Week 10 (Lecture 20)
Diagonalization

The preceding section discussed the **eigenvalue problem**. In this section, we will look at another classic problem in linear algebra called the **diagonalization problem**. Expressed in terms of matrices, the problem is this: "*For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?*"

Definition of Similar Matrices

For two square matrices A and B of size $n \times n$, B is said to be **similar** to A if there exists an invertible $n \times n$ matrix P such that

$$B = P^{-1}AP$$

■

For example, if

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

B is similar to A since there exists

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Such that

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Properties of Similar Matrices

- Every square matrix A is similar to itself.

  Proof. There exists $P = I$ such that

$$A = I^{-1}AI.$$

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- If B is similar to A, then A is similar to B.

  Proof. Since B is similar to A, there exists an invertible matrix P such that

  $$B = P^{-1}AP$$

  Now, we also have

  $$A = PBP^{-1}$$

  This shows A is also similar to B.

- If C is similar to B and B is similar to A, then C is similar to A.

  Proof. We have invertible matrices P and Q such that

  $$B = P^{-1}AP, \qquad C = Q^{-1}BQ$$

  Therefore,

  $$C = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ)$$

  This shows that C is similar to A.

Matrices that are similar to diagonal matrices are called **diagonalizable**.

Definition

An $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix. That is, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

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Example The matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable since there exists an invertible matrix

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

■

[Note. The eigenvalues of a diagonal matrix/upper-triangular matrix/lower-triangular matrix are the diagonal elements.

For example, the eigenvalues of

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

are: $\lambda_1 = 4, \lambda_2 = -2$ and $\lambda_3 = -2$.**]**

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Diagonalization

Theorem 1 If A and B are similar $n \times n$ matrices, then they have the same characteristic equations and eigenvalues.

Example The matrices A and D are similar.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Since D is a diagonal matrix, its eigenvalues are simply the diagonal elements—that is $\lambda_1 = 4, \lambda_2 = -2$ and $\lambda_3 = -2$.

Moreover, because A is similar to D, A has the same characteristic equation and eigen values of D. Therefore, the eigenvalues of A are: $\lambda_1 = 4, \lambda_2 = -2$ and $\lambda_3 = -2$.

Condition for Diagonalization

Theorem 2 An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Example The matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable since it has three linearly independent eigenvectors corresponding to its eigenvalues, namely

$$\lambda_1 = 4: \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -2: \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \lambda_3 = -2: \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Diagonalization

Example

The matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable since it has only one linearly independent eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, corresponding to its eigenvalue $\lambda_1 = \lambda_2 = 1$.

Sufficient Condition for Diagonalization

Theorem 3 If an $n \times n$ matrix A has *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent, and A is diagonalizable.

Example

The matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$$

is diagonalizable. Since A is a lower triangular matrix, the eigenvalues are the elements on the main diagonal, namely 2, 3, and 4. Also, every eigenvalue has multiplicity 1. Therefore, A is diagonalizable.

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Diagonalization

Steps for Diagonalizing an $n \times n$ Square Matrix

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots, \mathbf{p}_\mathrm{n}$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

2. If A has n linearly independent eigenvectors, let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is,

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \dots \mathbf{p}_n]$$

3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ on its main diagonal (and zeros elsewhere). Note that the order of the eigenvectors used to form Pwill determine the order in which the eigenvalues appear on the main diagonal of D.

Problem. Determine, whether $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ is diagonalizable.

If it is, identify an invertible matrix P, such that A is diagonalizable and find $P^{-1}AP$.

Solution. The characteristic polynomial is given by

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 2)(\lambda - 3)$$

So, the characteristic equation $\det(A - \lambda I) = 0$ gives the eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 3$. Since A is a 3×3 matrix and has 3 different eigenvalues, by Theorem 3, A is diagonalizable.

From these eigenvalues, we can calculate the corresponding eigenvectors as follows:

$$A-\lambda_1 I = A-2I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A-\lambda_2 I = A+2I = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$A-\lambda_3 I = A-3I = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Now we construct the invertible matrix P, by putting the linearly independent eigenvectors as columns to obtain

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

Therefore, we finally find

$$P^{-1}AP = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 1/5 & 0 & 1/5 \\ 1/5 & 1 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 1/5 & 0 & 1/5 \\ 1/5 & 1 & 1/5 \end{bmatrix} \begin{bmatrix} -2 & -2 & -3 \\ 0 & 2 & 3 \\ 2 & -8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Example

Although $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ does not have distinct eigenvalues, it is diagonalizable.

