MAT 216

Linear Algebra & Fourier Analysis

Week 3 Lecture 6

Lecture Note

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Contents:

- **Linear combination and Linear Dependency**
- **Gaussian Elimination on Linearly Dependent Column**
- **Linear Dependency in Square and non-Square matrices.**

Reference Book:

Introduction to Linear Algebra, 5th Ed. Gilbert Strang

Linear combination and Linear Dependency

A vector V is called a ***linear combination*** of vectors $v_1, v_2, \dots \dots, v_n$ if it can be expressed in the following form:

$$V = k_1 v_1 + k_2 v_2 + \cdots \cdots + k_n v_n$$

$$=> \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix} = k_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{1m} \end{pmatrix} + k_2 \begin{pmatrix} v_{21} \\ \vdots \\ v_{2m} \end{pmatrix} + \cdots \cdots + k_n \begin{pmatrix} v_{n1} \\ \vdots \\ v_{nm} \end{pmatrix}$$

Here $k_1, k_2, \cdots \cdots, k_n$ are scalars.

If $n = 1$, then the above equation will be reduce to $V = k_1 v_1$. It implies V is a linear combination of single vector v_1 if k_1 is a scalar multiple of v_1.

Consider: $\begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix} = k_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{1m} \end{pmatrix}$.

We evaluate the coefficients of $k_1, k_2, \cdots \cdots$ where

$$V = k_1 v_1 + k_2 v_2 + k_3 v_3 \cdots \cdots$$

There exists no linear combination if there is no solution of the system (when the system is inconsistent).

Example: consider the vectors $v_1 = (2, 1, 4), v_2 = (1, -1, 3), v_3 = (3, 2, 5)$ *in* $\mathbb{R}^3$ *(three dimensional vector space). Show that* $V = (5, 9, 5)$ *is a linear combination of* v_1, v_2, v_3.

Solution: In order to show that V is a linear combination of v_1, v_2, v_3 there must be scalars k_1, k_2, k_3 in a vector field such that

$$V = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$=> (5, 9, 5) = k_1(2, 1, 4) + k_2(1, -1, 3) + k_3(3, 2, 5)$$

$$=> \begin{pmatrix} 5 \\ 9 \\ 5 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + k_3 \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\boldsymbol{OR}\ (5, 9, 5) = (2k_1 + k_2 + 3k_3,\ k_1 - k_2 + 2k_3, 4k_1 + 3k_2 + 5k_3)$$

Equating corresponding components & formatting linear equations:

$$2k_1 + k_2 + 3k_3 = 5$$

$$k_1 - k_2 + 2k_3 = 9$$

$$4k_1 + 3k_2 + 5k_3 = 5$$

Augmented Matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 1 & -1 & 2 & 9 \\ 4 & 3 & 5 & 5 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 3 & 5 \\ 4 & 3 & 5 & 5 \end{array}\right) R_1, R_2\ interchange$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -1 & -13 \\ 0 & 7 & -3 & -31 \end{array}\right) \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -1 & -13 \\ 0 & 0 & -2 & -2 \end{array}\right) R_3 \rightarrow 3R_3 - 7R_2$$

$$\begin{gathered} k_1 - k_2 + 2k_3 = 9 \\ 3k_2 - k_3 = -13 \\ -2k_3 = -2 \end{gathered}$$

By Back substitution we have: $k_3 = 1$, $k_2 = -4$, $k_1 = 3$

$$\therefore (5, 9, 5) = k_1(2, 1, 4) + k_2(1, -1, 3) + k_3(3, 2, 5)$$

$$=> (5, 9, 5) = 3(2, 1, 4) - 4(1, -1, 3) + 1(3, 2, 5)$$

Hence $(5,9,5)$ is a linear combination of $(2,1,4)$, $(1, -1, 3)$ and $(3,2,5)$.

Linear Dependency

If $S = \{v_1, v_2, \cdots\cdots, v_n\}$ is non-empty set of vectors, then the vector equation

$k_1v_1 + k_2v_2 + \cdots\cdots + k_nv_n = 0$ or $\boldsymbol{Ax} = \mathbf{0}$ has at least one solution, namely

$k_1 = 0,\ k_2 = 0\,,\ \cdots, k_n = 0.$

- If there is a **unique solution**, then S is called a **linearly independent** set.
- If there are **many solutions**, then S is called **linearly dependent** set

Gaussian Elimination on Linearly Dependent Column

To test for linear dependence/independence we can write the corresponding matrix in echelon form.

Example: Are the following vectors linearly dependent/ Independent?

a) $\left\{\begin{pmatrix}1\\-2\\0\end{pmatrix}, \begin{pmatrix}4\\0\\8\end{pmatrix}, \begin{pmatrix}3\\-1\\5\end{pmatrix}\right\}$

b) $\left\{\begin{pmatrix}1\\2\\1\\4\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\7\end{pmatrix}\right\}$

Solution:

a) We have $k_1\begin{pmatrix}1\\-2\\0\end{pmatrix} + k_2\begin{pmatrix}4\\0\\8\end{pmatrix} + k_3\begin{pmatrix}3\\-1\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

The augmented matrix:

$$\left(\begin{array}{ccc|c}\mathbf{1} & 4 & 3 & 0\\ -2 & 0 & -1 & 0\\ 0 & 8 & 5 & 0\end{array}\right)$$

$$= \left(\begin{array}{ccc|c}1 & 4 & 3 & 0\\ 0 & \mathbf{8} & 5 & 0\\ 0 & 8 & 5 & 0\end{array}\right) \quad R_2 \to R_2 + 2R_1$$

$$= \left(\begin{array}{ccc|c}1 & 4 & 3 & 0\\ 0 & 8 & 5 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \quad R_3 \to R_3 - R_2$$

$$= \left(\begin{array}{ccc|c}\mathbf{1} & 4 & 3 & 0\\ 0 & \mathbf{1} & {}^5/_8 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \quad R_2 \to R_2 \div 8$$

The pivot in the last column is missing. Therefore C_3 is a dependent column.

