MAT 216

Linear Algebra & Fourier Analysis

Week 2 Lecture 3

Lecture Note

Contents:

- ***Introduction to Linear Transformation & Visualization***
- ***Matrix form of Linear Transformation***
- ***Linear Transformation as Matrix Vector Multiplication***
- ***Loss of dimension in Linear transformation***

Reference Book:

Elementary Linear Algebra, Larson | Edwards | Falvo | 6^{th} Ed.

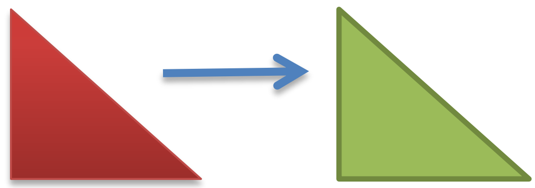
Linear Transformation

Transformation:

Transformation is actually mapping each element of the domain into some element of the codomain. It acts like a function.

Definition: A transformation T is a function that maps a set X to itself.

Example: $T: X \to X$



Linear Transformation:

A linear transformation is a function T that maps a vector space U into another vector space X:

$$T: U \to X$$

U: Domain of T

X: Codomain of T

Two Properties of Linear Transformation:

1. $T(u+v) = T(u) + T(v),\ u, v \epsilon U$
2. $T(cu) = cT(u),\ c\epsilon\mathbb{R}$

The elements of the set of U are u and v. The elements of the set of X are $T(u), T(v)$, and $T(u+v)$

We can combine the two properties into a single one:

$T(cu + dv) = cT(u) + dT(v); c, d\ \epsilon\ \mathbb{R}$

∴ Linear transformations preserve linear combination.

Consider few examples:

1. *Verify a linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$:*

$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$ or $T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + 2v_2 \end{pmatrix}$.

Proof:

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : vector\ in\ \mathbb{R}^2$

$c: any\ real\ number$

Addition law:

$$u + v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

$$\begin{aligned} T(u + v) &= T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \\ &= \begin{pmatrix} (u_1 + v_1) - (u_2 + v_2) \\ (u_1 + v_1) + 2(u_2 + v_2) \end{pmatrix} \quad \text{Since } T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + 2v_2 \end{pmatrix} \\ &= \begin{pmatrix} (u_1 - u_2) + (v_1 - v_2) \\ (u_1 + 2u_2) + (v_1 + 2v_2) \end{pmatrix} \\ &= \begin{pmatrix} u_1 - u_2 \\ u_1 + 2u_2 \end{pmatrix} + \begin{pmatrix} v_1 - v_2 \\ v_1 + 2v_2 \end{pmatrix} \\ &= T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T(u) + T(v) \end{aligned}$$

Scalar Multiplication law:

$$cu = c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$$

$$\begin{aligned} T(cu) &= T\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} \\ &= \begin{pmatrix} cu_1 - cu_2 \\ cu_1 + 2cu_2 \end{pmatrix} \\ &= c\begin{pmatrix} u_1 - u_2 \\ u_1 + 2u_2 \end{pmatrix} \\ &= c\,T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = cT(u) \end{aligned}$$

$\therefore T$ is a linear transformation.

2. *Let* $\boldsymbol{T}$ *be the function defined by* $T(u) = (x, x+y, x-y)$ *where* $u = (x, y)$ *is a vector in* $\mathbb{R}^2$*. Then* T *is a linear transformation.*

Let $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Then $u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$$T(u+v) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ (x_1 + x_2) + (y_1 + y_2) \\ (x_1 + x_2) - (y_1 + y_2) \end{pmatrix} \because T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x-y \end{pmatrix} \textit{given}$$

$$= \begin{pmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_2 + y_2 \\ x_2 - y_2 \end{pmatrix}$$

$$= T(u) + T(v)$$

If c is a scalar then $cu = c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix}$

$$T(cu) = T\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_1 + cy_1 \\ cx_1 - cy_1 \end{pmatrix} = c\begin{pmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{pmatrix} = cT(u)$$

