Basis and Dimension of a Vector Space

Definition 11.1.1. A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V is called a **basis** for V if the following conditions hold true.

1. S spans V. **2.** S is linearly independent.

Remark. This definition tells us that a basis has two features:

It

- must have *enough* vectors to span the space;
- should not have *so many* vectors that one of them could be written as a linear combination of the other vectors in it.

Example 11.1.1. Show that $S = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

Solution. In Example 10.2.3 and Example 10.2.6, we proved that $\operatorname{span}(S) = \mathbb{R}^3$ and S is linearly independent, respectively. Thus S is a basis for $\mathbb{R}^3$. This basis is called the **standard basis** for $\mathbb{R}^3$.

Similarly, the set $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\\\vdots\\\vdots\\\vdots\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\\\vdots\\\vdots\\\vdots\\0\end{bmatrix}, \quad \cdots \quad \mathbf{e}_k = \begin{bmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix}0\\0\\\vdots\\\vdots\\\vdots\\1\end{bmatrix}.$$

$\mathbf{e}_i$ has 1 in the ith position and 0's elsewhere. The set S is called the **standard basis** for $\mathbb{R}^n$.

Example 11.1.2. Show that $S = \left\{\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}}_{\mathbf{v}_3}\right\}$ is a basis for $\mathbb{R}^3$.

Solution. In Example 10.2.4 and Example 10.2.7, we obtained that $\operatorname{span}(S) = \mathbb{R}^3$ and S is linearly independent, respectively. Hence S is a basis for $\mathbb{R}^3$.

Example 11.1.1 and Example 11.1.2 show that basis for a vector space is <u>not unique</u>.

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Example 11.1.3. Show that the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $M_{2,2}$. This is called **the standard basis** for $M_{2,2}$.

Solution. Take an arbitrary matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2,2}$.

Clearly,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $\text{span}(S) = M_{2,2}$.

To show the linear independence, we solve the vector equation for c_1, c_2, c_3, and c_4

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So,

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This equation is true if and only if $c_1 = c_2 = c_3 = c_4 = 0$. This shows that S is linearly independent. Thus S is basis for $M_{2,2}$.

Example 11.1.4. Show that the set $S = \{1, x, x^2\}$ is a basis for $\mathbf{P}_2(x)$ where

$$\mathbf{P}_2(x) = \{a_2x^2 + a_1x + a_0 |\ a_2, a_1, a_0 \in \mathbb{R}\}.$$

Solution. Let us denote

$$\mathbf{v}_1 = 1, \qquad \mathbf{v}_2 = x, \qquad \mathbf{v}_3 = x^2$$

Take an arbitrary polynomial $a_2x^2 + a_1x + a_0$ from $\mathbf{P}_2(x)$. Then obviously,

$$a_2x^2 + a_1x + a_0 = a_0\mathbf{v}_1 + a_1\mathbf{v}_2 + a_2\mathbf{v}_3$$

This shows that $\text{span}(S) = \mathbf{P}_2(x)$. Now we will show the linear independence of S.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1 + c_2x + c_3x^2 = 0$$

By differentiating both sides with respect to x, we obtain

$$c_2 + 2c_3x = 0$$

Again differentiating, we get

$$2c_3 = 0$$

In Example 10.2.1, we observed that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of vectors in the set $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_3} \right\}$ in various ways.

For example,

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \quad \text{or} \quad \mathbf{v} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

The next theorem tells us that under what condition this representation is unique.

Theorem 11.1.1. (Uniqueness of Basis Representation)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof. The existence portion of the proof is straightforward. Since S is a basis for V, S spans V, thus an arbitrary vector $\mathbf{v}$ in V can be expressed as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{1}$$

for some scalars $c_1, c_2, \dots, c_n$.

To prove uniqueness (that a vector can be represented in only one way), assume $\mathbf{v}$ has another representation

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n \tag{2}$$

Subtracting eq. (2) from eq. (1), we obtain

$$(c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n = \mathbf{0}.$$

S is linearly independent, however, so the only solution to this equation is the trivial solution

$$c_1 - d_1 = 0, \qquad c_2 - d_2 = 0, \qquad \cdots, \qquad c_n - d_n = 0,$$

which means that $c_i = d_i$ for all $i = 1, 2, \dots, n$ and $\mathbf{v}$ has only one representation for the basis S. ■

Theorem 11.1.2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent, hence cannot be a basis.

Proof. Consult the textbook.

Question. Is $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_3}, \underbrace{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}}_{\mathbf{v}_4} \right\}$ a basis for $\mathbb{R}^3$?

