Week 10 (Lecture 19)
Eigenvalues and Eigenvectors

Contents:

- What are eigenvalues and eigenvectors
- Verifying eigenvalues and eigenvectors for a given matrix
- Eigenvalues and eigenvectors from the perspective of linear transformation
- Eigenspace
- Theorem on finding eigenvalues and eigenvectors
- Finding eigenvalues and eigenvectors

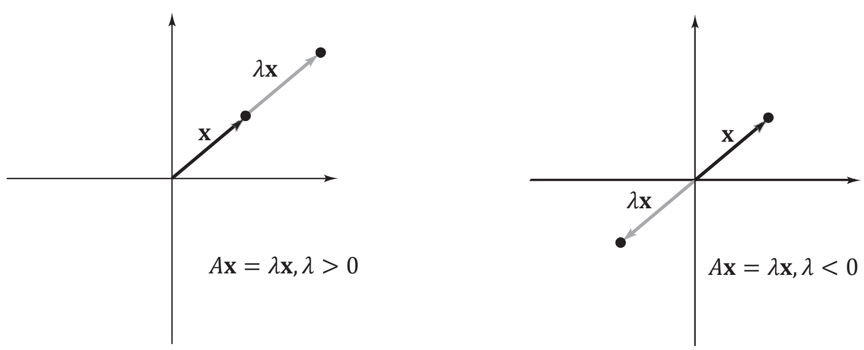
Week 10 (Lecture 19)
Eigenvalues and Eigenvectors

This lecture represents one of the most important problems in the linear algebra, the **eigenvalue and eigenvector problems**. The central question can be stated as follows, if a nonzero vector $\mathbf{x}$ is multiplied by a square matrix A, in most of the cases $\mathbf{x}$ will change its direction. But there are some exceptions, some vectors $\mathbf{x}$ remain in the same direction as $A\mathbf{x}$, and have scaled by a factor λ. This scalar λ (Greek letter lambda), is called an eigenvalue of the matrix A and the nonzero vector $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

So, we have

$$A\mathbf{x} = \lambda\mathbf{x}$$

The eigenvalue λ tells whether the special vector $\mathbf{x}$ is stretched or shrunk or reversed or left unchanged—when it is multiplied by A.



Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is called an **eigenvalue** of A if there is a *nonzero* vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Week 10 (Lecture 19)
Eigenvalues and Eigenvectors

Verifying Eigenvalues and Eigenvectors

Example 1

determine whether **x** is an eigenvector of A.

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, (b) $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (c) $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, (d) $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Solution

(a)

$$A\mathbf{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

Conclusion: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector.

(b)

$$A\mathbf{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Conclusion: $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

(c)

$$A\mathbf{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Conclusion: $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector.

(d)

$$A\mathbf{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Conclusion: $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not an eigenvector.

Example 2 For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Verify that $\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvector of A corresponding to the eigenvalues $\lambda_1 = 2$ and $\lambda_1 = -1$ respectively.

Solution

Multiplying $\mathbf{x_1}$ by A produces

$$A\mathbf{x_1} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{x_1}$$

So, $\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$.

Similarly,

$$A\mathbf{x_2} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1\mathbf{x_2}$$

So, $\mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = -1$. ■

Example 3 For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that $\mathbf{x_1} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are eigenvectors of A and find their corresponding eigenvalues.

Solution

Here,

$$A\mathbf{x_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{x_1}$$

So, $\mathbf{x_1} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.

Similarly,

$$\boldsymbol{A}\mathbf{x_2} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x_2}$$

So, $\mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of corresponding to the eigenvalue $\lambda_2 = 1$.■

Eigenvalues and Eigenvectors of Linear Transformations

This Lecture began with definitions of eigenvalues and eigenvectors in terms of matrices. They can also be defined in terms of linear transformations. A number λ is called an eigenvalue of a linear transformation $T: V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ.

Eigenspaces

If A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A. This may be seen by letting c be a nonzero scalar, which then produces

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x}).$$

Week 10 (Lecture 19)

Eigenvalues and Eigenvectors

In example 2, we saw that

For the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with corresponding eigenvalue $\lambda_1 = 2$.

But we can verify that $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ are also the eigenvectors of A.

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

In this way, for each eigenvalue λ, a square matrix A has infinite number of eigenvectors.

It is also true that if $\mathbf{x_1}$ and $\mathbf{x_2}$ are eigenvectors corresponding to the *same* eigenvalue λ, then their sum is also an eigenvector corresponding to λ, because

$$A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \lambda\mathbf{x_1} + \lambda\mathbf{x_2} = \lambda(\mathbf{x_1} + \mathbf{x_2})$$

Again, using the above example.

We have $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ are eigenvectors of $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ for the same eigenvalue $\lambda = 2$.

