MAT216: Linear Algebra & Fourier Analysis

Lecture Note: Week 8_Lecture 16_Part 1 and Part 2

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Contents:

References:

- Introduction to Linear Algebra, 5th Ed. Gilbert Strang
- https://bux.bracu.ac.bd/courses/course-v1:buX+MAT216+2020_Summer/course/

Part 1: Least Square Method

Let's start with reviewing a couple of things first. We know that, if the vector space V is a subspace of the vector space S, then the orthogonal complement of V is the set of all vectors that are perpendicular to V. This orthogonal subspace is denoted by $V^{\perp}$. (pronounced "V perp").

If υ is orthogonal to the nullspace, it must be in the row space, i.e. nullspace is the orthogonal complement of the row space. Every x that is perpendicular to the rows satisfies $Ax = 0$, and lies in the nullspace.

- $C(A)$ and $LN(A)$ are orthogonal complements.
- $N(A)$ is the orthogonal complement of the row space $C(A^T)$.
- $N(A^T)$ is the orthogonal complement of the column space $C(A)$.

- Every system of linear equation has no solutions, or has exactly one solution, or has infinitely many solutions.

Sometimes for the system $Ax = b$ we don't get any solution, reason behind it is too many equations. There are more equations than unknowns. Augmented matrix has more rows than columns ($m > n$). The n columns span a small part of m-dimensional apace. If we don't have a perfect measurement than we will see b is outside that column space. It often happens that the system $Ax = b$ becomes inconsistent and has no solution.

When x become an exact solution of the system, we get error $e = b - Ax$ down to zero. If the length e is as small as possible, $\hat{x}$ is a least square solution.

In this particular section, our main objective to find, $\hat{x}$ and use it to find answer of some real-world problems.

- When $Ax = b$ has no solution, multiply by A^T and solve $A^T A\hat{x} = A^T b$.

Let us now consider the following system $Ax = b$:

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad (1)$$

Reduced Row Echelon Form of (1) is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (2)$$

Which is inconsistent and has no solution.

Multiplying (1) by A^T, $A^T A\hat{x} = A^T b \ \Rightarrow$

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} 6 & 1 & 7 \\ 1 & 6 & 7 \\ 7 & 7 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 13 \end{pmatrix} \qquad (3)$$

Reduced Row Echelon Form of (3) is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{10}{7} \\ \frac{3}{7} \\ 0 \end{pmatrix}$$

We can see that our system is consistent, and it will definitely produce some solution, $\hat{x}$ (is a least square solution). The equation $A^T A\hat{x} = A^T b$ fully determines the best vector $\hat{x}$.

