Subspaces of Vector Spaces

Definition 10.1.1 A nonempty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V.

Note that the set $\{\mathbf{0}\}$ and V are subspaces of V. $\{\mathbf{0}\}$ is called the **zero subspace**. These subspaces are called **trivial subspaces** of V.

In general, if a nonempty subset W of V is given, then one needs to verify all the ten listed axioms for the elements of W to be a subspace. However, the following theorem reduces this task to only two steps.

Theorem 10.1.1 (**Test for a subspace**)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u} \in W$ and $\mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.

2. If $\mathbf{u} \in W$ and c is any scalar, then $c\mathbf{u} \in W$.

Proof. See Appendix.

Example 10.1.1. Show that the set $W = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$ with the standard operations.

In other words, the xz-plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Solution. Take $\mathbf{u} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}$ from W. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ z_1 + z_2 \end{bmatrix} \in W.$$

Similarly, if $c \in \mathbb{R}$, then

$$c\mathbf{u} = c \begin{bmatrix} x_1 \\ 0 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \\ cz_1 \end{bmatrix} \in W.$$

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This shows that W is closed under addition and scalar multiplication. Therefore, by the Theorem 10.1.1, W is a subspace of $\mathbb{R}^3$. ■

Recall that an $n \times n$ matrix A is called *symmetric* if $A^T = A$.

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 5 \end{bmatrix}$$

then $A^T = A$. Hence A is a 3×3 symmetric matrix.

But

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 5 \end{bmatrix}$$

is not a symmetric matrix since $B^T \neq B$. (Please verify.)

Example 10.1.2. Let S_2 be the set of all 2×2 real symmetric matrices. Show that S_2 is a subspace of $M_{2,2}$.

Solution. Let

$$S_2 = \{A \in M_{2,2} | \, A^T = A\}.$$

We must show that S_2 is closed under addition and scalar multiplication.

Take $A, B \in S_2$. Since

$$(A + B)^T = A^T + B^T = A + B.$$

This shows that $A + B$ is a symmetric matrix. Hence $A + B \in S_2$.

Similarly, if $A \in S_2$ and $c \in \mathbb{R}$, then

$$(cA)^T = cA^T = cA.$$

This shows that cA is also a symmetric matrix. Hence $cA \in S_2$. Therefore, by Theorem 10.1.1 S_2 is a subspace of $M_{2,2}$. In fact, the set of all $n \times n$ symmetric matrices is a subspace of $M_{n,n}$.

Non-Example. The set of all 2×2 singular matrices is NOT a subspace of $M_{2,2}$.

Let

$$S = \{A \in M_{2,2} \mid \det A = 0\}.$$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then clearly, $A, B \in S$ since $\det A = 0 = \det B$.

But

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S,$$

since

$$\det(A + B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Example 10.1.3. Let $D[a, b]$ denote the set of all differentiable real-valued functions defined on the bounded and closed interval $[a, b]$ with $a < b$. Show that $D[a, b]$ is a subspace of $C[a, b]$.

Solution. In Example 9.2.4, we proved that $C[a, b]$, the set of all continuous functions defined on the bounded and closed interval $[a, b]$, is a vector space. From Calculus we know that every differentiable function is continuous. Certainly, $D[a, b] \subset C[a, b]$. We want to show that $D[a, b]$ is subspace of $C[a, b]$.

First of all, $D[a, b] \neq \emptyset$ because the zero function $0 \in C[a, b]$ defined by $0(x) = 0$ is differentiable on $[a, b]$. Therefore $0 \in D[a, b]$.

Take two differentiable functions $f, g \in D[a, b]$. Then from Calculus we know that $f + g$ is also differentiable on $[a, b]$. So, $f + g \in D[a, b]$.

Similarly, if $f \in D[a, b]$ then for any scalar $c \in \mathbb{R}$, cf is also differentiable on $[a, b]$. Therefore,

$$cf \in D[a, b].$$

Therefore, by Theorem 10.1.1, $D[a, b]$ is a subspace of $C[a, b]$.

