CSE 230 Bonus Assignment

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SECTION: 3

CSE 230 SPRING 2022 BONUS ASSIGNMENT
(RECURSION AND PROBABILITY DISTRIBUTION)

(1) $a_n = 2a_{n-1} + 5$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)} = a_n - 2a_{n-1}$

$a_n - 2a_{n-1} = 0$

$r^n - 2r^{n-1} = 0 \quad (\div r^{n-1})$

$r - 2 = 0$

$r_1 = 2$

$a_n = \alpha_1 r_1^n$

$a_n = \alpha_1 (2)^n$ —— (1)

$a_n^{(p)} = A$

$A = 2A + 5$

$-A = 5$

$A = -5$

$a_n^{(p)} = -5$

$a_n = \alpha_1 (2)^n - 5$

$a_1 = 3$

$\alpha_1 (2) - 5 = 3$

$\alpha_1 = 4$

$a_n = 4(2)^n - 5$ (Ans)

(2) $a_n = 2a_{n-1} + a_{n-2} + 5n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)} = a_n - 2a_{n-1} - a_{n-2}$

$a_n - 2a_{n-1} - a_{n-2} = 0$

$r^n - 2r^{n-1} - r^{n-2} = 0 \quad (\div r^{n-2})$

$r^2 - 2r - 1 = 0$

$r_1 = 1+\sqrt{2}, \quad r_2 = 1-\sqrt{2}$

$a_n^{(h)} = \alpha_1 r_1^n + \alpha_2 r_2^n$

$= (1+\sqrt{2})^n \alpha_1 + (1-\sqrt{2})^n \alpha_2$

$a_n^{(p)} = A_1 n + A_0$

$a_n^{(p)} = 2(A_1(n-1) + A_0) + A_1(n-2) + A_0 + 5n$

$A_1 n + A_0 = 2A_1 n - 2A_1 + 2A_0 + A_1 n - 2A_1 + A_0 + 5n$

$\cancel{A_1 n + A_0 = n(2A_1 + A_1 + 5) + (2A_1 + 2A_0 - 2A_1 + A_0)}$

$\cancel{= n(3A_1 + 5) + 3A_0}$

$A_1 n + A_0 = 3A_1 n + 3A_0 + 5n - 4A_1$

$-2A_1 n + 2A_0 - 5n \cancel{= 0} + 4A_1 = 0$

$n(-2A_1 - 5) + 2A_0 \cancel{= 0} + 4A_1 = 0$

$n(-2A_1 - 5) = 0$ | $2A_0 \cancel{= 0} + 4A_1 = 0$

$A_1 = \frac{-5}{2}$ | $2A_0 \cancel{= 0} + 4\left(\frac{5}{2}\right) = 0$

| $A_0 = -5$

$a_n^{(p)} = -\frac{5}{2} n - 5$

$a_n = (1+\sqrt{2})^n \alpha_1 + (1-\sqrt{2})^n \alpha_2 - \frac{5}{2} n - 5$

$a_1 = 3$

$(1+\sqrt{2})\alpha_1 + (1-\sqrt{2})\alpha_2 - \frac{15}{2} = 3$

$(1+\sqrt{2})\alpha_1 + (1-\sqrt{2})\alpha_2 = \frac{21}{2}$ — ①

$\cancel{(1+\sqrt{2})^2\alpha_1 + (1-\sqrt{}}$

$a_2 = 4$

$(1+\sqrt{2})^2\alpha_1 + (1-\sqrt{2})^2\alpha_2 - \frac{5}{2}(2) - 5 = 4$

$(1+\sqrt{2})^2\alpha_1 + (1-\sqrt{2})^2\alpha_2 = 14$ —— (ii)

$$\alpha_1 = \frac{\frac{21}{2} - (1-\sqrt{2})\alpha_2}{(1+\sqrt{2})}$$ —— (iii)

$$(1+\sqrt{2})^2\left[\frac{\frac{21}{2} - (1-\sqrt{2})\alpha_2}{(1+\sqrt{2})}\right] + (1-\sqrt{2})^2\alpha_2 = 14$$

$$(1+\sqrt{2})\left(\frac{21}{2} - (1-\sqrt{2})\alpha_2\right) + (1-\sqrt{2})^2\alpha_2 = 14$$

$$(4-2\sqrt{2})\alpha_2 = \frac{7-21\sqrt{2}}{2}$$

$$\alpha_2 = \frac{-28-35\sqrt{2}}{8}$$

$$\alpha_1 = \frac{\frac{21}{2} - (1-\sqrt{2})\left(\frac{-28-35\sqrt{2}}{8}\right)}{(1+\sqrt{2})}$$

$$\alpha_1 = \frac{-28+35\sqrt{2}}{8}$$

$$a_n = \frac{-28+35\sqrt{2}}{8}(1+\sqrt{2})^n - \frac{28-35\sqrt{2}}{8}(1-\sqrt{2})^n - \frac{5}{2}n - 5$$ (Ans)

