Week 7 (Lecture 14)

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Solutions of Nonhomogeneous Linear Systems

If A be an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{b} \neq \mathbf{0}$ is called a *nonhomogeneous system* of linear equations.

The matrix equation

$$A\mathbf{x} = \mathbf{0} \tag{2}$$

is called the *associated homogeneous system* of linear equations to (1).

For example, the following is a nonhomogeneous system of linear equations

$$\begin{aligned} x_1 \quad\quad\; - 2x_3 + x_4 &= 5 \\ 3x_1 + x_2 - 5x_3 \quad\quad &= 8 \\ x_1 + 2x_2 \quad\quad - 5x_4 &= -9 \end{aligned}$$

The associated homogeneous system to the above is

$$\begin{aligned} x_1 \quad\quad\; - 2x_3 + x_4 &= 0 \\ 3x_1 + x_2 - 5x_3 \quad\quad &= 0 \\ x_1 + 2x_2 \quad\quad - 5x_4 &= 0 \end{aligned}$$

In the previous lecture, we saw that the set of all solution vectors of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a subspace (nullspace or solution space).

Is this true also of the set of all solution vectors of the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$?

The answer is NO. Because the zero vector is <u>never</u> a solution of a nonhomogeneous linear system.

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Solutions of Nonhomogeneous Linear Systems

Particular Solution

The vector $\mathbf{x} = \mathbf{x}_p$ is called *a* particular solution of (1) if the vector satisfies $A\mathbf{x}_p = \mathbf{b}$.

For example,

$$\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

is a particular solution of

$$\begin{aligned} x_1 \qquad\quad - 2x_3 + x_4 &= 5 \\ 3x_1 + \ x_2 - 5x_3 \qquad\ &= 8 \\ x_1 + 2x_2 \qquad\quad - 5x_4 &= -9 \end{aligned}$$

To verify this—

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix}$$

$$A\mathbf{x}_p = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix} = \mathbf{b}$$

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Solutions of Nonhomogeneous Linear Systems

There is a relationship between the sets of solutions of the two systems $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.

Theorem 1 If $\mathbf{x}_{p_1}$ and $\mathbf{x}_{p_2}$ be particular solutions of a nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_{p_1} - \mathbf{x}_{p_2}$ is a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. Define $\mathbf{v} = \mathbf{x}_{p_1} - \mathbf{x}_{p_2}$. Now

$$\begin{aligned} A\mathbf{v} = A\left(\mathbf{x}_{p_1} - \mathbf{x}_{p_2}\right) &= A\mathbf{x}_{p_1} - A\mathbf{x}_{p_2} \\ &= \mathbf{b} - \mathbf{b} \\ &= \mathbf{0} \end{aligned}$$

This shows that $\mathbf{x}_{p_1} - \mathbf{x}_{p_2}$ is a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. ■

Theorem 2 Let A be an $m \times n$ matrix. Also let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_k\}$ is a basis of the nullspace of A. If $\mathbf{x}_p$ is any particular solution and $\mathbf{u}$ is an arbitrary solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then there exist scalars $c_1, c_2, \dots c_k$ such that

$$\mathbf{u} = \mathbf{x}_p + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

Proof. Since $\mathbf{u}$ and $\mathbf{x}_p$ are solutions of $A\mathbf{x} = \mathbf{b}$, then by the Theorem 1, $\mathbf{u} - \mathbf{x}_p$ is a solution of $A\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{u} - \mathbf{x}_p \in N(A)$. Since B is a basis of $N(A)$, there exist scalars $c_1, c_2, \dots c_k$ such that

$$\mathbf{u} - \mathbf{x}_p = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

$$\mathbf{u} = \mathbf{x}_p + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

This completes the proof of the theorem. ■

Naturally, the expression $\mathbf{x}_p + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ is called the general/complete solution of $A\mathbf{x} = \mathbf{b}$.