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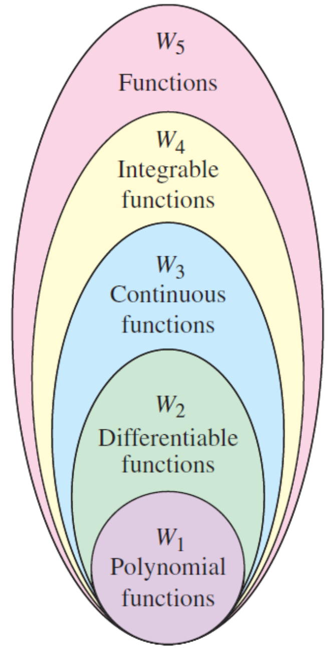
Subspaces of Functions (Calculus)



$W_1 = P[a,b] =$ The set of all polynomial functions that are defined on $[a,b]$.

$W_2 = D[a,b] =$ The set of all differentiable functions defined on $[a,b]$.

$W_3 = C[a,b] =$ The set of all continuous functions defined on $[a,b]$.

$W_4 = R[a,b] =$ The set of all (Riemann) integrable functions defined on $[a,b]$.

$W_5 = F[a,b] =$ The set of all functions defined on $[a,b]$.

$$W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$$

Theorem 10.1.2. (Intersection of two subspaces is a subspace)

If U and W be two subspaces of V, then the intersection $U \cap W$ is also a subspace of V.

Proof. See Appendix.

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Definition 10.2.1. A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

where $c_1, c_2, \dots, c_n$ are scalars.

Example 10.2.1. If $\mathbf{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, then $\mathbf{v}$ is a linear combination of vectors in the set S where

$$S = \left\{ \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\mathbf{v}_3} \right\}.$$

Solution. To see if $\mathbf{v}$ is a linear combination of vectors in S, consider the following equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$$

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}c_1 - c_3\\2c_1 + c_2\\3c_1 + 2c_2 + c_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

The augmented matrix becomes

$$\left[\begin{array}{ccc|c}1 & 0 & -1 & 1\\2 & 1 & 0 & 1\\3 & 2 & 1 & 1\end{array}\right] \qquad [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$$

Row reduction gives

$$\left[\begin{array}{ccc|c}1 & 0 & -1 & 1\\2 & 1 & 0 & 1\\3 & 2 & 1 & 1\end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 0 & -1 & 1\\0 & 1 & 2 & -1\\0 & 2 & 4 & -2\end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & -1 & 1\\0 & 1 & 2 & -1\\0 & 0 & 0 & 0\end{array}\right]$$

The reduced row echelon form shows that c_3 is free. Therefore, if $c_3 = 1$, then

$$c_1 = 2, \qquad c_2 = -3.$$

Therefore, $\mathbf{v} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$.

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Remark This linear combination is not unique. For example, if $c_3 = 0$, then $c_1 = 1, c_2 = -1$. This gives $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.

Example 10.2.2. Show that $\mathbf{v} = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$ is a linear combination of the vectors in the set

$$S = \left\{ \underbrace{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}_{\mathbf{v}_3} \right\}.$$

Solution. We want to solve the vector equation for c_1, c_2 and c_3.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$$

$$c_1 \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} + c_3 \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -c_2 - 2c_3 & 2c_1 + 3c_2 \\ c_1 + c_2 + c_3 & 2c_2 + 3c_3 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$$

The system becomes

$$-c_2 - 2c_3 = 0$$

$$2c_1 + 3c_2 = 8$$

$$c_1 + c_2 + c_3 = 2$$

$$2c_2 + 3c_3 = 1$$

Therefore, the reduced row echelon form may be obtained as follows:

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ 2 & 3 & 0 & 8 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 0 & 8 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[-2R_2 + R_4 \to R_4]{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 7 & -7 \end{array}\right] \xrightarrow[\frac{1}{4}R_4 \to R_4]{-\frac{1}{4}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{-R_3 + R_4 \to R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\xrightarrow{-R_2+R_1\to R_1}\left[\begin{array}{ccc|c}1&0&3&-2\\0&1&-2&4\\0&0&1&-1\\0&0&0&0\end{array}\right]\xrightarrow[-3R_3+R_1\to R_1]{2R_3+R_2\to R_2}\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{array}\right]$$

The solution is given by $c_1 = 1,\ c_2 = 2,\ c_3 = -1$.

$$\therefore \mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3.$$

Non-Example. The vector $\mathbf{v} = \begin{bmatrix}1\\-2\\2\end{bmatrix}$ is not a linear combination of vectors in the set S of Example 10.2.1.

