_{rms} = \frac{155.56}{\sqrt{2}} \times \frac{2.16}{\sqrt{2}} = 168\ (VA)$$

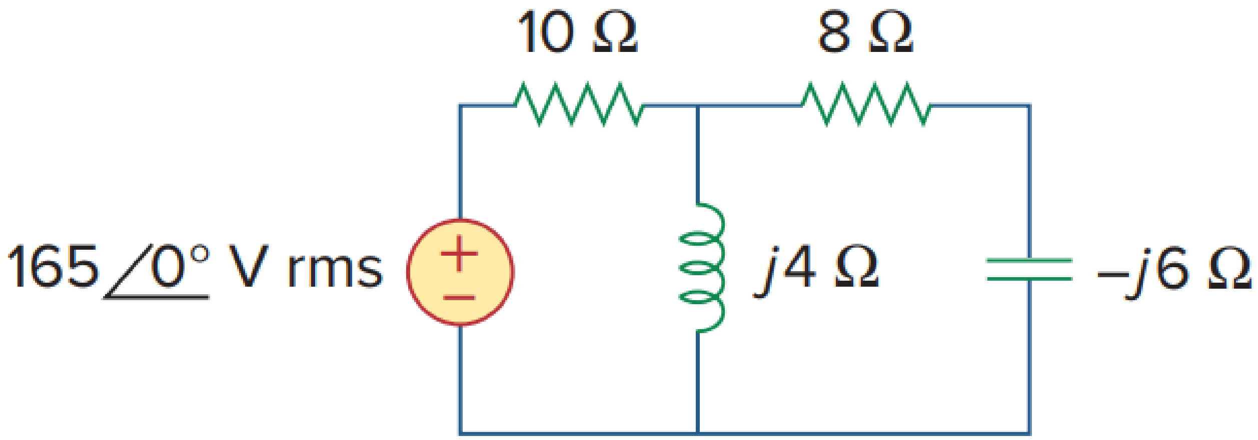
Problem 26

- A series-connected load draws a current $i(t) = -4\sin(400t - 120°)\,A$ when the applied voltage is $v(t) = 100\cos(400t - 60°)\,V$. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Ans: $\mathbf{200\,VA;\ 0.866\ (leading);\ R = 21.65\,\Omega;\ C = 200\,\mu F}$

Example 7

- Calculate the power factor of the entire circuit as seen by the source. What is the average power supplied by the source?



Let's convert the circuit to one that contains only the voltage source in series with an impedance.

The equivalent impedance as seen from the source side is,

$$\boldsymbol{Z_{eq}} = 10 + \{(8 - j6)\,||\,j4\} = 12.7\angle 20.62°$$

$$pf = \cos(20.62°) = 0.936, lagging$$

Current supplied by the source is,

$$\boldsymbol{I_{rms}} = \frac{\boldsymbol{V_{rms}}}{\boldsymbol{Z_{eq}}} = \frac{165}{12.7\angle 20.62°} = 13\angle -20.62°$$

So, power supplied by the source is,

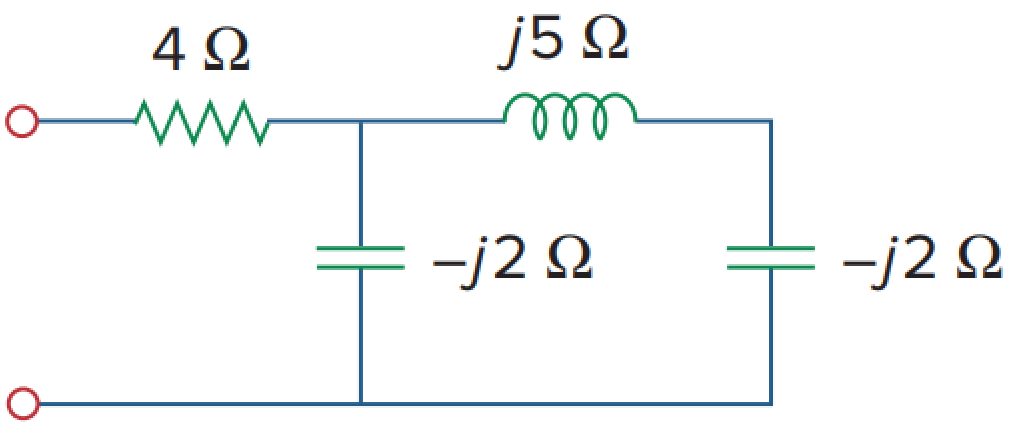
$$P = V_{rms} \times I_{rms} \times pf$$

