MAT216: Linear Algebra & Fourier Analysis

Lecture Note: Week 2 Lecture 4

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Contents:

- Introduction to System of Linear Equation

Reference Book:

- Introduction to Linear Algebra, 5th Ed. Gilbert Strang

A linear equation is one that does not have any variable whose power is greater than 1 i.e. a linear equation is always a polynomial of degree 1. For an example, $5x + 6y = 30$. In the two-dimensional case, they always form lines, in other dimension, they might also form planes, points or hyperplanes. Linear equations are always perfect straight line. Nonlinear equations are higher degree polynomials, looks like curve in a graph and has a variable slope value.

Linear equations have some useful properties, mostly in that they are very easy to manipulate and solve. Nonlinear equation equations, for the most part, are much harder to solve and manipulate.

Example of linear equation:

- $3x + 4y = 12$
- $4y = 12$
- $x + y = 1$

Example of nonlinear equation:

- $3x^2 = 12$
- $x^2 + x + 2 = 25$
- $x^2 + 12xy = 12$

When building linear models to solve problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let us assume Frodo and Samwise are going on an adventure. As food for their journey, Frodo has to bring x number of Chickens and Samwise has to bring y number of Cakes. Each Chicken cost 4 silver coins and each Cake cost 2 silver coins. Let's say, together they spent 14 silver coins. Now we can start modeling i.e. we have

$$\mathbf{4x + 2y = 14} \quad \textbf{(1)}$$

Is a linear equation/ linear model. Solution of this linear equation are the values for which, the equation statement (1) becomes TRUE. For instant, if we consider, $x = 3\ and\ y = 1$

(1)$\Rightarrow$ $\mathbf{4 \times 3 + 2 \times 1 = 14.}$

We may have other solutions for this linear equation/ linear model, like $\boldsymbol{x = 2\ and\ y = 3}$.

An example of a linear equation in two unknowns is $2x + 7y = 5$. A solution of this equation is $x = -1\ and\ y = 1$ and it is also true that, this equation has many more solutions. The graph of this equation is a line.

An example of a linear equation in three unknowns is $2x + y + \pi z = \pi$. A solution of this equation is $x = 0, y = 0, z = 1$. This equation has many more solutions. The graph of this equation is a plane (in 3-space).

Definition 1: A linear equation in n (Unknowns) variables $x_1, x_2, \ldots\ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots\ldots\ldots\ldots + a_nx_n = b \quad (2)$$

here $a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots, a_n$, b are real numbers, we say b is the constant term and a_i is the coefficient of x_i. For real numbers $\beta_1,\ \beta_2, \ldots\ldots\ldots\ldots, \beta_n$ if

$$a_1\beta_1 + a_2\beta_2 + \ldots\ldots\ldots\ldots + a_n\beta_n = b,$$

We say that $x_1 = \beta_1, x_2 = \beta_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_n = \beta_n$ is a solution of this equation.

Example:

- The equation $2x_1 + x_2 - 7x_3 = \sqrt{5}$ is linear.
- The equation $3x_1x_2 + 2x_3^2 = 1$ is NOT linear

Definition 2: A system of linear equations is a collection of one or more linear equations.

Example:

$$3x + 2y + 7z + 2w = 6$$

$$5x + 11y - 5w = 6$$

$$2y + 7z = -1$$

It is possible to solve this system using the elimination or substitution method, but it is also possible to do it with matrix operation. Before we start setting up matrices, it important to remember that each equation in the system becomes a row, each variable in the system becomes a column. The variables are dropped and the coefficients are placed into a matrix. If the right-hand side is included it is called an augmented matrix. if the right-hand side is not included, it's called a **coefficient** matrix.

