MAT216 ASSIGNMENT 2

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SECTION: 07

MAT216 ASSIGNMENT 2

(1) $S = \left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 1 & a \\ 4 & -1 & 1 & b \end{array}\right] \xrightarrow{R_2 \to R_2 + 4R_1} \left[\begin{array}{ccc|c} -1 & 4 & 1 & a \\ 0 & 15 & 5 & 4a+b \end{array}\right]$$

c_3 is a free variable.

$c_3 = t$

$15c_2 + 5c_3 = 4a + b$

$$c_2 = \frac{4a+b-5t}{15}$$

$-c_1 + 4c_2 + c_3 = a$

$$-c_1 + 4\left(\frac{4a+b-15t}{15}\right) + t = a$$

$$-c_1 = \frac{-a}{15} - \frac{4b}{15} + 3t$$

$$c_1 = \frac{a}{15} + \frac{4b}{15} + 3t$$

if $t = 0$,

$c_3 = 0$

$$c_2 = \frac{4a+b}{15}$$

$$c_1 = \frac{a+4b}{15}$$

$$\cancel{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (a+4b)/15 \\ (4a+b)/15 \\ 0 \end{bmatrix}} \quad \frac{a+4b}{15} \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \frac{4a+b}{15} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^2$

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & a \\ 2 & -1 & 1 & b \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \left[\begin{array}{ccc|c} -1 & 2 & 1 & a \\ 0 & 3 & 3 & \cancel{b}\ 2a+b \end{array}\right]$$

c_3 is a free variable.

$c_3 = t$

$3c_2 + 3c_3 = 2a+b$

$$c_2 = \frac{2a+b-3t}{3}$$

$-c_1 + 2c_2 + c_3 = a$

$$-c_1 + 2\left(\frac{2a+b-3t}{3}\right) + t = a$$

$$-c_1 = -\frac{a}{3} - \frac{2b}{3} + t$$

$$c_1 = \frac{a}{3} + \frac{2b}{3} - t$$

if $t = 0$,

$c_3 = 0$

$$c_2 = \frac{2a+b}{3}$$

$$c_1 = \frac{a+2b}{3}$$

$$\frac{a+2b}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{2a+b}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^2$

(2) $S = \left\{ \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & a \\ 7 & 2 & -3 & b \\ 3 & 6 & 5 & c \end{array}\right] \xrightarrow{R_2 \leftarrow 4R_2 - 7R_1} \left[\begin{array}{ccc|c} 4 & -1 & 2 & a \\ 0 & 15 & -26 & 4b - 7a \\ 3 & 6 & 5 & c \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 4R_3 - 3R_1} \left[\begin{array}{ccc|c} 4 & -1 & 2 & a \\ 0 & 15 & -26 & 4b - 7a \\ 0 & 27 & 14 & 4c - 3a \end{array}\right] \xrightarrow{R_3 \leftarrow R_3/9 - R_2/15}$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & a \\ 0 & 15 & -26 & 4b - 7a \\ 0 & 0 & \frac{304}{45} & \frac{4}{9}c + \frac{16}{15}a - \frac{4b}{15} \end{array}\right]$$

$$\frac{304}{45} c_3 = \frac{16}{15} a - \frac{4}{15} b + \frac{4}{9} c \Rightarrow c_3 = \frac{3}{19} a - \frac{9}{76} b + \frac{5}{76} c$$

$$15c_2 - 26\left(\frac{3}{19} a - \frac{9}{76} b + \frac{5}{76} c\right) = 4b - 7a \Rightarrow c_2 = -\frac{11}{57} a + \frac{7}{114} b + \frac{13}{114} c$$

$$4c_1 + \frac{11}{57} a - \frac{7}{114} b - \frac{13}{114} c + 2\left(\frac{3}{19} a - \frac{9}{76} b + \frac{5}{76} c\right) = a$$

$$c_1 = \frac{7}{57} a + \frac{17}{228} b - \frac{1}{228} c$$

$$\frac{7}{57} a + \frac{17}{228} b - \frac{1}{228} c \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} - \frac{11}{57} a + \frac{7}{114} b + \frac{13}{114} c \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$$

$$+ \frac{3}{19} a - \frac{9}{76} b + \frac{5}{76} \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 6 & 3 & 1 & a \\ 7 & 2 & 3 & b \\ 6 & -4 & 2 & c \end{array}\right] \xrightarrow{R_2 \leftarrow 6R_2 - 7R_1} \left[\begin{array}{ccc|c} 6 & 3 & 1 & a \\ 0 & -9 & 11 & 6b - 7a \\ 6 & -4 & 2 & c \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 6 & 3 & 1 & a \\ 0 & -9 & 11 & 6b - 7a \\ 0 & -7 & 1 & c - a \end{array}\right] \xrightarrow{R_3 \leftarrow 9R_3 - 7R_2}$$

$$\left[\begin{array}{ccc|c} 6 & 3 & 1 & a \\ 0 & -9 & 11 & 6b - 7a \\ 0 & 0 & -68 & 9c + 40a - 42b \end{array}\right]$$

