BRAC UNIVERSITY

Inspiring Excellence

MAT 216

Linear Algebra and Fourier Analysis

Lecture 13

Prepared by : Zadid Hasan (MZH)

Reference Book:

- Introduction to Linear Algebra, 5th Edition by Gilbert Strang.

Table of Contents

Column Space of a Matrix

Definition: The column space of a matrix is the subspace spanned by the columns of a matrix. For example, if A is an $m \times n$ matrix with columns vectors $c_1, c_2, \cdots, c_n$. Then the column space of A is $C(A) = span\{c_1, c_2, \cdots, c_n\}$. It is also written as $col(A)$.

Suppose, we have this linear function $f(x) = Ax$, This is a relation from $\mathbb{R}^n \to \mathbb{R}^m$. The domain of this function is $\mathbb{R}^n$ and the co-domain is $\mathbb{R}^m$. But the range or image of this function is the column space $C(A)$.

Because, $f(x) = Ax = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} c_i x_i$.

Since, we are taking a linear combination of the columns of A, this output vector will surely belong to the column space of A.

Consider the following matrix, $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix}$

To find the column space for this matrix, we only need to find the linearly independent rows of A^T.

$A^T = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 5 & 2 \\ 1 & 3 & 3 & 0 \\ 4 & 9 & 1 & 8 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -3 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 \end{bmatrix}$

Since, in the row echelon form of the matrix A^T, we have 3 independent rows, it means row 1, 2 and 4 of A^T are linearly independent, which means column 1, 2, 4 of A are linearly independent. So, the basis of $C(A)$ is:

$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 1 \\ 8 \end{bmatrix}$

Row Space of a Matrix

Definition: The row space of a matrix is the subspace spanned by the rows of a matrix. For example, if A is an $m \times n$ matrix with row vectors $r_1, r_2, \cdots, r_m$. Then the column space of A is $R(A) = span\{r_1, r_2, \cdots, r_m\}$. It is also written as $row(A), C(A^T)$. We can write $R(A)$ as $C(A^T)$ because, the rows of A are the columns of A^T.

Suppose you are given a system of equations,
x + 3y + 2z = 8
2x + 7y + 4z = 17
x + 5y + 2z = 10
we can represent each of these equations as a row vector in the following matrix,

$$\begin{bmatrix} 1 & 3 & 2 & 8 \\ 2 & 7 & 4 & 17 \\ 1 & 5 & 2 & 10 \end{bmatrix}$$

If some of the rows are linearly dependent on the rows above it, that row as an equation doesn't put any extra restraint on the system and we can get rid of that row or equation. For example, if we perform row operations in this matrix like we do in gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 2 & 7 & 4 & 17 \\ 1 & 5 & 2 & 10 \end{array}\right) r_2' = r_2 - 2r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 1 & 5 & 2 & 10 \end{array}\right) r_3' = r_3 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right) r_3' = r_3 - 2r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So, it looks like the 3rd equation didn't add anything new to the system of equations since it was a linear combination of the previous equations. So, we got rid of it. Now we have two equations only. From here, we can say the we could just work with the first 2 equations in the given system and the basis for the row space would be,

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 2 & 7 & 4 & 17 \end{array}\right)$$

Null Space of a Matrix

Definition: The kernel or the null space of a matrix A is the set of vectors in $\mathbb{R}^n$ so that $Ax = 0$. So, the subspace that these vectors span is called the null space of a matrix.

Say, we want to find the null space of the following matrix,

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} r_3' = r_3 - 3r_1, r_2' = r_2 - 2r_1$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} r_3' = r_3 - r_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} r_2' = \frac{r_2}{4}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} r_1' = r_1 - 2r_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ From here, we get two equations,

$x_1 + x_2 + x_4 = 0$

$x_3 + x_4 = 0$

Notice that for any value of x_2 and x_4, the following solution satisfies the above equations.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_4 \\ x_2 + 0.x_4 \\ 0.x_2 - x_4 \\ 0.x_2 + x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$ This is the general solution for $Ax = 0$.

The basis for the null-space for A is:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Left Null Space

If we want to find for which vectors $[x_1, x_2, x_3]$ the following product is 0.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0.$$

The sub-space spanned by all these vectors is called the left null space of A. So, we basically want to solve $x^T A = 0$. By transposing we get, $A^T x = 0$. So, any vector x from the null space of A^T is in the left null space of A and vice versa. So, $LN(A) = N(A^T)$.

Row space and column space both have the same dimension

Theorem: The dimensions of the row space and the column space of a matrix are equal.

Proof: Suppose $c_1, c_2, \cdots, c_n$ are the column vectors of A.

$A = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$

And suppose, only $k \leq n$ of them are linearly independent. Let's call them $v_1, v_2, \cdots, v_k$. So, every c_i can be written as a linear combination of these k basis vectors.

$c_i = \gamma_{i,1} v_1 + \gamma_{i,2} v_2 + \cdots + \gamma_{i,k} v_k$

So, we can write,

$$A = BC = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \begin{bmatrix} \gamma_{1,1} & \gamma_{2,1} & \cdots & \gamma_{n,1} \\ \gamma_{1,2} & \gamma_{2,2} & \cdots & \gamma_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{1,k} & \gamma_{2,k} & \cdots & \gamma_{n,k} \end{bmatrix}$$

If we group $\gamma_{1,i} \gamma_{2,i} \cdots \gamma_{n,i}$ as a row vector γ_i and expand the columns v_i element-wise, then we get,

$$A = \begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{k,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{k,2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{1,m} & v_{2,m} & \cdots & v_{k,m} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_k \end{bmatrix}$$

So, the i'th row in A will be,
$v_{1,i}\gamma_1 + v_{2,i}\gamma_2 + v_{3,i}\gamma_3 + \cdots + v_{k,i}\gamma_k$
Which means that, the rows of A are linear combinations of the rows of C.
So, $dim(R(A)) \leq dim(R(C)) \leq k = dim(C(A))$.

