Week 4 (Lecture 7)

Contents:

- $A = LU$ Decomposition/Factorization
- Solving a system of linear equations by LU Factorization
- Using Gaussian Elimination to solve various types of system of linear equations
- Determining the number of solutions of a linear system

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

$A = LU$ Factorization

Just as it is natural to factor a natural number into a product of other natural numbers—for example, $10 = 2 \cdot 5$—it is also frequently helpful to factor matrices as products of other matrices. Any representation of a matrix as a product of two or more other matrices is called a *matrix factorization*.

Some factorizations are more useful than others. In this lecture, we introduce a matrix factorization called *LU factorization* that arises in the solution of systems of linear equations by Gaussian elimination and is particularly well suited to computer implementation.

Definition of LU Factorization:

If an $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an $\boldsymbol{LU}$ **factorization/decomposition** of A.

For example,

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}}_{U}$$

is an LU factorization of matrix A as the product of the lower triangular matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$ and the upper triangular matrix $U = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$.

How to find an LU decomposition?

Example 1 Find the LU-factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$, if exists.

Solution. We begin by row reducing A to upper triangular form while keeping track of the elementary matrices used for each row operation.

$$\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\2 & -10 & 2\end{bmatrix}\xrightarrow{-2R_1+R_3\to R_3}\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & -4 & 2\end{bmatrix}\xrightarrow{4R_2+R_3\to R_3}\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & 0 & 14\end{bmatrix}=U$$

It follows that $E_2E_1A=U$. Since elementary matrices are invertible, we have

$$A=E_1^{-1}E_2^{-1}U$$

where

$$E_1^{-1}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix},\qquad E_2^{-1}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -4 & 1\end{bmatrix}.$$

Observe that E_1^{-1} and E_2^{-1} are lower triangular matrices and the product of two lower triangular matrices is again a lower triangular matrix

$$E_1^{-1}E_2^{-1}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & -4 & 1\end{bmatrix}=L.$$

$$\therefore\ \underbrace{\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\2 & -10 & 2\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & -4 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & 0 & 14\end{bmatrix}}_{U}$$

Thus, the factorization $A=LU$ is complete. ■

A Faster Method

Since the inverse of an elementary matrix is an elementary matrix, instead of finding the inverses E_1^{-1}, E_2^{-1} and the product $E_1^{-1}E_2^{-1}$, we can use $A=(E_1^{-1}E_2^{-1}I)U$ as follows

$$A=\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\2 & -10 & 2\end{bmatrix}\xrightarrow{-2R_1+R_3\to R_3}\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & -4 & 2\end{bmatrix}\xrightarrow{4R_2+R_3\to R_3}\begin{bmatrix}1 & -3 & 0\\0 & 1 & 3\\0 & 0 & 14\end{bmatrix}=U$$

Now apply these elementary row operations to the identity matrix I with multiplier **2** and **−4** in a reverse order as follows:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} = L$$

$$\therefore \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}. \quad \blacksquare$$

Remarks

- Observe that the matrix A in Example 1 had an LU factorization because no row interchanges were needed in the row reduction of A. Hence, all of the elementary matrices that arose were (unit) lower triangular. Thus, L was guaranteed to be (unit) lower triangular because inverses and products of unit lower triangular matrices are also unit lower triangular. If a zero had appeared in a pivot position at any step, we would have had to swap rows to get a nonzero pivot. This would have resulted in L no longer being (unit) lower triangular. Therefore, row interchanging is not allowed while LU factorizing.

- Not every square matrix has an LU decomposition. For example, the following 2×2 matrix has no such factorization

  $$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

  For a square matrix A, if row interchange is required to get the upper triangular matrix U, then LU decomposition does not exist for matrix A.

- A singular matrix may have LU decomposition. For example,

  $$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1/2 & 0 \\ 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}}_{U}.$$

- In some books, an LU factorization of a square matrix A is defined to be any factorization $A = LU$, where L is lower triangular and U is upper triangular. Hence, the factor matrices L and U are not unique. For example,

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix}}_{U}$$

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{U}$$

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}}_{U}$$

Example 2 Find the LU-factorization of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow{1R_1+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{2R_2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \xrightarrow{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = L$$

$$\therefore \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

■

Example 3 Use an LU factorization to solve the following system

$$2x_1 + x_2 + 3x_3 = 1$$

$$4x_1 - x_2 + 3x_3 = -4$$

$$-2x_1 + 5x_2 + 5x_3 = 9$$

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

Solution.

