Week 4 (Lecture 8)

Contents:

- Reduced Row Echelon Form (RREF)
- Gauss–Jordan Elimination Method (An extension of Gaussian Elimination)
- Solving System of Linear Equations by Gauss–Jordan Elimination
- Inverse of a matrix
- Properties of inverse matrices
- Finding inverse of a matrix by Gauss–Jordan Elimination
- Solving system of linear equations by using the inverse matrix

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Reduced Row Echelon Form

We recall that

A matrix is in ***row echelon form*** if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom.
2. In each nonzero row, the first nonzero entry (called the leading entry) is in a column to the left of any leading entries below it.

Examples of matrices in row echelon form:

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Definition

A matrix is in ***reduced row echelon form*** if it satisfies the following properties:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a **leading** 1).
3. Each column that has a leading 1 has zeros in every position above and below its leading 1.

For example, the following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 3 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 5 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For 2×2 matrices, the possible reduced row echelon forms are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Note. Unlike the row echelon form, the reduced row echelon form of a matrix is ***unique.***

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Gauss–Jordan Elimination

In **Gauss–Jordan elimination**, we proceed as in Gaussian elimination (with **leading 1** in each nonzero row) but reduce the augmented matrix to reduced row echelon form.

Steps for Gauss-Jordan Elimination:

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to reduced row echelon form.
3. If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

Example 1. Solve by using Gauss–Jordan elimination

$$\begin{aligned} 2x_1 + 5x_2 + 3x_3 &= 11 \\ -x_1 + 3x_2 + x_3 &= 5 \\ x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

Solution. The system has its matrix form

$$A\mathbf{x} = \mathbf{b}$$

where,

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and } \mathbf{b} = \begin{bmatrix} 11 \\ 5 \\ -3 \end{bmatrix}$$

The augmented matrix of the above system is

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 5 & 3 & 11 \\ -1 & 3 & 1 & 5 \\ 1 & 1 & -2 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ -1 & 3 & 1 & 5 \\ 2 & 5 & 3 & 11 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 4 & -1 & 2 \\ 0 & 3 & 7 & 17 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -8 & -15 \\ 0 & 3 & 7 & 17 \end{array}\right]$$

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$$\xrightarrow{R_3-3R_2\to R_3}\left[\begin{array}{ccc|c}1 & 1 & -2 & -3\\ 0 & 1 & -8 & -15\\ 0 & 0 & 31 & 62\end{array}\right]\xrightarrow{\frac{1}{31}R_3\to R_3}\left[\begin{array}{ccc|c}1 & 1 & -2 & -3\\ 0 & 1 & -8 & -15\\ 0 & 0 & 1 & 2\end{array}\right]$$

$$\xrightarrow[R_1+2R_3\to R_1]{R_2+8R_3\to R_2}\left[\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{array}\right]\xrightarrow{R_1-R_2\to R_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{array}\right]$$

Therefore, the required solution is

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 2\end{bmatrix}.$$

which is the unique solution to the given system. ■

Example 2. Solve the system using Gauss–Jordan elimination

$$\begin{aligned}x_1 - x_2 - x_3 + 2x_4 &= 1\\ 2x_1 - 2x_2 - x_3 + 3x_4 &= 3\\ -x_1 + x_2 - x_3 &= -3\end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{cccc|c}1 & -1 & -1 & 2 & 1\\ 2 & -2 & -1 & 3 & 3\\ -1 & 1 & -1 & 0 & -3\end{array}\right]\xrightarrow[R_3+R_1\to R_3]{R_2-2R_1\to R_2}\left[\begin{array}{cccc|c}1 & -1 & -1 & 2 & 1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & -2 & 2 & -2\end{array}\right]$$

$$\xrightarrow{2R_2+R_3\to R_3}\left[\begin{array}{cccc|c}1 & -1 & -1 & 2 & 1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]\xrightarrow{R_1+R_2\to R_1}\left[\begin{array}{cccc|c}1 & -1 & 0 & 1 & 2\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

The associated system is now

$$\begin{aligned} x_1 - x_2 + x_4 &= 2 \\ x_3 - x_4 &= 1 \end{aligned}$$

In this case, the leading variables are x_1 and x_3, and the free variables are x_2 and x_4.

