**MAT216: Linear Algebra & Fourier Analysis**

**Lecture Note: Week 1_Lecture 2**

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**Contents:**

- Linear Combination, Linear Combination of Vectors
- Vector Coordinate as Linear Combinations of Basis Vectors
- Alternate Basis Vectors for Coordinates

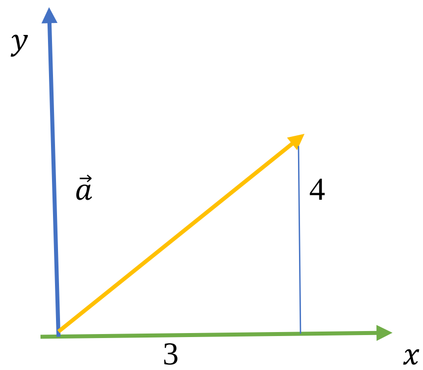
**Reference Book:**

Introduction to Linear Algebra, 5th Ed. Gilbert Strang

Let's begin by saying what scalar are: A scalar is a number. Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no "direction", other than perhaps positive or negative. One the other hand a vector is a list of numbers. There are at least two ways to interpret what this list of numbers mean: One way to think of the vector as being a point in a space. Then this list of numbers is a way of identifying that point in space, where each number represents the vector's component that dimension.

**If there are 2 numbers in the list, there is a natural correspondence to a point in a plane, determined by the choice of axes. If there are 3 numbers in the list, it corresponds to a point in 3-dimensional space.**

Another way to think of a vector is a magnitude and a direction, e.g. a quantity like velocity in a particular direction. In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers $\vec{a}$ = [3, 4]. Graphically, you can think of this vector as an arrow in the $x - y$ plane, pointing from the origin to the point at $x = 3,\ y = 4$ (see illustration).




In this example, the list of numbers was only two elements long, but in principle it could be any length. The dimensionality of a vector is the length of the list. So, our example $\vec{a}$ is 2-dimensional because it is a list of two numbers. Not surprisingly all 2-dimentional vectors live in a plane. A 3-dimensional vector would be a list of three numbers, and they live in a 3-D volume. A 27-dimensional vector would be a list of twenty seven numbers.

So, what are the obvious things we can do with vectors?

In $\mathbb{R}^2$ can the vector $(5, 3)$ be written in the form $(5, 3) = 1(2, 0) + 3(1, 1)$ ?

Since the answer is yes, therefore (2,0) and (1,1) is a linear combination of (5,3).

Yes, we can and each case $(5, 3)$is a **linear combination** of the two vectors on the right side. Generally, in mathematics, we say that a linear combination of things is a sum of multiples of those things. So, for example, one linear combination of the functions $f(x), g(x)$ and $h(x)$ is

$$2f(x) + 3g(x) - 4h(x).$$

***Definition***: consider S= $\{v_1, \dots, v_k\}$

A linear combination of vectors $v_1, v_2, \dots\dots\dots, v_k$ in a vector space $V$ is an expression of the form $V = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ where the $c_i$'s are scalars, that is, it's a sum of scalar multiples of them. More generally, if $S$ is a set of vectors in $V$ , not necessarily finite, then a linear combination of S refers to a linear combination of some finite subset of S. Of course, differences are allowed, too, since negations of scalars are scalars.

**Example:**

1. Consider the vector space $\mathbb{R}^3$. Let $v_1 = (0, 1, -1)$, $v_2 = (-1, 0, 1)$, $v_3 = (1, -1, 0)$, $v_4 = (3, 2, -5)$, $v_5 = (1, 1, 1)$. Here $v_4$ is a linear combination of $v_1$, $v_2$ because we can write $v_4 = 2v_1 - 3v_2$.