$-68c_3 = 9c + 40a - 42b \Rightarrow c_3 = \cancel{-\frac{9}{68}c - \frac{10}{17}} -\frac{10}{17}a + \frac{21}{34}b + \frac{9}{68}c$

$-9c_2 + 11\left(-\frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c\right) = 6b - 7a \Rightarrow c_2 = \frac{1}{17}a + \frac{3}{34}b - \frac{11}{68}c$

$6c_1 + 3\left(\frac{1}{17}a + \frac{3}{34}b - \frac{11}{68}c\right) - \frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c = a$

$c_1 = \frac{5}{17}a - \frac{5}{34}b + \frac{943}{8840}c$

$-68c_3 = 9c + 40a - 42b \Rightarrow c_3 = \cancel{-\frac{9}{68}} -\frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c$

$-9c_2 + 11\left(-\frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c\right) = 6b - 7a \Rightarrow c_2 = \frac{1}{17}a + \frac{3}{34}b - \frac{11}{68}c$

$6c_1 + 3\left(\frac{1}{17}a + \frac{3}{34}b - \frac{11}{68}c\right) + \left(-\frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c\right) = a$

$c_1 = \frac{4}{17}a - \frac{5}{34}b + \frac{7}{68}c$

$$\left(\frac{4}{17}a - \frac{5}{34}b + \frac{7}{68}c\right) \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix} + \left(\frac{1}{17}a + \frac{3}{34}b - \frac{11}{68}c\right) \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + \left(-\frac{10}{17}a + \frac{21}{34}b - \frac{9}{68}c\right) \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^3$

(3)(a) $S = \left\{ \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 1 & t & 1 & 0 \\ 1 & 1 & t & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow tR_2 - R_1} \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & t^2-1 & t-1 & 0 \\ 1 & 1 & t & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow tR_3 - R_1} \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & t^2-1 & t-1 & 0 \\ 0 & t-1 & t^2-1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & (t-1)(t+1) & (t-1) & 0 \\ 0 & t-1 & (t-1)(t+1) & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow (t+1)R_3 - R_1} \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & (t-1)(t+1) & (t-1) & 0 \\ 0 & 0 & (t-1)[(t+1)^2-1] & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3/(t-1)[(t+1)^2-1]]{R_1 \leftarrow R_1/t \quad R_2 \leftarrow R_2/(t-1)(t+1)}$$

$$\left[\begin{array}{ccc|c} 1 & 1/t & 1/t & 0 \\ 0 & 1 & 1/(t+1) & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - \frac{1}{t}R_2]{R_2 \leftarrow R_2 - \frac{1}{(t+1)}R_3 \quad R_1 \leftarrow R_1 - \frac{1}{t}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ∴ S is linearly independent if $t \neq 0, 1, -1, -2$.

$t = 0$

$(t-1)(t+1) = 0$

$t = 1, t = -1$

$(t-1)[(t+1)^2 - 1] = 0$

$t = 1$ | $(t+1)^2 = 1$

$t^2 + 2t + 1 = 1$

$t(t+2) = 0$

$t = 0, t = -2$

∴ $t = 0, 1, -1, -2$ ~~[in these cases, S is linearly depen.~~

(b) $S = \left\{ \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3t \end{bmatrix} \right\}$

$$\left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3t & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow tR_2 - R_1} \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & -1 & t-1 & 0 \\ 1 & 1 & 3t & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow tR_3 - R_1} \left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & -1 & t-1 & 0 \\ 0 & t-1 & 3t^2-1 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + (t-1)R_2}$$

$$\left[\begin{array}{ccc|c} t & 1 & 1 & 0 \\ 0 & -1 & t-1 & 0 \\ 0 & 0 & (3t^2-1)-(t-1)^2 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3/((3t^2-1)-(t-1)^2)]{R_1 \leftarrow R_1/t,\ R_2 \leftarrow R_2/-1}$$

$$\left[\begin{array}{ccc|c} 1 & 1/t & 1/t & 0 \\ 0 & 1 & -(t-1) & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \leftarrow R_1 - 1/t\,R_2]{R_2 \leftarrow R_2 + (t-1)R_3,\ R_1 \leftarrow R_1 - 1/t\,R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ S is linearly independent if $t \neq 0$, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$