Therefore k_3 is a free variable and we are having many solution of the system.

Let $k_3 = p$ any ***arbitrary value*** (*real number*)

In R_2 we have: $k_2 + {}^5/_8\, k_3 = 0 \;=>\; k_2 = -{}^5/_8\, p$

In R_1 we have: $k_1 + 4k_2 + 3k_3 = 0 => \; k_1 = -4\left(-{}^5/_8\, p\right) - 3(p) => \; k_1 = -\frac{1}{2}p.$

$\therefore$ Solution of the system is $(k_1, k_2, k_3) = (-\frac{1}{2}p, -{}^5/_8\, p,\; p)$ while $p \epsilon \mathbb{R}$ (any real number).

- ⇨ Non trivial solution
- ⇨ The set of vectors is linearly dependent.

Trivial solution: The only **solution** to $Ax = 0$ is $x = 0$.

For example $\boldsymbol{k_1 v_1 + k_2 v_2 + \cdots \cdots + k_n v_n = 0}$ has at least one solution, namely $k_1 = 0$,

$k_2 = 0\,, \;\; \cdots,\; k_n = 0.$

Non-trivial solution: There exists x for which $Ax = 0$ where $x \neq 0$.

b) We have $k_1\begin{pmatrix}1\\2\\1\\4\end{pmatrix}+k_2\begin{pmatrix}0\\1\\1\\1\end{pmatrix}+k_3\begin{pmatrix}2\\0\\1\\7\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

The augmented matrix:

$$\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\2 & 1 & 0 & 0\\1 & 1 & 1 & 0\\4 & 1 & 7 & 0\end{array}\right)$$

$$=\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -4 & 0\\0 & 1 & -1 & 0\\0 & 1 & -1 & 0\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-R_1\\R_4\to R_4-4R_1\end{array}$$

$$=\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -4 & 0\\0 & 0 & 3 & 0\\0 & 0 & 3 & 0\end{array}\right)\begin{array}{l}R_3\to R_3-R_2\\R_4\to R_4-R_2\end{array}$$

$$=\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -4 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right)\ R_4\to R_4-R_3$$

We will omit R_4 in our next step since it is entirely “0”

$$=\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -4 & 0\\0 & 0 & 3 & 0\end{array}\right)$$

$$=\left(\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & -4 & 0\\0 & 0 & 1 & 0\end{array}\right)\ R_3\to R_3\div 3$$

Hence by **back substitution** we have $k_1=0,\ k_2=0,\ k_3=0.$

- ⇨ Trivial solution
- ⇨ The set of vectors is linearly independent.

Linear Dependency in Square and non-Square matrices

We have observed linear dependency in a square matrix while solving Example (a) from ***Gaussian Elimination on Linearly Dependent Column.***

Let's consider non-Square matrices or Rectangular matrices:

Few Facts about Rectangular Matrices

- **Consider the following system of linear Equation:**

$$x + y = a$$

$$x + 2y = b$$

$$-2x - 3y = c$$

Converting the system to augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \\ -2 & -3 & c \end{array}\right)$$

It is a **tall matrix** with dimension 3×2 {Number of variables are less than the number of equations, i.e. 2 variables x, y and 3 equations}.

After applying Gaussian Elimination Method we have

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-a \\ 0 & -1 & c+2a \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-a \\ 0 & 0 & a+b+c \end{array}\right) R_3 \to R_3 + R_2$$

$Considering\ R_3: 0 = a + b + c$

- If $a + b + c = 0 \to R_3: 0 = 0$, then the system has unique solution.
  By back substitution we have:
  $R_2: y = b - a$
  $R_1: x + y = a => x + (b - a) = a => x = 2a - b$
  $$\therefore (x, y) = (2a - b, b - a)$$

- If $a + b + c \neq 0$ then the system is inconsistent, which means no solution.
  In that case $R_3 : 0 = a + b + c, but\ a + b + c \neq 0$ (Hence it is a false statement).

- **Consider another system of linear Equation:**

$$x + y + z = a$$

$$x - y + z = b$$

Converting the system to augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 1 & b \end{array}\right)$$

It is a **wide matrix** with dimension 2×3 {Number of variables are more than the number of equations, i.e. 3 variables x, y, z and 2 equations}.

Applying Gaussian Elimination method:

$$\left(\begin{array}{ccc|c} \mathbf{1} & 1 & 1 & a \\ 0 & \mathbf{-2} & 0 & b-a \end{array}\right) R_2 \to R_2 - R_1$$

C_3 is a dependent column (as there is no pivot in C_3) and hence z is a free variable. The system will have many solutions in this case.

Let $z = p$ any arbitrary value.

$R_2 \Rightarrow -2y = b - a \quad \therefore y = -\frac{1}{2}(b - a)$

$R_1 \Rightarrow x + y + z = a \ \ or\ x + \left\{-\frac{1}{2}(b - a)\right\} + p = a,$

$$x = \frac{1}{2}b + \frac{1}{2}a - p + a$$

$$x = \frac{3}{2}a + \frac{1}{2}b - p$$

Solution of the system $(x, y, z) = (\frac{3}{2}a + \frac{1}{2}b - p, -\frac{1}{2}(b - a), p)$ while a, b, p are real numbers