Vector Spaces

Topics

- Vectors in $\mathbb{R}^n$;
- Vector Spaces: Vectors in Abstract/General Spaces;
- Subspaces of Vector Spaces; Linear Combinations
- Spanning Sets and Linear Independence, Span of a Set;
- Basis and Dimension of a Vector Space, Dimension of Subspaces;
- Row Space and Column Space of a Matrix;
- Rank of a Matrix
- Coordinates and Change of Basis

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Vectors in $\mathbb{R}^n$

Definition 9.1.1. For $n \in \mathbb{N}$, we define

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n \right\}.$$

For example,

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \in \mathbb{R}^2, \qquad \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^3, \qquad \begin{bmatrix} -2 \\ 4 \\ 0 \\ 12 \\ -3 \end{bmatrix} \in \mathbb{R}^5.$$

The objects in $\mathbb{R}^n$ are usually called *vectors*, more specifically *geometric vectors*. In this lecture, we will generalize the concepts of vectors and explore vectors other than the geometric ones. In other words, not every vector looks like an n-tuple.

Definition 9.1.2. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$. We say $\mathbf{x} = \mathbf{y}$ if and only if $x_i = y_i$ for all $i = 1, 2, \dots, n$

The **sum** of two vectors in $\mathbb{R}^n$ and the **scalar multiple** of a vector in $\mathbb{R}^n$ are called the **standard operations** in $\mathbb{R}^n$ and are defined as follows:

Definition 9.1.3. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$ and c be a real number. Then the sum $\mathbf{x} + \mathbf{y}$ and scalar multiple $c\mathbf{x}$ are defined as

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}. \tag{9.1.1}$$

Clearly, the vectors $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ are in $\mathbb{R}^n$.

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Definition 9.1.4. Zero Vector

The element in $\mathbb{R}^n$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is called the **zero vector** in $\mathbb{R}^n$.

For example, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

Observe that $\mathbf{0} \in \mathbb{R}^2$ and $\mathbf{0} \in \mathbb{R}^3$ are not the same zero vectors.

Definition 9.1.5. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$, then the vector $-\mathbf{x}$ is called the **negative** of $\mathbf{x}$ defined as

$$-\mathbf{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}.$$

For example, if $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ -3 \\ 2 \end{bmatrix} \in \mathbb{R}^4$, then $-\mathbf{x} = \begin{bmatrix} 1 \\ -4 \\ 3 \\ -2 \end{bmatrix}$.

Theorem 9.1.1. (**Properties of vector addition and scalar multiplication in** $\mathbb{R}^n$)

Let $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ be any vectors in $\mathbb{R}^n$, and $c, d \in \mathbb{R}$. Then

1. $\mathbf{x} + \mathbf{y} \in \mathbb{R}^n$ — Closure Property: $\mathbb{R}^n$ is closed under addition

2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. — Commutative Property of $\mathbb{R}^n$: addition is commutative

3. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. — Associative Property of $\mathbb{R}^n$: addition is associative

4. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ — Existence of Additive Identity

5. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$ — Existence of Additive Inverse

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6. $c\mathbf{x} \in \mathbb{R}^n$ Closure Property: $\mathbb{R}^n$ is closed under scalar multiplication

7. $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ Distributive Property: Scalar distributes over vectors in $\mathbb{R}^n$

8. $(c+d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ Distributive Property: vector distributes over scalars

9. $(cd)\mathbf{x} = c(d\mathbf{x})$ Associative Property of Scalars

10. $1\mathbf{x} = \mathbf{x}$. Multiplicative Identity

Proof.

1. The proof immediately follows from eq. (9.1.1).

2. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$, then

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} = \begin{bmatrix} y_1+x_1 \\ y_2+x_2 \\ \vdots \\ y_n+x_n \end{bmatrix} = \mathbf{y}+\mathbf{x}$$

since $x_i + y_i = y_i + x_i$ for all $i = 1, 2, \dots, n$.

3. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$, then

$$(\mathbf{x}+\mathbf{y})+\mathbf{z} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1+z_1 \\ x_2+y_2+z_2 \\ \vdots \\ x_n+y_n+z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ \vdots \\ y_n+z_n \end{bmatrix}$$

$$= \mathbf{x}+(\mathbf{y}+\mathbf{z})$$

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4. By using Definition 9.1.1, we can write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be two vectors in $\mathbb{R}^n$, then

$$\mathbf{x} + \mathbf{0} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}$$

5. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $-\mathbf{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ and

$$\mathbf{x} + (-\mathbf{x}) = \begin{bmatrix} x_1 + (-x_1) \\ x_2 + (-x_2) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

6. The proof immediately follows from eq. (9.1.1).

7. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be two vectors in $\mathbb{R}^n$, then

$$c(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} c(x_1 + y_1) \\ c(x_2 + y_2) \\ \vdots \\ c(x_n + y_n) \end{bmatrix} = \begin{bmatrix} cx_1 + cy_1 \\ cx_2 + cy_2 \\ \vdots \\ cx_n + cy_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} cy_1 \\ cy_2 \\ \vdots \\ cy_n \end{bmatrix} = c\mathbf{x} + c\mathbf{y}.$$

8. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and c, d be scalars, then

$$(c + d)\mathbf{x} = \begin{bmatrix} (c + d)x_1 \\ (c + d)x_2 \\ \vdots \\ (c + d)x_n \end{bmatrix} = \begin{bmatrix} cx_1 + dx_1 \\ cx_2 + dx_2 \\ \vdots \\ cx_n + dx_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} = c\mathbf{x} + d\mathbf{x}.$$

9. The proof immediately follows from eq. (9.1.1).

10. The proof immediately follows from eq. (9.1.1).

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Theorem 9.1.2. (**Left Cancellation Property in** $\mathbb{R}^n$)

Let $\mathbf{x}, \mathbf{y}$ be two vectors in $\mathbb{R}^n$. If $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$, then $\mathbf{y} = \mathbf{z}$.

Proof. Since $\mathbf{y} \in \mathbb{R}^n$, by Property 4 of the Theorem 9.1.1, we can write

$$\begin{aligned}
\mathbf{y} &= \mathbf{0} + \mathbf{y} && \text{Property 4} \\
&= (-\mathbf{x} + \mathbf{x}) + \mathbf{y} && \text{Property 5} \\
&= -\mathbf{x} + (\mathbf{x} + \mathbf{y}) && \text{Property 3} \\
&= -\mathbf{x} + (\mathbf{x} + \mathbf{z}) && \text{Hypothesis} \\
&= (-\mathbf{x} + \mathbf{x}) + \mathbf{z} && \text{Property 3} \\
&= \mathbf{0} + \mathbf{z} && \text{Property 5} \\
&= \mathbf{z} && \text{Property 4}
\end{aligned}$$

Corollary Let $\mathbf{x}, \mathbf{y}$ be two vectors in $\mathbb{R}^n$. If $\mathbf{y} + \mathbf{x} = \mathbf{z} + \mathbf{x}$, then $\mathbf{y} = \mathbf{z}$. This is called the **right cancellation property**.

Proof. By commutativity of addition, we can write

$$\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$$

Now the result follows from the left cancellation.

Theorem 9.1.3. (**Properties of additive identity and additive inverse in** $\mathbb{R}^n$)

Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and let c be a real number. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{x} + \mathbf{y} = \mathbf{x}$ then $\mathbf{y} = \mathbf{0}$.

2. The additive inverse is unique. That is, if $\mathbf{x} + \mathbf{y} = \mathbf{0}$ then $\mathbf{y} = -\mathbf{x}$.

3. $0\mathbf{x} = \mathbf{0}$

4. $c\mathbf{0} = \mathbf{0}$

5. If $c\mathbf{x} = \mathbf{0}$ then $c = 0$ or $\mathbf{x} = \mathbf{0}$.

6. $-(-\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$

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Proof.