If we assign parameters $x_2 = s$ and $x_4 = t$, where $s, t \in \mathbb{R}$. we get

$$x_1 = 2 + s - t$$

$$x_3 = 1 + t$$

Therefore, the solution can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}; \text{ where } s, t \in \mathbb{R}.$$

Thus, the system has infinitely many solutions. ∎

Example 3. Solve the system using Gauss–Jordan elimination

$$2x_1 + 3x_2 - x_3 = 0$$

$$-x_1 + 5x_2 + 2x_3 = 0$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ -1 & 5 & 2 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 8 & 1 & 0 \\ -1 & 5 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 8 & 1 & 0 \\ 0 & 13 & 3 & 0 \end{array}\right] \xrightarrow{\frac{1}{13} R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 8 & 1 & 0 \\ 0 & 1 & 3/13 & 0 \end{array}\right] \xrightarrow{R_1 - 8R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -11/13 & 0 \\ 0 & 1 & 3/13 & 0 \end{array}\right]$$

Now, the associated system is

$$\begin{aligned} x_1 - \frac{11}{13} x_3 &= 0 \\ x_2 + \frac{3}{13} x_3 &= 0 \end{aligned}$$

Here x_1 and x_2 are leading variables, and x_3 is free variable.

Set, $x_3 = t$, where $t \in \mathbb{R}$. Then

$$x_1 = \frac{11}{13}t, \qquad x_2 = -\frac{3}{13}t.$$

Therefore, the solution to the given homogeneous system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{13}t \\ -\frac{3}{13}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{11}{13} \\ -\frac{3}{13} \\ 1 \end{bmatrix};\ t \in \mathbb{R}.$$

This system of equations has infinitely many solutions, one of which is the trivial solution (given by $t = 0$). ∎

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Inverse of a Matrix

Definition 8.2.1 An $n \times n$ matrix A is called **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n. \tag{1}$$

Here I_n is the identity matrix of order n which has 1's in the main diagonal and 0's everywhere else, namely

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If the order of I_n is clear from the context, we will simply write I for I_n.

Here B is called *an* inverse of A. Since the eq. (1) is symmetric in A and B, we can also say A is an inverse of B.

For example, $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is invertible since there exists a matrix $B = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix}$ such that

$$AB = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$BA = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Not every square matrix is invertible. If A is not an invertible matrix, then we say A is **noninvertible** (or **singular**).

For example, the zero matrix $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is noninvertible. Clearly, there cannot exist any matrix B such that $OB = BO = I$ since $OB = BO = O$.

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Theorem (Uniqueness of an Inverse)

If A is an invertible matrix, then its inverse is unique.

Proof. Let A be an $n \times n$ invertible matrix. So, there exists an $n \times n$ matrix B such that

$$AB = BA = I.$$

Let us assume that there exists another $n \times n$ matrix C such that

$$AC = CA = I.$$

To show that $B = C$, we use

$$\begin{aligned} B &= BI && \text{[Definition of Identity matrix]} \\ &= B(AC) && [C \text{ is an inverse of } A] \\ &= (BA)C && \text{[Associative Property for Matrix Multiplication]} \\ &= IC && [B \text{ is an inverse of } A] \\ &= C && \text{[Definition of Identity matrix]} \end{aligned}$$

The proof is complete. ■

Notation: If A is invertible and since its inverse is unique, we will use the symbol A^{-1} to denote *the* inverse of A.

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Properties of Inverse of a Matrix

Theorem If A be an $n \times n$ invertible matrix, then the following are true.

1. $(A^{-1})^{-1} = A$.

2. $(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{k \text{ many factors}} = (A^{-1})^k$

3. $(cA)^{-1} = c^{-1}A^{-1}$ if $c \neq 0$.

4. $(A^T)^{-1} = (A^{-1})^T$

Proof.

1. By definition,

$$A^{-1}(A^{-1})^{-1} = I$$

Now by multiplying both sides by A from the left, we get the result as follows

$$A(A^{-1}(A^{-1})^{-1}) = AI$$

$$(AA^{-1})(A^{-1})^{-1} = A$$

$$I(A^{-1})^{-1} = A$$

$$(A^{-1})^{-1} = A$$

2. By definition,

$$A^k(A^k)^{-1} = I$$

Since k is a positive integer,

$$\underbrace{(AA \dots A)}_{k}(A^k)^{-1} = I$$

Now multiplying both sides by A^{-1} from the left by k many times we obtain the desired result.