Answer. NO. Because in Example 11.1.1 we found that one basis of $\mathbb{R}^3$ contains three vectors, since S has four vectors, it must be linearly dependent, hence S is not a basis for $\mathbb{R}^3$.

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Corollary (to Theorem 11.1.2) If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Proof. Let us take $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ and $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are two bases for V. We want to show that $m = n$.

Since S_1 is a basis, then by Theorem 11.1.2, S_2 cannot have more than m vectors. That is, $n \leq m$.

By a similar argument, since S_2 is a basis, the set S_1 cannot have more than n vectors. That is, $m \leq n$. These two inequalities prove that $m = n$. ■

Dimension of a Vector Space

Definition 11.1.2 *If* a vector space V has a basis consisting of n vectors, then the number n is called the **dimension** of V, denoted by $\dim(V)$. If V consists of the zero vector alone, the dimension of V is defined as zero. That is, if $V = \{\mathbf{0}\}$, then $\dim(V) = 0$.

In other words, if a vector space is spanned by a set of finitely many vectors of V, then it is called **finite-dimensional**, otherwise it is called an **infinite-dimensional** vector space.

By using Examples 11.1.1, 11.1.3, and 11.1.4, we can write

$$\dim(\mathbb{R}^3) = 3, \qquad \dim\left(M_{2,2}\right) = 4, \qquad \dim(\mathbf{P}_2) = 3.$$

In general, $\dim(\mathbb{R}^n) = n$, $\dim\left(M_{m,n}\right) = mn$, $\dim(\mathbf{P}_n) = n + 1$.

Some examples of finite-dimensional vector spaces are $\mathbb{R}^n$, $\mathbf{P}_n$, $M_{m,n}$. On the other hand, $C[a, b]$, the vector space of all continuous functions defined over $[a, b]$ with $a < b$, is an example of an infinite-dimensional vector space.

Question. Does every vector space have a basis?

Answer. Yes. Unfortunately, it is beyond the scope of our discussion.

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Dimension of a Subspace

Example 11.1.5. Find the dimension of $W = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$, a subspace of $\mathbb{R}^3$.

Solution. In Example 10.1.1, we showed that W is subspace of $\mathbb{R}^3$. To find its dimension, we take an arbitrary vector $\mathbf{v} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \in W$. This vector can be written as

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that $\text{span}(S) = W$ where $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2} \right\}$.

Furthermore, S is linearly independent. To show this, consider the following equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last equation is true if and only if $c_1 = c_2 = 0$. Therefore, S is linearly independent. Hence S is a basis for W. Thus, $\dim(W) = 2$.

Example 11.1.6. Find the dimension of the subspace S_2 that is a set of all 2×2 real symmetric matrices.

Solution. Let

$$S_2 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in M_{2,2} \mid a, b, c \in \mathbb{R} \right\}.$$

In Example 10.1.2, we showed that S_2 is subspace of $M_{2,2}$. To find its dimension, we take an arbitrary matrix A from S_2. Then

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $S = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{v}_3} \right\}$ spans S_2.

To show that S is linearly independent, consider the following equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The last equation is true if and only if $c_1 = c_2 = c_3 = 0$. This proves that S is linearly independent. Therefore, S is a basis for S_2. Thus, $\dim(S_2) = 3$.

Theorem 11.1.3. (Basis Test in an n-Dimensional Space)

Let V be a vector space of dimension n and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in V.

1. If S is linearly independent, then S is a basis for V.
2. If S spans V, that is $\text{span}(S) = V$, then S is a basis for V.

Proof. See Appendix.

Example 11.1.7. Show S is a basis for $\mathbb{R}^3$.

$$S = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution. Since $\dim(\mathbb{R}^3) = 3$, we just need to check if S is either linearly independent or spans $\mathbb{R}^3$. This is left for practice. Please try.

Row Space, Column Space, and Rank of a Matrix

Let A be an $m \times n$ matrix over $\mathbb{R}$.

$$A = \begin{array}{cc} \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & & \mathbf{c}_n \end{array} & \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \leftarrow \mathbf{r}_1 \\ \leftarrow \mathbf{r}_2 \\ \\ \leftarrow \mathbf{r}_m \end{array} \end{array} \qquad A^T = \begin{array}{cc} \begin{array}{cccc} \mathbf{r}_1^T & \mathbf{r}_2^T & & \mathbf{r}_m^T \end{array} & \\ \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \leftarrow \mathbf{c}_1^T \\ \leftarrow \mathbf{c}_2^T \\ \\ \leftarrow \mathbf{c}_n^T \end{array} \end{array}$$

Definition 11.2.1.