Thus T is a linear transformation

Functions that are not Linear Transformations

- $f(x) = sinx$
  $\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$
  {Check: $\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) \neq \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{3}\right)$}
  $\therefore f(x) = sinx$ is not a linear transformation

- $f(x) = x^2$
  $(x_1 + x_2)^2 \neq {x_1}^2 + {x_2}^2$
  {Check: $(1+2)^2 \neq 1^2 + 2^2$}
  $\therefore f(x) = x^2$ is not a linear transformation

- $f(x) = x + 1$
  $f(x_1 + x_2) = x_1 + x_2 + 1$
  $f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$
  $\therefore f(x_1 + x_2) \neq f(x_1) + f(x_2)$
  Hence $f(x) = x + 1$ is not a linear transformation
  Note:
- $f(x) = x + 1$ is called a linear function because its' graph is a straight line.
- $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither addition nor scalar multiplication.

Zero Transformation

$T: U \to X,\ T(u) = 0, while\ u \epsilon\ U$

Identity Transformation

$T: U \to U,\ T(u) = u, while\ u \epsilon\ U$

Properties of Linear Transformation

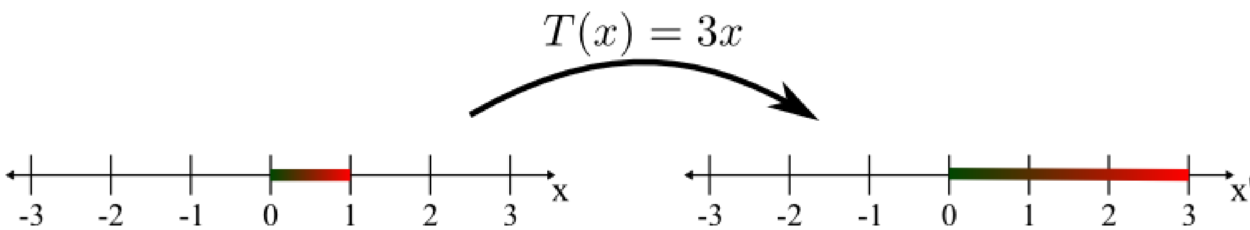
$T: U \to X,\ while\ \ u, v\ \epsilon U$

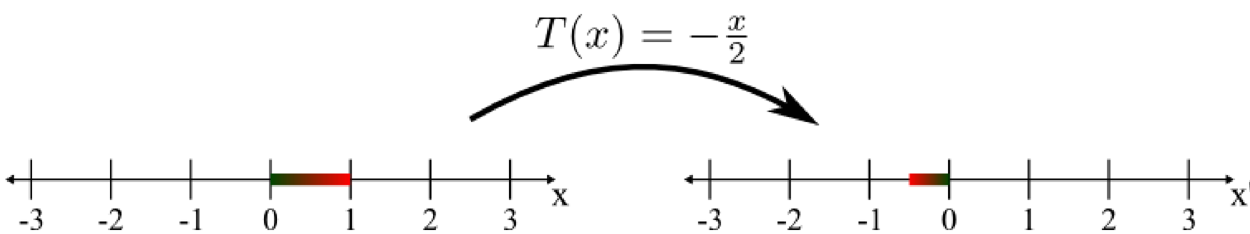
I. $T(0) = 0$
II. $\mathrm{T}(-\mathrm{u}) = -\mathrm{T}(\mathrm{u})$, $T(-2) = -T(2)$
III. $\mathrm{T}(\mathrm{u} - \mathrm{v}) = \mathrm{T}(\mathrm{u}) - \mathrm{T}(\mathrm{v})$
IV. If $u = c_1u_1 + c_2u_2 + \cdots\cdots + c_nu_n$
    Then $T(u) = T(c_1u_1 + c_2u_2 + \cdots\cdots + c_nu_n)$
    $= c_1T(u_1) + c_2T(u_2) + \cdots c_nT(u_n)$.

