# REAL NUMBERS

## 1. Introduction

In this note we will give some idea about the real number system and its properties.

## 2. Rational Numbers

**Definition 2.1. Rational Numbers** A rational number is an expression of the form $\frac{a}{b}$ where $a, b$ are integers and $b \neq 0$. $\frac{a}{0}$ is not considered as rational number. Two rational numbers $\frac{a}{b}$ $(b \neq 0)$ and $\frac{c}{d}$ $(d \neq 0)$ are considered to be equal iff $\frac{a}{b} = \frac{c}{d}$. The set of rationals is denoted by $\mathbb{Q}$ i.e

$$\mathbb{Q} = \{\ldots\ldots\ldots\ldots - \frac{1}{1}, -\frac{1}{2}, \ldots\ldots.0, \ldots\ldots\ldots\ldots, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \ldots\ldots\ldots.\}.$$

The operations of addition (and subtraction), multiplication and $<$ extend to this set in a natural way.

### 2.1. Reductio ad absurdum.

We shall see, from a very simple situation that numbers other than rational numbers are needed. Consider a right-angled triangle whose two sides are of unit length. Then by Pythagoras Theorem, the length $l$ of the hypotenuse must satisfy $l^2 = 2$ (we write $l = \sqrt{2}$). What is $l$? The following theorem shows that $l$ can not be a rational number that is $l$ can not be of the form $\frac{p}{q}$ where $p,\ q \in \mathbb{N}$

**Theorem 2.1. Reductio ad absurdum or reduction to the absurd or Proof by contradiction** *There is no $a \in \mathbb{Q}$ such that $a^2 = 2$.*

*Proof.* Assume to the contrary that there is $a \in \mathbb{Q}$ with $a^2 = 2$. Then there are $p,\ q \in \mathbb{N}$ such that $a = \frac{p}{q}$ with $p$ **and** $q$ **relatively prime.** Now,

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow p^2 = 2q^2 \tag{1}$$

shows $p^2$ is even. Since the square of an odd number is odd, $p$ must be even; i.e., $p = 2r$ for some $r \in \mathbb{N}$. Substituting this into (1), shows

$$4r^2 = 2q^2 \quad \Rightarrow 2r^2 = q^2.$$

The same argument as above establishes $q$ is also even. This contradicts the assumption that $p$ and $q$ are relatively prime. So we arrive at a contradiction. Therefore, our assumption is wrong. So no such $a$ exists. □

Apparently there is a positive length whose square is 2, which we write as $\sqrt{2}$. The above discussion shows that $\sqrt{2}$ is not a rational number. Of course, $\sqrt{2}$ can be approximated by rational numbers. There are rational numbers whose squares are close to 2; for example, $(1.4142)^2 = 1.99996164$ and $(1.4143)^2 = 2.00024449$.

as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, ........ . Such numbers ( the numbers which can not be written off the form $\frac{p}{q}$ will be called **irrational numbers**. The problem now is to define these numbers from $\mathbb{Q}$. We will not present the mathematical definition of irrational numbers here, since it is bit involved.

**Theorem 2.2. Rational Zeros Theorem.** *Suppose $c_0, c_1, \cdots, c_n$ are integers and $r$ is a rational number satisfying the polynomial equation*

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{2}$$

*where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where $c, d$ are integers having no common factors and $d \neq 0$. Then $c$ divides $c_0$ and $d$ divides $c_n$.*

In other words, the only rational candidates for solutions of (2) have the form $\frac{c}{d}$ where $c$ divides $c_0$ and $d$ divides $c_n$.