1. Since $\mathbf{x}+\mathbf{0}=\mathbf{x}$, the right-hand side of the given equation can be written as

$$\mathbf{x}+\mathbf{y}=\mathbf{x}+\mathbf{0}$$

Now the left cancellation property (Theorem 9.1.2) gives

$$\mathbf{y}=\mathbf{0}.$$

2.

$$\mathbf{x}+\mathbf{y}=\mathbf{0}$$

$$\mathbf{x}+\mathbf{y}=\mathbf{x}+(-\mathbf{x})$$

The left cancellation property (Theorem 9.1.2) gives

$$\mathbf{y}=-\mathbf{x}.$$

3.

$$0\mathbf{x}+0\mathbf{x}=(0+0)\mathbf{x}$$

$$=0\mathbf{x}$$

$$0\mathbf{x}+0\mathbf{x}=0\mathbf{x}+\mathbf{0}$$

The left cancellation property gives

$$0\mathbf{x}=\mathbf{0}.$$

4.

$$c\mathbf{0}+c\mathbf{0}=c(\mathbf{0}+\mathbf{0})$$

$$=c\mathbf{0}$$

$$=c\mathbf{0}+\mathbf{0}$$

The left cancellation property gives

$$c\mathbf{0}=\mathbf{0}.$$

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5. If $c = 0$, then we are done. Let's consider the case $c \neq 0$. Then c^{-1} exists and

$$
\begin{aligned}
\mathbf{x} &= 1\mathbf{x} \\
&= (c^{-1}c)\mathbf{x} \\
&= c^{-1}(c\mathbf{x}) \\
&= c^{-1}\mathbf{0} \\
&= \mathbf{0}
\end{aligned}
$$

6.

$$
\begin{aligned}
-(-\mathbf{x}) + (-\mathbf{x}) &= \mathbf{0} \\
&= \mathbf{x} + (-\mathbf{x})
\end{aligned}
$$

Right Cancellation gives

$$-(-\mathbf{x}) = \mathbf{x}.$$

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9.2.1 Vector Spaces

In Theorem 9.1.1, ten special properties of vector addition and scalar multiplication in $\mathbb{R}^n$ were listed. Suitable definitions of addition and scalar multiplication reveal that many other mathematical quantities (such as matrices, polynomials, functions, sequences, and many others) also share these ten properties.

Any set that satisfies these properties (or **axioms**) is called a **vector space**, and the objects in the set are called **vectors**.

Definition 9.2.1 Let V be a set of "objects" on which two operations (addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and every scalar[*] c and d in a field $\mathbb{F}$, then V is called a **vector space** over $\mathbb{F}$.

Addition	Scalar Multiplication
1. $\mathbf{u} + \mathbf{v} \in V$	6. $c\mathbf{u} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.	7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.	8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	9. $(cd)\mathbf{u} = c(d\mathbf{u})$
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	10. $1\mathbf{u} = \mathbf{u}$.

If $\mathbb{F} = \mathbb{R}$, then V is called a **vector space over** $\mathbb{R}$ or simply a **real vector space**. Similarly, if $\mathbb{F} = \mathbb{C}$, then V is called a **complex vector space**. In physics and engineering, complex vector spaces play important role.

In this lecture, we will only consider real vector spaces.

Example 9.2.1 $\mathbb{R}^n$ is a (real) vector space.

[*] A **scalar** is an element of a *field*. The definition of a *field* may be found in the following link: https://youtu.be/UFdl4HI4Wdo?list=PLx0IgmhgjODDDqXQ9drwIwfhFe8auxGXU

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Example 9.2.2. Let $M_{2,3}(\mathbb{R})$ be the set of all 2×3 matrices over $\mathbb{R}$. To show that $M_{2,3}(\mathbb{R})$ is a real vector space under the standard matrix addition and scalar multiplication.