$$(A^k)^{-1} = \underbrace{(A^{-1}A^{-1} \dots A^{-1})}_{k} = (A^{-1})^k$$

3. By definition,

$$(cA)(cA)^{-1} = I$$

$$A(cA)^{-1} = c^{-1}I$$

$$(cA)^{-1} = A^{-1}(c^{-1}I) = c^{-1}(A^{-1}I) = c^{-1}A^{-1}$$

4. By definition,

$$AA^{-1} = I$$

Taking the transpose of both sides, we obtain

$$(AA^{-1})^T = I^T$$

$$(A^{-1})^T A^T = I$$

Here we have used $(AB)^T = B^T A^T$ and $I^T = I$.

$$(A^{-1})^T A^T = I$$

$$\therefore (A^T)^{-1} = (A^{-1})^T.$$

Theorem. If A and B are two $n \times n$ invertible matrices, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(This is known as **Socks-shoes property**. You might be surprised to see that taking the multiplicative inverse reverses the order of multiplication. So interpret A as putting on socks, and B as putting on shoes. To reverse the operation AB of putting on both socks and shoes, you must reverse the order: you take off shoes first, then the socks, and so the inverse operation is $B^{-1}A^{-1}$.)

Proof. The proof is straightforward. Observe that

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= \left(B^{-1}(A^{-1}A)\right)B \\ &= (B^{-1}I)B \\ &= B^{-1}B \\ &= I\end{aligned}$$

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Similarly,

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= \left(A(BB^{-1})\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

Therefore, AB is invertible and $B^{-1}A^{-1}$ is an inverse of AB. Since inverse of an invertible matrix is unique, $B^{-1}A^{-1}$ is the inverse of AB. Thus

$$(AB)^{-1} = B^{-1}A^{-1}. \blacksquare$$

Corollary: If $A_1, A_2, \ldots A_n$ are $n \times n$ invertible matrices, then

$$(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1}A_1^{-1}.$$

In general, $AC = BC$ does not imply $A = B$. In other words, cancellation property does not hold for matrices. For example, if

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

then $AC = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = BC$. However, $A \neq B$.

The next theorem tells us under what condition matrix equation enjoys the cancellation property.

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Properties of Inverse of a Matrix

Theorem Let A, B, and C be $n \times n$ matrices. If C is invertible, then

1. If $AC = BC$, then $A = B$. This is called the **right cancellation property**.
2. If $CA = CB$, then $A = B$. This is called the **left cancellation property**.

Proof.

1. Since C^{-1} exists, we have

$$A = AI = A(CC^{-1}) = (AC)C^{-1} = (BC)C^{-1} = B(CC^{-1}) = BI = B$$

The proof of 2. is similar. ■

Caution that $AC = CB$ does not imply $A = B$ even if C is invertible unless $AC = CA$.

Properties of Inverse of a Matrix

Theorem. If A is invertible, then the system $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

Proof.

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

The proof is complete. ■

Definition Two matrices A and B are called **row equivalent** if one can be obtained from the other by a sequence of elementary row operations.

For example, the matrices $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & -4 \\ 1 & 2 & 3 \end{bmatrix}$ are row equivalent since

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{4R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & -4 \\ 1 & 2 & 3 \end{bmatrix} = B.$$

Theorem. A is invertible if and only if A is row equivalent to I.

Proof. $(\Rightarrow)$ Let A be invertible, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $I\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{b}$. Therefore, by Gauss–Jordan elimination process, A can be reduced to I by a sequence of elementary row operations. Hence, A is row equivalent to I.

Conversely, if A is row equivalent to I, then there exists a sequence of elementary matrices $E_1, E_2, \dots, E_n$ corresponding to each elementary row operations such that

$$E_n \cdots E_2 E_1 A = I$$

Define $B = E_n \cdots E_2 E_1$. Hence A is invertible and B is the inverse of A.

How to find A^{-1}?

$$\underbrace{(E_n \cdots E_2 E_1)}_{A^{-1}} A = I$$

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Therefore, by the uniqueness of inverse, we have

$$A^{-1} = E_n \cdots E_2 E_1 I.$$

Therefore, the formula says that one needs to apply the same sequence of elementary row operations to an identity matrix I that helped A reduces to I.

Example. Find A^{-1} for

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix},$$

if exists.

Solution.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 2 & 7 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1 \to R_1]{-4R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & -2 \\ 0 & 1 & 0 & 2 & -7 & 4 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

Verification

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ -2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

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Application

Example. Solve the following system

$$x_1 - 2x_3 = 3$$

$$2x_1 + x_2 = 1$$

$$3x_1 + 2x_2 + x_3 = -2$$

Solution. Let us put the system into the following form $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

Since A is invertible, the given system has a unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

From the previous example, we have

$$A^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

So,

$$\mathbf{x} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ 1 \end{bmatrix}.$$