(3) $a_n = 2a_{n-1} + a_{n-2} + n^2 - 1$

$a_n^{(t)} = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)} = a_n - 2a_{n-1} - a_{n-2}$

$a_n - 2a_{n-1} - a_{n-2} = 0$

$r^n - 2r^{n-1} - r = 0 \quad (\div r^{n-1})$

$r^2 - 2r - 1 = 0$

$r_1 = 1+\sqrt{2}, \; r_2 = 1-\sqrt{2}$

$a_n^{(h)} = \alpha_1 r_1^n + \alpha_2 r_2^n$

$a_n^{(h)} = (1+\sqrt{2})^n \alpha_1 + (1-\sqrt{2})^n \alpha_2$

$a_n^{(p)} = A_2 n^2 + A_1 n + A_0$

$A_2 n^2 + A_1 n + A_0 = 2(A_2(n-1)^2 + A_1(n-1) + A_0) + A_2(n-2)^2 + A_1(n-2) + A_0$
$+ n^2 - 1$

$A_2 n^2 + A_1 n + A_0 = 2(A_2 n^2 - 2A_2 n + A_2 + A_1 n - A_1 + A_0) + A_2 n^2 - 4A_2 n + 4A_2$
$+ A_1 n - 2A_1 + A_0$

$A_2 n^2 + A_1 n + A_0 = 3A_2 n^2 - 8A_2 n + 6A_2 + 3A_1 n - 4A_1 + 3A_0 + n^2 - 1$

$-2A_2 n^2 + 8A_2 n - 6A_2 - 2A_1 n + 4A_1 - 2A_0 = n^2 - 1$

$n^2(-2A_2) + n(8A_2 - \cancel{6A_2}\, 2A_1) + (4A_1 - 6A_2 - 2A_0) = n^2 - 1$

$-2A_2 = 1$	$8A_2 - 2A_1 = 0$	$4A_1 - 6A_2 - 2A_0 = -1$
$A_2 = -\frac{1}{2}$	$8(-\frac{1}{2}) - 2A_1 = 0$	$4(-2) - 6(-\frac{1}{2}) - 2A_0 = -1$
	$A_1 = -2$	$A_0 = -2$

$a_n^{(p)} = -\frac{1}{2}n^2 - 2n \cancel{+6} - 2$

$a_n = (1+\sqrt{2})^n \alpha_1 + (1-\sqrt{2})^n \alpha_2 - \frac{1}{2}n^2 - 2n - 2$

$a_1 = 3$

$(1+\sqrt{2})\alpha_1 + (1-\sqrt{2})\alpha_2 - \frac{1}{2}(1)^2 - 2(1) - 2 = 3$

$(1+\sqrt{2})\alpha_1 + (1-\sqrt{2})\alpha_2 = \frac{15}{2}$ —— ①

$a_2 = 4$

$(1+\sqrt{2})^2\alpha_1 + (1-\sqrt{2})^2\alpha_2 - \frac{1}{2}(2)^2 - 2(2) - 2 = 4$

$(1+\sqrt{2})^2\alpha_1 + (1-\sqrt{2})^2\alpha_2 = 12$ —— (ii)

$$\alpha_1 = \frac{\frac{15}{2} - (1-\sqrt{2})\alpha_2}{(1+\sqrt{2})} \text{ —— (i)}$$

$$(1+\sqrt{2})^2\left[\frac{\frac{15}{2} - (1-\sqrt{2})\alpha_2}{(1+\sqrt{2})}\right] + (1-\sqrt{2})^2\alpha_2 = 12$$

$$(1+\sqrt{2})\left(\frac{15}{2} - (1-\sqrt{2})\alpha_2\right) + (1-\sqrt{2})^2\alpha_2 = 12$$

$$(4-2\sqrt{2})\alpha_2 = \frac{9-15\sqrt{2}}{2}$$

$$\alpha_2 = \frac{-12-21\sqrt{2}}{8}$$

$$\alpha_1 = \frac{\frac{15}{2} - (1-\sqrt{2})\left(\frac{-12-21\sqrt{2}}{8}\right)}{(1+\sqrt{2})}$$

$$\alpha_1 = \frac{-12+21\sqrt{2}}{8}$$

$$a_n = \frac{-12+21\sqrt{2}}{8}(1+\sqrt{2})^n - \frac{12-21\sqrt{2}}{8}(1-\sqrt{2})^n - \frac{1}{2}n^2 - 2n - 2 \text{ (Ans)}$$