$$= 165 \times 13 \times 0.936$$

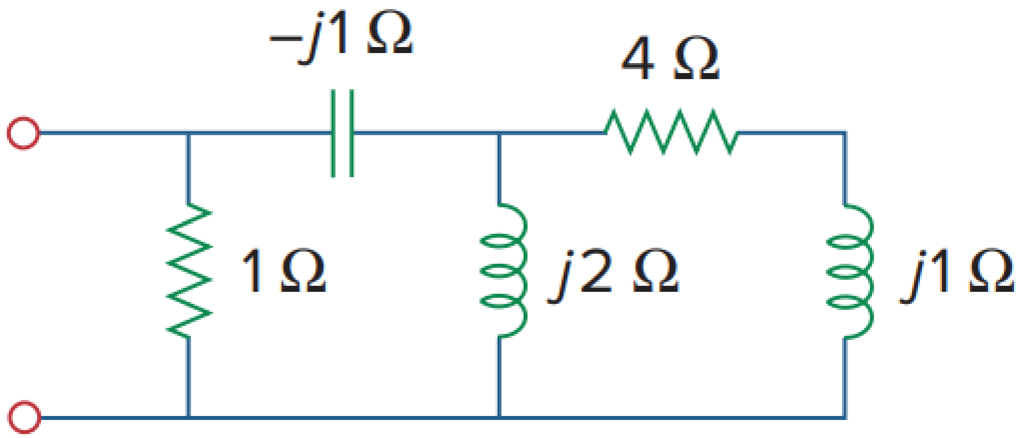
$$= 2.008\ (kW)$$

Problem 27

- Obtain the power factor for each of the circuits shown below. Specify each power factor as leading or lagging.



(a)

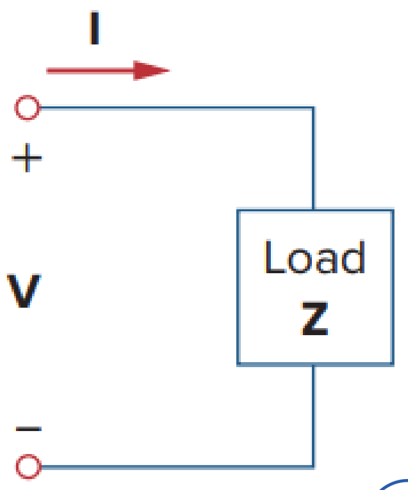


(b)

<u>Ans</u>: $(a)\mathbf{0.5547}\ (\boldsymbol{leading});(b)\ \mathbf{0.9304}\ (\boldsymbol{lagging})$

Complex Power

- Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains all the information pertaining to the power absorbed by a given load.
- Consider an ac load with voltage $v(t)$ and current $i(t)$ in phasor form $\boldsymbol{V} = V_m\angle\theta_v$ and $\boldsymbol{I} = I_m\angle\theta_i$, the complex power **S** absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or



$$\boldsymbol{S} = \boldsymbol{V_{rms}}\boldsymbol{I_{rms}^*} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* \text{ (in } VA \text{ units)}$$

Recall that, apparent power, $S = |\boldsymbol{V_{rms}}||\boldsymbol{I_{rms}}| = |\boldsymbol{S}|$

$$\boldsymbol{S} = |\boldsymbol{I_{rms}}|^2\boldsymbol{Z} = \frac{|\boldsymbol{V_{rms}}|^2}{\boldsymbol{Z}^*}$$

Real and Reactive Power

- The complex power, $\boldsymbol{S}$, is a complex number that may be expressed as $x + jy$.