Matrix Transformation

Transformation refers to some sort of motion. Therefore the word ***"Transformation"*** is more appropriate than the word ***"Function"***. In linear transformation, the multiplication of a vector by a matrix "transforms" the input vector into an output vector, by performing a linear combination:

Let's consider few scenarios:





Matrix Vector Multiplication:

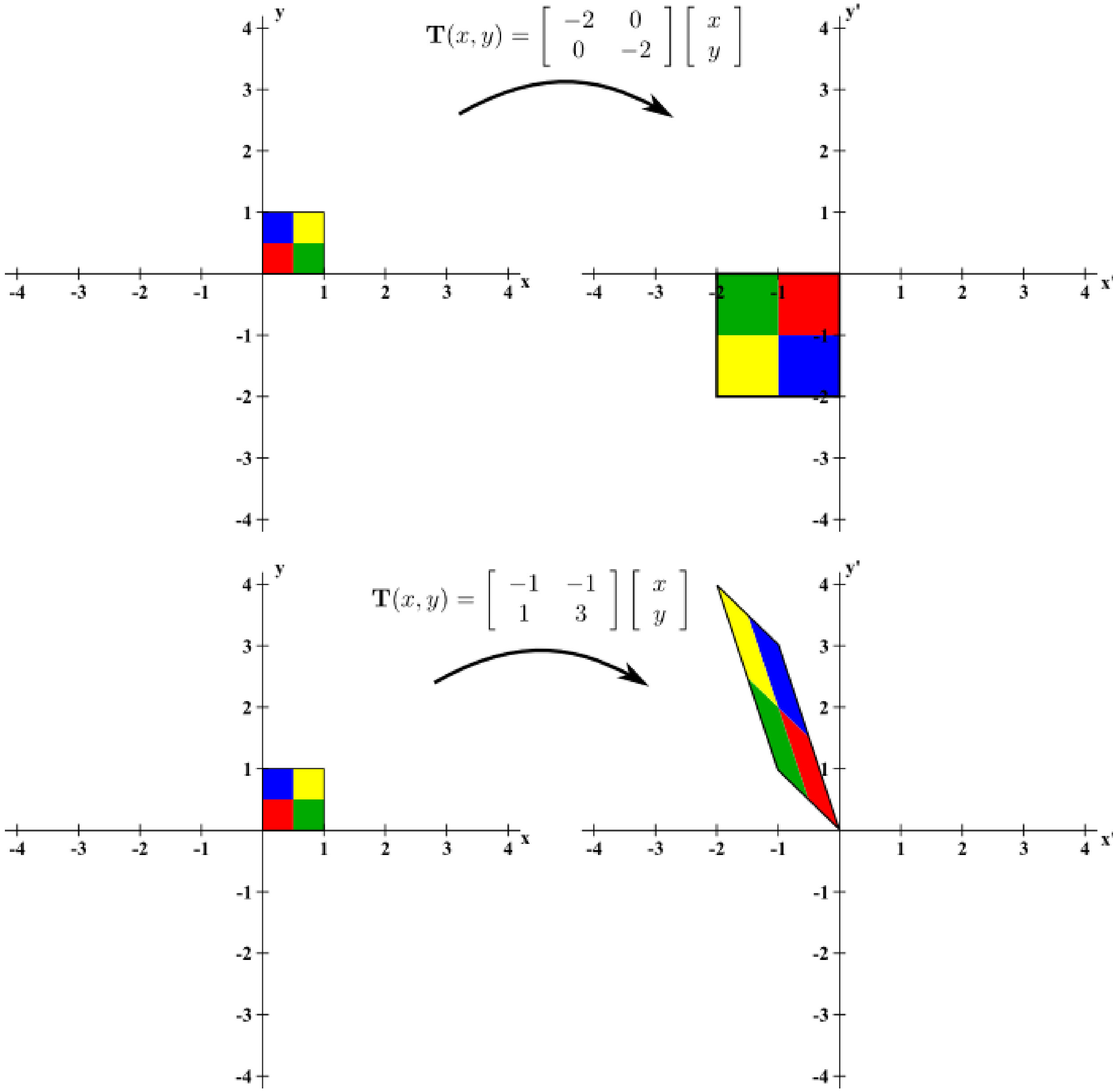
Every matrix performs a transformation. We define the **matrix-vector multiplication** only for the case when the number of columns in A equals the number of rows in x. So, if A is a ($m \times n$) **matrix** (i.e., with n columns), then the **product** Ax is defined for n×1 column **vectors** x. If we let $Ax = b$, then b is an m×1 column **vector**.