$t = 0$

$(3t^2-1) - (t-1)^2 = 0$

$(3t^2-1) = (t-1)^2$

$3t^2 - 1 = t^2 - 2t + 1$

$2t^2 + 2t - 2 = 0$

$2(t^2 + t - 1) = 0$

$t^2 + t - 1 = 0$

$t = \frac{-1+\sqrt{5}}{2}, t = \frac{-1-\sqrt{5}}{2}$

$\therefore t = 0, t = \frac{-1+\sqrt{5}}{2}, t = \frac{-1-\sqrt{5}}{2}$

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(4)(a) $S = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\left[\begin{array}{ccc|c} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/t} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 t = 0$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ S is linearly independent if $t \neq 0$

$\therefore$ $t = 0$

(b) $S = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\left[\begin{array}{ccc|c} t & t & t & 0 \\ t & 1 & 0 & 0 \\ t & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} t & t & t & 0 \\ 0 & 1-t & -t & 0 \\ t & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} t & t & t & 0 \\ 0 & 1-t & -t & 0 \\ 0 & -t & 1-t & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow \cancel{t(1-t)R_3 +} (1-t)R_3 + tR_2}$$

$$\left[\begin{array}{ccc|c} t & t & t & 0 \\ 0 & 1-t & -t & 0 \\ 0 & 0 & (1-t)^2 - t^2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} t & t & t & 0 \\ 0 & 1-t & -t & 0 \\ 0 & 0 & 1-2t & 0 \end{array}\right] \xrightarrow[\substack{R_2 \leftarrow R_2/1-t \\ R_3 \leftarrow R_3/1-2t}]{R_1 \leftarrow R_1/t}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{t}{t-1} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{R_1 \leftarrow R_1 - R_3 \\ R_1 \leftarrow R_1 - (1-t)R_2}]{R_2 \leftarrow R_2 - \frac{t}{t-1}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$t = 0$

$1 - t = 0$

$t = 1$

$1 - 2t = 0 \Rightarrow t = \frac{1}{2}$

$(1-2t)c_3 = 0 \Rightarrow c_3 = 0$

$(1-t)c_2 - tc_3 \Rightarrow c_2 = 0$

$(t)c_1 + (t)c_2 + (t)c_3 \Rightarrow c_1 = 0$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ S is linearly independent if $t \neq 0, t \neq 1, t \neq \frac{1}{2}$

(5)

$$S = \left\{ \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}-2\\0\\1\end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix}3\\1\\-2\end{bmatrix}}_{v_4} \right\} \Rightarrow k_1\begin{bmatrix}1\\2\\3\end{bmatrix} + k_2\begin{bmatrix}0\\1\\2\end{bmatrix} + k_3\begin{bmatrix}-2\\0\\1\end{bmatrix} + k_4\begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & a \\ 2 & 1 & 0 & 1 & b \\ 3 & 2 & 1 & 2 & c \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & a \\ 0 & 1 & 4 & -5 & b-2a \\ 3 & 2 & 1 & 2 & c \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & a \\ 0 & 1 & 4 & -5 & b-2a \\ 0 & 2 & 7 & -7 & c-3a \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & a \\ 0 & 1 & 4 & -5 & b-2a \\ 0 & 0 & -1 & 3 & c+a-2b \end{array}\right]$$

k_4 is a free variable

$k_4 = t$

$-k_3 + 3t = c + a - 2b \Rightarrow k_3 = 3t - c - a + 2b$

$k_2 + 4k_3 - 5k_4 = b - 2a$

$k_2 + 4(3t - c + a + 2b) - 5t = b - 2a \Rightarrow k_2 = -2t + 4c - 6a - 7b$

$k_1 - 2k_3 + 3k_4 = a \Rightarrow k_1 - 2(3t - c + a + 2b) + 3t = a$

$\Rightarrow k_1 = 3t + 3a + 4b - 2c$

if $t = 0$,

$k_4 = 0$

$k_3 = \cancel{3t} - c + a + 2b$

$k_2 = 4c - 6a - 7b$

$k_1 = 3a + 4b - 2c$

$$3a+4b-2c \begin{bmatrix}1\\2\\3\end{bmatrix} - 6a - 7b + 4c \begin{bmatrix}0\\1\\2\end{bmatrix} + a + 2b - c \begin{bmatrix}-2\\0\\1\end{bmatrix} + 0 \begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^3$