To see this—we consider the following equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$$

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\2\end{bmatrix} \tag{1}$$

As in the previous example, the augmented matrix becomes

$$\left[\begin{array}{ccc|c}1&0&-1&1\\2&1&0&-2\\3&2&1&2\end{array}\right] \qquad [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$$

Row reduction gives

$$\left[\begin{array}{ccc|c}1&0&-1&1\\2&1&0&-2\\3&2&1&2\end{array}\right]\xrightarrow[-3R_1+R_3\to R_3]{-R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1&0&-1&1\\0&1&2&-4\\0&2&4&-5\end{array}\right]\xrightarrow{-2R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&0&-1&1\\0&1&2&-1\\0&0&0&3\end{array}\right]$$

The last row shows that the eq. (1) has no solution. Therefore, there exist no scalars c_1, c_2 and c_3 such that the eq. (1) holds true.

Definition 10.2.2 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a subset of a vector space V. The set S is called a **spanning set** of V if every vector $\mathbf{v}$ in V can be written as a linear combination of vectors in S. In such cases, we say S **spans** V.

Example 10.2.3. The set $S = \left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\mathbf{v}_3} \right\}$ spans $\mathbb{R}^3$.

Solution. Take an arbitrary vector $\mathbf{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$ where $a, b, c \in \mathbb{R}$. It is easy to see that

$$a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

This shows that any vector $\mathbf{v} \in \mathbb{R}^3$ can be written as a linear combination of vectors in S.

Example 10.2.4. Show that $S = \left\{ \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}}_{\mathbf{v}_3} \right\}$ spans $\mathbb{R}^3$.

Solution. Take an arbitrary vector $\mathbf{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3$ where $a, b, c \in \mathbb{R}$. We want to solve the following equation for c_1, c_2, c_3.

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}.$$

The augmented matrix for the corresponding system of equations of the above takes the following form $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$ and the row reduction gives

$$\left[\begin{array}{ccc|c}1 & 0 & -2 & a\\2 & 1 & 0 & b\\3 & 2 & 1 & c\end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 0 & -2 & a\\0 & 1 & 4 & -2a+b\\0 & 2 & 7 & -3a+c\end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & -2 & a\\0 & 1 & 4 & -2a+b\\0 & 0 & -1 & a-2b+c\end{array}\right]$$

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$$\xrightarrow{-R_3\leftrightarrow R_3}\left[\begin{array}{ccc|c}1 & 0 & -2 & a\\ 0 & 1 & 4 & -2a+b\\ 0 & 0 & 1 & -a+2b-c\end{array}\right]\xrightarrow[2R_3+R_1\to R_1]{-4R_3+R_2\to R_2}\left[\begin{array}{ccc|c}1 & 0 & 0 & -a+4b-2c\\ 0 & 1 & 0 & 2a-7b+4c\\ 0 & 0 & 1 & -a+2b-c\end{array}\right]$$

Therefore, the solution is given by

$$c_1=-a+4b-2c,\qquad c_2=2a-7b+4c,\qquad c_3=-a+2b-c.$$

For example, if $a=1, b=1, c=2$, then $c_1=-1$, $c_2=3$, and $c_3=-1$. So, the vector $\mathbf{v}=\begin{bmatrix}1\\1\\2\end{bmatrix}$ can be written as

$$(-1)\begin{bmatrix}1\\2\\3\end{bmatrix}+3\begin{bmatrix}0\\1\\2\end{bmatrix}+(-1)\begin{bmatrix}-2\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}.$$

Similarly, if $a=0, b=0, c=0$, then the equation

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}0\\1\\2\end{bmatrix}+c_3\begin{bmatrix}-2\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

has the only trivial solution $c_1=0, c_2=0, c_3=0$.

Example 10.2.5. The set $S=\left\{\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{v}_1},\ \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\mathbf{v}_2},\ \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\mathbf{v}_3}\right\}$ does not span $\mathbb{R}^3$. In particular, the vector $\mathbf{v}=\begin{bmatrix}1\\1\\2\end{bmatrix}$ cannot be written as a linear combination of vectors in S.

Solution. Take an arbitrary vector $\mathbf{v}=\begin{bmatrix}a\\b\\c\end{bmatrix}\in\mathbb{R}^3$ where $a,b,c\in\mathbb{R}$. We want to solve the following equation for c_1, c_2, c_3.

