Shortest Path with Multiples of k

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1 Problem Statement

This is a modification of Q15 of the problem set. Given a graph $G = (V, E)$, with vertices V, edges E, a source vertex s, a target vertex t, and edge weights $c(uv)$, where each weight $c(uv) > 0$ represents the cost associated with traveling from vertex u to v or from v to u, the objective is to find the shortest path from s to t such that the total cost of the path is a multiple of a given positive integer k. Shortest path from u to v is defined as the path that contains the minimum number of vertices and has nothing to do with edge weights.

2 Algorithm

The proposed algorithm is based on modifying the graph and applying the Breadth-First-Search algorithm. Here's the pseudocode:

Algorithm 1 ShortestPathWithMultiplesOfK(G, s, t, k)

Let R be the set $\{0, 1, 2, ..., k-1\}$
Define a new graph $G' = (V', E')$
  where $V' = V \times R$
$E' := \{(u, r) \to (v, (r + c(uv)) \mod k) | (u, v) \in E, r \in R\}$
Weight of each edge $(u, r) \to (v, r')$ is 1
Compute shortest path from source vertex $(s, 0)$ via BFS
return shortest distance from $(s, 0)$ to $(t, 0)$

3 Implementation

Below is the C++ implementation of the algorithm:

```
// Code snippet: https://pastebin.com/3hFWucVG
```

4 Time Complexity

The time complexity of the algorithm is $O(E' + V') = O(mk + nk) = O(m + n)$, since k is treated as a constant factor.