Introduction to Robotics
CSE 461

Chapter 2: Lecture 6 (Forward Kinematics)

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Last Class

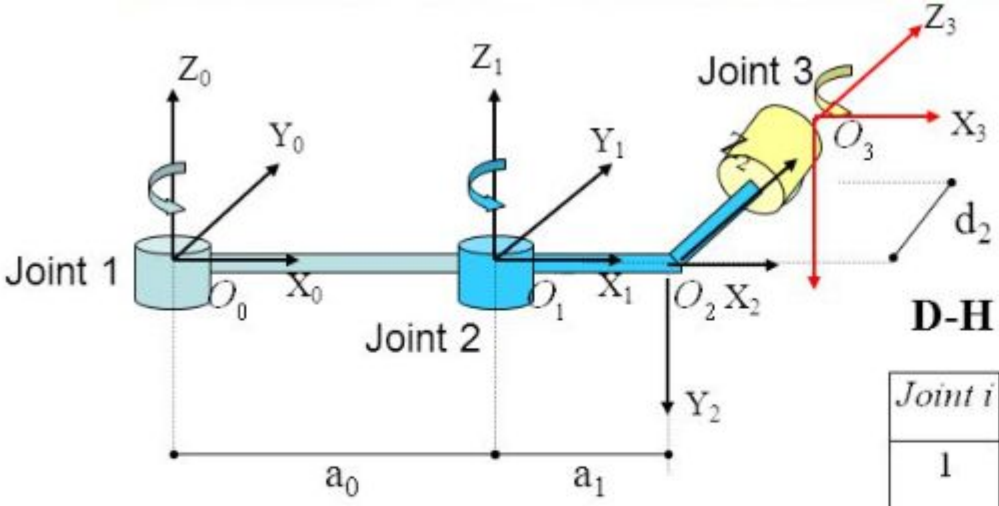
Kinematics

Degrees of Freedom (DOF)

D-H Parameters

Recall D-H Parameters

D-H Parameters



D-H Link Parameter Table

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

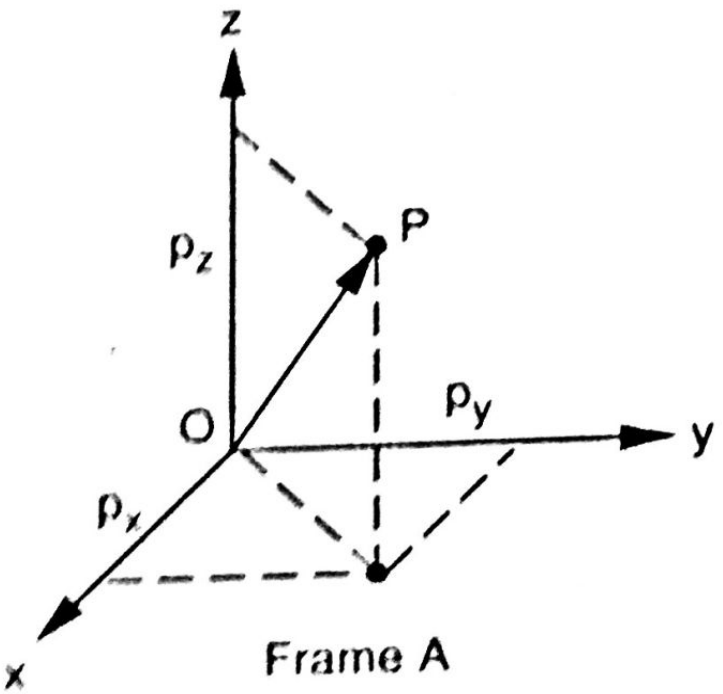
α_i : rotation angle from Z_{i-1} to Z_i about X_i

a_i : distance from intersection of Z_{i-1} & X_i to origin of i coordinate along X_i

d_i : distance from origin of (i-1) coordinate to intersection of Z_{i-1} & X_i along Z_{i-1}

