Coordinates and Change of Basis

Let V be a real vector space. In Theorem 11.1.1, we saw that if B is a basis for V, then every vector $\mathbf{v}$ in V can be expressed in one and only one way as a linear combination of vectors in B. The coefficients in the linear combination are the **coordinates** of $\mathbf{v}$ relative to B.

In the context of coordinates, the *order* of the vectors in the basis is important, so this will sometimes be emphasized by referring to the basis B as an ***ordered basis***.

For example, $B = \left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\mathbf{e}_1}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\mathbf{e}_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\mathbf{e}_3} \right\}$ and $B' = \left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\mathbf{e}_1}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\mathbf{e}_3}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\mathbf{e}_2} \right\}$ are <u>same sets</u> and each set is a basis for $\mathbb{R}^3$, however, we will consider B and B' are <u>different ordered bases</u>.

Definition 12.1.1 Suppose $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an ordered basis for a real vector space V. If $\mathbf{v} \in V$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

then the scalars $c_1, c_2, \dots, c_n$ are called the **coordinates of v relative to the basis** B. The **coordinate matrix** (or **coordinate vector**) **of v relative to the basis** B is the column matrix in $\mathbb{R}^n$ whose components are the coordinates of $\mathbf{v}$,

$$[\mathbf{x}(\mathbf{v})]_B = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}.$$

In $\mathbb{R}^n$, column notation is used for the coordinate matrix. For the vector $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} \in \mathbb{R}^n$, the x_i's are the coordinates of **x relative to the standard basis** $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ since

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

So, we have

$$\therefore [\mathbf{x}]_S = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}.$$

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.1.1. Find the coordinate matrices of $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ relative to the following ordered bases for $\mathbb{R}^3$.

i) $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{e}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{e}_3} \right\}$ (standard).

ii) $B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\mathbf{v}_2}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_3} \right\}$.

Solution.

i) The coordinate matrix of $\mathbf{v}$ relative to the standard ordered basis S is

$$[\mathbf{x}(\mathbf{v})]_S = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

ii) Since B is a basis, there exist unique scalars c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \tag{1}$$

The augmented matrix corresponding the matrix equation (1) takes the following form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 3 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 7 & 6 \end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_3\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[2R_3+R_1\to R_1]{-4R_3+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the coordinate matrix of $\mathbf{v}$ relative to the standard ordered basis B is

$$[\mathbf{x}(\mathbf{v})]_B = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}. \tag{2}$$

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

This example shows that the procedure for finding the coordinate matrix relative to a standard basis is straightforward. It is more difficult, however, to find the coordinate matrix relative to a *nonstandard basis* B.

Example 12.1.2. Find the coordinate matrices of $A = \begin{bmatrix} -2 & 7 \\ 3 & -4 \end{bmatrix} \in M_{2,2}$ relative to the following ordered bases for $M_{2,2}$.

i) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$;

ii) $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

Solution.

i) Since

$$\begin{bmatrix} -2 & 7 \\ 3 & -4 \end{bmatrix} = (-2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore [\mathbf{x}(A)]_S = \begin{bmatrix} -2 \\ 7 \\ 3 \\ -4 \end{bmatrix}.$$

ii) To find the coordinate matrix of A relative to the basis B, we need to solve the following matrix equation for c_1, c_2, c_3, and c_4.

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 + c_3 + c_4 & c_1 + c_3 + c_4 \\ c_1 + c_2 + c_4 & c_2 + c_3 + c_4 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & -4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 & 7 \\ 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$\therefore [\mathbf{x}(A)]_B = \begin{bmatrix} 2 \\ -9 \\ -5 \\ 10 \end{bmatrix}.$$

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.1.3. Find the coordinate matrices of $p(x) = -6x^2 + x - 3 \in \mathbf{P}_2$ relative to the following ordered bases for $\mathbf{P}_2$.

i) $S = \{1, x, x^2\}$ (standard).

ii) $B = \{1 + x, x + x^2, 1 + x^2\}$.

Solution.

i) Since

$$-6x^2 + x - 3 = (-3) \cdot 1 + 1 \cdot x + (-6) \cdot x^2$$

$$\therefore [\mathbf{x}(p)]_S = \begin{bmatrix} -3 \\ 1 \\ -6 \end{bmatrix}$$

ii) To find the coordinate matrix of $p(x)$ relative to the basis B, we need to solve the following equation for c_1, c_2, and c_3.