$$Ax = b \Rightarrow A.\hat{u} = T(u)$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & m \times n & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix}$$

Here we have performed a transformation that is denoted by: $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Our input vectors are n dimensional and our output vectors are m dimensional. Therefore matrix transforms vectors.



Matrix transformation is linear:

If T is a matrix transformation $T(\vec{v}) = A.\vec{v}$, where A is a matrix. $(\because Ax = b => A.\hat{u} = T(u))$

All linear transformation satisfy $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

Proof:

$$\begin{aligned} L.H.S. &= T(c\vec{u} + d\vec{v}) \\ &= A.(c\vec{u} + d\vec{v}) \quad \because\ T(\vec{v}) = A.\vec{v} \\ &= A.c\vec{u} + A.d\vec{v} \\ &= cA.\vec{u} + dA.\vec{v} \\ &= c\,T(\vec{u}) + dT(\vec{v}) = R.H.S. \end{aligned}$$

∴Every matrix transformation is linear transformation.

Note: Every linear transformation is not a matrix transformation. Since linear transformation can occur in infinite set but matrices have to be finite. Hence matrix transformation cannot be described in terms of infinite linear transformation.

In this course we will consider only the finite sets of linear transformations. In that case every linear transformation can be matrix transformation as well.

Every matrix performs a linear transformation

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & m \times n & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix} => Ax = b$$

The inverse of the matrix restores the transformation

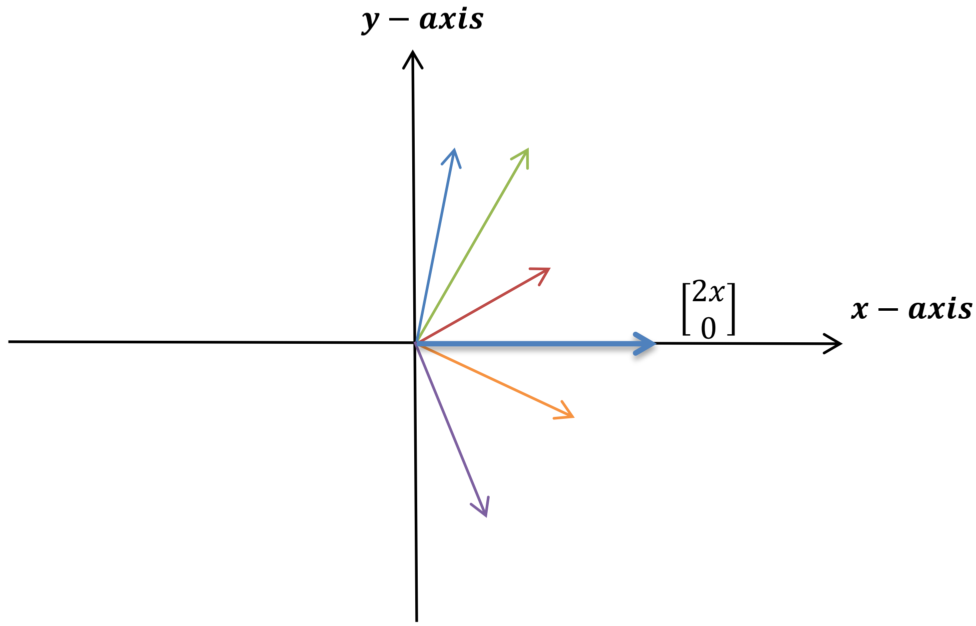
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & m \times n & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}^{-1} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ x_n \end{bmatrix} => A^{-1}b = x$$

All matrices are not invertible

Consider $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix}$

If we multiply $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ with any 2 dimensional vector of an $xy - plane$, the whole plane will collapse into a single line. In the graph we have indicated that line on x-axis (horizontal line). Once a vector collapse in a line, it is not possible to identify from which part of the plane it is coming from. Hence we cannot restore the transformation.