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}0\\1\\2\end{bmatrix}+c_3\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}.$$

The augmented matrix for the corresponding system of equations of the above takes the following form $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$ and the row reduction gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 2 & 1 & 0 & b \\ 3 & 2 & 1 & c \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 2 & -2a+b \\ 0 & 2 & 4 & -3a+c \end{array}\right] \xrightarrow{-2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 2 & -2a+b \\ 0 & 0 & 0 & a-2b+c \end{array}\right]$$

This shows that the system has no solution (inconsistent) if $a - 2b + c \neq 0$. In particular, if $a = 1, b = 1, c = 2,$ then $a - 2b + c \neq 0$ and hence the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is not a linear combination of vectors in S. In other words, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \notin \operatorname{span}(S)$. Thus S does not span $\mathbb{R}^3$.

Span of a Set

Definition 10.2.3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a subset of a vector space V. The **span** of S is defined by

$$\operatorname{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n\}$$

In other words, the span of S is the set of all linear combinations of vectors in S.

If $\operatorname{span}(S) = V$, then we say S **spans** V, or the vector space V is **spanned** by S.

A subset S of a vector space need not be a subspace. The following theorem, however, tells us that there exists a *smallest* subspace containing the set S.

Theorem 10.2.1. If $S \subseteq V$, then $\operatorname{span}(S)$ is a subspace of V.

Proof. See Appendix.

Remarks

- $S \subseteq \operatorname{span}(S)$;
- $\operatorname{span}(S)$ is the smallest subspace of V containing the set S in a sense that if W be any subspace of V containing the set S, then $\operatorname{span}(S) \subseteq W$.

Linear Dependence and Linear Independence

Definition 10.2.4. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of vectors in a vector space V is called **linearly independent** if the following vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

has the only trivial solution $c_1 = c_2 = \cdots = c_n = 0$. If the vector equation has any nontrivial solution, then S is called **linearly dependent**.

In other words, S is called **linearly independent** if none of the vectors in S is a linear combination of other vectors in S. Otherwise, S is called **linearly dependent**.

Example 10.2.6. Show that $S = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is linearly independent.

Solution. Consider the following equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Therefore, $c_1 = c_2 = c_3 = 0$. Hence S is linearly independent.

Example 10.2.7. Show that $S = \left\{\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}}_{\mathbf{v}_3}\right\}$ is linearly independent.

Solution. Consider the following equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

The augmented matrix takes the form

$$\left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\2 & 1 & 0 & 0\\3 & 2 & 1 & 0\end{array}\right] \qquad [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$$

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By reducing it to the reduced row echelon form, we obtain

$$\left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & 4 & 0\\ 0 & 2 & 7 & 0\end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & 4 & 0\\ 0 & 0 & -1 & 0\end{array}\right]$$

$$\xrightarrow{-R_3\leftrightarrow R_3} \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & 4 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \xrightarrow[2R_3+R_1\to R_1]{-4R_3+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

This shows the system has the only trivial solution $c_1 = c_2 = c_3 = 0$. Therefore, S is linearly independent.

Example 10.2.8. Check if $S = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ is linearly independent.

Solution. To solve the following vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

we apply the same procedure as in Example 10.2.7 to the following matrix. By reducing it to the reduced row echelon form, we obtain

$$\left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 2 & 4 & 0\end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

This shows that c_3 is free variable, hence the system has a nontrivial solution.

$$c_1 - c_3 = 0$$

$$c_2 + 2c_3 = 0$$

For example, if $c_3 = 1$, then $c_1 = 1$ and $c_2 = -2$. That is

$$\mathbf{v}_1 + (-2)\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\1\\2\end{bmatrix} + \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence S is linearly dependent.

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Remark If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{0}\}$, then S is linearly dependent. In other words, if a set contains the zero vector, it must be linearly dependent. Because the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{0} = \mathbf{0}.$$

has nontrivial solution, namely

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k + (1)\mathbf{0} = \mathbf{0}.$$

Theorem 10.2.2. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_i, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors in S.

Proof. See Appendix.

Corollary A set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ of two nonzero vectors in V is linearly dependent if and only if $\mathbf{v}_2 = c\mathbf{v}_1$ for some scalar c.

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