We put the system into the following form $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$$

Now we use the LU factorization of A we obtained in Example 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{y}} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$$

where $U\mathbf{x} = \mathbf{y}$ with $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then the system reduces to $L\mathbf{y} = \mathbf{b}$. This is just a linear system in matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$$

or in equation form

$$y_1 = 1$$

$$2y_1 + y_2 = -4$$

$$-y_1 - 2y_2 + y_3 = 9$$

Forward substitution (that is, working from top to bottom) yields

$$y_1 = 1, \qquad y_2 = -6, \qquad y_3 = -2.$$

Therefore, $\mathbf{y} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix}$. Now we solve the system $U\mathbf{x} = \mathbf{y}$ which is in matrix form

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix}$$

or

$$2x_1 + x_2 + 3x_3 = 1$$

$$-3x_2 - 3x_3 = -6$$

$$2x_3 = -2$$

Backward substitution (that is, working from bottom to top) yields

$$x_3 = -1, \qquad x_2 = 3, \qquad x_1 = \frac{1}{2}$$

Therefore, the solution to the given system $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \begin{bmatrix} 1/2 \\ 3 \\ -1 \end{bmatrix}$.

■

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

Number of Solutions of a system of Linear Equations

We recall that…

It is possible for a system of linear equations to have exactly one solution, or infinitely many solutions, or no solution.

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

Next, we will see some figures which represent the above-mentioned systems.

Figure 1 shows three planes that intersect at a single point, and it represents a system of three linear equations in three variables with a unique solution.

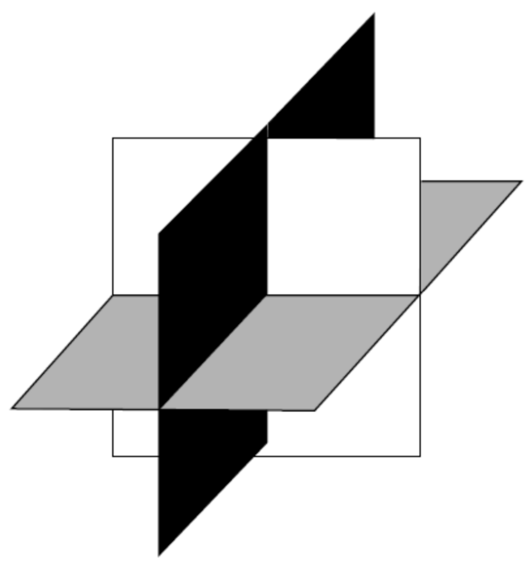

Figure 1

Figures 2 and 3 show systems of planes that have no points that lie on all three planes; each figure depicts a different system of three linear equations in three unknowns with no solutions.

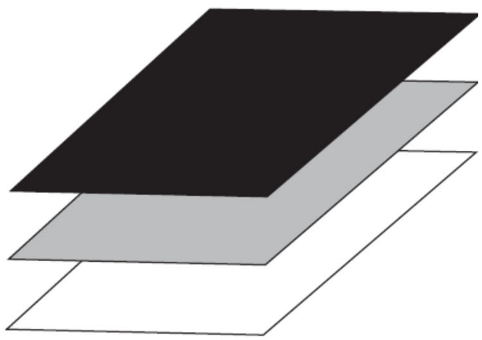

Figure 2

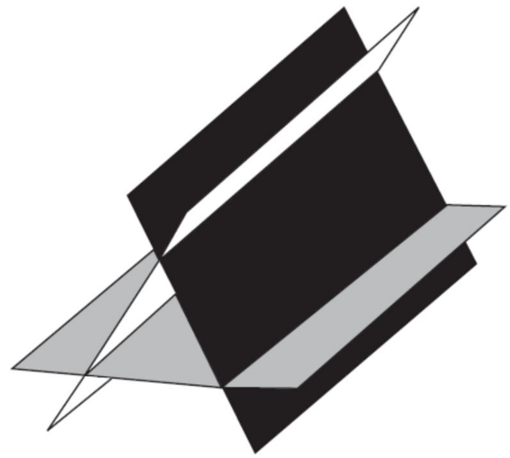

Figure 3

Figure 4 shows three planes intersecting at a line, and it represents a system of three equations in three variables with infinitely many solutions, one solution corresponding to each point on the line.