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = -6x^2 + x - 3$$

Rearrangement of the terms gives

$$(c_2 + c_3)x^2 + (c_1 + c_2)x + (c_1 + c_3) = -6x^2 + x - 3$$

Since the last equation is an identity, by equating the coefficients like powers of x, we obtain

$$c_2 + c_3 = -6$$

$$c_1 + c_2 = 1$$

$$c_1 + c_3 = -3$$

The augmented matrix for the system becomes

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -6 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -3 \end{array}\right] \xrightarrow{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\therefore [\mathbf{x}(p)]_B = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}.$$

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

From the eqs. (1) and (2) in Example 12.1.1, we observe that

$$\underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{P_{B\to S}} \underbrace{\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}}_{[\mathbf{x}(\mathbf{v})]_B} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}_{[\mathbf{x}(\mathbf{v})]_S}$$

The matrix $P_{B\to S}$ is called the ***transition matrix from** B **to** S*. Multiplication by the transition matrix $P_{B\to S}$ changes a coordinate matrix relative to B into a coordinate matrix relative to S.

$$P_{B\to S}[\mathbf{x}(\mathbf{v})]_B = [\mathbf{x}(\mathbf{v})]_S$$

We can generalize this idea.

Assume that $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are two ordered bases (note that the number of vectors in each basis has to be the same, why?) for a vector space V, then the transition matrix $P_{B'\to B}$ from B' to B is a matrix such that

$$P_{B'\to B}[\mathbf{x}(\mathbf{v})]_{B'} = [\mathbf{x}(\mathbf{v})]_B$$

where $[\mathbf{x}(\mathbf{v})]_B$ and $[\mathbf{x}(\mathbf{v})]_{B'}$ are coordinate matrices relative to the bases B and B', respectively.

The next theorem tells us that the transition matrix $P_{B'\to B}$ is invertible and its inverse is the transition matrix from B to B'. That is, $P_{B'\to B}^{-1}[\mathbf{x}(\mathbf{v})]_B = [\mathbf{x}(\mathbf{v})]_{B'}$.

Before we prove the theorem, we need to prove the following lemma.

Lemma 12.2.1. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are two ordered bases for a vector space V. If

$$\mathbf{u}_1 = c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n$$

$$\mathbf{u}_2 = c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n$$

$$\vdots$$

$$\mathbf{u}_n = c_{1n}\mathbf{v}_1 + c_{2n}\mathbf{v}_2 + \cdots + c_{nn}\mathbf{v}_n$$

then the transition matrix from B to B' is given by

$$P_{B\to B'} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

Proof. See Appendix.

Observe that

$$P_{B\to B'} = [[\mathbf{x}(\mathbf{u}_1)]_{B'} \quad [\mathbf{x}(\mathbf{u}_2)]_{B'} \quad \cdots \quad [\mathbf{x}(\mathbf{u}_n)]_{B'}].$$

Theorem 12.2.1. If $P_{B\to B'}$ is the transition matrix from a basis B to a basis B' for V, then $P_{B\to B'}$ is invertible and the transition matrix from B' to B is given by $P^{-1}_{B\to B'}$.

Proof. (Optional)

$$P_{B\to B'}[\mathbf{x}(\mathbf{v})]_B = [\mathbf{x}(\mathbf{v})]_{B'} \tag{1}$$

Let $P_{B'\to B}$ be the transition matrix from a basis B' to a basis B. Then

$$P_{B'\to B}[\mathbf{x}(\mathbf{v})]_{B'} = [\mathbf{x}(\mathbf{v})]_B \tag{2}$$

By substituting $[\mathbf{x}(\mathbf{v})]_{B'}$ from (1) into (2), we get

$$P_{B'\to B}\underbrace{P_{B\to B'}[\mathbf{x}(\mathbf{v})]_B}_{[\mathbf{x}(\mathbf{v})]_{B'}} = [\mathbf{x}(\mathbf{v})]_B \tag{3}$$

Since equation (3) is true for any $[\mathbf{x}(\mathbf{v})]_B \in \mathbb{R}^n$, we must have $P_{B'\to B}P_{B\to B'} = I$. Therefore, $P_{B\to B'}$ is invertible and $P_{B'\to B} = P^{-1}_{B\to B'}$. Thus

$$P^{-1}_{B\to B'}[\mathbf{x}(\mathbf{v})]_{B'} = [\mathbf{x}(\mathbf{v})]_B.$$

This concludes the proof. ■

The next examples show how to find transition matrices.