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* = \boldsymbol{V_{rms}}\boldsymbol{I_{rms}^*} = V_{rms}\angle\theta_v \times (I_{rms}\angle\theta_i)^* = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = V_{rms}I_{rms}e^{j(\theta_v - \theta_i)}$$

$$= V_{rms}I_{rms}cos(\theta_v - \theta_i) + j\, V_{rms}I_{rms}sin(\theta_v - \theta_i) = P + jQ$$

$$or, \qquad \boldsymbol{S} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* = \boldsymbol{V_{rms}}\boldsymbol{I_{rms}^*} = \frac{|\boldsymbol{V_{rms}}|^2}{\boldsymbol{Z}^*} = |\boldsymbol{I_{rms}}|^2\boldsymbol{Z} = I_{rms}^2(R + jX) = P + jQ$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = Re\{\boldsymbol{S}\} = I_{rms}^2 R = V_{rms}I_{rms}\cos(\theta_v - \theta_i) \text{ and}$$

$$Q = Im\{\boldsymbol{S}\} = I_{rms}^2 X = V_{rms}I_{rms}sin(\theta_v - \theta_i)$$

- P is the *average or real power* we've encountered so far. It depends on the load's resistance R.
- Q depends on the load's reactance X and is called the *reactive (or quadrature) power*.
- Let's find out what Q actually means.

Significance of Reactive Power (Q)

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)\,;\; Q = V_{rms} I_{rms} sin(\theta_v - \theta_i)$$

- The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load.

- The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the volt-ampere reactive (VAR) to distinguish it from the real power, whose unit is the watt.

- We know that energy storage elements (capacitors and inductors) neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source.

- Power providers still have to supply even though reactive power is not used, which results in wasteful heat loss because the current is higher than the requirement.

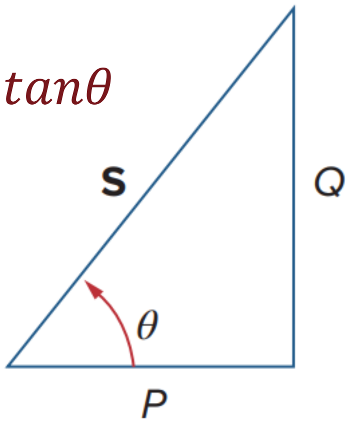
Reactive Power: pros and cons

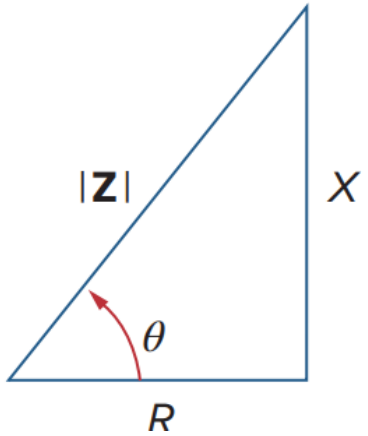
- **Cons**
  - Reactive power is the wattless component and results in unproductive work, especially in terms of overheating of the equipment, thus derating them.
  - Unlike active power, reactive power cannot be transmitted over long distances; it has to be generated locally.
  - Due to reactive power, there is a drop in the power factor, the capacity of transmission lines for active power reduces, causes a voltage drop at the terminal due to increased current flowing through the line, causes increased losses.
- **Pros**
  - Reactive power is needed for a proper functioning AC power system to keep the voltage up.
  - Importance of reactive power lies in the magnetizing components of the electrical machines. Without this there won't be any magnetic linkage, no electromagnetic induction, no motion and hence no induced emfs and currents.
  - Reactive power control enables more operating flexibility and increased reliability.