All the vectors of a 2-demensional vector space have been transformed into 1-dimension.



Loss of Dimension in Linear Transformation

When we are not using a square matrix, a transformation can cause a loss of dimension:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We are multiplying a $3 - dimensional$ vector with a $(2 \times 3) - dimensional$ matrix and producing a $2 - dimensional$ vector.

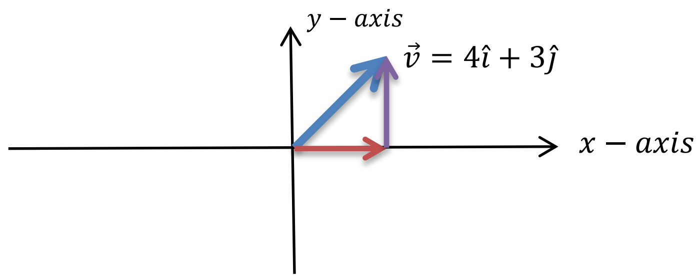
Hence we have $T: \mathbb{R}^3 \to \mathbb{R}^2$

- **<u>So far we have established:</u>**
  - Multiplying with a matrix results in a transformation.
  - This transformation is a linear transformation

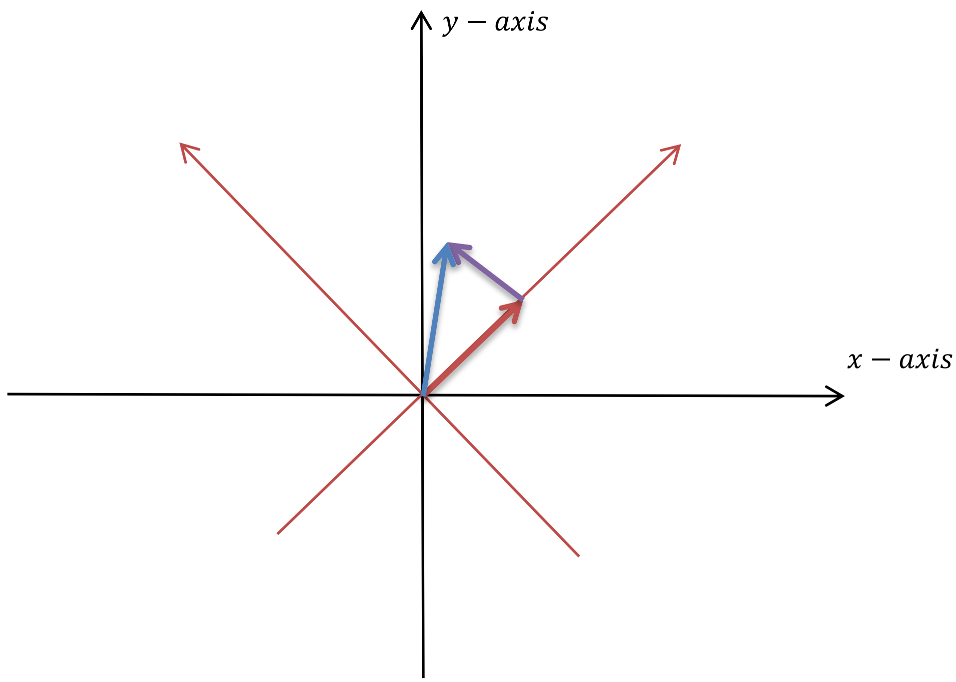
- **<u>A Transformation is Linear if:</u>**
  - The origin [$(0,0)$for $\mathbb{R}^2$ and $(0,0,0)$for $\mathbb{R}^3$ and so on] does not move from its' place.
  - All straight lines remain straight, Or in other words, Gridlines remain evenly spaced.
- Review of the Key Ideas:
  - Multiplying a vector with a matrix causes a transformation.
  - A space is made of infinite vectors, and when we multiply a matrix with each and every one of them, the whole space transform into something new.
  - These transformations are always linear transformation in nature.
  - Linear transformations keep the origin in place, and the gridlines parallel and evenly spaced.