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.2.1. Find the transition matrix from B to B' for the following ordered bases for $\mathbb{R}^2$.

$$B = \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\mathbf{u}_1}, \quad \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\mathbf{u}_2} \right\}, \quad B' = \left\{ \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\mathbf{v}_1}, \quad \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\mathbf{v}_2} \right\}.$$

Solution. Let

$$c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 = \mathbf{u}_1$$

$$\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}c_{11}\\c_{21}\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \tag{1}$$

Similarly, for the equation

$$c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 = \mathbf{u}_2$$

we have

$$\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}c_{12}\\c_{22}\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \tag{2}$$

Equations (1) and (2) can be put into the following matrix equation:

$$\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}c_{11} & c_{12}\\c_{21} & c_{22}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}P_{B\to B'} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \tag{3}$$

Now we begin by forming the matrix $[B' \mid B]$ and use the Gauss–Jordan elimination to obtain transition matrix $P_{B\to B'}$ as $[I \mid P_{B\to B'}]$. The details follow.

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 2 & 0 & 1\end{array}\right] \xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & 0 & 1/2\end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{cc|cc}1 & 0 & 1 & -1/2\\0 & 1 & 0 & 1/2\end{array}\right]$$

$$\therefore P_{B\to B'} = \begin{bmatrix}1 & -1/2\\0 & 1/2\end{bmatrix}.$$

Observe that $P_{B\to B'}$ can be obtained from eq. (3) as follows

$$P_{B\to B'} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1/2\\0 & 1/2\end{bmatrix}.$$

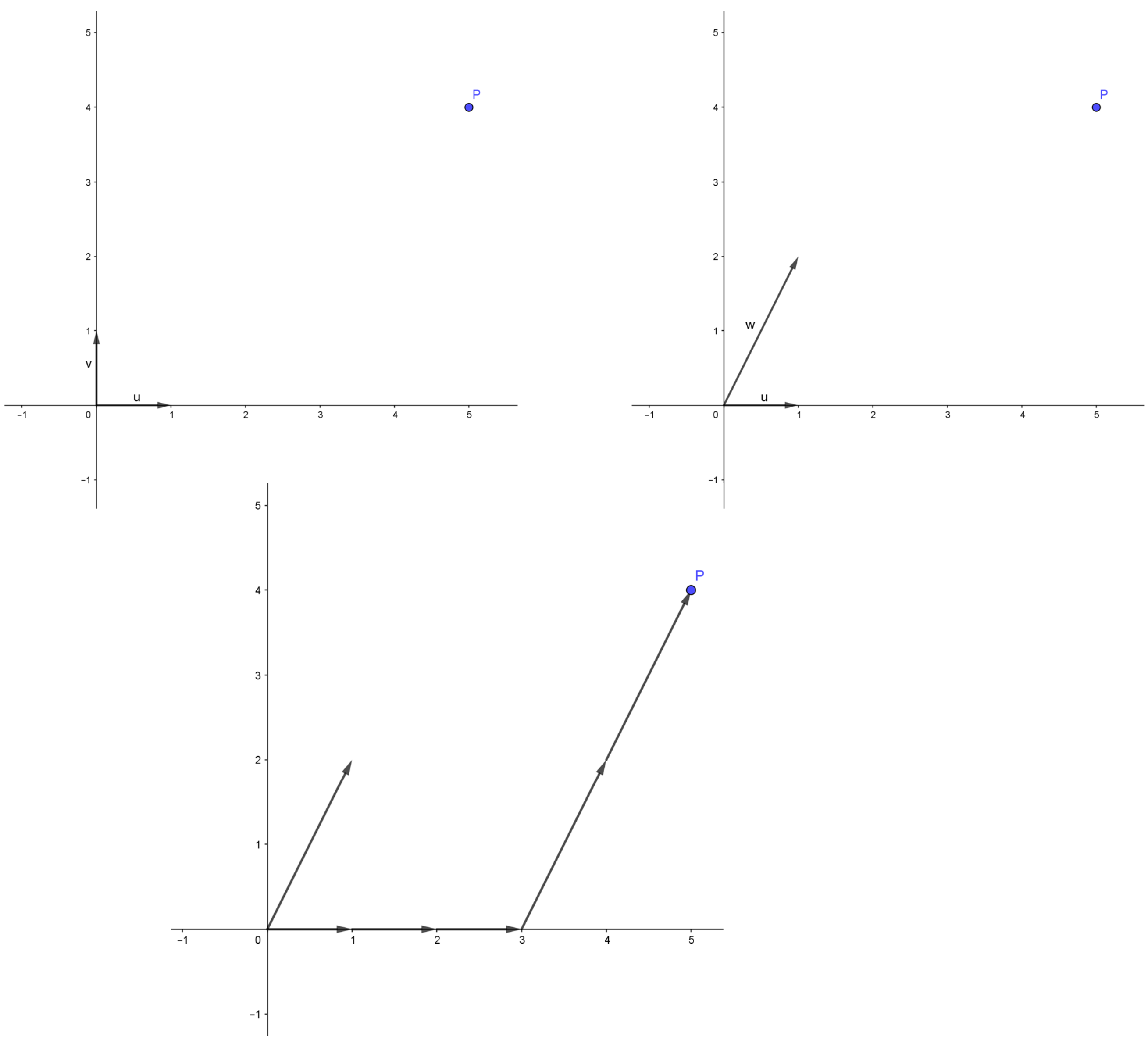
Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.2.2. Find the coordinate matrix of $[\mathbf{x}(\mathbf{v})]_B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ relative to B' given in Example 12.2.1. Also plot the vectors in $\mathbb{R}^2$.