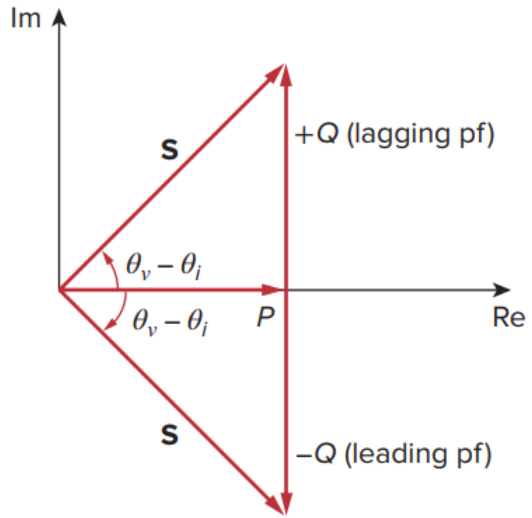
Power Triangle

- It is a standard practice to represent $\boldsymbol{S}, P$, and Q in the form of a triangle, known as the *power triangle*. This is similar to the impedance triangle showing the relationship between $\boldsymbol{Z}, R$, and X.
- The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle.
- When $\boldsymbol{S}$ lies in the first quadrant, we have an inductive load and a lagging pf. When **S** lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

$Q = P\ tan\theta$







Summary of AC Power relations

$$\text{Complex Power} = \boldsymbol{S} = P + jQ = \boldsymbol{V}_{\boldsymbol{rms}}\boldsymbol{I}^*_{\boldsymbol{rms}} = \frac{|\boldsymbol{V}_{\boldsymbol{rms}}|^2}{\boldsymbol{Z}^*} = |\boldsymbol{I}_{\boldsymbol{rms}}|^2\boldsymbol{Z}$$

$$= |\boldsymbol{V}_{\boldsymbol{rms}}||\boldsymbol{I}_{\boldsymbol{rms}}|\angle(\theta_v - \theta_i)$$

$$\text{Apparent Power} = S = |\boldsymbol{S}| = |\boldsymbol{V}_{\boldsymbol{rms}}||\boldsymbol{I}_{\boldsymbol{rms}}| = \sqrt{P^2 + Q^2}$$

$$\text{Real Power} = P = Re\{\boldsymbol{S}\} = S\cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = Im\{\boldsymbol{S}\} = S\sin(\theta_v - \theta_i) = P\,tan\theta$$

$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

Example 8

The voltage across a load is $v(t) = 20\cos(10t - 30°)\,V$ and current through the element in the direction of voltage drop is $i(t) = -8\sin(10t - 70°)\,A$. Determine with appropriate units,

(i) the complex power,

(ii) the apparent power,

(iii) the real and reactive powers. Also, specify for each whether the load is supplying or absorbing,

(iv) the power factor of the load, and

(v) the load impedance.

$$\boldsymbol{V} = 20\angle -30°\ (\mathrm{V})$$

$$i(t) = -8\,sin(10t - 70°)\,(A)$$

$$= 8\cos(10t - 70° + 90°) = 8\cos(10t + 20°)$$

$$\boldsymbol{I} = 8\angle 20°\ (\mathrm{A})$$

(i) Complex power

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^* = \frac{1}{2} \times 20\angle -30° \times 8\angle -20°$$

$$= 51.42 - j61.28 = 80\angle -50°\ (VA)$$

(ii) Apparent power

$$S = |\boldsymbol{S}| = 80\ (VA)$$

(iii) Real power

$$P = Re\{\boldsymbol{S}\} = 78.78\ (W)$$

Example 8 (contd … 2)

The voltage across a load is $v(t) = 20\cos(10t - 30°)\,V$ and current through the element in the direction of voltage drop is $i(t) = -8\sin(10t - 70°)\,A$. Determine with appropriate units,

(i) the complex power,

(ii) the apparent power,

(iii) the real and reactive powers. Also, specify for each whether the load is supplying or absorbing,

(iv) the power factor of the load, and

(v) the load impedance.