*Proof.* We are given

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0 \tag{3}$$

We multiply (3) through by $d^n$ and obtain

$$c_n c^n + c_{n-1} c^{n-1} d + c_{n-2} c^{n-2} d^2 + \cdots + c_1 c d^{n-1} + c_0 d^n = 0 \tag{4}$$

If we solve for $c_0 d^n$, we obtain

$$c_0 d^n = -c[c_n c^{n-1} + c_{n-1} c^{n-2} d + c_{n-2} c^{n-3} d^2 + \cdots + c_1 d^{n-1}]$$

It follows that $c$ divides $c_0 d^n$. But $c$ and $d^n$ have no common factors,so $c$ divides $c_0$. This follows from the basic fact that if an integer $c$ divides a product $ab$ of integers, and if $c$ and $b$ have no common factors, then $c$ divides $a$.2 Now we solve (4) for $c_n c^n$ and obtain

$$c_n c^n = -d[c_{n-1} c^{n-1} + c_{n-2} c^{n-2} d + \cdots + c_1 c d^{n-1} + c_0 d^{n-1}].$$

Thus d divides $c_n c^n$. Since $c^n$ and $d$ have no common factors, $d$ divides $c_n$. $\square$

**Corollary 2.3.** *Consider the polynomial equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{5}$$

*where the coefficients $c_0, c_1, \cdots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides $c_0$.*

*Proof.* We apply Theorem 2.2, with $c_n = 1$ and if $r = \frac{c}{d}$ is a solution, then $d$ must divide $c_n = 1$ that implies $d = 1$. And $c$ must divide $c_0$. Hence the result. $\square$

**Example 2.1.** *Using the above corollary we can prove $\sqrt{2}$ and $\sqrt{3} \cdots \sqrt{17}$ are not rational numbers. Consider the equation $x^2 - 2 = 0$ and $x^2 - 3 = 0$ and $x^2 - 17 = 0$ and apply the above argument. For example, The only possible rational solutions of $x^2 - 17 = 0$ are $\pm 1$, $\pm 17$, and none of these numbers are solutions.*

**Example 2.2.** $6^{\frac{1}{3}}$ *is not a rational number.*

*Proof.* If the solution exists then it must satisfy the equation $x^3 = 6$. If the solution is a rational number then by corollary 2.3 it must be an integer say $c$ and $c|6$. So $c$ must be either $\pm 1$ or $\pm 2$ or $\pm 3$ or $\pm 6$. It is easy to verify that none of these eight numbers satisfies the equation $x^3 - 6 = 0$. $\square$

**Example 2.3.** $a = \sqrt{2 + 5^{\frac{1}{3}}}$ *is not a rational number.*

*Proof.* Use Corollary 2.3. $\square$

**Example 2.4.** *There are even more exotic numbers such as $\pi$ and $e$ that are not rational numbers, but which come up naturally in mathematics. The number $\pi$ is basic to the study of circles and spheres, and $e$ arises in problems of exponential growth. They do not satisfy equations with integer coefficient like* (2).

**Definition 2.2. Irrational Numbers** So far we proved that $\sqrt{2}$ and $\sqrt{3} \cdots \sqrt{17}$ and so on are not rational numbers. Such numbers ( the numbers which can not be written off the form $\frac{p}{q}$) will be called **irrational numbers**. The problem now is to define these numbers from $\mathbb{Q}$

**Definition 2.3. Real Numbers** Real numbers are the numbers which include both rational and irrational numbers. It is denoted by $\mathbb{R}$.

We will give a rough idea about real numbers.

On a straight line, if we mark off segments....... ..., $[-1, 0]$, $[0, 1]$, $[1, 2]$... then all the rational numbers can be represented by points on this straight line. The set of points representing rational numbers seems to fill up this line (rational number $r + s$ lies in between the rational numbers $r$ and $s$). But we have seen above that the rationals do not cover the entire straight line. Intuitively we feel that there should be a larger set of numbers, say $\mathbb{R}$ such that there is a correspondence between $\mathbb{R}$ and the points of this straight line. Indeed, one can construct such a set of numbers from the rational number system $\mathbb{Q}$, called the set of real numbers $\mathbb{R}$, which contains the set of rationals and also numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$,...... and more. Moreover, on this set we can define operations of addition and multiplication, and an order in such a way that when these operations are restricted to the set of rationals, they coincide with the usual operations and the usual order. The set $\mathbb{R}$ with these operations is called the real number system.