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For example, consider the following system.

$$\begin{aligned} x_1 \qquad\quad - 2x_3 + x_4 &= 5 \\ 3x_1 + \ x_2 - 5x_3 \qquad\quad &= 8 \\ x_1 + 2x_2 \qquad\quad - 5x_4 &= -9 \end{aligned}$$

If $\mathbf{x}_p$ be a particular solution and B be a basis of the nullspace of the associated homogeneous system, where

$$\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}, \qquad B = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

then the general solution is given by

$$\mathbf{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

■

Example Solve the following system

$$\begin{aligned} x_1 \qquad\quad - 2x_3 + x_4 &= 5 \\ 3x_1 + \ x_2 - 5x_3 \qquad\quad &= 8 \\ x_1 + 2x_2 \qquad\quad - 5x_4 &= -9 \end{aligned}$$

Solution Let's write the system in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 3 & 1 & -5 & 0 \\ 1 & 2 & 0 & -5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ -9 \end{bmatrix}$$

By applying the Gauss–Jordan elimination to the augmented matrix we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 2 & 2 & -6 & -14 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The above matrix gives

$$x_1 \quad - 2x_3 + x_4 = 5$$

$$x_2 + x_3 - 3x_4 = -7$$

Since x_3 and x_4 are free variables, set $x_3 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$. Then

$$x_1 = 5 + 2s - t, \quad x_2 = -7 - s + 3t, \quad x_3 = s, \quad x_4 = t$$

Therefore, the general solution is given by

$$\mathbf{x} = \begin{bmatrix} 5 + 2s - t \\ -7 - s + 3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

■

In the next example, we will establish a condition, for which a nonhomogeneous system will be consistent.

Example

$$\begin{aligned} x_1 + x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \\ -2x_1 - 3x_2 &= b_3 \end{aligned}$$

Let's reduce the augmented matrix,

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_1 + b_2 + b_3 \end{array}\right]$$

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The above system will be consistent if

$$b_1 + b_2 + b_3 = 0$$

Considering the above condition, the given system provides the following solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}.$$

■

Next, we are going to discuss on the dependency of the solution of $A\mathbf{x} = \mathbf{b}$ on rank r.

Case I. A is a square invertible matrix

In $A\mathbf{x} = \mathbf{b}$, if A is a square invertible matrix, i.e., $m = n = r$, then the particular solution is the one and only solution $\mathbf{x}_p = A^{-1}\mathbf{b}$. There will exist no free variables and hence no special solution.

For example, consider the system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

Here,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{x}_p = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Since the nullspace $N(A)$ consists only the trivial solution, the complete solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

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Solutions of Nonhomogeneous Linear Systems

Case II. A has full column rank

For an $m \times n$ matrix A (with $m > n$), if the column rank$(A) = n$, then A is called a matrix with **full column rank** ($r = n$). For $A\mathbf{x} = \mathbf{b}$, if A has full column rank, then the system has the following properties:

- All columns of A are pivot columns.
- There are no free variables, and hence no special solutions.
- The nullspace $N(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$.
- $A\mathbf{x} = \mathbf{b}$ may have only one solution, or none.

Let's have an example.

Consider the following system,

$$\begin{aligned} x_1 + \ \ x_2 &= \ 2 \\ -2x_1 - 3x_2 &= -3 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

It can be written as,

$$A\mathbf{x} = \mathbf{b}$$

where, $\quad A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Applying row reduction to augmented matrix, we obtain

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & -3 & -3 \\ 1 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution will be $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. ■

Note. Since, $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, this implies that, the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has the trivial solution only, i.e. $N(A)$ contains the zero vector only.

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Consider another system,

$$\begin{aligned} x_1 + \ \ x_2 &= \ 2 \\ -2x_1 - 3x_2 &= -3 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

After reduction, augmented matrix becomes

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & -3 & -3 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Since the last row provides a false statement, the system is inconsistent.