(4) $a_n = 2a_{n-1} + b_{n-1}$

$b_n = b_{n-1} + a_{n-1}$

$a_n - b_n = 2a_{n-1} + b_{n-1} - (b_{n-1} + a_{n-1})$

$= 2a_{n-1} + \cancel{b_{n-1}} - \cancel{b_{n-1}} - a_{n-1}$

$a_n - b_n = a_{n-1}$ —— (i)

Put $n = n+1$ in $b_n = b_{n-1} + a_{n-1}$

$b_{n+1} = b_{n+1-1} + a_{n+1-1}$

$a_n + b_n = b_{n+1}$ —— (ii)

$a_n - \cancel{b_n} = a_{n-1}$

$a_n + \cancel{b_n} = b_{n+1}$

$2a_n = a_{n-1} + b_{n+1}$ —— (iii)

Put $n = n-2$ in ~~$a_n = 2a_{n-1} + b_{n-1}$~~ eq (iii)

~~$a_{n-2} = 2a_{n-2-1} + b_{n-2-1}$~~

~~$a_{n-2} = 2a_{n-3} + b_{n-3}$ —— (iv)~~

$2a_{n-2} = a_{n-2-1} + b_{n-2+1}$

$2a_{n-2} = a_{n-3} + b_{n-1}$

$b_{n-1} = 2a_{n-2} - a_{n-3}$ —— (iv)

Put eq (iv) in $a_n = 2a_{n-1} + b_{n-1}$

$a_n = 2a_{n-1} + (2a_{n-2} - a_{n-3})$

$a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}$

$a_n - 2a_{n-1} - 2a_{n-2} + a_{n-3} = 0$

$r^n - 2r^{n-1} - 2r^{n-2} + r^{n-3} = 0 \quad (\div r^{n-3})$

$r^3 - 2r^2 - 2r + 1 = 0$

$(r+1)$ is a factor of $r^3 - 2r^2 - 2r + 1$

$$r+1 \overline{) r^3 - 2r^2 - 2r + 1} \left(r^2 - 3r + 1 \right.$$

$$\cancel{r^3} + r^2$$

$$-3\cancel{r^2} - 2r$$

$$-\cancel{3r^2} - 3r$$

$$\cancel{r} + \cancel{1}$$

$$\cancel{r} - \cancel{1}$$

$(r+1)(r^2 - 3r + 1) = 0$

$r_1 = -1$, $r_2 = \dfrac{3+\sqrt{5}}{2}$, $r_3 = \dfrac{3-\sqrt{5}}{2}$

~~$a_n = \alpha_3 r_3^n + \alpha_2 r_2^n + \alpha_1 r_1^n$~~

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$

$a_n = (-1)^n \alpha_1 + \left(\dfrac{3+\sqrt{5}}{2}\right)^n \alpha_2 + \left(\dfrac{3-\sqrt{5}}{2}\right)^n \alpha_3$

$a_1 = 1$

$-\alpha_1 + \left(\dfrac{3+\sqrt{5}}{2}\right)\alpha_2 + \left(\dfrac{3-\sqrt{5}}{2}\right)\alpha_3 = 1$ —— (V)

$a_2 = 2$

$(-1)^2 \alpha_1 + \left(\dfrac{3+\sqrt{5}}{2}\right)^2 \alpha_2 + \left(\dfrac{3-\sqrt{5}}{2}\right)^2 \alpha_3 = 2$

$\alpha_1 + \left(\dfrac{3+\sqrt{5}}{2}\right)^2 \alpha_2 + \left(\dfrac{3-\sqrt{5}}{2}\right)^2 \alpha_3 = 2$ —— (VI)

$\cancel{a_3 = ?}$

$\cancel{\alpha_3 = 2a_{3-1} + 2a_{3-2} - a_{3-3}}$

$\cancel{= 2a_2 + 2a_1 - a_0}$

$\cancel{= 2 + 1 -}$

$a_3 = ?$

$a_n = 2a_{n-1} + b_{n-1}$

$a_3 = 2a_{3-1} + b_{3-1}$

$= 2a_2 + b_2 = 2(2) + 1 = 5$

$a_3 = 5$

$$(-1)^3\alpha_1 + \left(\frac{3+\sqrt{5}}{2}\right)^3\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)^3\alpha_3 = 5$$

$$-\alpha_1 + \left(\frac{3+\sqrt{5}}{2}\right)^3\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)^3\alpha_3 = 5 \quad \text{——} \quad (VII)$$