An important property of $\mathbb{R}$, which is missing in $\mathbb{Q}$ is the following. Before that we need to define some notations:

### 2.2. **Notations.**

- An **open interval** denoted by $(a, b)$ (where $a, b \in \mathbb{R}$) is the set defined by $\{x \in \mathbb{R}; a < x < b\}$.
- A **closed interval** denoted by $[a, b]$ (where $a, b \in \mathbb{R}$) is the set defined by $\{x \in \mathbb{R}; a \leq x \leq b\}$.
- An **half-open interval or half closed interval** denoted by $[a, b)$ (where $a, b \in \mathbb{R}$) is the set defined by $\{x \in \mathbb{R}; a \leq x < b\}$ Or denoted by $(a, b]$ (where $a, b \in \mathbb{R}$) is the set defined by $\{x \in \mathbb{R}; a < x \leq b\}$.

## 3. Completeness Axiom of real number system:

In this section, we give the completeness axiom for $\mathbb{R}$. This is the axiom that will assure us $\mathbb{R}$ has no gaps It has far-reaching consequences and almost every significant result in this course relies on it. Most theorems in this course would be false if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers. Let $A$ be a non-empty subset of $\mathbb{R}$.

**Definition 3.1. Maximum and Minimum** (a) If $A$ contains a largest element $a_0$ [that is, $a_0$ belongs to $S$ and $a \leq a_0$ for all $a \in A$], then we call $a_0$ the maximum of $A$ and write $s_0 = maxS$.
(b) If $A$ contains a smallest element, then we call the smallest element the minimum of $A$ and write it as $minA$.

**Example 3.1.** *Every finite nonempty subset of $R$ has a maximum and a minimum. Thus*

$$max\{1, 2, 3, 4, 5\} = 5 \text{ and } min\{1, 2, 3, 4, 5\} = 1.$$

**Example 3.2.** *$[a, b]$ the closed interval of $\mathbb{R}$, observe $max[a, b] = b$ and $min[a, b] = a$. $(a, b)$ the open interval of $\mathbb{R}$, the set $(a, b)$ has no maximum and no minimum, since the endpoints $a$ and $b$ do not belong to the set.*
*While $[a, b)$ and $(a, b]$ half-open or semi-open intervals, Observe $max(a, b] = b$ and $min[a, b) = a$.*

**Example 3.3.** *The sets $\mathbb{Z}$ and $\mathbb{Q}$ have no maximum or minimum. The set $\mathbb{N}$ has no maximum, but min $\mathbb{N} = 1$.*

### 3.1. **Supremum and Infimum.**

**Definition 3.2. Upper bound/ Bounded above** If a real number $M$ satisfies $a \leq M$ for all $a \in A$, then $M$ is called an upper bound of $A$ and the set $A$ is said to be bounded above.

**Definition 3.3. Lower Bound/ Bounded below** Similarly, If a real number $m$ satisfies $m \leq a$ for all $a \in A$, then $m$ is called a lower bound of $A$ and the set $A$ is said to be bounded below.

**Remark 3.1.** Let $A$ be a non-empty set of real numbers. Suppose $m$ is a lower bound of $A$ and $M$ is an upper bound of $A$. As $m \leq x \leq M$ for all $x \in A$. Hence $m \leq M$.

**Definition 3.4.** The set $A$ is said to be bounded if it is bounded above and bounded below. Thus $A$ is bounded if there exist real numbers $m$ and $M$ such that $A \subseteq [m, M]$.

**Definition 3.5. Least upper bound or l.u.b or Supremum** An upper bound $x_0$ of $A$ is said to be a least upper bound (l.u.b.) or supremum (sup) of $A$ if whenever $z$ is an upper bound of $A$, $x_0 \leq z$. A greatest lower bound (g.l.b.) or infimum (inf) is defined similarly.