Case III. A has full row rank

For an $m \times n$ matrix A (with $m < n$), if the row rank$(A) = m$, then A is called a matrix with **full row rank** $(r = m)$. In such case $A\mathbf{x} = \mathbf{b}$ has the following properties:

- All rows have pivots, i.e., the reduced row echelon form, R has no zero rows.
- There will exist free variable(s), and hence special solution(s).
- The column space is the whole space $\mathbb{R}^m$.
- There are $n - r = n - m$ special solutions in the nullspace of A.
- $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Let's see an example.

Consider the following system

$$\begin{aligned} x_1 + \ \ x_2 + x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= 4 \end{aligned}$$

Applying elimination in the augmented matrix

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

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we get,

$$\begin{aligned} x_1 + \quad\quad 3x_3 &= 2 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

since x_3 is free variable, set $x_3 = s$, where $s \in \mathbb{R}$. Then

$$x_1 = 2 - 3s, \quad x_2 = -1 + 2s, \quad x_3 = s$$

So, the complete solution is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 3s \\ -1 + 2s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

■

Case IV. A is not full rank (i.e. $r < m, r < n$)

In such case, $A\mathbf{x} = \mathbf{b}$ has no solution, or has infinitely many solutions.

Let's have examples.

Consider

$$\begin{aligned} x_1 + \quad\quad 2x_3 + 3x_4 &= 7 \\ 2x_1 + x_3 + 3x_3 + 8x_4 &= 17 \\ 3x_1 + x_3 + 5x_3 + 11x_4 &= 24 \end{aligned}$$

Augmented matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 2 & 1 & 3 & 8 & 17 \\ 3 & 1 & 5 & 11 & 24 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot variables are x_1, x_2; and free variables are x_3, x_4.

So, set $x_3 = s,\ x_4 = t$, where $s, t \in \mathbb{R}$. Then

$$x_1 = 7 - 2s - 3t, \quad x_2 = 3 + s - 2t, \quad x_3 = s, \quad x_4 = t$$

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$$\mathbf{x} = \begin{bmatrix} 7 - 2s - 3t \\ 3 + s - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

■

Consider another example,

$$\begin{aligned} x_1 + \qquad 2x_3 + 3x_4 &= 7 \\ 2x_1 + x_3 + 3x_3 + 8x_4 &= 8 \\ 3x_1 + x_3 + 5x_3 + 11x_4 &= 24 \end{aligned}$$

Augmented matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 2 & 1 & 3 & 8 & 17 \\ 3 & 1 & 5 & 11 & 24 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Last row of the above matrix gives a false statement. Hence, the system is inconsistent. ■

Dependency of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ on rank r :

Relation among $\boldsymbol{r}, \boldsymbol{m}, \boldsymbol{n}$	**Type/shape of matrix $\boldsymbol{A}$**	**No. of solution to $\boldsymbol{A}\mathbf{x} = \mathbf{b}$**
$r = m = n$	Square, Invertible	One solution
$r = n$ and $m > n$	Tall, Thin	No solution, or One solution
$r = m$ and $m < n$	Short, Wide	Infinitely many solutions
$r < m$ and $r < n$	Not full rank	No solution or, Infinitely many solutions

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Solutions of Nonhomogeneous Linear Systems

Exercise

1. (a) Determine whether the nonhomogeneous system $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ is consistent, and (b) if the system is consistent, then find the complete solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is a solution of $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ and.

i) $$\begin{aligned} x + 3y + 10z &= 18 \\ -2x + 7y + 32z &= 29 \\ -x + 3y + 14z &= 12 \\ x + y + 2z &= 8 \end{aligned}$$

ii) $$\begin{aligned} 3x_1 - 2x_2 + 16x_3 - 2x_4 &= -7 \\ -x_1 + 5x_2 - 14x_3 + 18x_4 &= 29 \\ 3x_1 - x_2 + 14x_3 + 2x_4 &= 1 \end{aligned}$$

2. (a) Under what condition on b_1, b_2 and b_3, the following system will be solvable (or, consistent)? (b) Find $\mathbf{x}$ in that case.

$$\begin{aligned} x + 3y + z &= b_1 \\ 3x + 8y + 2z &= b_2 \\ 2x + 4y &= b_3 \end{aligned}$$