Example: Consider a linear system

$$x + 3y - 2z = 5$$

$$3x + 5y + 6z = 7$$

$$2x + 4y + 3z = 8$$

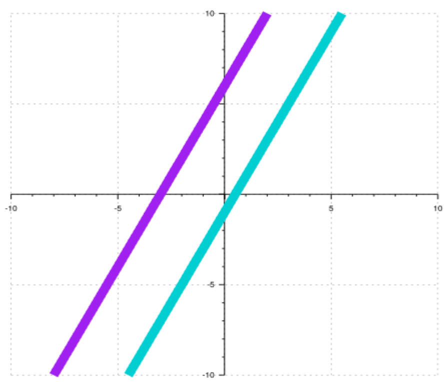
The matrix $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$ is called the coefficient matrix of the given linear system and

The matrix $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$ is called the augmented matrix of the system.

There are three types of solution which are possible when solving a system of linear equations. Systems of equations fall into two categories: **consistent systems and inconsistent systems**.

A system of equation that has **no solution** is said to be **inconsistent**, **which is one in which the equations represent parallel lines.**

Inconsistent System [No solution]

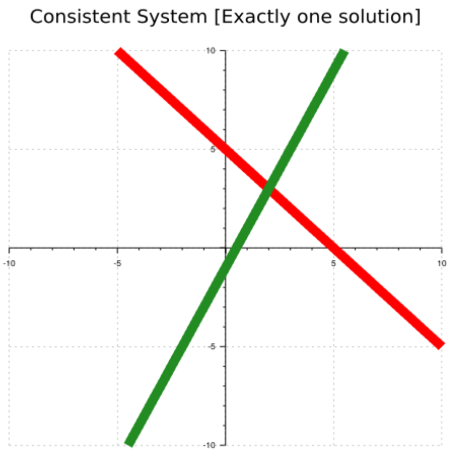


INCONSISTENT:

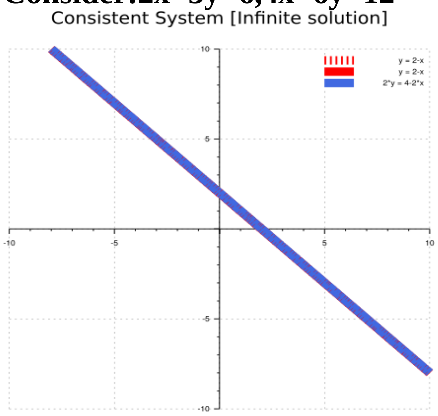
2x+3y=6, 4x+6y=5

If there is at least one solution of the system, it is called **consistent.**

A consistent system is considered to be an **independent system** if it has a **single solution**, such lines have different slopes and intersect at one particular point.



Consider:2x+3y=6,4x+6y=12



$$\left(\begin{matrix}2 & 3\\4 & 6\end{matrix}\middle|\begin{matrix}6\\12\end{matrix}\right)=\left(\begin{matrix}\mathbf{2} & 3\\0 & 0\end{matrix}\middle|\begin{matrix}6\\0\end{matrix}\right) R_2 \to R_2-2R_1$$

$$Let\ y=a, a: any\ real\ number$$

$$R_1 \to 2x+3y=6, \qquad 2x=6-3a, \qquad x=3-\frac{3}{2}a$$

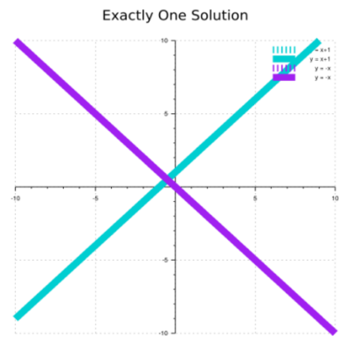
$$(x,y)=(3-\tfrac{3}{2}a,a),\ a\epsilon R$$

A consistent system is considered to be a **dependent system** if the lines coincide (overlap with each other) so the equations represent the same line. Every point on the line represents a coordinate that satisfies the system. Thus, there are an infinite number of solutions.

