Assignment - 2

* Let A be a non-empty subset of $\mathbb{R}$. Then A is countable iff there exists a one-to-one (injective) f^n $f: A \to \mathbb{N}$ (not necessarily onto)
* A finite set is always countable.
* A countable set which is infinite is called countably infinite.

Q1. $\mathbb{Z}$ is countably infinite.

Pf That means we have to find an injective map $f: \mathbb{Z} \to \mathbb{N}$

Consider the f^n $f: \mathbb{Z} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 1 & , x = 0 \\ 2x & , x > 0 \\ -2x+1 & , x < 0 \end{cases}$$

Claim - f is injective

Case-I - If x_1, x_2 are (+) ve

Now, $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

So f is injective

Case-II - If x_1 & x_2 are (-) ve

$f(x_1) = f(x_2)$

$\Rightarrow -2x_1+1 = -2x_2+1$

$\Rightarrow x_1 = x_2$

So f is injective

Case-III If $x_1 > 0$ & $x_2 < 0$ then $f(x_1) = 2x_1$ is even

~~$f(x_1) = f(x_2)$~~ and $f(x_2) = -2x_2+1$ is odd

So $f(x_1) = f(x_2)$ is not possible.

Case-4 If $x_1 > 0$ & $x_2 = 0$

$f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 1$ which is not possible

$\Rightarrow x_1 = \frac{1}{2}$ ~~Since $2x_1$~~

Case-5 If $x_1 < 0$ & $x_2 = 0$ ~~same~~ One can do in a same way

Q.2 $\mathbb{Q}$ is countably infinite.

$\mathbb{Q} = \{ \ldots, \frac{-1}{1}, 0, \frac{1}{1}, \frac{1}{2}, \ldots \}$

RTP - an injective map $f: \mathbb{Q} \to \mathbb{N}$

Let us define the map $f: \mathbb{Q} \to \mathbb{N}$

{Any rational $r \in \mathbb{Q}$ can be written in its lowest form

$r = \frac{p}{q}$, $\gcd(p,q) = 1$ & $q \neq 0$

If both p & q are -ve, that is

$p = -p_1$ $(p_1 \geq 0)$ & $q = -q_1$ $(q_1 > 0)$

then $r = \frac{p}{q} = \frac{-p_1}{-q_1} = \frac{p_1}{q_1} \geq 0$

So both p & q can not be -ve if r is -ve. One of them (p or q) will be -ve.

With out loss of generality we can take p to be -ve if r is -ve.

& Define $f(r) = f\left(\frac{p}{q}\right) = 2^p 3^q 5^s$

Now, $s = 1$ if $r > 0$

$s = 2$ if $r < 0$

When $r = 0$, write $r = \frac{0}{1}$ $(p=0, q=1)$

then $f(0) = 3 \cdot 5 = 15$ $(s=1)$

The f^n $f : \frac{p}{q} \longmapsto 2^p 3^q 5^s$ is injective as prime factorisations are unique.

$$f\left(\frac{p}{q}\right) = f\left(\frac{a}{b}\right)$$

$$\Rightarrow 2^p 3^q 5^{s_1} = 2^a 3^b 5^{s_2}$$

$$\Rightarrow 2^{p-a} 3^{q-b} 5^{s_1 - s_2} = 1$$

LHS product is 1 iff $p = a$, $q = b$ & $s_1 = s_2$

So $p = a$ & $q = b$

Hence f is injective.

Q3. $\mathbb{R}$ is uncountable.

Q4. TP - $\lim_{n \to \infty} \frac{b}{n^2} = 0$ for any real no. b

RTP - for every $\varepsilon > 0$, there exists (+)ve integer N_ε (depend on ε) s.t.

$$\left|\frac{b}{n^2} - 0\right| < \varepsilon \quad \forall\ n \geq N_\varepsilon$$

or $$\left|\frac{b}{n^2}\right| < \varepsilon \quad \forall\ n \geq N_\varepsilon$$

By Archimedean property (Take $x = \varepsilon > 0$ & $y = |b|$)
there exists N_ε s.t. $N_\varepsilon x > y$

$$\Rightarrow N_\varepsilon \varepsilon > |b| \Rightarrow \frac{|b|}{N_\varepsilon} < \varepsilon \quad \text{— ①}$$

Consider,

$$\left|\frac{b}{n^2} - 0\right| = \frac{|b|}{n^2} \le \frac{|b|}{n} \le \frac{|b|}{N_\varepsilon} \quad \forall\ n \ge N_\varepsilon$$

$$< \varepsilon \quad \text{by ①}$$

$$\Rightarrow \left|\frac{b}{n^2} - 0\right| < \varepsilon \quad \text{for all } n \ge N_\varepsilon$$

Hence, $\lim_{n\to\infty} \frac{b}{n^2} = 0$

(5.) TP - $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Consider,

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\le \frac{1}{\sqrt{n}} \quad \text{— (*)} \qquad \text{Since } \sqrt{n+1} + \sqrt{n} \ge \sqrt{n}$$

RTP - for every $\varepsilon > 0$, there exists (+)ve integer N_ε (depend on ε) such that

$$\left|(\sqrt{n+1} - \sqrt{n}) - 0\right| < \varepsilon \quad \forall\ n \ge N_\varepsilon$$