Linear transformation preserves linear combination

If $c\hat{\imath} + d\hat{\jmath} = \vec{v}$, then $cT(\hat{\imath}) + dT(\hat{\jmath}) = T(\vec{v})$



After transformation: $\vec{v} = 4(not\ \hat{\imath}) + 3(not\hat{\jmath})$



Therefore if we rotate our axis, even if the graph gets skewed, the linear combination will still exist and also it will satisfy the schemes of linear transformation.

Note:

$(not\ \hat{\imath})$is the transformation of $\hat{\imath}\ vector,$ similartly $(not\ \hat{\jmath})$is the transformation of $\hat{\jmath}\ vector$.

Understanding a Transformation following the basis vectors:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Also

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So if we multiply a 2×2 matrix with $\hat{\imath}$ it will always produce the 1^{st} column of the original matrix as a result of the product.

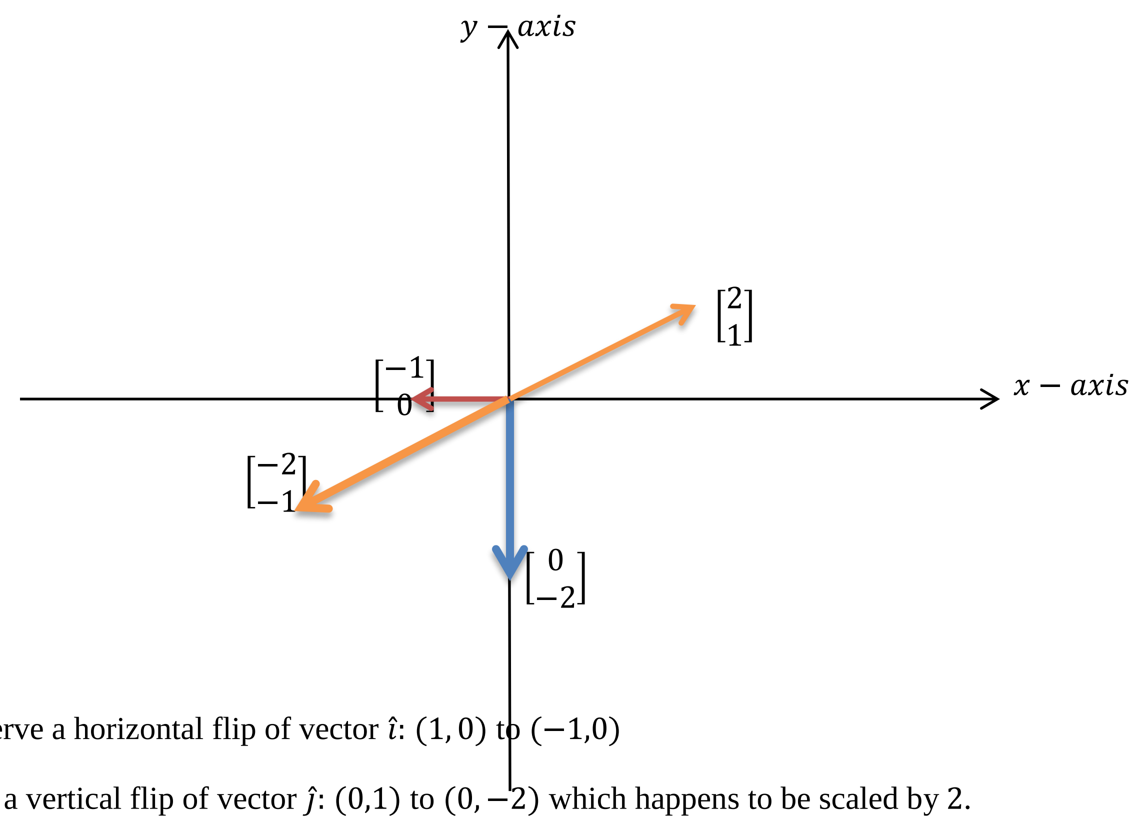
Similarly if we multiply a 2×2 matrix with $\hat{\jmath}$ it will always produce the 2^{nd} column of the original matrix as a result of the

