MAT 216

Linear Algebra & Fourier Analysis

Week 3 Lecture 5

Lecture Note

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Contents:

- **Solving a system using Gaussian Elimination & Back Substitution**
- **Elementary row operations & their corresponding Elementary matrices**
- **Inverse of an elementary matrix.**

Reading Module:

Introduction to Linear Algebra, 5th Ed. Gilbert Strang

Solving a system using Gaussian Elimination & Back Substitution

Let us consider the following system of linear equations:

$$\begin{gathered} \boldsymbol{x + 2y + z = 2} \\ \boldsymbol{3x + 8y + z = 12} \\ \boldsymbol{4y + z = 2} \end{gathered}$$

We will find the solutions by simplifying the above system through row operations. We can consider the following techniques of row operations:

1. Multiply one equation by a nonzero constant
2. Add one equation to another.

We will describe a technique that will help us to perform row operations more systematically and with greater clarity.

This particular technique is known as ***Gaussian Elimination method***, which can also be stated as ***row echelon form***.

The ***Augmented Matrix*** of the above linear system is denoted by:

$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{8} & \mathbf{1} & \mathbf{12} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} & \mathbf{2} \end{array}\right).$$

In this case "Augmented" indicates to tag something.

The ***Coefficient Matrix*** of the above system is denoted by:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}.$$

Pivot Position:

A **pivot** position in a **matrix** A is a location in A that corresponds to a leading 1 in the echelon form of A. A **pivot column** is a **column** of A that contains a **pivot** position.

If there is a row of all zeros, then it is at the bottom of the **matrix**. The first non-zero element of any row is a **1**. That element is called the **leading 1**. The **leading 1** of any row is to the right of the **leading 1** of the previous row.

Considering an example:

$$A = \begin{bmatrix} 1 & 4 & 5 & 9 & 7 \\ 0 & 2 & 4 & 6 & 6 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Columns

The matrix A is in echelon form and thus reveals that columns 1, 2, and 5 are pivot columns.

The pivots in this example are 1, 2, and 5.

In the ***Gaussian Elimination Method***, the structure of the matrix after elimination should be:

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ 0 & \boldsymbol{a_{22}} & a_{23} \\ 0 & 0 & \boldsymbol{a_{33}} \end{bmatrix}$$

Process:

1. If $a_{11} = 0$ then we have to interchange R_1 with any other row.
$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} => \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} R_1, R_2 \ interchange$$
2. If $a_{11} \neq 0$, then it is either 1 or any other numerical value known as ***Pivot***.

   Ex: $\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 5 \\ -1 & 7 & 3 \\ 3 & 1 & -1 \end{bmatrix}$.

3. If there exists any row that is entirely zero, should be shifted at the bottom.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

4. All the elements under each ***pivot*** should be 0.

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ 0 & \boldsymbol{a_{22}} & a_{23} \\ 0 & 0 & \boldsymbol{a_{33}} \end{bmatrix}$$

Let us consider the given exam again.

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

We will solve $Ax = b$.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}, x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 2 \\ 12 \\ 2 \end{pmatrix}.$$

We will start with *Augmented Matrix* of the system:

$$\left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 2 \\ 3 & \mathbf{8} & 1 & 12 \\ 0 & 4 & \mathbf{1} & 2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 2 \\ 0 & \mathbf{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$= \left(\begin{array}{ccc|c} \mathbf{1} & 2 & 1 & 2 \\ 0 & \mathbf{2} & -2 & 6 \\ 0 & 0 & \mathbf{5} & -10 \end{array}\right) R_3 \to R_3 - 2R_2$$

Our pivots are 1, 2 and 5.

Now we will solve the system in reverse order since the system is a triangle. We call it ***Back Substitution***.

Here $\boldsymbol{U} = \boldsymbol{upper\ triangle} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 2 \\ 6 \\ -10 \end{pmatrix}$.

After the row elimination, the matrix $\boldsymbol{A}$ becomes $\boldsymbol{U}$ and $\boldsymbol{b}$ becomes $\boldsymbol{c}$.

Let's write the matrix in terms of system of linear equation:

$$\begin{aligned} &x + 2y + z = 2 && (1) \\ &2y - 2z = 6 && (2) \\ &5z = -10 && (3) \end{aligned}$$

$$\begin{aligned} (3) &\to 5z = -10,\ \therefore z = -2 \\ (2) &\to 2y - 2(-2) = 6,\ \therefore y = 1 \\ (1) &\to x + 2(1) + (-2) = 2,\ \therefore x = 2 \end{aligned}$$

$\therefore$ Solution of the system $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

Row Echelon Form Of Matrix U

$$\boldsymbol{U} = \boldsymbol{upper\ triangle} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

Row Echelon Form: $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 \div 2 \\ R_3 \to R_3 \div 5 \end{matrix}$

Elementary row operations & their corresponding Elementary matrices

Consider the upper triangular matrix that we have evaluated after row operation:

$\boldsymbol{U} = \boldsymbol{upper\ triangle} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix}$. It contains *three* ***pivots***: 1, 2 and 5.