**Definition 3.6. Greatest lower bound or g.l.b or Infimum** A lower bound $y_0$ of $A$ is said to be the greatest lower bound (g.l.b) or infimum (inf) of $A$ if whenever $v$ is a lower bound of $A$, $v \leq y_0$.

**Remark 3.2.** Suppose $A$ be a non-empty bounded subset of $\mathbb{R}$. Then $infA \leq x \leq supA$ for all $x \in A$ by definition. Hence $infA \leq supA$.

**Remark 3.3.** If $y$ is a real number which is not an upper bound of $A$ that implies there exists an element $r_y \in A$ such that $y < r_y$.
If $z$ is a real number which is not a lower-bound of $A$ that implies there exists an element $s_z \in A$ such that $s_z < z$.

**Remark 3.4. For finite sets** *For finite sets in $\mathbb{R}$ Supremum is the same as maximum and Infimum is same as the minimum. (Use the definition of Supremum and Infimum to ensure that you are correct).*

In the following two examples, we will derive the Supremum and Infimum of a set by using the definitions.

**Example 3.4.** *If $a, b \in \mathbb{R}$ and $a < b$,then*

$$sup(a, b) = b.$$

*Let us take $A = (a, b)$. Obviously, $b$ is an upper bound of $A$ (because $x < b$ for all $x \in (a, b)$.*
*Required to prove that $b$ is the supremum or lub of $A$.*
*Suppose $u_0$ is an upper bound of $A$ (implies $a < x \leq u_0$ for all $x \in (a, b) = A$). Then we need to prove $b \leq u_0$.*
*Because if $u_0 < b$ ($b - u_0 > 0$) then we have $a < u_0 < u_0 + \frac{(b-u_0)}{2} = \frac{(u_0+b)}{2} < b$. So we get an element $u_0 + \frac{(b-u_0)}{2} \in A$ which is greater than the upper bound $u_0$ which is a contradiction. So we can not have $u_0 < b$, so $b \leq u_0$. Hence $b$ is the lub or supremum of $(a, b)$.*
*Similarly, the infimum.*

**Remark 3.5. When Supremum and Infimum belong to the Set:** *If $M$ is an upper bound of the set $A$ and $M \in A$, then $M$ is the lub or supremum of $A$ (because if $u$ is any upper bound of $A$, then $x \leq u$ for all $x \in A$ this implies $M \leq u$ as $M \in A$, hence by definition of lub or supremum $M$ is the lub or supremum of $A$).*
*If $m$ is a lower bound of the set $A$ and $m \in A$, then $m$ is the glb or infimum of $A$ (similar to the previous case).*
*And in this case, also Supremum is the same as the Maximum and Infimum is the same as the Minimum.*

**Example 3.5.** *For example take $[a, b]$. 1 is an upper bound and $b \in [0, 1]$. So $1b$ is the supremum as well as maximum. Similarly $a$ is the infimum as well as minimum.*

**Example 3.6.** *Consider the set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$. (a) Is $A$ bounded from above? If it is bounded from above then the supremum? Is this supremum a maximum of $A$? (b) Is $A$ is bounded from below. Find the infimum. Is this infimum a minimum of $A$?*

*Proof.* (a) Clearly, $\frac{1}{2}$ is an upper bound of $A$ as $\frac{(-1)^n}{n} \leq \frac{1}{2}$ for all $n \in \mathbb{N}$. And $\frac{1}{2} \in A$(take $n = 2$). By Remark 3.5 hence $\frac{1}{2}$ is the Supremum. Since the supremum is an element of $A$ we conclude that $\frac{1}{2}$ is also the maximum of $A$.
(b) Clearly, $-1$ is a lower bound of $A$. And $-1 \in A$(take $n = 1$)

By Remark 3.5 hence $-1$ is the Infimum. Since $-1$ is in $A$, it is the minimum of $A$. □