Solution. The equation $[\mathbf{x}(\mathbf{v})]_{B'} = P_{B \to B'}[\mathbf{x}(\mathbf{v})]_B$ gives

$$[\mathbf{x}(\mathbf{v})]_{B'} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

This is clear from the following figure.



Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.2.3.

a) Find the transition matrix from B to B' if $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and

$B' = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ are the ordered bases for $\mathbb{R}^3$.

b) Find the coordinate matrix of $[\mathbf{x}(\mathbf{v})]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ relative to B'.

Solution.

a) Following the procedure, shown in Example 12.2.1, we begin by forming the augmented matrix $[B' \mid B]$ and reduce it to the reduced row echelon form to obtain $[I \mid P_{B\to B'}]$ as follows.

$$[B' \mid B] = \left[\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & 0 & -2 \\ 2 & 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -4 & -5 \\ 0 & 1 & 0 & 3 & 3 & 4 \\ 0 & 0 & 1 & \underbrace{7}_{[x(\mathbf{u}_1)]_{B'}} & \underbrace{6}_{[x(\mathbf{u}_2)]_{B'}} & \underbrace{6}_{[x(\mathbf{u}_3)]_{B'}} \end{array}\right]$$

$$\therefore P_{B\to B'} = \begin{bmatrix} -4 & -4 & -5 \\ 3 & 3 & 4 \\ 7 & 6 & 6 \end{bmatrix}$$

b)

$$[\mathbf{x}(\mathbf{v})]_{B'} = P_{B\to B'}[\mathbf{x}(\mathbf{v})]_B$$

$$[\mathbf{x}(\mathbf{v})]_{B'} = \begin{bmatrix} -4 & -4 & -5 \\ 3 & 3 & 4 \\ 7 & 6 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}.$$

In other words, if the coordinate matrix of a vector $\mathbf{v}$ relative to the basis B is $[\mathbf{x}(\mathbf{v})]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, then its coordinate matrix relative to the basis B' is $[\mathbf{x}(\mathbf{v})]_{B'} = \begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}$.

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

Example 12.2.4.

a) Find the transition matrix from S to B if $S = \{1, x, x^2\}$ and $B = \{1 + x, x + x^2, 1 + x^2\}$ are the ordered bases for $\mathbf{P}_2$.

b) Find the coordinate matrix of $p(x) = -6x^2 + x - 3$ relative to B.

Solution. a) Recall that $P_{S\to B} = [[\mathbf{x}(1)]_B \quad [\mathbf{x}(x)]_B \quad [\mathbf{x}(x^2)]_B]$

$$c_{11}(1 + x) + c_{21}(x + x^2) + c_{31}(1 + x^2) = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$c_{12}(1 + x) + c_{22}(x + x^2) + c_{32}(1 + x^2) = x$$

$$c_{13}(1 + x) + c_{23}(x + x^2) + c_{33}(1 + x^2) = x^2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Putting these three

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} P_{S\to B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Following the procedure, shown in Example 12.2.1, we begin by forming the following augmented matrix and reduce it to the reduced row echelon form to obtain $[I \mid P_{S\to B}]$ as follows.

$$[B \mid S] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\therefore P_{S\to B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University

b)

$$[\mathbf{x}(p)]_{B'} = P_{S\to B}[\mathbf{x}(p)]_S = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

In other words, if the coordinate matrix of the polynomial $p(x) = -6x^2 + x - 3$ relative to the basis $B = \{1 + x, x + x^2, 1 + x^2\}$ is $[\mathbf{x}(p)]_B = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$.

The result is exactly the same as in Example 12.1.3.

Mohammad Hassan Murad
Assistant Professor
Department of Mathematics and Natural Sciences
Brac University