Week 7 (Lecture 13)

Contents:

- Recall to different types of system of linear equations
- Recall to homogeneous system of linear equations
- Nullspace of a matrix
- *Theorem* 1: The Nullspace is a subspace
- Basis and dimension of Nullspace
- Rank of a matrix
- *Theorem* 2: Row rank = column rank
- *Theorem* 3: Rank + Nullity = Number of columns
- *Theorem* 4: Homogeneous linear system with $n > m$ has at least one free variable
- Worked out examples
- Exercises

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Different types of System of Linear Equations

We recall that

It is possible for a system of linear equations to have exactly one solution, or an infinite number of solutions, or no solution.

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

Depending upon the number of variables and the number of equations in a system of linear equations, there are two more criteria.

A system of linear equations is said to be **underdetermined** if there are more variables than equations.

Example:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 2x_1 - \ x_2 + 4x_3 &= 3 \end{aligned}$$

On the other hand, a system of linear equations is said to be is **overdetermined** if there are more equations than variables.

Example:

$$\begin{aligned} x_1 + \ x_2 &= \ 2 \\ -2x_1 - 3x_2 &= -3 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

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Homogeneous System of Linear Equations

We recall that

A system of m equations in n variables is called a **homogeneous system of linear equations**, if it has the following form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

The above system can be written as a matrix equation

$$A\mathbf{x} = \mathbf{0} \tag{2}$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$

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Note. Every homogeneous system is *consistent* since $A\mathbf{0} = \mathbf{0}$ where $\mathbf{0} \in \mathbb{R}^n$.

For example, consider the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 2x_3 \quad + x_4 &= 0 \\ 3x_1 + 6x_2 - 5x_3 + 4x_4 &= 0 \\ x_1 + 2x_2 \qquad\quad + 3x_4 &= 0 \end{aligned} \tag{1}$$

This system can be expressed in terms of matrices as

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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or,

$$A\mathbf{x} = \mathbf{0} \tag{2}$$

where, $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The system (1) or (2) is known as **homogeneous system of linear equations**.

It is easy to see that a homogeneous system must have at least one solution. If we set $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, i.e., $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$ in the above system, we find that each of the equations is satisfied. Therefore, $\mathbf{x} = \mathbf{0}$ is a solution to this system. This solution is called **trivial** solution.

A nonzero solution, if any, is called a **nontrivial** solution.

For example, $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is a nontrivial solution of (1) because

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Nullspace of a matrix

If A is an $m \times n$ matrix, then the set

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^m\}$$

is called the **nullspace** of A. In other words, the nullspace of a matrix A is the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$.

For example, let

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

The nullspace of A is the set

$$N(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^3\}.$$

Consider

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -5 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

We see that

$$A\mathbf{x}_1 = \mathbf{0} = A\mathbf{x}_2, \qquad \text{but} \quad A\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \neq \mathbf{0}.$$

Therefore, the vectors $\mathbf{x}_1, \mathbf{x}_2 \in N(A)$ but $\mathbf{x}_3 \notin N(A)$.■

Now we want to prove that the nullspace of A is a subspace of $\mathbb{R}^n$.

Theorem 1. If A be an $m \times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof. Take $\mathbf{x}_1, \mathbf{x}_2 \in N(A)$. Then $A\mathbf{x}_1 = \mathbf{0} = A\mathbf{x}_2$. Thus

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

This proves that $\mathbf{x}_1 + \mathbf{x}_2 \in N(A)$.

And for all scalars c,

$$A(c\mathbf{x}_1) = cA\mathbf{x}_1 = c\mathbf{0} = \mathbf{0}.$$

This proves that $c\mathbf{x}_1 \in N(A)$. This completes the proof. ∎

Note. The nullspace of matrix A is also known as the *solution space* of $A\mathbf{x} = \mathbf{0}$.

Now we will find the nullspace of a matrix.

Consider,

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Recall that $N(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^3\}$. Consider $\mathbf{x} \in \mathbb{R}^4$ as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Then we apply eliminations to A, which gives the following

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the last matrix gives the following homogeneous system

$$x_1 + 2x_2 \quad + 3x_4 = 0$$

$$x_3 + x_4 = 0$$

Since x_2 and x_4 are free variables, we set $x_2 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$. Then

$$x_1 = -2s - 3t, \qquad x_2 = s, \qquad x_3 = -t, \qquad x_4 = t.$$

Therefore, the solution vector **x** can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

which gives the nullspace $N(A) = \left\{ s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

■

Note. All the solutions of the above homogeneous system $A\mathbf{x} = \mathbf{0}$ are linear combination of two vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, i.e. these two vectors span the solution space. When a homogeneous system is solved from the row-echelon form or, from the reduced row-echelon form, the spanning set is always independent.

- A basis of $N(A)$ is the set of linearly independent vectors in $N(A)$.

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

- The dimension of the nullspace of A is called **nullity**. The nullity$(A) = 2$.