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$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right]$$

$k_4 = t$

$\cancel{k} - k_3 + 3k_4 = 0$

$k_3 = 3t$

$k_2 + 4(3t) + t = 0$

$k_2 = -13t$

$k_1 - 2(3t) + 3(t) = 0$

$k_1 = 3t$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = t \begin{bmatrix} \cancel{1} \\ 3 \\ -13 \\ 3 \end{bmatrix}$$

$\therefore$ S is linearly dependent

$\therefore$ S is not a basis for $\mathbb{R}^3$

(b) $S = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow k_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & a \\ 2 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & b \\ 0 & -2 & 1 & a \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow{R_3 \leftarrow 2R_3 - R_1}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & b \\ 0 & -2 & 1 & a \\ 0 & -1 & 1 & 2c-b \end{array}\right] \xrightarrow{R_3 \leftarrow 2R_3 - R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & b \\ 0 & -2 & 1 & a \\ 0 & 0 & 1 & 4c-2b-a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & b/2 \\ 0 & 1 & -1 & -a \\ 0 & 0 & 1 & 4c-2b-a \end{array}\right] \xrightarrow[\substack{R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 - 1/2 R_2}]{R_1 \leftarrow R_1 + 1/2 R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2b - 1/2a \\ 0 & 1 & 0 & 4c-2b \\ 0 & 0 & 1 & 4c-2b-a \end{array}\right]$$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1/2b - 1/2a \\ 4c - 2b \\ 4c - 2b - a \end{bmatrix}$$

$\therefore$ S spans $\mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ S is linearly independent

$\therefore$ S is a basis for $\mathbb{R}^3$

(7) ~~(a)~~ $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$

$$K_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + K_3 \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 3 & 1 & 3 & b \\ 0 & 2 & -2 & c \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -11 & 11 & b-3a \\ 0 & 2 & -2 & c \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 11R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -11 & 11 & b-3a \\ 0 & 0 & 0 & 11c+2b-6a \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -11 & 11 & b-3a \end{array}\right]$$

$\therefore$ ~~No solution~~ If $11c + 2b - 6a = 0$, consistent

$\therefore S$ ~~does not~~ spans $\mathbb{R}^3$ but is linearly dependent due to K_3 as free variable

$\therefore S$ is not a basis for $\mathbb{R}^3$

~~(b)~~ $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} \right\}$

$$K_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + K_2 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + K_3 \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 4 & a \\ 1 & -1 & 2 & b \\ -2 & 2 & -4 & c \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & -2 & 4 & a \\ 0 & 3 & -6 & b-2a \\ -2 & 2 & -4 & c \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_1} \left[\begin{array}{ccc|c} 2 & -2 & 4 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 0 & 0 & c+a \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 4 & a \\ 0 & 3 & -6 & b-2a \end{array}\right]$$

$\therefore$ ~~No solution~~ If $c + a = 0$, consistent

$\therefore S$ ~~does not~~ spans $\mathbb{R}^3$ but is linearly dependent due to K_3 as a free variable

$\therefore S$ is not a basis for $\mathbb{R}^3$

$S = \left\{ \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} \right\}$

$$k_1 \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 7 & 8 & a \\ 0 & -4 & b \\ 3 & 1 & c \end{array}\right] \xrightarrow{R_3 \leftarrow 7R_3 - 3R_1} \left[\begin{array}{cc|c} 7 & 8 & a \\ 0 & -4 & b \\ 0 & -17 & 7c-3a \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 4R_3 - 17R_2} \left[\begin{array}{cc|c} 7 & 8 & a \\ 0 & -4 & b \\ 0 & 0 & 28c-12a-17b \end{array}\right] \cancel{\rightarrow \left[\begin{array}{cc|c} 7 & 8 & a \\ 0 & -4 & b \end{array}\right]}$$

∴ No solution

~~∴ No solution If $28c - 12a - 17b = 0$, consistent~~

∴ S ~~does not~~ does not spans $\mathbb{R}^3$ ~~but is linearly dependent~~ due to k_2 ~~as free variable~~

∴ S is not a basis for $\mathbb{R}^3$

$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

$$k_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & 2 & b \\ 2 & 1 & c \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b-a \\ 2 & 1 & c \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b-a \\ 0 & 1 & c-2a \end{array}\right] \xrightarrow{R_3 \leftarrow 2R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b-a \\ 0 & 0 & 2c-3a-b \end{array}\right]$$

∴ No solution

∴ S does not span $\mathbb{R}^3$

∴ S is not a basis for $\mathbb{R}^3$