   **Exercise:**

   a. Verify that $v_4$is a linear combination of $v_1$, $v_3$.

$$v_4 = c_1 v_1 + c_3 v_3$$

$$(3,2,-5) = 5(0,1,-1) + 3(1,-1,0)$$

2. Let $\mathbb{R}^2$ be the underlying vector space. Is $w = (1, 2)$ a linear combination of $u = (1, 3)$ and $v = (4, 1)$?

$$(1, 2) = \frac{7}{11}(1,3) + \frac{1}{11}(4,1)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 1 \\ 3 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -11 & -1 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$\therefore c_2 = \frac{1}{11} \quad hence\ c_1 + 4c_2 = 1\ \to c_1 = \frac{7}{11}$$

3. Draw $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v + w$ and $v - w$ in a single $xy$ plane.

4. Compute $u + v + w$ and $2u + 2v + w$. How do you know $u, v, w$ lie in a plane? [Hints: These lie in a plane because $\boldsymbol{w = cu + dv}$, Find $c$ and $d$]

**Theorem:** If $B = \{v_1, v_2, \ldots\ldots\ldots, v_k\}$ is a **basis** (set of vectors) for a subspace $V$, then any vector $x$ in $V$ can be written as a liner combination $x = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ in exactly one way.

**Example:** Consider the standard basis of $\mathbb{R}^3$: $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

According to the theorem, every vector in $\mathbb{R}^3$can be written as a linear combination of $e_1, e_2, e_3$ with unique coefficient. For example,

$$v = \begin{bmatrix}3\\5\\-2\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\end{bmatrix} = 3e_1 + 5\,e_2 - 2\,e_3$$

**In this case, the coordinates of v are exactly the coefficients of $\boldsymbol{e_1, e_2, e_3}$.**

Any vector has vector coordinates and these vectors has basis vector, let us assume two basis vectors $\hat{\imath}$ along the x- axis and $\hat{\jmath}$ goes along y-axis and usually these basis vectors are 1, but we can change them anytime. We can scale the vector coordinates with the basis vectors for instance

$$\vec{v} = \begin{bmatrix}2\hat{\imath}\\5\hat{\jmath}\end{bmatrix}$$

Then if we assume our basis vectors to be standard, the equation would be

$$\vec{v} = \begin{bmatrix}2\\5\end{bmatrix}$$

We could scale it by multiplying with 5 on both vector coordinates

$$3\vec{v} = 3\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}10\\25\end{bmatrix}$$

**Therefore, basis vectors enable us to reach any point. The two exceptions of this is if we have all vectors are struck at origin or on the exact same line, meaning all vectors can facing the exact same or opposite way.**

Let V be a subspace of $\mathbb{R}^n$for some $n$. If V has a basis containing exactly r vectors, then every basis for V contains exactly r vectors. That is, the choice of basis vectors for a given space is not unique, but the number of basis vectors is unique. This fact permits the following notion to be well defined:

**The number of vectors in a basis for a vector space $V \subseteq \mathbb{R}^n$ is called the** dimension **of V, denoted dim V.**

**Example**: Since the standard basis for $\mathbb{R}^2$, {i, j}, contains exactly 2 vectors, every basis for $\mathbb{R}^2$ contains exactly 2 vectors, so dim $\mathbb{R}^2 = 2$. Similarly, since {i, j, k} is a basis for $\mathbb{R}^3$ that contains exactly 3 vectors, every basis for $\mathbb{R}^3$ contains exactly 3 vectors, so dim $\mathbb{R}^3 = 3$. In general, dim $\mathbb{R}^n = n$ for every natural number n.

Let us again consider two vectors $\vec{u}$ and $\vec{v}$, if we add $\vec{u}$ and $\vec{v}$ together, we call it a linear combination of two vectors. If we take those two vectors, $\vec{u}$ and $\vec{v}$, and find every possible combination, then we call **that the set** of **<u>all possible linear combinations</u>**, which is also called the **<u>span</u>**.

# Review of the Key Ideas

- A vector v in two-dimensional space has two components $v_1$ and $v_2$.
- A linear combination of three vectors u, v, and w is $cu + dv + ew$. All the linear combinations of these three vectors fall a plane in $\mathbb{R}^3$.
- If we have a set of N vectors, the maximum dimension we can cover with these vectors is N-dimension.
- We can never span a three-dimensional space using two dimensional vectors, regardless of how many vectors we use, i.e. vectors with n components, spanning dimension that are higher than n is impossible.