MAT216: Linear Algebra & Fourier Analysis

Lecture Note: Week 8_Lecture 15_Part 1 and Part 2

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References:

- Introduction to Linear Algebra, 5th Ed. Gilbert Strang
- https://www.youtube.com/watch?v=0MtwqhIwdrI&t=2411s
- https://bux.bracu.ac.bd/courses/course-v1:buX+2020_SummerMAT216+/course/

Part 1: Orthogonal and Orthonormal Basis

Let's start with the definition of the basis of a vector space one more time: A basis for a vector space is a sequence of vectors with two properties:

i. The basis vectors are linearly independent and
ii. They span the space.

Usually, we have to check both properties. When the count is right, one property implies the other.

i. Any n independent vectors in R_n must span R_n. So, they are a basis.
ii. Any n vectors that span R_n must be independent. So, they are a basis.

Therefore, a set of linearly independent vectors that can span the entire space is a basis of a space.

Definition: A set of linearly independent vectors that can span the entire space, and each of the vector is perpendicular to all other vectors is an orthogonal basis of the space.

Definition: An orthogonal basis for a subspace V of R^n is a basis for V that is also an orthogonal set.

i.e. $V = \{v_1, v_2\}$ is an orthogonal basis if the vectors that form it are perpendicular. In other words, v_1 and v_2 form an angle of 90° [or, $\langle v_1, v_2 \rangle = 0$].

Definition: A set of vectors $V = \{v_1, v_2, \ldots\ldots, v_n\}$ are mutually orthogonal if every pair of vectors is orthogonal. i.e. $\vec{v_i}.\vec{v_j} = 0$, for all $i \neq j$.

Example 1: The standard basis vectors are orthogonal.

$$e_i.e_j = e_i^T e_j = 0 \text{ when } i \neq j.$$

Example 2: Show that $\{v_1,\ v_2,\ v_3\}$ is an orthogonal set, where

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} \frac{-1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

Solution:

Here, we have $v_1.v_2 = v_2.v_3 = v_3.v_1 = 0$.

Thus, each pair of distinct vectors is orthogonal, and so $\{v_1,\ v_2,\ v_3\}$ is an orthogonal set.

- Two subspaces V and W of a vector space are orthogonal if every vector v in V is perpendicular to every vector w in W. i.e. $v^T w = 0$ for all v in V and all w in W. [**For example, please find Introduction to Linear Algebra, 5th Edition by Gilbert Strang, page 195**]

- Every vector x in the null space is perpendicular to every row of A, because $Ax = 0$. The null space $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of R^n.

Definition: When the basis is orthogonal and also the length of the vectors of the basis is 1, then the basis is called orthonormal.

- A set of vectors $V = \{v_1, v_2, \ldots\ldots v_n\}$ are orthonormal if:

$$v_i^T v_j = \begin{cases} 0 \; if \;\; i \neq j \\ 1 \;\; if \;\; i = j \end{cases};$$

  In other words, they all have length 1 ($i.e. \|\vec{v_i}\| = 1$) and are perpendicular to each other. Orthonormal vectors are always independent.

- A square orthonormal matrix A is called an orthogonal matrix. If A is square then $A^T A = I$ tells us that $A^T = A^{-1}$

Example: The standard basis vectors are orthogonal and each of them has unit length, therefore they form an orthonormal basis.

Example: Determine which of the following sets of vector form orthonormal basis:

a. $u = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ -2 \end{bmatrix};$

b. $u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \; w = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix};$

c. $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ -1 \end{bmatrix};$

d. $u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix};$

Why are orthonormal bases more convenient?

- When we need a basis to do calculation, it is convenient to use and orthonormal basis. For instance, the formula for a vector space projection is much simpler with an orthonormal basis.
- They use for making good coordinate system or good coordinate bases.

Part 2: The Gram Schmidt Process

Let's consider a set of vectors $V = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots\ldots, \overrightarrow{v_n}\}$which may not be orthonormal. Our goal in this particular section is to produce a set of orthonormal vectors $U = \{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots\ldots, \overrightarrow{u_n}\}$ by Gram Schmidt process with the same span as the original set of vectors V.

Gram Schmidt process always start with:

Step 1: $\overrightarrow{u_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_\iota}\|}$

Remember that dividing a vector by its length always produced a unit vector, so $\overrightarrow{u_1}$ has length 1 and points in the same direction as $\overrightarrow{v_1}$ [Find video lecture 15, part 2].

Step 2: Compute $\overrightarrow{w_2} = \overrightarrow{v_2} - (\overrightarrow{v_2} \,.\, \overrightarrow{u_1})\overrightarrow{u_1}$ and find $\overrightarrow{u_2} = \frac{\overrightarrow{w_2}}{\|\overrightarrow{w_2}\|}$

After doing the first step we could produce a vector $\overrightarrow{w_2}$ which is orthogonal to $\overrightarrow{u_1}$ and by second step we get the length 1.

Step 3: Let us first find $\overrightarrow{w_3} = \overrightarrow{v_3} - (\overrightarrow{v_3} \,.\, \overrightarrow{u_2})\overrightarrow{u_2} - (\overrightarrow{v_3} \,.\, \overrightarrow{u_1})\overrightarrow{u_1}$ and then set $\overrightarrow{u_3} = \frac{\overrightarrow{w_3}}{\|\overrightarrow{w_3}\|}$

Thus $\overrightarrow{w_3}$ is orthogonal to $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ and by dividing by its length, we get something of length one.

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$\vec{w}_n = \vec{v}_n - (\vec{v}_n.\vec{u}_{n-1})\vec{u}_{n-1}$- …………… - $(\vec{v}_n.\vec{u}_1)\vec{u}_1$ and set $\overrightarrow{u_n} = \frac{\overrightarrow{w_n}}{\|\overrightarrow{w_n}\|}$

At the final step we take $\overrightarrow{v_\iota}$ and subtract off its projections onto all the previous $\overrightarrow{u_J}$'s constructed thus far, and divide the result by its length.

- Span $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots\ldots\ldots, \overrightarrow{u_n}\}$ = span $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots\ldots\ldots, \overrightarrow{v_n}\}$ and $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots\ldots\ldots, \overrightarrow{u_n}\}$ is an orthonormal set, thus $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots\ldots\ldots, \overrightarrow{u_n}\}$ is an orthonormal basis V.

Example: Find an orthonormal basis spanned by a set of vectors:

$$v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$$

Solution:

Step 1: $\overrightarrow{u_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_\iota}\|} = \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}$

Step 2: Compute $\overrightarrow{w_2} = \overrightarrow{v_2} - (\overrightarrow{v_2} \,.\, \overrightarrow{u_1})\overrightarrow{u_1}$ and find $\overrightarrow{u_2} = \frac{\overrightarrow{w_2}}{\|\overrightarrow{w_2}\|}$

$$\overrightarrow{w_2} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{\sqrt{30}}\overrightarrow{u_1} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{-5}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$\overrightarrow{u_2} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Therefore, according to the definition $U = \{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ form an orthonormal basis

Exercise:

1. Find an orthonormal basis of $\mathcal{H}: x - 2y - 3z = 0$.
2. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$. Find an orthonormal basis for span $(v_1, v_{2\,,} v_3)$.
3. Construct an orthonormal basis of $\mathbb{R}^3$ by applying Gram-Schmidt orthogonalization to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.