Example:

a. Let us consider the following system of linear equations and its graph:

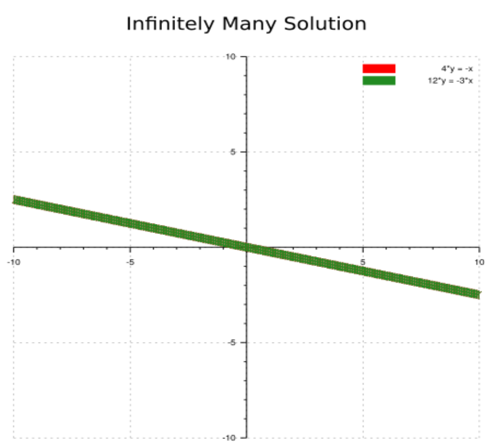
$$\left.\begin{aligned} y &= x + 1 \\ y &= -x \end{aligned}\right\}$$



From the graph we can see the solution or the point of intersection of the system of equation and which is $(x, y) = \left(\frac{-1}{2}, \frac{1}{2}\right)$. Since line intersect so this linear system is independent and there exist a solution which is unique/one solution/ Exactly one solution.

b. Again, consider the following liner system of equations and its graph:

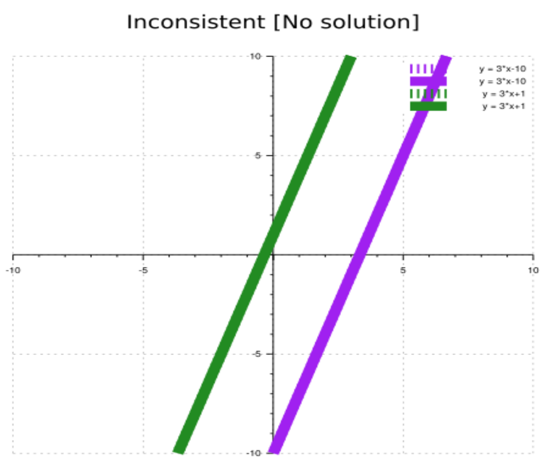
$$\left.\begin{aligned} 4y &= -x \\ 12y &= -3x \end{aligned}\right\}$$



We see that two linear equations have the same slope and the same y-intercept, they meet everywhere. Since they meet everywhere, there are infinitely many solutions.

c. Finally consider the following linear system:

$$\left.\begin{aligned} y &= 3x - 10 \\ y &= 3x + 1 \end{aligned}\right\}$$



From the graph we see that the lines have the same slope but different y-intercepts, they never meet in space, they are just parallel to each other. Since they never meet, there are no solution. This kind of system is known as inconsistent systems.

Review of the Key Ideas

- **Every system of linear equation has no solutions, or has exactly one solution, or has infinitely many solutions.**
- **A system of equation just 'more than 1 equation, A system of linear equations is just more than 1 line.**
- **The solution is where the equation 'meets' or intersect**
- **A system of equation consists of the values of the variables that make all of the equations in the system true.**
- **An independent linear system has exactly one solution. The point where the lines intersect is the only solution**
- **A dependent linear system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate on the line is a solution to the system**
- **When a system of linear equations has the same slope and the same y-intercept, they meet everywhere. Since they meet everywhere, there are infinitely many solutions.**

When multiplying two matrices, there's a manual procedure we all know how to go through. Each result cell is computed separately as the dot-product of a row in the first matrix with a column in the second matrix. While it's the easiest way to compute the result manually, it may obscure a very interesting property of the operation: *multiplying A by B is the linear combination of A's columns using coefficients from B*. Another way to look at it is that it's a *linear combination of the rows of B using coefficients from A*.

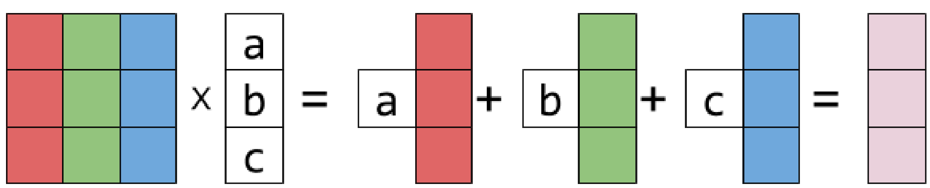
In this quick post I want to show a colorful visualization that will make this easier to grasp.