We can see that $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is also an eigenvector of the matrix with eigenvalue $\lambda = 2$.

Let us verify,

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Week 10 (Lecture 19)
Eigenvalues and Eigenvectors

Definition of Eigenspace

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors correspond to λ, together with the zero vector

$$\{\mathbf{0}\} \cup \{\mathbf{x}: \mathbf{x} \text{ is an eigenvector correspond to } \lambda\}$$

is subspace of $\mathbb{R}^n$. This subspace is called the eigenspace of λ.

■

For example,

Let, $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Geometrically, multiplying a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ by the matrix A corresponds to a reflection in the y-axis. That is, if $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

The figure illustrates that the only vectors reflected onto scalar multiples of themselves are those lying on either the x-axis or the y-axis.

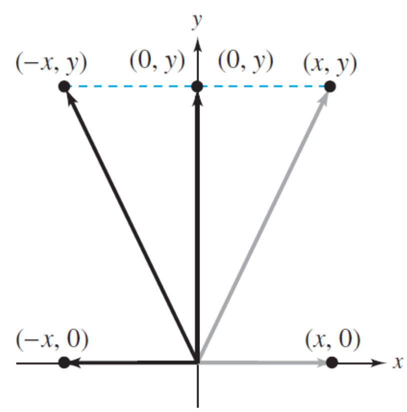


Figure: A reflects vectors in the y-axix

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For a vector on the x-axis, we have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

So, the eigenvalue is $\lambda_1 = -1$.

Again, for a vector on the y-axis, we have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

So, the eigenvalue is $\lambda_2 = 1$.

Therefore, the eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the x-axis, and the eigenvectors corresponding to $\lambda_2 = 1$ are the nonzero vectors on the y-axis. This implies that the eigenspace corresponding to $\lambda_1 = -1$ is the x-axis, and that the eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

Week 10 (Lecture 19)

Eigenvalues and Eigenvectors

Finding eigenvalues and eigenvectors

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A, let I be the $n \times n$ identity matrix. Writing the equation $A\mathbf{x} = \lambda\mathbf{x}$ in the form $A\mathbf{x} = \lambda I\mathbf{x}$ then produces

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(A - \lambda I)$ is *not* invertible–that is, if and only if the determinant of $(A - \lambda I)$ is zero. This is formally stated in the next theorem.

Theorem

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that
$$\det(A - \lambda I) = 0$$
2. The eigenvectors of A corresponding to λ are the nonzero solutions of
$$(A - \lambda I)\mathbf{x} = \mathbf{0} \qquad \blacksquare$$

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A. Moreover, when expanded to polynomial form, the polynomial

$$|A - \lambda I| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

is called the **characteristic polynomial** of A. This definition tells us that the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A. Because the characteristic polynomial of A is of degree n , A can have at most n distinct eigenvalues.

Week 10 (Lecture 19)

Eigenvalues and Eigenvectors

Steps for finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $\det(A - \lambda I) = 0$. It will be a polynomial equation of degree n in the variable λ.
2. Find the real roots of the characteristic equation. These are the eigenvalues of A.
3. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$. This requires row reducing of an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

Example

Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

or, Find the eigenvalues and corresponding eigenvectors of $T: \mathbb{R}^2 \to \mathbb{R}^2$ where the transformation is given by $T(x_1, x_2) = (2x_1 - 12x_2,\ x_1 - 5x_2)$.

Solution

Let us calculate,

$$(A - \lambda I) \equiv \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 - \lambda & -12 \\ 1 & -5 - \lambda \end{bmatrix}$$

So, the characteristic equation

$$\det(A - \lambda I) = 0$$

$$(2 - \lambda)(-5 - \lambda) - (-12) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

gives,

$$\lambda = -1, -2$$

To find the corresponding eigenvectors, we use Gauss-Jordan elimination to solve the homogeneous linear system represented by $(A - \lambda I)\mathbf{x} = \mathbf{0}$

For $\lambda_1 = -1$ the coefficient matrix is

$$(A - \lambda I) = \begin{bmatrix} 2-(-1) & -12 \\ 1 & -5-(-1) \end{bmatrix} = \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix}$$

Apply row reduction, we get

$$\begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

And the corresponding eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now,

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which shows that

$$x_1 - 4x_2 = 0$$

$$x_1 = 4x_2$$

x_2 is a free variable. Let,

$$x_2 = t$$

then,

$$x_1 = 4t$$

Therefore, the corresponding eigenvector is

$$\mathbf{x} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

So, we can conclude that every eigenvector of $\lambda_1 = -1$ is of the form

$$\mathbf{x} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$$

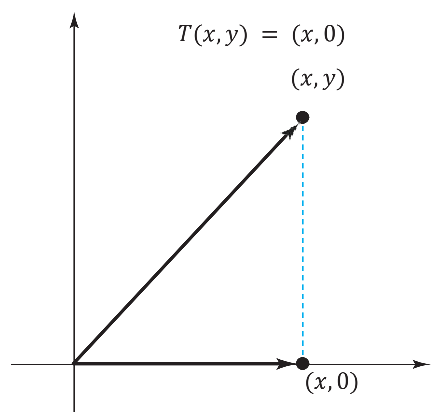
Similarly, for $\lambda_2 = -2$,

$$\mathbf{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, t \neq 0$$

■

Example

The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x, y) = (x, 0)$ represents the projection of each point in $\mathbb{R}^2$ onto the x-axis, as shown in figure.