Solution.

$$M_{2,3} = \left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \text{ for all } i, j = 1, 2, 3\right\}$$

We will verify the first few axioms in the Definition 9.1.5.

1. Take $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

Clearly,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} \in M_{2,3}.$$

2. Since matrix addition is commutative, the axiom is easily verified. For example,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} & b_{13} + a_{13} \\ b_{21} + a_{21} & b_{22} + a_{22} & b_{23} + a_{23} \end{bmatrix} = B + A.$$

3. Since matrix addition is associative, the axiom is easily verified. That is, if $A, B, C \in M_{2,3}$,

$$(A + B) + C = A + (B + C).$$

4. Take $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{2,3}$. Then clearly, $A + O = A$ for all $A \in M_{2,3}$.

5. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, then $-A$ is defined as

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}.$$

Then clearly,

$$A + (-A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

6.

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix} \in M_{2,3}$$

In this fashion, we can prove the rest of the axioms.

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In general, the set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices is a real vector space.

Example 9.2.3. Let $\mathbf{P}_2(x)$ be the set of all polynomials over $\mathbb{R}$ of degree less than or equal to 2. Show that $\mathbf{P}_2(x)$ is a vector space.

$$\mathbf{P}_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_2, a_1, a_0 \in \mathbb{R}\}.$$

Solution. Take $p(x), q(x) \in \mathbf{P}_2(x)$ as follows:

$$p(x) = a_2x^2 + a_1x + a_0$$

$$q(x) = b_2x^2 + b_1x + b_0$$

Then

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0).$$

If $c \in \mathbb{R}$, then

$$cp(x) = (ca_2)x^2 + (ca_1)x + (ca_0).$$

Therefore, $p(x) + q(x) \in \mathbf{P}_2(x)$ and $cp(x) \in \mathbf{P}_2(x)$. Similarly, we can verify the rest of the axioms in the Definition 9.1.5. Hence $\mathbf{P}_2(x)$ is a vector space.

Example 9.2.4. Let $C[a,b]$ denotes the set of all continuous real-valued functions defined on the bounded and closed interval $[a,b]$ with $a < b$. One can show that $C[a,b]$ is a vector space.

Solution. Take two continuous functions $f, g \in C[a,b]$. Then from Calculus we know that the sum of continuous functions defined on $[a,b]$ is also continuous. So,

$$f + g \in C[a,b]$$

Similarly, for any scalar $c \in \mathbb{R}$, cf is continuous on $[a,b]$. Therefore,

$$cf \in C[a,b].$$

The **zero function** $0(x) = 0$ for all $x \in [a,b]$. Clearly, the zero function $0 \in C[a,b]$.

In a similar fashion, we can verify the rest of the axioms in the Definition 9.2.1. Thus $C[a,b]$ is a vector space.

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Theorem 9.2.1 (Properties of Scalar Multiplication)

Let V be a vector space.

1. $0\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$.

2. $c\mathbf{0} = \mathbf{0}$ for any scalar c.

3. If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.

4. $(-1)\mathbf{v} = -\mathbf{v}$ for any $\mathbf{v} \in V$.

Proof. The proofs are exactly the same as in Theorem 9.1.3.

Non-Example. The set of integers, $\mathbb{Z}$, is not a real vector space.

Note that the set $\mathbb{Z}$ is closed under addition. But if we take $c = 1/2$ and $3 \in \mathbb{Z}$, then

$$\frac{1}{2} \times 3 \notin \mathbb{Z}$$

Non-Example. The set of polynomials of degree 2 is not a real vector space.

Observe that the set of polynomials of degree 2 is not closed under addition. For example, if $p(x) = 2x^2 - 4x + 5$ and $q(x) = -2x^2 + 5x - 6$

$$p(x) + q(x) = x - 1$$

which is not a polynomial of degree 2. Hence the set of polynomials of degree 2 is not closed under addition.

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