Right-Multiplication: Combination of Columns

Let's begin by looking at the right-multiplication of matrix X by a column vector:

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 + cz_1 \\ ax_2 + by_2 + cz_2 \\ ax_3 + by_3 + cz_3 \end{pmatrix}$$

Representing the columns of X by colorful boxes will help visualize this:

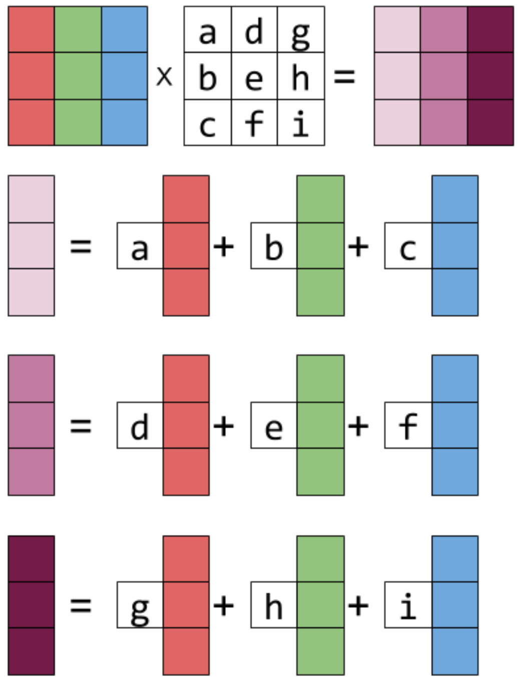


Sticking the white box with a in it to a vector just means: multiply this vector by the scalar a. The result is another column vector - a linear combination of X's columns, with a, b, c as the coefficients.

Right-multiplying X by a matrix is more of the same. Each resulting column is a different linear combination of X's columns:

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} * \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 + cz_1 & dx_1 + ey_1 + fz_1 & gx_1 + hy_1 + iz_1 \\ ax_2 + by_2 + cz_2 & dx_2 + ey_2 + fz_2 & gx_2 + hy_2 + iz_2 \\ ax_3 + by_3 + cz_3 & dx_3 + ey_3 + fz_3 & gx_3 + hy_3 + iz_3 \end{pmatrix}$$

Graphically:



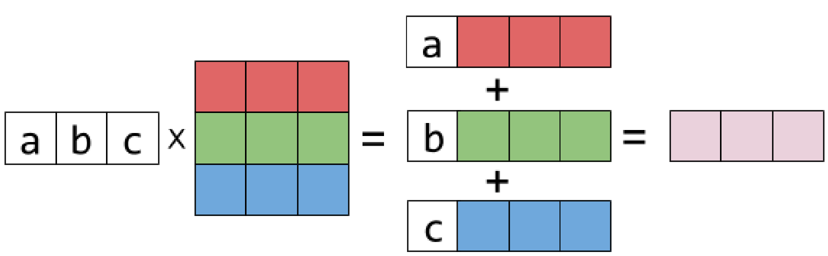
If you look hard at the equation above and squint a bit, you can recognize this column-combination property by examining each column of the result matrix.

Left-Multiplication: Combination of Rows

Now let's examine left-multiplication. Left-multiplying a matrix X by a row vector is a linear combination of X's *rows*:

$$\begin{pmatrix} a & b & c \end{pmatrix} * \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 + cx_3 & ay_1 + by_2 + cy_3 & az_1 + bz_2 + cz_3 \end{pmatrix}$$