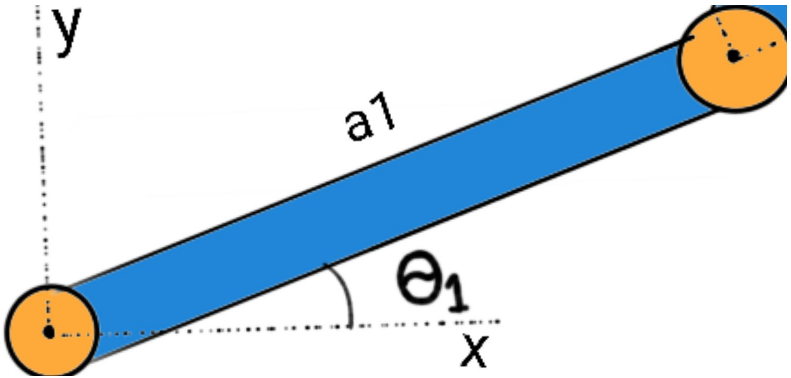
θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Position



$$^{A}\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Rotation

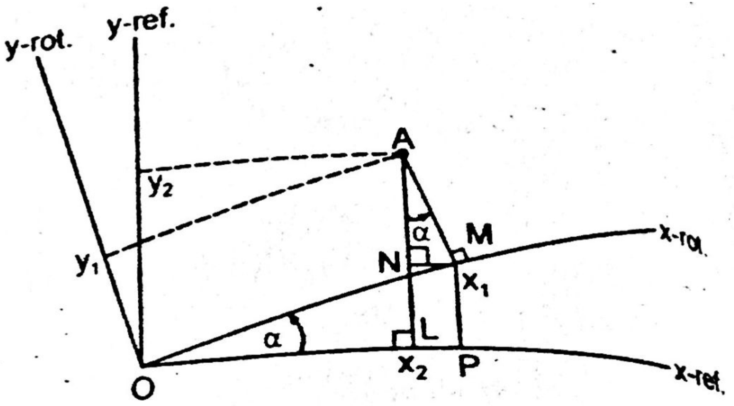


$$\text{Z-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\text{X-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\text{Y-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Rotation



$$x_2 = |OL| = |OP| - |LP|$$
$$y_2 = |LA| = |LN| + |NA|$$
$$|OP| = |OM| \cos\alpha$$
$$= x_1 \cos\alpha$$
$$|LN| = |PM| = |OM| \sin\alpha$$
$$= x_1 \sin\alpha$$
$$|LP| = |NM| = |MA| \sin\alpha$$
$$= y_1 \sin\alpha$$
$$|NA| = |MA| \cos\alpha$$
$$= y_1 \cos\alpha$$

Combining these eqns., we get

$$\left.\begin{aligned} x_2 &= x_1 \cos\alpha - y_1 \sin\alpha \\ y_2 &= x_1 \sin\alpha + y_1 \cos\alpha \end{aligned}\right\}$$

Homogeneous Transformation Matrix

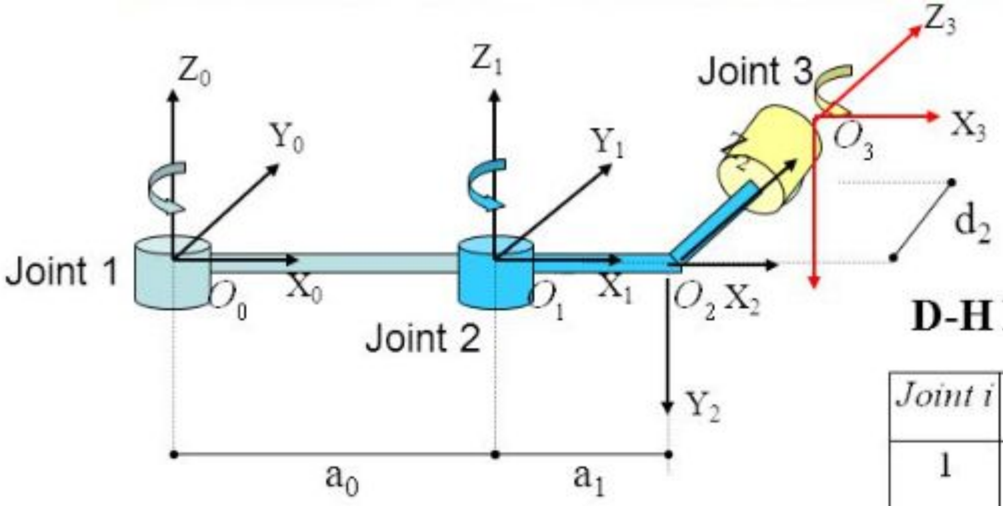
Homogeneous Transformation Matrices are commonly used to represent linkages (for robot arm) in a systematic way for representing *Position Vector* and *Rotation* Matrices {4X4 Matrix}.