(V) + (VII)

$$-\cancel{\alpha_1} + \left(\frac{3+\sqrt{5}}{2}\right)^2\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)^2\alpha_3 = 2$$

$$\cancel{\alpha_1} + \left(\frac{3+\sqrt{5}}{2}\right)\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)\alpha_3 = 1$$

$$(5+2\sqrt{5})\alpha_2 + (5-2\sqrt{5})\alpha_3 = 3$$

$$\alpha_2 = \frac{3-(5-2\sqrt{5})\alpha_3}{(5+2\sqrt{5})}$$

(V) − (VII)

$$-\cancel{\alpha_1} + \left(\frac{3+\sqrt{5}}{2}\right)\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)\alpha_3 = 1$$

$$-\cancel{\alpha_1} + \left(\frac{3+\sqrt{5}}{2}\right)^3\alpha_2 + \left(\frac{3-\sqrt{5}}{2}\right)^3\alpha_3 = 5$$

$$-\frac{15-7\sqrt{5}}{2}\alpha_2 - \frac{15+7\sqrt{5}}{2}\alpha_3 = -4$$

$$-\frac{15-7\sqrt{5}}{2}\left(\frac{3-(5-2\sqrt{5})\alpha_3}{5+2\sqrt{5}}\right) - \frac{15+7\sqrt{5}}{2}\alpha_3 = -4$$

$$\cancel{2+3-2\sqrt{5}}$$

$$-\frac{1-\sqrt{5}}{2}\left(3-(5-2\sqrt{5})\alpha_3\right) - \frac{15+7\sqrt{5}}{2}\alpha_3 = -4$$

$$-10+5\sqrt{5}\,\alpha_3 = \frac{-5+3\sqrt{5}}{2}$$

$$\alpha_3 = \frac{5+\sqrt{5}}{10}$$

$$\alpha_2 = \frac{3-(5-2\sqrt{5})\left(\frac{5+\sqrt{5}}{10}\right)}{(5+2\sqrt{5})} = \frac{5-\sqrt{5}}{10}$$

$$-\alpha_1 + \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{5-\sqrt{5}}{10}\right) + \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{5+\sqrt{5}}{10}\right) = 1$$

$$-\alpha_1 + 1 = 1$$

$$\alpha_1 = 0$$

$$a_n = \left(\frac{5-\sqrt{5}}{10}\right)^n \left(\frac{3+\sqrt{5}}{10}\right)^n + \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{10}\right)^n \quad \text{(Ans)}$$

(5) $E(X) = \sum_{x=0}^{n} x \cdot {}^{n}C_x p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}$

$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$

$y = x-1$, $m = n-1$

$= \sum_{y=0}^{m} \frac{(m+y)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y}$

$= (m+1) p \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$

$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$

~~$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^m \cdot (1-p)^y$~~

~~$u = p, v = 1-p$~~

~~$np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p u$~~

$= np \sum_{y=0}^{m} \frac{m!}{0!(m-0)!} p^0 (1-p)^{m-0}$

$= np \sum_{y=0}^{m} \frac{\cancel{m!}}{\cancel{m!}} p (p)^0 (1-p)^{m-0}$

$= np (\cancel{p} + 1 - \cancel{p})^m = np(1)$

$E(X) = np$ (shown)

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\,{}^{n}C_{x}\, p^{x} (1-p)^{n-x} \qquad \Bigg|\ \begin{array}{l} y = x-2 \\ m = n-2 \end{array}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^{x} (1-p)^{n-x}$$

$$= n(n-1)p^{2} \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x}$$

$$= n(n-1)p^{2} \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^{y} (1-p)^{m-y}$$

$$= n(n-1)p^{2} (p + (1-p))^{m} = n(n-1)p^{2}(1)$$

$$= n(n-1)p^{2}$$

$$\mathrm{Var}(X) = E(X(X-1)) + E(X) - E(X)^{2}$$

$$= n(n-1)p^{2} + np - (np)^{2}$$

$$= (n^{2}-n)p^{2} + np - n^{2}p^{2}$$

$$= \cancel{n^{2}p^{2}} - np^{2} + np - \cancel{n^{2}p^{2}}$$

$$\mathrm{Var}(X) = np(1-p) \quad \text{(shown)}$$