**Remark 3.6.** Note that, unlike $maxA$ and $minA$, $supA$ and $infA$ need not belong to $A$. For example, 1 is the l.u.b of the sets $\{x : 0 < x < 1\}$,

**Remark 3.7.** A set may not have Supremum and Infimum. They are called unbounded sets. Here are some examples:

**Example 3.7.** *The set of natural numbers $\mathbb{N} = \{1, 2, \cdots\}$ is bounded below 1 is the infimum as well as minimum but it is not a bounded above subset of $\mathbb{R}$. (why it is not bounded above? because if it is bounded above then there would be an upper bound say $a$ that means $n \leq a$ for all $n \in \mathbb{N}$ but $a < [a] + 1$ ($[x]$ is the floor function of $x$) and $[a] + 1$ is in $\mathbb{N}$. So upper bound $a$ is smaller that a natural number $[a] + 1$. Hence a contradiction.*

**Example 3.8.** *For example the set $C = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ isn't bounded above or below (so it has no maximum or minimum)(Prove it H.W). Proof similar to the above.*

**Theorem 3.1. Equivalent criterion for Supremum or lub** *Let $A$ be a non empty subset of $\mathbb{R}$ which is bounded from above. The upper bound $u$ is said to be the Supremum or lub of $A$ iff for every $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $u - \varepsilon < x_\varepsilon \leq u$.*

*Proof.* Given $A$ be a non empty subset of $\mathbb{R}$ which is bounded above.
Let $u$ is the the Supremum or least upper bound of $A$. As $u - \varepsilon < u$. So $u - \varepsilon$ is not an upper bound of $A$ so there must exist an $x_\varepsilon$ in $A$ such that $u - \varepsilon < x_\varepsilon$. And $x_\varepsilon \leq u$ by definition.
Conversely suppose $u$ is an upper bound of $A$ with the condition: for every $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $u - \varepsilon < x_\varepsilon \leq u$. Required to prove $u$ is the supremum of $A$. Suppose $u$ is not the the supremum of $A$, then there must exist an upper bound $v$ of $A$ such that $v < u$. Choose $\varepsilon = u - v$. then by the assumption, for this choice of $\varepsilon$, there exist an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $v = u - \varepsilon < x_\varepsilon$. So $v$ can not be the upper bound of $A$ which is a contradiction to the fact that $v$ is an upper of $A$. Hence our assumption $u$ is not the the supremum of $A$ is wrong. So $u$ has to be the supremum. □

Similarly

**Corollary 3.2. Equivalent criterion for Infimum or glb** *Let $A$ be a non empty subset of $\mathbb{R}$ which is bounded from below. The lower bound $l$ is said to be the glb or Infimum of $A$ iff for every $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $l \leq x_\varepsilon < l + \varepsilon$.*

*Proof.* Proof is similar to the above. □

**Remark 3.8. (depending on $\varepsilon$) meaning** Consider the set $S = \{1, 1.5, 1.7, 1.8, 1.9, 2\}$ so $u = 2$ is the supremum. Now if I choose $\varepsilon = 0.11$, then $u - \varepsilon = u - 0.11 = 2 - 0.11 = 1.89$. Now 2 and 1.9 are in $S$ so my $x_\varepsilon = 2$ or 1.9 (as $u - \varepsilon = 1.89 < 1.9, 2 \leq 2$) that is it satisfies $u - \varepsilon < x_\varepsilon \leq u$. Now if I choose $\varepsilon = 0.1$, then $u - \varepsilon = u - 0.1 = 2 - 0.1 = 1.9$. Now 2 is in $S$ so my $x_\varepsilon = 2$ (as $u - \varepsilon = 1.9 < 2 \leq 2$) that is it satisfies $u - \varepsilon < x_\varepsilon \leq u$.

**Remark 3.9.** One can use Theorem 3.1 or 3.2 to derive the Supremum or Infimum of a set respectively. Let us discuss this the following Example.