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Rank of a matrix

The number of linearly independent vectors in the row space of a matrix A is known as **row rank**.

In other words, the dimension of the row space of a matrix A is called the row rank of A.

Similarly, the number of linearly independent vectors in column space of a matrix A is known as **column rank**, i.e. the dimension of the column space of matrix A.

Theorem 2 If A is an $m \times n$ matrix, then the row space and column space of A have the same dimension. ■

The above theorem provides that,

$$\text{row rank}(A) = \text{column rank}(A) = \text{rank}(A)$$

Since we generally perform the row reduction to achieve the row echelon form, so the number of nonzero rows in row echelon form of the matrix A gives the **rank** of A.

Alternatively, we can say, rank(A) is obtained by the number of pivots in row echelon form of A.

Observe that, the reduced row echelon form of $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$ is

Pivots $$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has two nonzero rows, so the **rank** of A is 2. Alternatively, we can say that there are 2 pivots in the reduced row echelon form of A, so rank(A) is 2. Here, x_1, x_3 are **pivot variables**. On the other hand, x_2, x_4 are **free variables**, so the dimension of the nullspace i.e. nullity is also 2. So, we have the following observation:

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns in } A.$$

This statement is officially known as the **rank-nullity theorem**.

Theorem 3 If A be an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

For example, to find the rank and nullity of

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$$

We reduce the matix A as follows:

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & -3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has three nonzero rows, so

$$\text{rank}(A), r = 3.$$

Also, the number of columns in A is $n = 5$, then by **rank-nullity theorem**

$$\begin{aligned} \text{nullity}(A) &= n - r \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

■

Theorem 4 Homogeneous linear system with $n > m$ has at least one free variable.■

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Worked Out Examples

Example 1 Find the nullspace, rank and nullity of the following matrices:

$$\text{i) } A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \qquad \text{ii) } B = \begin{bmatrix} A \\ 2A \end{bmatrix}, \qquad \text{iii) } C = [A \quad 2A]$$

Solution

i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

Here, $N(A) = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^2\}$. Consider $\mathbf{x} \in \mathbb{R}^2$ as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then we apply eliminations to A,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The last matrix gives the following system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. $x_1 = 0, x_2 = 0$.

Hence, $N(A) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

Rank$(A), r = 2$.

Nullity$(A) = n - r = 2 - 2 = 0$. ■

Note. For the above matrix A, the nullspace contains only one solution, which is trivial solution.

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ii) $B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}$

Here, $N(B) = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^4\}$. Consider $\mathbf{x} \in \mathbb{R}^2$ as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then we apply eliminations to B,

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \xrightarrow[-6R_1+R_4 \to R_4]{\substack{-3R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow{-4R_2+R_4 \to R_4} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The last matrix gives the following system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. $x_1 = 0, x_2 = 0$.

Hence, $N(B) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Rank$(B), r = 2$.

Nullity$(B) = n - r = 2 - 2 = 0$. ■

iii) $C = [A \quad 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$

Here, $N(C) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^2\}$. Consider $\mathbf{x} \in \mathbb{R}^4$ as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

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Then we apply eliminations to C,

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

$$\xrightarrow{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_2\to R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

The last matrix gives the following system

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.

$x_1 + 2x_3 = 0$

$x_2 + 2x_4 = 0.$

Since x_3 and x_4 are free variables, we set $x_3 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$. Then

$$x_1 = -2s, \qquad x_2 = -2t, \qquad x_3 = s, \qquad x_4 = t.$$

Hence, $N(C) = \left\{ \begin{bmatrix} -2s \\ -2t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$

$\text{Rank}(C), r = 2.$

$\text{Nullity}(C) = n - r = 4 - 2 = 2.$ ■

Example 2 For the following matrix A, find (a) a basis for, and (b) the dimension of, the solution space of $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}$$

We know that the solution space of $A\mathbf{x} = \mathbf{0}$ is the nullspace $N(A)$.

Here, $N(A) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^3\}$

Now we apply elimination to A up to the reduced row-echelon form, and obtain the following

$$A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 7 \\ -2 & 4 & -14 \\ 3 & -6 & 21 \end{bmatrix} \xrightarrow[-3R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix gives

$$\begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies $x_1 - 2x_2 + 7x_3 = 0$.

Here, x_1 is pivot variable, and x_2, x_3 are free variables.