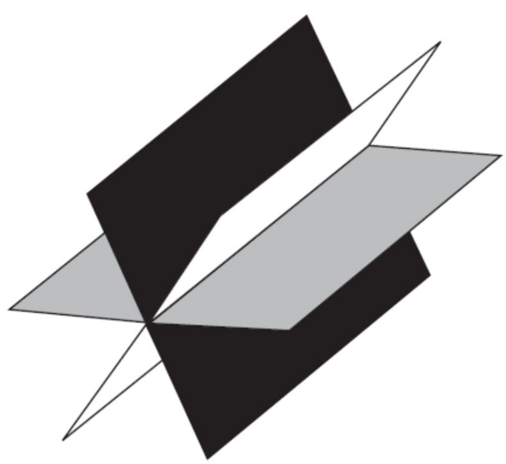

Figure 4

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

Number of Solutions of a system of Linear Equations

A different example of infinitely many solutions is obtained by collapsing the three planes in Figure 2 onto each other so that each plane is an exact copy of the others. Then every point on one plane is also on the other two.

Now, we will observe the number of solutions of different systems by solving them.

Example 1

Solve

$$\begin{aligned} 2x_1 + 5x_2 + 3x_3 &= 11 \\ -x_1 + 3x_2 + x_3 &= 5 \\ x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

Solution The system has its matrix form

$$A\mathbf{x} = \mathbf{b}$$

where,

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -1 & 3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and } \mathbf{b} = \begin{bmatrix} 11 \\ 5 \\ -3 \end{bmatrix}$$

The augmented matrix of the above system is

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 5 & 3 & 11 \\ -1 & 3 & 1 & 5 \\ 1 & 1 & -2 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ -1 & 3 & 1 & 5 \\ 2 & 5 & 3 & 11 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 4 & -1 & 2 \\ 0 & 3 & 7 & 17 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -8 & -15 \\ 0 & 3 & 7 & 17 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -8 & -15 \\ 0 & 0 & 31 & 62 \end{array}\right] \xrightarrow{\frac{1}{31}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -8 & -15 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

The associated system is now

$$\begin{aligned} x_1 + x_2 - 2x_3 &= -3 \\ x_2 - 8x_3 &= -15 \\ x_3 &= 2 \end{aligned}$$

By back substitution, we get

$$x_2 = -15 + 8x_3 = -15 + (8 \times 2) = 1$$

$$x_1 = -3 - x_2 + 2x_3 = -3 - 1 + (2 \times 2) = 0$$

Therefore, the required solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Thus, the system has unique solution.

■

Example 2 Solve the system

$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 &= 3 \\ -x_1 + x_2 - x_3 \qquad &= -3 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[R_3+R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The associated system is now

$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 1 \\ x_3 - x_4 &= 1 \end{aligned}$$

In this case, the leading variables are x_1 and x_3, and the free variables are x_2 and x_4.

Here, we will proceed to use back substitution, writing the variables corresponding to the leading entries (the *leading variables*) in terms of the other variables (the *free variables*).

Thus,

$$x_3 = 1 + x_4$$

and from this we obtain,

$$\begin{aligned} x_1 &= 1 + x_2 + x_3 - 2x_4 \\ &= 1 + x_2 + (1 + x_4) - 2x_4 \\ &= 2 + x_2 - x_4 \end{aligned}$$

If we assign parameters $x_2 = s$ and $x_4 = t$, where $s, t \in \mathbb{R}$. we get

$$x_1 = 2 + s - t$$

$$x_2 = s$$

$$x_3 = 1 + t$$

$$x_4 = t$$

Therefore, the solution can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}; \text{ where } s, t \in \mathbb{R}.$$

Thus, the system has infinitely many solutions.

■

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University

Example 3 Solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 - 2x_3 &= 1 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

(We could also have performed $R_3 - \frac{2}{3} R_2$ as the second elementary row operation, which would have given us the same contradiction but a different row echelon form.)

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Now, the associated system is

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ x_2 - x_2 &= -2 \\ 0 &= 5 \end{aligned}$$

The equation $0 = 5$ gives a false statement.

Therefore, the system has no solution. It is inconsistent.

■

Number of Solutions of a system of Linear Equations

Theorem (Number of solutions of a system of linear equations)

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution.
2. The system has an infinite number of solutions.
3. The system has no solution.

■

Theorem A homogeneous system of linear equations is consistent.

(A system of linear equations is called ***homogeneous*** if the constant term in each equation is zero.)

Example

$$\begin{aligned} 2x - 3y - z &= 0 \\ -x + 5y + 2z &= 0 \\ 3x - 8y - 3z &= 0 \end{aligned}$$

This system has an obvious solution, which is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and this solution is known as ***trivial*** solution.

Therefore, a homogeneous linear system is consistent.

■

Saba Fatema
Lecturer
Department of Mathematics and Natural Sciences
Brac University