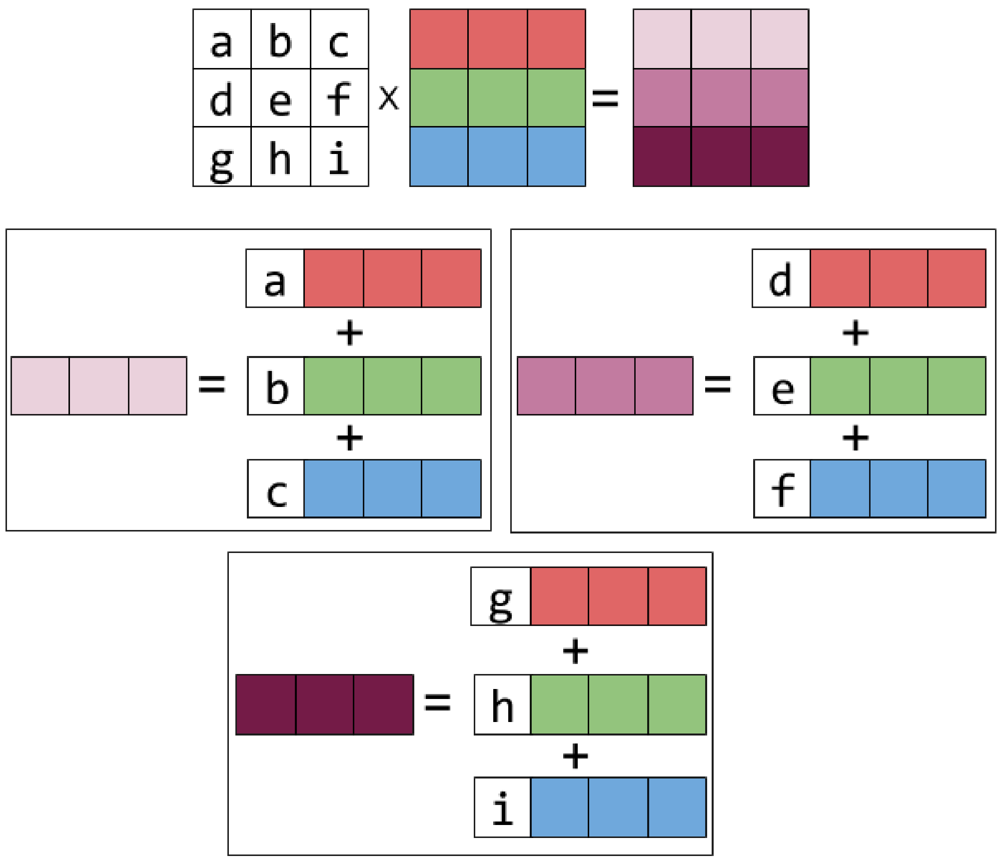
Is represented graphically thus:



And left-multiplying by a matrix is the same thing repeated for every result row: it becomes the linear combination of the rows of X, with the coefficients taken from the rows of the matrix on the left. Here's the equation form:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} * \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 + cx_3 & ay_1 + by_2 + cy_3 & az_1 + bz_2 + cz_3 \\ dx_1 + ex_2 + fx_3 & dy_1 + ey_2 + fy_3 & dz_1 + ez_2 + fz_3 \\ gx_1 + hx_2 + ix_3 & gy_1 + hy_2 + iy_3 & gz_1 + hz_2 + iz_3 \end{pmatrix}$$

And the graphical form:

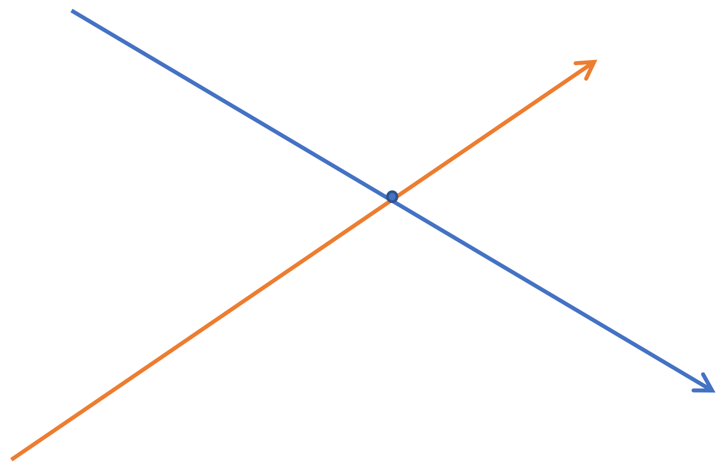


Row Picture:

$$\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\mathbf{-1x + 1y = 1}$$

$$\mathbf{1x + 2y = 5}$$



Here we have a point of intersection.

Column Picture

$$\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} => x\begin{bmatrix} -1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Or

$1\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} + 2\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ Linear combination