Projection onto the x-axis

The standard matrix for this transformation can be obtained as

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1.x + 0.y \\ 0.x + 0.y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix for T is given by $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Projection matrix A has two eigenvalues: 1 and 0.

- Every nonzero vector in the column space of A is an eigenvector corresponding to the eigenvalue 1 .
- Every nonzero vector orthogonal to the column space of A will be an eigenvector corresponding to the eigenvalue 0 .

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Week 10 (Lecture 19)

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}.$$

Solution

The characteristic equation is given by

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -3 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(1-\lambda) - 6 = 0$$

$$\lambda^2 - 7\lambda = 0$$

$$\lambda(\lambda - 7) = 0$$

So, the eigenvalues are: $\lambda = 0, 7$.

For $\lambda = 0$, we have

$$A - 0I = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector, the reduced row echelon form is showing that $-2x_1 + x_2 = 0$. Since x_2 is a free variable, letting $x_2 = 2t$ gives $x_1 = t$. Therefore, the eigenvector becomes

$$\mathbf{x} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad t \neq 0.$$

Similarly, for $\lambda = 7$, we have

$$A - 7I = \begin{bmatrix} -1 & -3 \\ -2 & -6 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} -1 & -3 \\ -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

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If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector, the reduced row echelon form is showing that $x_1 + 3x_2 = 0$. Letting $x_2 = -t$ gives $x_1 = 3t$. Therefore,

$$\mathbf{x} = \begin{bmatrix} 3t \\ -t \end{bmatrix} = t\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad t \neq 0.$$

Example

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

Solution

The characteristic equation

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda) - 2] = 0$$

$$(2-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$

$$(\lambda - 2)(\lambda - 1)(\lambda - 4) = 0$$

So, the eigenvalues are: $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 4$.

For $\lambda_1 = 1$, we have

$$A - 1I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\mathbf{x} = [x_1\ x_2\ x_3]^T$ is an eigenvector, the reduced row echelon form is showing that

$$x_1 \qquad + x_3 = 0$$

$$x_2 - x_3 = 0$$

Considering x_3 as a free variable, we let $x_3 = t$, and this gives $x_1 = -t$, $x_2 = t$. Therefore,

$$\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad t \neq 0.$$

Similarly, for $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\mathbf{x} = [x_1\ x_2\ x_3]^T$ is an eigenvector, the reduced row echelon form is showing that

$$x_2 \qquad = 0$$

$$x_3 = 0$$

Since x_1 is a free variable, we let $x_1 = t$. Therefore,

$$\mathbf{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad t \neq 0.$$

Similarly, for $\lambda_3 = 4$, we have

$$A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{x} = [x_1\ x_2\ x_3]^T$ is an eigenvector, the reduced row echelon form is showing that

$$x_1 \qquad -\frac{7}{2}x_3 = 0$$

$$x_2 + 2x_3 = 0$$

Considering x_3 as a free variable, we let $x_3 = 2t$, this gives $x_1 = 7t$ and $x_2 = -4t$. Therefore,

$$\mathbf{x} = \begin{bmatrix} 7t \\ -4t \\ 2t \end{bmatrix} = t\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}, \qquad t \neq 0.$$

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Week 10 (Lecture 19)
Eigenvalues and Eigenvectors

Notes.

- Eigenvalues and eigenvectors are only for square matrices.
- An eigenvector cannot be zero. (Allowing $\mathbf{x}$ to be the zero vector would render the definition meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ is true for all real values of λ.)
- Eigenvalues may be equal to zero.
- A matrix can have more than one eigenvalue.
- If A is an identity matrix, then $I\mathbf{x} = \mathbf{x}$**,** i.e. all vectors are eigenvectors of I and all eigenvalues are $\lambda = 1$.
- Sum of the eigenvalues of a matrix equals the trace of the matrix.
- The eigenvalues of an upper or lower triangular matrix are the elements on the main diagonal.
- A matrix is singular if and only if it has a zero eigenvalue.
- The product of all the eigenvalues of a matrix (counting multiplicity) equals the determinant of the matrix.
- If λ is an eigenvalue of A, then λ also an eigenvalue of A^T.

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