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R}_{3x3} & | & \boldsymbol{P}_{3x1} \\ -- & | & -- \\ \boldsymbol{f}_{1x3} & | & 1x1 \end{bmatrix} = \begin{bmatrix} \textbf{\textit{Rotation}} & | & \textbf{\textit{Position}} \\ \textbf{\textit{Matrix}} & & \textbf{\textit{Matrix}} \\ \text{---------} & | & \text{--------} \\ \textbf{\textit{Perspective}} & | & \textbf{\textit{Scaling}} \\ \textbf{\textit{Transformation}} & & \end{bmatrix}$$

Denavit-Hartenberg (D-H) Homogeneous Transformation matrices

$$
\begin{aligned}
A_i &= R_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}R_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

D-H Parameters



D-H Link Parameter Table

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θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Transformation Matrices

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

$$T_{i-1}^{i} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = (T_0^1)(T_1^2)(T_2^3)$$

$$T_0^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_0 \cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_0 \sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

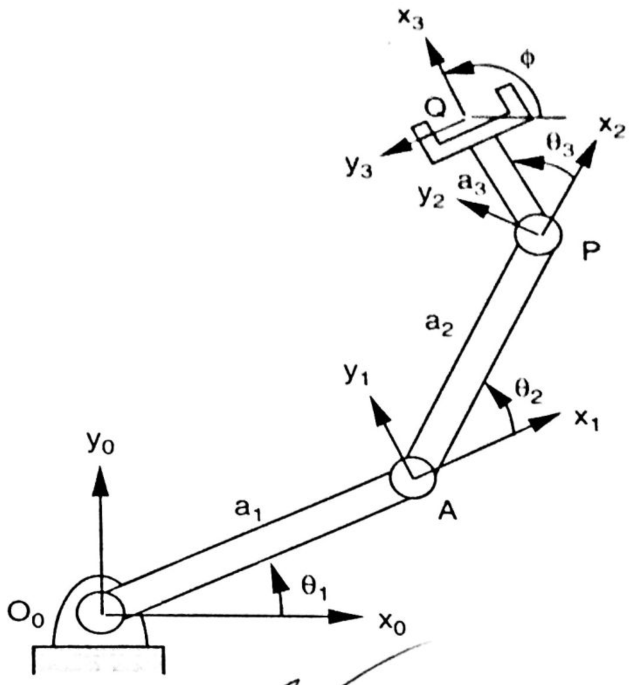
$$T_1^2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & a_1 \cos\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 & a_1 \sin\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3



$$
{}^0A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

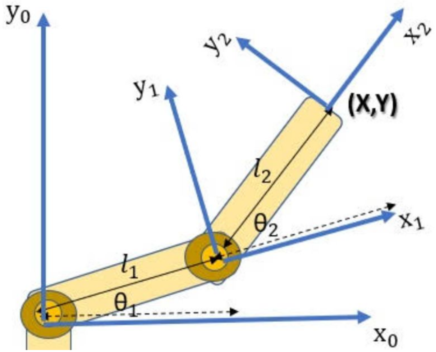
Overall transformation matrix?

$$
{}^{0}A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Overall transformation matrix?

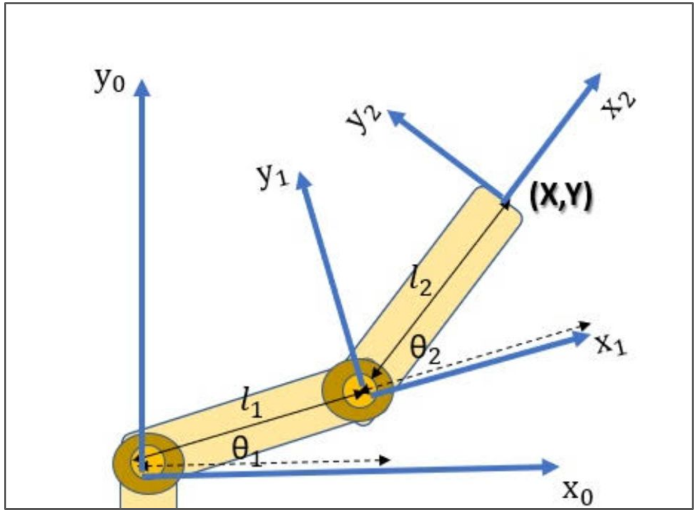
$$
\begin{array}{cc} & \text{Orientation} \qquad\qquad\qquad \text{Position} \end{array}
$$