So, for any matrix A, $dim(R(A)) \leq dim(C(A)) \cdots (1)$.
If we substitute A with A^T, we get,
$dim(R(A^T)) \leq dim(C(A^T))$
But, $R(A^T) = C(A)$ and $C(A^T) = R(A)$.
So, $dim(C(A)) \leq dim(R(A)) \cdots (2)$
From, 1 and 2, we can say that,
$dim(R(A)) = dim(C(A))$.

Dimensions of the subspaces

Rank of a matrix: The number of pivots in the echelon form of a matrix is called the rank of that matrix. It is written as $rank(A)$ or $\rho(A)$ or simply r.
So, here are the dimensions of the four sub-spaces:

1. **dim(R(A)) = r:** As in the examples we have seen above, if we convert the matrix to an echelon matrix, the number of independent rows is equal to the number of pivots present in that matrix. So, $dim(R(A)) = r$.

2. **dim(C(A)) = r:** Since, we know that $dim(R(A)) = dim(C(A))$, sim $dim(C(A)) = r$.

3. **dim(N(A)) = n - r:** In the section for null-space we have seen that, for each free variable we get a basis for the null space. Since there are $n - r$ free variables, the dimension for the null space should be $n - r$.

4. **dim(LN(A)) = m - r:** Since, there are $m-r$ free variables in A^T, $dim(LN(A)) = dim(N(A^T)) = m - r$.

Reduced Row Echelon Form

A reduced row echelon form is the state of a matrix after Gauss Jordan Elimination is performed and all rows have been scaled to make the pivots 1. For example, the following are the reduced row echelon forms of the 4 matrices we used as examples in the previous section. It is also called rref.

$$\begin{pmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & 0 & -11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced row echelon form of a matrix has the following properties:

- It doesn't violate the conditions of row echelon forms.
- All leading coefficients(pivots) of each row has to be 1.
- The column that has a pivot will have 1 in the position of the pivot and will have 0 everywhere else.

From these conditions, we can see that the reduced row echelon form of a matrix has to be unique.

Rref and the four sub-spaces

Here are some relationships with $rref(A)$ and the four subspaces of A:

1. **R(A) = R(rref(A)**: Since, row operations doesn't change the row space, so the row-space for A and $rref(A)$ are the same. So, $dim(A) = dim(rref(A)) = rank(A) = rank(rref(A))$
2. **N(A) = N(rref(A))**: Since, $Ax = 0 \iff rref(A)x = 0$. That says that both of these matrices have the same null space.
3. **dim(C(A)) = dim(C(rref(A))**: Since, $R(A) = R(rref(A))$, and $dim(R(A)) = dim(R(rref(A))$. So,
   $dim(C(A)) = dim(R(A))$
   $= dim(R(rref(A)))$
   $= dim(C(rref(A))$
   But, notice that, $C(A) \neq C(rref(A))$, because row operations doesn't preserve the column space.
4. **dim(LN(A)) = dim(LN(rref(A)))**: $dim(LN(A)) = dim(N(A^T)) = dim(N(rref(A^T))) = dim(LN(rref(A)))$
5. **Finding basis vectors of R(A) and C(A)**:
   $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 1 & 5 & 2 \end{pmatrix}$
   $rref(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
   Since the non-zero rows of $rref(A)$ are row 1 and row 2, the basis for the row space of A should be row 1 and row 2 as well. So, $basis\{R(A)\} = \{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \}$
   Also, since column 1 and column 2 are the pivot columns, the basis for the column space of A should be constituted of the first 2 columns of A.
   $basis\{C(A)\} = \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \}$

R(A) and N(A) are orthogonal complements

Vectors in the row space are from $\mathbb{R}^n$. Vectors in the null-space are from $\mathbb{R}^n$ as well. Notice that, Since $Ax = 0$, so $\begin{bmatrix} r_1.x \\ r_2.x \\ \cdots \\ r_m.x \end{bmatrix} = 0$

So, for any row r_i, $r_i.x = 0$. Here, . means dot-product.
Since any vector in the row space can be written as,
$v = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_m r_m$
so, $v.x = \alpha_1 r_1.x + \alpha_2 r_2.x + \cdots + \alpha_m r_m.x$
$v.x = \alpha_1 0 + \alpha_2 0 + \cdots + \alpha_m 0 = 0$
So, any vector from $N(A)$ is perpendicular to any vector from $R(A)$. So, $N(A)$ and $R(A)$ are orthogonal and it is expressed as $N(A) \perp R(A)$. And since $dim(N(A)) + dim(R(A)) = n$, $N(A)$ and $R(A)$ are orthogonal complements in $\mathbb{R}^n$.

C(A) and LN(A) are orthogonal complements

Since, according to the previous section, $N(A^T) \perp R(A^T)$
and since $N(A^T) = LN(A)$ and $R(A^T) = C(A)$,
$LN(A) \perp C(A)$.