The vectors $\mathbf{r}_1 = (a_{11}, a_{12}, \dots, a_{1n})$, $\mathbf{r}_2 = (a_{21}, a_{22}, \dots, a_{2n}), \dots, \mathbf{r}_m = (a_{m1}, a_{m2}, \dots, a_{mn})$ are called the **row vectors** of A.

Similarly, the vectors $\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$ are called the **column vectors** of A.

Note that $\mathbf{r}_i \in \mathbb{R}^n$ for $i = 1, 2, \dots, m$ and $\mathbf{c}_i \in \mathbb{R}^m$ for $i = 1, 2, \dots, n$.

Definition 11.2.2.

1. The **row space** of A, $R(A)$, is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.

2. The **column space** of A, $C(A)$, is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

In other words, if $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ and $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the row and column vectors of A, respectively, then

$$R(A) = \text{rowspace}(A) = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\},$$

$$C(A) = \text{colspace}(A) = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}.$$

Observe that $C(A) = R(A^T)$. In other words, the column space of A is same as the row space of A^T,

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Row Space, Column Space, and Rank of a Matrix

Recall that two matrices are called **row equivalent** if and only if one can be obtained from another by a sequence of elementary row operations.

For example, the matrices

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & -4 \\ 1 & 2 & 3 \end{bmatrix}$$

are row equivalent, since

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{4R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 1 & -4 \\ 1 & 2 & 3 \end{bmatrix} = B.$$

Theorem 11.2.1. If A is row equivalent to B, then $R(A) = R(B)$.

Proof. See Appendix.

Example 11.2.1. Finding a basis and dimension of a row space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

Solution. Reducing A by applying a sequence of elementary row operations to its row echelon form, we obtain

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{\substack{3R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4 \\ -2R_1+R_5\to R_5}} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & 9 & 8 \\ 0 & -5 & -5 & -8 \\ 0 & -6 & -6 & -8 \end{bmatrix} \xrightarrow{\substack{-9R_2+R_3\to R_3 \\ 5R_2+R_4\to R_4 \\ 6R_2+R_5\to R_5}} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{8}R_3\to R_3} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -8 \end{bmatrix} \xrightarrow{\substack{8R_3+R_4\to R_4 \\ 8R_3+R_5\to R_5}} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The non-zero row vectors of B in row echelon form are:

$$\mathbf{w}_1 = (1,3,1,3), \qquad \mathbf{w}_2 = (0,1,1,0), \qquad \mathbf{w}_3 = (0,0,0,1)$$

Hence $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a basis for $R(B)$. Thus, by the Theorem 11.2.1, S is a basis for $R(A)$. And $\dim\big(R(A)\big) = 3$.

Example 11.2.2. Find a basis and dimension of the column space of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

Solution. Since the column space of A is same as the row space of A^T, by reducing A^T to its row echelon form, we obtain

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & 2 \end{bmatrix} \xrightarrow[-3R_1+R_3\to R_3]{\substack{-3R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 8 & -8 & -4 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & -8 & -4 \end{bmatrix} \xrightarrow{R_3\leftrightarrow R_4} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 8 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The non-zero row vectors of B in row echelon form are:

$$\mathbf{w}_1 = (1,0,-3,3,2), \qquad \mathbf{w}_2 = (0,1,9,-5,-6), \qquad \mathbf{w}_3 = (0,0,8,-8,-4).$$

Hence $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a basis for $R(B)$. Thus, by the Theorem 11.2.1, S is a basis for $R(A^T)$. This is equivalent to saying that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $C(A) = R(A^T)$ where

$$\mathbf{v}_1 = \mathbf{w}_1^T = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \mathbf{w}_2^T = \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \qquad \mathbf{v}_3 = \mathbf{w}_3^T = \begin{bmatrix} 0 \\ 0 \\ 8 \\ -8 \\ -4 \end{bmatrix}.$$

And $\dim\big(C(A)\big) = 3$.

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Row Space, Column Space, and Rank of a Matrix

Definition 11.2.3. The dimension of row space of a matrix A is called the **row rank** of A, and the dimension of column space of a matrix A is called the **column rank** of A.

Theorem 11.2.2. For any $m \times n$ matrix A,

$$\dim\big(R(A)\big) = \dim\big(C(A)\big).$$

In other words, row rank of a matrix is equal to its column rank.

Proof. Please consult the textbook.

Definition 11.2.4. The dimension of the row (or column) space of a matrix is called the **rank** of A and is denoted by $\text{rank}(A)$.

Example 11.2.3. Find the rank of the matrix of

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}.$$

Solution. From Example 11.2.1 and Example 11.2.2, we find that

$$\dim\big(R(A)\big) = \dim\big(C(A)\big) = 3.$$

Hence $\text{rank}(A) = 3$.

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