$$
{}^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$



Steps

1. Define z axis
2. Define x,y using R.H rule

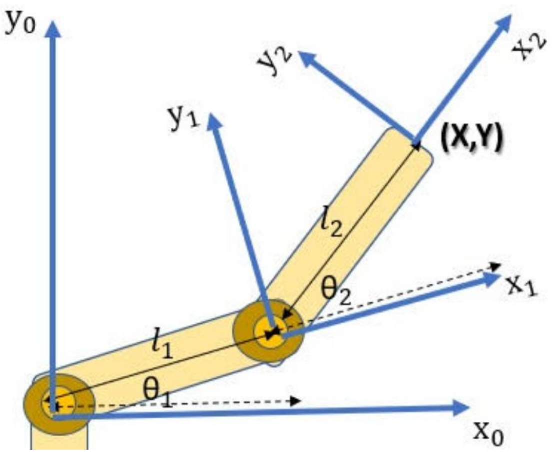


Steps

1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2



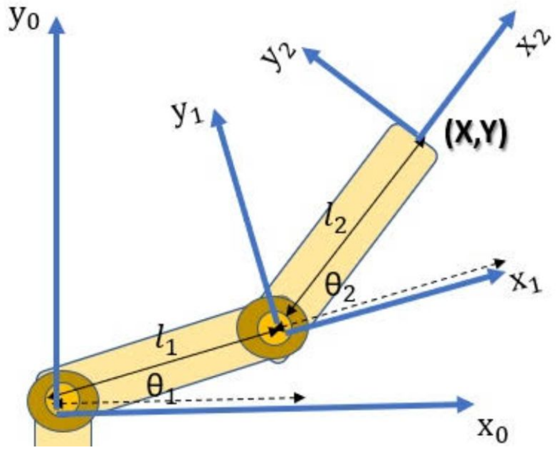
Steps

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters
4. Put the parameters to the homogeneous transformation matrices



Steps

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
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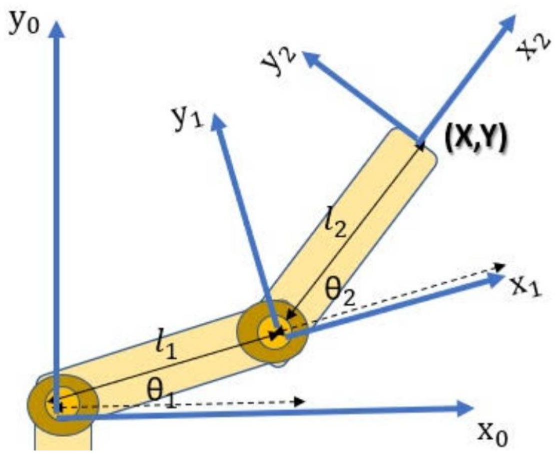
$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters
4. Put the parameters to the homogeneous transformation matrices
5. Multiply for each joints

$$\boxed{T_0^3 = (T_0^1)(T_1^2)(T_2^3)}$$



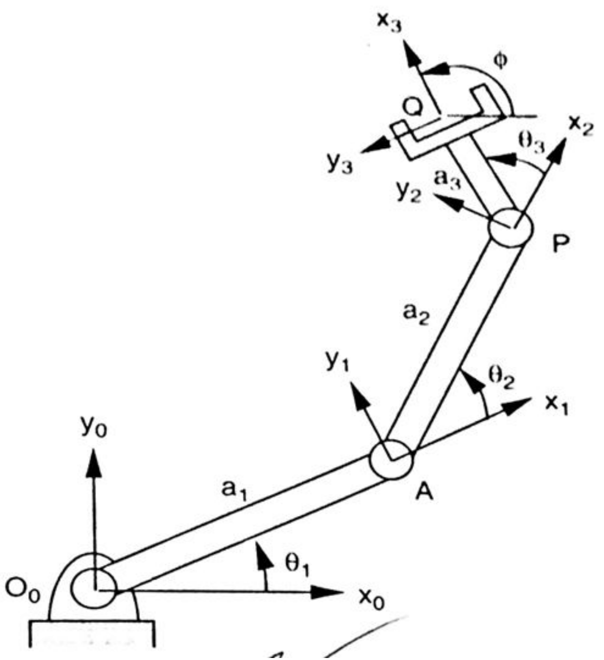
Forward Kinematics

Initial position vector : $^{3}\mathbf{q} = [0, 0, 0, 1]^{T}$

Final position vector?

$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^{0}A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

Assignment: Calculate position of end effector of this 3 DOF Manipulator



Angles of three joints are **30 degree, 45 degree and 10 degree.**

$\mathbf{a_1 = 10}$
$\mathbf{a_2 = 8}$
$\mathbf{a_3 = 3}$

Next class

Inverse Kinematics

Thanks