By Archimedean property (Take $x = \varepsilon^2 > 0$ & $y = 1$)
there exists N_ε s.t. $N_\varepsilon \varepsilon^2 > 1$

$$\Rightarrow \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon \quad \text{— (**)}$$

~~Conside~~ Let $\varepsilon > 0$ be given

Consider,

$$|(\sqrt{n+1} - \sqrt{n}) - 0| = \sqrt{n+1} - \sqrt{n}$$

$$\leq \frac{1}{\sqrt{n}} \qquad (\text{by } *)$$

$$\leq \frac{1}{\sqrt{N_\varepsilon}} \qquad \forall \; n \geq N_\varepsilon$$

$$< \varepsilon \qquad \text{by } (**)$$

$$\Rightarrow \quad |(\sqrt{n+1} - \sqrt{n}) - 0| < \varepsilon \qquad \forall \; n \geq N_\varepsilon$$

Hence, $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Q6. Given $\lim_{n\to\infty} x_n = x > 0$

TP - there exists a natural no. K s.t. if $n \geq K$, then $\frac{x}{2} < x_n < 2x$

(Home work)

Q7. Given $\lim_{n\to\infty} a_n = 0$

If for some constant $C > 0$ & some $m \in \mathbb{N}$, we have

$$|x_n - x| < C a_n \qquad \forall \; n \geq m \qquad (1)$$

TP - $\lim_{n\to\infty} x_n = x$

Since $\lim_{n\to\infty} a_n = 0$

for every $\frac{\varepsilon}{C} > 0$, there exists (+)ve integer N_ε (depend on ε) s.t.

$|a_n - 0| < \varepsilon/c \quad \forall \quad n \geq N_\varepsilon$

$\Rightarrow \quad a_n < \varepsilon/c \quad \forall \quad n \geq N_\varepsilon$ —— ② (Since $\{a_n\}$ is a seq^n of (+)ve real no.s)

Consider, by ①

$|x_n - x| < c a_n < c \cdot \frac{\varepsilon}{c} = \varepsilon \quad \forall \quad n \geq \max\{m, N_\varepsilon\}$ (by ②)

$\Rightarrow \quad |x_n - x| < \varepsilon \quad \forall \quad n \geq \max\{m, N_\varepsilon\} = N'$

Hence, $\lim\limits_{n\to\infty} x_n = x$

Q8. If $c>0$ TP - $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$

Case 1 - If $c=1$, we have nothing to prove.

Case 2 - If $c>1$, then $c^{\frac{1}{n}} > 1 \quad \forall \; n \in \mathbb{N}$

Consider, $c^{\frac{1}{n}} - 1 = a_n > 0 \quad , n \in \mathbb{N}$ —— ①

$\Rightarrow \quad c = (1+a_n)^n$

$= 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \cdots$

$> n a_n \qquad \left[\binom{n}{1} = n\right]$

$\Rightarrow \quad \frac{c}{n} > a_n \quad \forall \; n \in \mathbb{N}$ —— ②

So, $\{a_n\}$ is a seq^n which satisfies

$0 < a_n < \frac{c}{n} \quad \forall \; n \in \mathbb{N}$ (by ① & ②)

Now by Sandwich theorem, $x_n = 0$ & $z_n = \frac{c}{n} \quad \forall \; n \in \mathbb{N}$

$\lim x_n = 0$ & $\lim z_n = 0$

$\Rightarrow \lim a_n = 0$

$\Rightarrow \lim_{n\to\infty} (c^{\frac{1}{n}} - 1) = \lim a_n = 0$

$\Rightarrow \lim_{n\to\infty} c^{\frac{1}{n}} - \lim_{n\to\infty} 1 = 0$ (by Algebra of limits)

$\Rightarrow \lim_{n\to\infty} c^{\frac{1}{n}} = 1$

Case 2 - $c < 1$, then $c^{\frac{1}{n}} < 1 \quad \forall\, n \in \mathbb{N}$

Take $c^{\frac{1}{n}} = \frac{1}{1+h_n}$, $h_n > 0 \quad \forall\, n \in \mathbb{N}$ —— (3)

$$c = \frac{1}{(1+h_n)^n} \quad \text{—— (4)}$$

Now, $(1+h_n)^n = 1 + nh_n + \binom{n}{2} h_n^2 + \cdots$

$\geq nh_n \quad \forall\, n \in \mathbb{N}$

So, $\frac{1}{nh_n} > \frac{1}{(1+h_n)^n} = c$ by (4)

$\Rightarrow c < \frac{1}{nh_n} \quad \forall\, n \in \mathbb{N}$

So $h_n < \frac{1}{cn} \quad \forall\, n \in \mathbb{N}$ —— (5)

by (3) & (5), $0 < h_n < \frac{1}{cn} \quad \forall\, n \in \mathbb{N}$

By Sandwich thm, $x_n = 0$ & $z_n = \frac{1}{cn} \quad \forall\, n \in \mathbb{N}$

$\lim x_n = 0 = \lim z_n$

$\Rightarrow \lim_{n\to\infty} h_n = 0$

So, $\lim_{n\to\infty} c^{\frac{1}{n}} = \frac{1}{\lim(1+h_n)}$ (by Algebra of limits)

$= \frac{1}{1+\lim h_n}$ (by Algebra of limits)

$= \frac{1}{1+0} = 1$

$\Rightarrow \lim_{n\to\infty} c^{\frac{1}{n}} = 1$