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A has eigenvalues $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = -1$. But for $\lambda_1 = \lambda_2 = 1$, the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and for $\lambda_3 = -1$, the eigenvector is $\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}$. Therefore, A has three linearly independent eigenvectors, and that is why A is diagonalizable.■

Note. To be diagonalizable, it is not necessary for a square matrix to have distinct eigenvalues.

Example If A is a diagonalizable matrix with P an invertible matrix such that $P^{-1}AP = D$, where D is a diagonal matrix, show that $A^k = PD^kP^{-1}$ for all $k \in \mathbb{N}$.

Proof.

$$A = PDP^{-1}$$

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)(DP^{-1}) = P(DD)P^{-1} = PD^2P^{-1}$$

By the principle of mathematical induction, we can show $A^k = PD^kP^{-1}$ is true for all $k \in \mathbb{N}$. ■

Example Calculate A^{10}

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Solution

Since A has eigenvalues $\lambda_1 = 2, \lambda_2 = -2$, $\lambda_3 = 3$ with the corresponding eigenvectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, respectively.

So, A is diagonalizable with

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$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix},$$

we have

$$P^{-1}AP = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

$$\therefore A = PDP^{-1}$$

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & (-2)^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{5} \begin{bmatrix} 6 \cdot 2^{10} - 3^{10} & 5 \cdot 2^{10} - 5 \cdot 3^{10} & 2^{10} - 3^{10} \\ -2^{10} + 3^{10} & 5 \cdot 3^{10} & -2^{10} + 3^{10} \\ -2^{10} + 3^{10} & -5 \cdot 2^{10} + 5 \cdot 3^{10} & 4 \cdot 2^{10} + 3^{10} \end{bmatrix}$$

$$= \begin{bmatrix} -10581 & -58025 & -11605 \\ 11605 & 59049 & 11605 \\ 11605 & 58025 & 12629 \end{bmatrix}$$

Diagonalization and Linear Transformations

So far in this section, the diagonalization problem has been considered in terms of matrices. In terms of linear transformations, the diagonalization problem can be stated as follows.

For a linear transformation $T: V \to V$, does there exist a basis B for V such that the matrix for T relative to B is diagonal?

The answer is "yes," provided the standard matrix for T is diagonalizable.

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Example 4. Let be the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represented by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

If possible, find a basis B for $\mathbb{R}^3$ such that the matrix for T relative to B is diagonal.

The standard matrix for T is represented by

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From one of the previous examples, we know that A is diagonalizable. So, the three linearly independent eigenvectors (found earlier) can be used to form the basis. That is,

$$B = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$$

The matrix for T relative to this basis is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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System of Differential Equations

A system of first-order linear differential equations has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned} \tag{1}$$

If we define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

and the system of differential equations (1) can be put into the following form:

$$\mathbf{y}' = A\mathbf{y} \tag{2}$$

where

$$A = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Example 5. Solve the following system of first-order linear differential equations.

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$$y_1' = 3y_1 + 2y_2$$

$$y_2' = 6y_1 - y_2$$

Solution. Define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \therefore \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

Now the given system can be written as

$$\mathbf{y}' = A\mathbf{y}$$

where

$$A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}.$$

We first find a matrix P that diagonalizes A. The eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 5$ (Please verify.) with corresponding eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore,

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} = D$$

Or $A = PDP^{-1}$. Now we define

$$\mathbf{x} = P^{-1}\mathbf{y}$$

$$\mathbf{x}' = P^{-1}\mathbf{y}' = P^{-1}A\mathbf{y} = (P^{-1}AP)(P^{-1}\mathbf{y}) = D\mathbf{x}$$

If $\mathbf{x} = [x_1 \; x_2]^T$, then

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Turning back into the scalar form, we obtain

$$x_1' = -3x_1$$

$$x_2' = 5x_2$$

Solving these equations, we get

$$x_1 = c_1 e^{-3t}$$

$$x_2 = c_2 e^{5t}$$

$$\mathbf{x} = \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{5t} \end{bmatrix}$$

$$\mathbf{y} = P\mathbf{x} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{5t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t} + c_2 e^{5t} \\ -3c_1 e^{-3t} + c_2 e^{5t} \end{bmatrix}$$

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Week 9 (Lecture 18)

Appendix

Theorem 1 If A and B are similar $n \times n$ matrices, then they have the same characteristic equations and eigenvalues.

Proof.

Theorem 3 If an $n \times n$ matrix A has *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent, and A is diagonalizable.

Proof.

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