This theory is also applicable for 3×3 matrix and so on⋯

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Examples:

a) $\begin{bmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{-1} \end{bmatrix}$

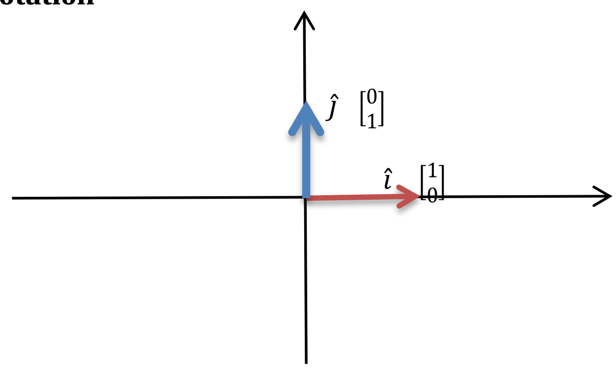


We observe a horizontal flip of vector $\hat{\imath}$: $(1, 0)$ to $(-1,0)$

There is a vertical flip of vector $\hat{\jmath}$: $(0,1)$ to $(0, -2)$ which happens to be scaled by 2.

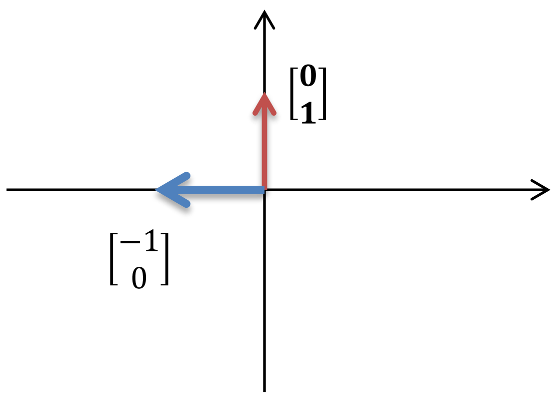
b) Which matrix will be useful to rotate the 2-D space 90^0 counterclockwise?

Before rotation



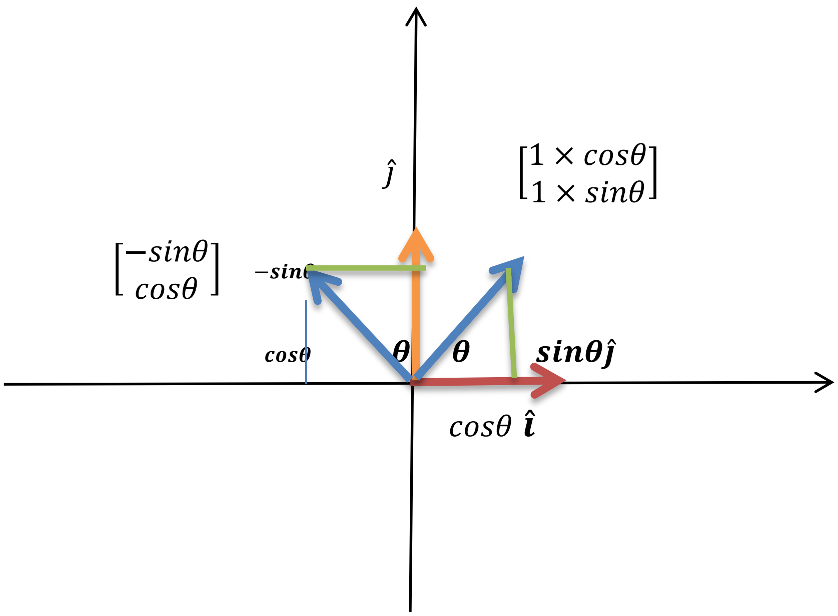
Now, both basis vectors $\hat{\imath} = (1,0)$ and $\hat{j} = (0,1)$ have to rotate $\mathbf{90^0}$ counterclockwise.