Set, $x_2 = s,\ x_3 = t$, where $s, t \in \mathbb{R}$. Then

$$x_1 = 2s - 7t, \qquad x_2 = s, \qquad x_3 = t.$$

Therefore, the solution vector $\mathbf{x}$ can be written as

$$\mathbf{x} = \begin{bmatrix} 2s - 7t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$$

(a) A basis for the solution space of $A\mathbf{x} = \mathbf{0}$ is given by the following set

$$\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-7\\0\\1\end{bmatrix}\right\}$$

(b) The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is 2. ■

Example 3 Find (a) a basis for, and (b) the dimension of, the solution space of the following homogeneous system of linear equations.

$$\text{i)}\quad \begin{aligned} 4x - y + 2z &= 0 \\ 2x + 3y - 2z &= 0 \\ 3x + y + z &= 0 \end{aligned} \qquad\qquad \text{ii)}\quad \begin{aligned} 2x_1 + 2x_2 + 4x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\ -x_1 + x_2 + 4x_3 - 2x_4 &= 0 \end{aligned}$$

Solution

i)

$$\begin{aligned} 4x - y + 2z &= 0 \\ 2x + 3y - 2z &= 0 \\ 3x + y + z &= 0 \end{aligned}$$

Consider the above system as $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix}4 & -1 & 2\\2 & 3 & -2\\3 & 1 & 1\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}x\\y\\z\end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We know that the solution space of $A\mathbf{x} = \mathbf{0}$ is the nullspace $N(A)$.

Here, $N(A) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^3\}$

Now we apply elimination to A, and obtain the following

$$A = \begin{bmatrix}4 & -1 & 2\\2 & 3 & -2\\3 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

The last matrix gives

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Which implies $x = 0,\ y = 0,\ z = 0$

Therefore, the solution vector $\mathbf{x}$ is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is the trivial solution.

(a) Since, all the unknowns x, y, z are pivot variables, and there is no free variable, so there is no linearly independent vector in the solution space of the given system $A\mathbf{x} = \mathbf{0}$. Hence, A basis for the solution space of $A\mathbf{x} = \mathbf{0}$ is given by the empty set $\{\emptyset\}$.

(b)The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is 0. ■

ii)
$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\ -x_1 + x_2 + 4x_3 - 2x_4 &= 0 \end{aligned}$$

Consider the above system as $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 2 & 2 & 4 & -2 \\ 1 & 2 & 1 & 2 \\ -1 & 1 & 4 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We know that the solution space of $A\mathbf{x} = \mathbf{0}$ is the nullspace $N(A)$.

Here, $N(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0} \in \mathbb{R}^3\}$

Applying elimination to the matrix A yields,

$$A = \begin{bmatrix} 2 & 2 & 4 & -2 \\ 1 & 2 & 1 & 2 \\ -1 & 1 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -5/8 \\ 0 & 1 & 0 & 15/8 \\ 0 & 0 & 1 & -9/8 \end{bmatrix}$$

The last matrix gives

$$\begin{bmatrix} 1 & 0 & 0 & -5/8 \\ 0 & 1 & 0 & 15/8 \\ 0 & 0 & 1 & -9/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies

$$\begin{gathered} x_1 - \frac{5}{8}x_4 = 0 \\ x_2 + \frac{15}{8}x_4 = 0 \\ x_3 - \frac{9}{8}x_4 = 0 \end{gathered}$$

Here, x_1, x_2, x_3 are pivot variables, and x_4 is free variable.

Set, $x_4 = t$, where $t \in \mathbb{R}$. Then

$$x_1 = \frac{5}{8}t, \qquad x_2 = -\frac{15}{8}t, \qquad x_3 = \frac{9}{8}t, \qquad x_4 = t$$

Therefore, the solution vector $\mathbf{x}$ can be written as

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{8}t \\ -\frac{15}{8}t \\ \frac{9}{8}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{8} \\ -\frac{15}{8} \\ \frac{9}{8} \\ 1 \end{bmatrix}$$

(a) A basis for the solution space of $A\mathbf{x} = \mathbf{0}$ is given by the following set

$$\left\{ \begin{bmatrix} -\frac{5}{8} \\ -\frac{15}{8} \\ \frac{9}{8} \\ 1 \end{bmatrix} \right\}$$

(b) The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is 1. ■

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Exercises

1. Find (a) the nullspace, (b) rank, and (c) nullity (or, dimension of the nullspace) of the following matrices:

i) $A = \begin{bmatrix} 2 & 6 & 3 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & -2 & 1 & 1 \\ 0 & 6 & 2 & 0 \end{bmatrix}$ ii) $A = \begin{bmatrix} -3 & 6 & -9 \\ 1 & -2 & 3 \end{bmatrix}$

2. For the following matrices, find (a) a basis for, and (b) the dimension of, the solution space of $A\mathbf{x} = \mathbf{0}$.

i) $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ ii) $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ iii) $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{bmatrix}$

3. Find (a) a basis for, and (b) the dimension of, the solution space of the following homogeneous system of linear equations.

i) $\begin{aligned} -x + y + z &= 0 \\ 3x - y &= 0 \\ 2x - 4y - 5z &= 0 \end{aligned}$ ii) $\begin{aligned} 3x_1 + 3x_2 + 15x_3 + 11x_4 &= 0 \\ x_1 - 3x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + 11x_3 + 8x_4 &= 0 \end{aligned}$