The whole purpose of this elimination was to get from $\boldsymbol{A}$ to $\boldsymbol{U}$.

Elementary Matrix "E": It is the matrix or product of two or more matrices that takes us from $\boldsymbol{A}$ to $\boldsymbol{U}$.

Consider the following steps:

1. Get a **leading 1** as the first entry of R_1 *(Note: Leading 1 can be any constant value except 0)*

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

So it is the matrix $\boldsymbol{A}$ itself.

2. Use the 1st Pivot to clear out C_1 $(Column\ 1)$ as follows: *(Everything under the 1st Pivot position in Column 1 should be 0)*

$$\left[\qquad ? \qquad\right]\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

The corresponding element that changes in the "?" matrix is $\boldsymbol{a_{21}}$ $\because$ we see the change in $\boldsymbol{R_2}$ by the row operation $\boldsymbol{R_2 - 3R_1}$, while R_1 and R_3 are unchanged in the right hand side.

Note: ***Identity matrix*** acts like **"1"** in the ***Matrix Multiplication***.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, since in the R.H.S. of step **2**, $\boldsymbol{R_1}$ and $\boldsymbol{R_3}$ are unchanged and $\boldsymbol{R_2}$ has been changed by the row operation $\boldsymbol{R_2 - 3R_1}$ **which will affect the element $\boldsymbol{a_{21}}$ by – 3,**

Hence:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

Let's call $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}$, while E stands for ***elementary*** or ***elimination***.

3. Use the 2nd Pivot to clear out C_2 $(Column\ 2)$ as follows:

$$\left[\qquad ? \qquad\right]\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} R_3 \to R_3 - 2R_2 = U$$

The corresponding element that changes in the "?" matrix is $\boldsymbol{a_{32}}$ $\because$ we see the change in $\boldsymbol{R_3}$ by the row operation $\boldsymbol{R_3 - 2R_2}$ **,** while R_1 and R_2 are unchanged in the right hand side.

Since in the R.H.S. of step **3**, $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ are unchanged and $\boldsymbol{R_3}$ has been changed by the row operation $\boldsymbol{R_3 - 2R_2}$ **which will affect the element** $\boldsymbol{a_{32}}$ **by – 2,**

Hence:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} R_3 \rightarrow R_3 - 2R_2$$

Let's call $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{32}$

So we have $\boldsymbol{E_{32}E_{21}A = U}$ **therefore EA=U**

Therefore in this case ***<u>Elementary Matrix</u>*** $\boldsymbol{E = E_{32}E_{21}}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$$

The basic concept of Inverse

Reverse of Elementary Row Operation

- We know that $EA = U$, while E represents elementary matrix.
- We have learnt how to reach U from A.
- But the question is how do we get back to A from U?
- ✓ We have to reverse our calculation steps. We can use the word "$Inverse$" to explain that situation.

Let's think ***"what is Inverse?"***

Consider:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We want to find a matrix that will undo the elimination and give us *Identity*.

$$\begin{pmatrix} & \cdots & \\ \vdots & ? & \vdots \\ & \cdots & \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{-3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We have to add 3 times $R_1$to R_2 $(R_2 + 3R_1)$ to get the Identity. As you can see R_1 and R_3 will remain unchanged.

$$\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{-3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_2 \rightarrow R_2 + \mathbf{3}R_1$$

$$\therefore E^{-1}E = I$$

So, basically when we change the signs of an elementary matrix it becomes inverted.

Inverse of an elementary matrix

Consider the following:
Since E_{21} is one of the elementary matrices

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We kept the signs opposite to each other for the element a_{21} since the product of these two matrices is I_3.

$$\therefore E_{21}^{-1}E_{21} = I$$

The matrix $\boldsymbol{A}$ is invertible if there exists $\boldsymbol{A^{-1}}$, that ***inverts A***.

Two sided inverse: $\boldsymbol{A^{-1}A = I}$ & $\boldsymbol{AA^{-1} = I}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} => E_{21}^{-1}E_{21} = I$$

Similarly

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} => E_{21}E_{21}^{-1} = I$$