After 90^0 rotation



$\therefore$ The matrix that will be useful to rotate the $2-D$ space counterclockwise is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

In computer graphics, rotation, scaling, skewing, perspective and translation are achieved by linear transformation.



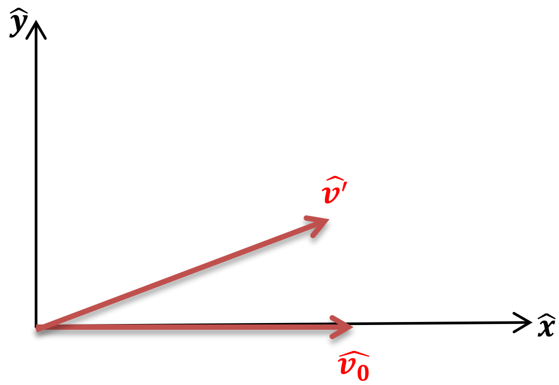
$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Rorating unit vector $\hat{\imath}$ and unit vector $\hat{\jmath}$ by θ^0.

$\therefore$ The matrix that will rotate vectors by θ^0 is $=\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$.

There are TWO possible ways of rotation:

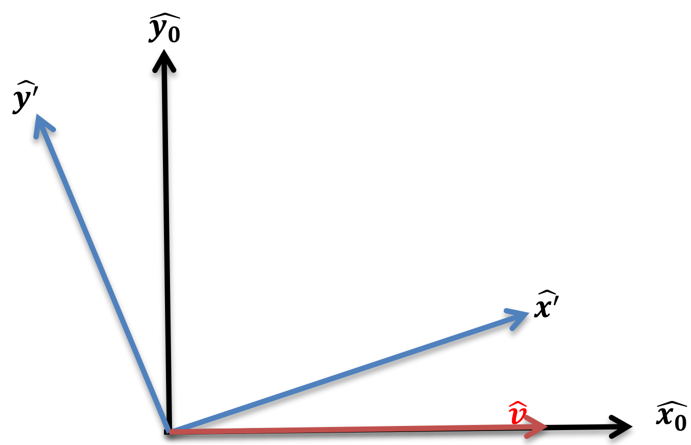
(i) Rotation of the axes:



In R^2, consider the matrix that rotates a given vector v_o by a counterclockwise angle θ in a fixed coordinate system. Then

$\boldsymbol{R_\theta} = \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{-sin\theta} \\ \boldsymbol{sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$. So $\boldsymbol{v'} = \boldsymbol{R_\theta v_0}$

(ii) Rotation of the *object* relative to fixed axes:



On the other hand, consider the matrix that rotates the *coordinate system* through a **counterclockwise** angle $\boldsymbol{\theta}$. The coordinates of the fixed vector $\boldsymbol{v}$ in the rotated coordinate system are now given by a rotation matrix which is the transpose of the fixed-axis matrix and, as can be seen in the above diagram, is equivalent to rotating the *vector* by a counterclockwise angle of $\boldsymbol{-\theta}$ relative to a fixed set of axes, giving $\boldsymbol{R'_\theta} = \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{sin\theta} \\ \boldsymbol{-sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$.

Rotational Matrix for 3-D:

Rotation around the $\boldsymbol{x-axis}$:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

Rotation around the $\boldsymbol{y-axis}$**:**

$$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

Rotation around the $\boldsymbol{z-axis}$:

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$