Reactive power

$$Q = Im\{\boldsymbol{S}\} = -13.89\ (VAR)$$

So, the load is absorbing real power and supplying reactive power.

(iv) Power factor

$$pf = \frac{P}{S} = \frac{51.42}{80} = 0.6428\ (leading)$$

$$or, pf = cos(\theta_v - \theta_i) = \cos(-30° - 20°)$$
$$= 0.6428\ (leading)$$

(v) Load impedance

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{20\angle -30°}{8\angle 20°} = 2.5\angle -50°$$

Problem 28

For a load, $\boldsymbol{V_{rms}} = 110\angle 85°\ V$, $\boldsymbol{I_{rms}} = 3\angle 15°\ A$. Determine:

(a) the complex and apparent powers,

(b) the real and reactive powers, and

(c) the power factor and the load impedance.

Ans: $(a)\ \mathbf{330\angle 70°\ VA, 44\ VA}, (b)\ \mathbf{112.87\ W, 310.1\ VAR}, (c)\ \mathbf{0.342\ lagging}, (\mathbf{12.541 + j34.46})\ \mathbf{\Omega}$.

Problem 29

A sinusoidal source supplies $100\ kVAR$ reactive power to load $\boldsymbol{Z} = 250\angle -75°\ \Omega$. Determine:

(a) the power factor,

(b) the apparent power delivered to the load, and

(c) the rms voltage.

Ans: $(a)\ \mathbf{0.2588}\ \boldsymbol{leading}, (b)\ \mathbf{103.53}\ \boldsymbol{kVA}, (c)\ \mathbf{5.087}\ \boldsymbol{kV}.$

Problem 30

A $110\,V\,rms,\ 60Hz$ source is applied to a load impedance $\boldsymbol{Z}$. The apparent power entering the load is $120\,VA$ at a power factor of $0.707\,lagging$.

(a) Calculate the complex power.

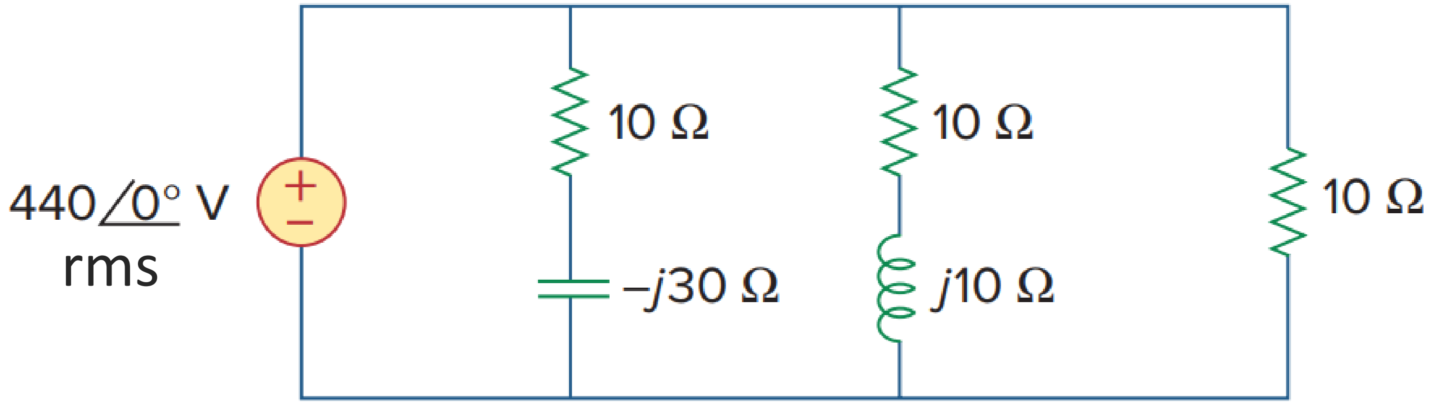
(b) Find the rms current supplied to the load.