**Example 3.9.** *Using Theorem 3.1 and 3.2, one can also prove if $a, b \in \mathbb{R}$ and $a < b$,then*

$$sup[a,b] = sup(a,b) = sup[a,b) = sup(a,b] = b$$

*. Why? Obviously, $b$ is an upper bound. Let $\varepsilon > 0$ be any arbitarily small real number (such that $a < b - \varepsilon$), then $b - \varepsilon < b_\varepsilon = b - \frac{\varepsilon}{2} < b$. Then $a < b_\varepsilon < b$ or $b_\varepsilon \in (a,b)$ is the desired element of the set. Hence Theorem 3.1 holds, proving $b$ is the supremum or lub.*

**Remark 3.10.** One can use Theorem 3.1 to deduce that if a set $A$ has $u$ as its supremum, then for every $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $u - \varepsilon < x_\varepsilon \leq u$. Similarly one can use Theorem 3.2 to deduce that if a set $A$ has $l$ as its infimum, then for every $\varepsilon > 0$, there exists an $x_\varepsilon \in A$ (depending on $\varepsilon$) such that $l \leq x_\varepsilon < l + \varepsilon$. Let us discuss this with the following Example.

**Example 3.10.** *For finite set say $D = \{2, 4, 8, \cdots, 2^{100}\}$. 2 is the minimum as well as infimum and $2^{100}$ is the maximum as well as the supremum. Then also Theorem 3.1 and Theorem 3.2 hold. As $2 \leq 2 < 2 + \varepsilon$ and $2 \in D$. Similarly for the supremum.*

## 4. **Completeness Axiom and Archimedean property:**

The Completeness Axiom or Least Upper Bound Property/ Greatest lower bound property, is one of the fundamental properties of the real numbers. The remaining properties are consequences of the Completeness Axiom, and you know how to deduce them from the Completeness Axiom. These properties may seem obvious (and you can use them without further justification when doing epsilon proofs), but they are closely tied to the real numbers, and there exist domains in which the properties fail.
Let us consider the set $A = \{x \in \mathbb{Q}; 0 < x < 1\}$. Obviously, $A \subset (0,1)$. 1 is an upper bound for $(0,1)$ as well as $A$. And we have already proved 1 is the supremum of $(0,1)$. But what is the guarantee that $SupA$ exists and if it exists what it is? To give the answer we need the following axiom.

**Completeness Axiom:** Every nonempty subset $A$ of $\mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$. In other words, Sup $A$ exists and is a real number. Similarly, every nonempty subset $A$ of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, Inf $A$ exists and is a real number. (Assuming the first one, we can prove the second or assuming the second we can prove the first (see for reference Ross). By completeness property we mean either l.u.b. property or g.l.b. property.

Assuming Completeness Property of $\mathbb{R}$, one can prove Archimedean property.

**Theorem 4.1. Archimedean property:** *If $x, y \in \mathbb{R}$ and $x > 0$, then there is an $n_0 \in \mathbb{N}$ such that $n_0 x > y$.*

*Proof.* Suppose that $nx \leq y$ for every $n \in \mathbb{N}$. Then $y$ is an upper bound of the set $A = \{nx : \ n \in \mathbb{N}\}$ (since $1.x \leq nx \leq y$, so $x \in A$ which implies $A$ is non empty). By the least upper bound property or Completeness Axiom, let $\alpha$ be a l.u.b. of $A$. If we choose $\varepsilon = x$ then by Theorem 3.1, there exists $n_0 x \in A$ such that $\alpha - x < m_0 x \leq \alpha$ whic implies $\alpha < (m_0 + 1)x$. But $(m_0 + 1)x \in A$ which is greater than the upper bound $\alpha$ of $A$. This is a contradiction. So our assumption $nx \leq y$ for every $n \in \mathbb{N}$ is wrong. So there exists an $n_0 \in \mathbb{N}$ such that $n_0 x > y$. $\square$

**Remark 4.1.** This result has been attributed to the great Greek mathematician (born in Syracuse in Sicily) Archimedes (287BC to 212BC) and appears in Book V of The Elements of Euclid (325BC to 265BC).