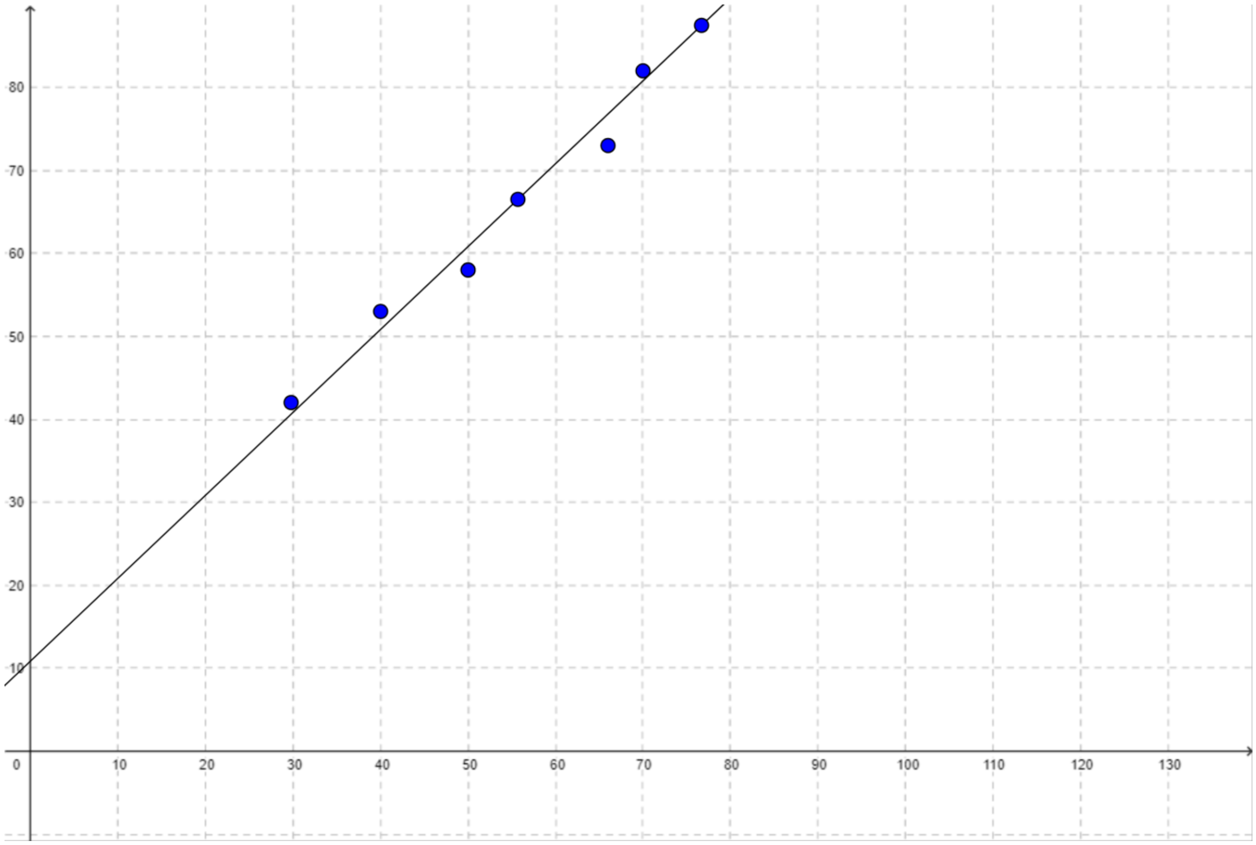
Part 2: Finding the Best Fit Line

Let's consider the following table-1

Price of Milk	Price of Milkshake
30	42
40	53
50	58
66	73
70	82

Consider the straight line $y = mx + c$, y =Price of Milkshake and x = Price of Milk.

Let's also consider the following graph by using table-1



In the above graph, we get an arbitrary line which could be our best fit for the given data. Our goal is to minimize the error between the actual data point and arbitrary line, which will be estimated by using the following formula

$$e_1^2 + e_2^2 + \ldots\ldots\ldots + e_n^2 = \|Ax - b\|^2 \text{ when } A^T A\hat{x} = A^T b \qquad (*)$$

Now we need to find the exact line to minimize the error, which ultimately represent our best fit line for the provided date set, and find the best fit line requires the method list square. Now using table-1 and $y = mx + c$, we get

$$\begin{bmatrix} 30 & 1 \\ 40 & 1 \\ 50 & 1 \\ 66 & 1 \\ 70 & 1 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 42 \\ 53 \\ 58 \\ 73 \\ 82 \end{bmatrix};$$

We shall be using list square method to solve it, i.e.

$A^T A\hat{x} = A^T b \;\Rightarrow$

$$\begin{bmatrix} 30 & 40 & 50 & 66 & 70 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 30 & 1 \\ 40 & 1 \\ 50 & 1 \\ 66 & 1 \\ 70 & 1 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 30 & 40 & 50 & 66 & 70 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 42 \\ 53 \\ 58 \\ 73 \\ 82 \end{bmatrix}$$

$$\begin{bmatrix} 14256 & 256 \\ 256 & 5 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 16838 \\ 308 \end{bmatrix}$$

From here we get the reduced echelon form as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 0.93 \\ 13.98 \end{bmatrix} \qquad \Rightarrow$$

$$c = 13.98 \; and \; m = 0.93 \qquad \Rightarrow$$

$$y = 0.93x + 13.98$$

is our best fit line and, which minimizes the error between our actual data point and this best fit line [Please watch video lecture-16 properly] .