(c) Determine $\boldsymbol{Z}$.

(d) Assuming that $\boldsymbol{Z} = R + j\omega L$, find the values of R and L.

Ans: $(a)\ \mathbf{120\angle 45°\ VA}, (b)\ \mathbf{\frac{12}{11}\angle -45°\ A}, (c)\ \mathbf{\frac{605}{6}\angle 45°\ \Omega}, (d)\ \boldsymbol{R = 71.3\ \Omega, L = 189.12\ mH}.$

Example 9



For the circuit shown, calculate:

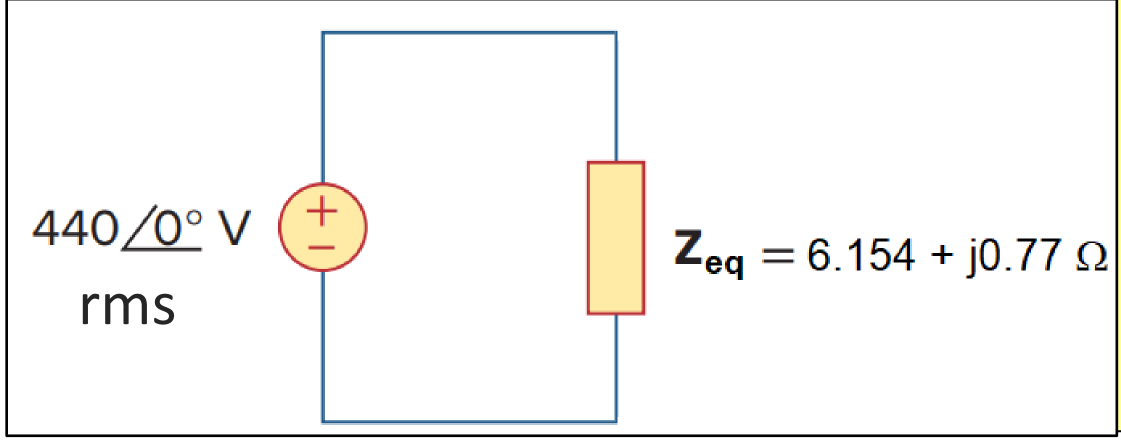
(a) the power factor,

(b) the average power delivered by the source,

(c) the reactive power,

(d) the apparent power,

(e) the complex power.



First convert the circuit to a single impedance in series with the voltage source.

$$\boldsymbol{Z_{eq}} = 10 \,||\, (10 + j10) \,||\, (10 - j30)$$

$$= 6.154 + j0.77 = 6.20\angle 7.125° \; (\Omega)$$

(a) Power Factor

$$pf = \cos(7.125°) = 0.9923 \; (lagging)$$

(A positive angle of impedance means the current lags the voltage across the impedance.)

Example 9 (contd … 2)

For the circuit shown, calculate:

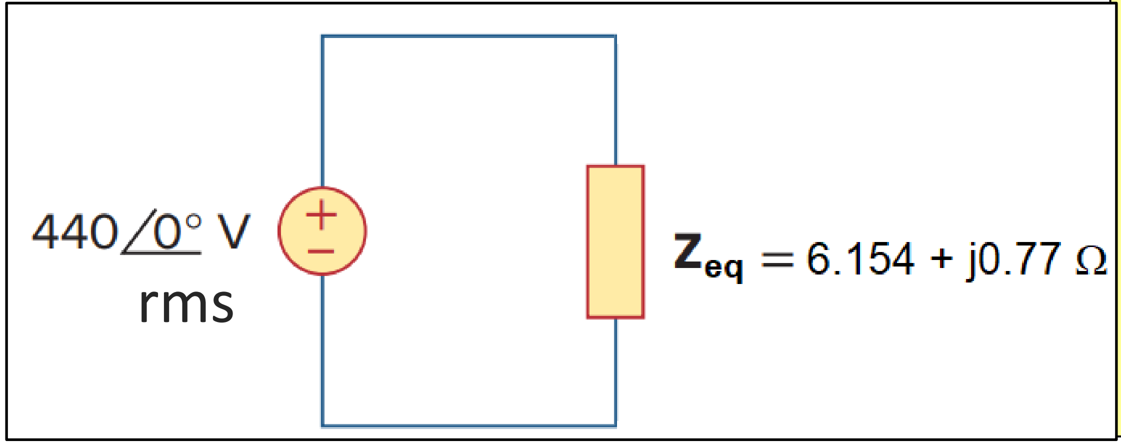
(a) the power factor,

(b) the average power delivered by the source,

(c) the reactive power,

(d) the apparent power,

(e) the complex power.



(b) The average power delivered by the source is equal to the power consumed by $\boldsymbol{Z_{eq}}$

The complex power of the load is,

$$\boldsymbol{S} = \frac{|\boldsymbol{V_{rms}}|^2}{\boldsymbol{Z_{eq}^*}} = \frac{440^2}{6.20\angle - 7.125°}$$

$$= 30984.68 + j3873.076\ (VA)$$

So, the average power delivered by the source is, $P = 30984.68\ (W)$

(c) $Q = 3873.076(VAR)$

(d) $S = |\boldsymbol{S}| = 31225.81\ (VA)$

(e) $\boldsymbol{S} = 30984.68 + j3873.076\ (VA)$

Problem 31

For the circuit shown, calculate:

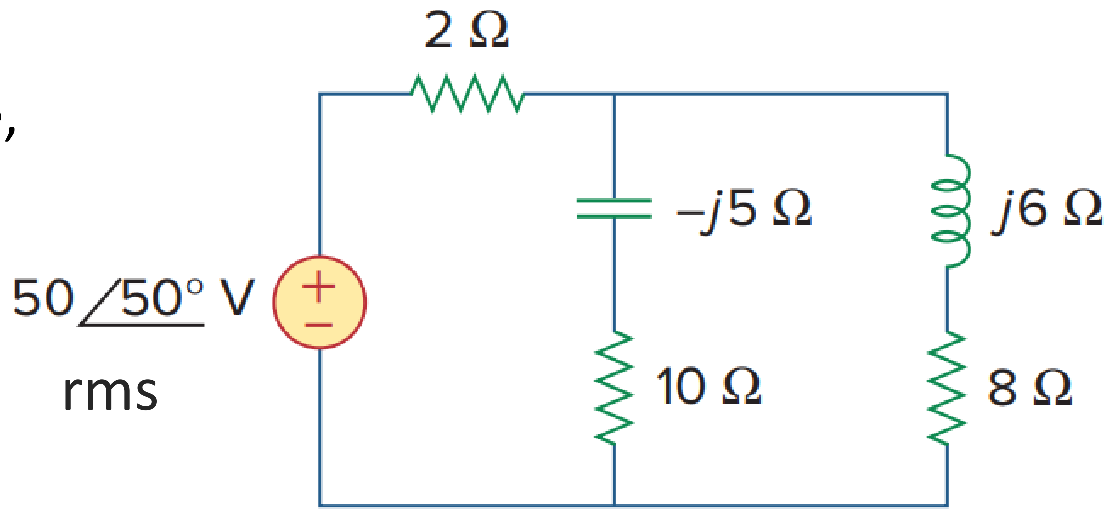
(a) the power factor,

(b) the average power delivered by the source,

(c) the reactive power,

(d) the apparent power,

(e) the complex power.



Ans: (a) $\mathbf{0.9956\ (lagging)}$, (b) $\mathbf{304\ W}$, (c) $\mathbf{28.64\ VAR}$, (d) $\mathbf{305.3\ VA}$, (e) $\mathbf{304\ +\ j28.64\ VA}$

Thank you for your attention