The following examples are the application of Archimedean property.

**Example 4.1.** *Let $S = \{\frac{1}{n} : \ n \in \mathbb{N}\}$. Then $inf S = 0$ and $Sup S = 1$.*

*$S$ is bounded below since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$ that means $0$ is a lower bound of $S$. Let us denote $\alpha = Inf S$. Since $0$ is a lower bound of $S$ and $\alpha$ is the greatest lower bound of $S$, we have $\alpha \geq 0$. Required to prove $\alpha = 0$.*
*Suppose $\alpha > 0$. Since $\alpha$ is the Infimum (lower bound) of $S$ and $\frac{1}{N} \in S$, we have*

$$\alpha \leq \frac{1}{N}. \tag{6}$$

*Now by Theorem 4.1, (with $x = \alpha$ and $y = 1$) there exists an $N \in \mathbb{N}$ such that $1 < \alpha N$. This implies*

$$\frac{1}{N} < \alpha. \tag{7}$$

*So (7) contradicts (6). So our assumption $\alpha > 0$ is wrong. So $\alpha = 0$.*

**Example 4.2.** *If $t > 0$ be a real number, then there exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.*

*Since $inf\{\frac{1}{n} : \ n \in \mathbb{N}\} = 0$ then $t$ is not a lower bound for the set $inf\{\frac{1}{n} : \ n \in \mathbb{N}\}$. Thus there exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.*

**Remark 4.2.** By slightly modifying the preceding argument, the reader can show that if $a > 0$, then there is a unique $b > 0$ such that $b^2 = a$. We call $b$ the positive square root of $a$ and denote it by $b = \sqrt{a}$ or $b = a^{\frac{1}{2}}$. A slightly more complicated argument involving the binomial theorem can be formulated to establish the existence of a unique positive $n$th root of $a$, denoted by $a^{\frac{1}{n}}$, for each $n \in \mathbb{N}$.

**Example 4.3.** *Let $S = \{\frac{2^n-1}{2^n} : n \in \mathbb{N}\}$. Show that sup $S = 1$ and inf $S = \frac{1}{2}$.*
*For all $n \in \mathbb{N}$,we trivially have $2^n - 1 \leq 2^n$ and so $\frac{2^n-1}{2^n} \leq 1$ for all $n \in \mathbb{N}$. Thus 1 is an upper bound for $S$.*
*Let $\varepsilon > 0$, be given. By the Archimedean Property we can find an $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ (Take $y = 1$ and $x = \varepsilon$ in Theorem 4.1). As $n < 2^n$ for all $n \geq 1$(a proof can be given by induction) we find that*

$$\frac{1}{2^{n_0}} < \frac{1}{n_0} < \varepsilon$$
$$-\varepsilon < -\frac{1}{2^{n_0}}$$
$$1 - \varepsilon < 1 - \frac{1}{2^{n_0}}$$
$$1 - \varepsilon < \frac{2^{n_0} - 1}{2^{n_0}}$$

*That is, we have found an element $\frac{2^{n_0}-1}{2^{n_0}}$ of the set $A$ greater than $1 - \varepsilon$. So by Theorem 3.1, sup$S = 1$.*

*As $\frac{1}{2} \in S$ (for $n = 1$) and $\frac{1}{2}$ is a lower bound of $S$, so by Remark 3.5, inf $S = \frac{1}{2}$.*

Finally, we have the following density theorem.

4.1. **Density Property.** Another application of Theorem 4.1 is the Density property of Real numbers:

**Theorem 4.2. Density Theorem** *Let $a, b$ are real numbers such that $a < b$. Then there exists a rational number $\beta$ such that $a < \beta < b$.*

*Proof.* Without loss of generality, assume that $a > 0$. Now by Theorem 4.1, (with $x = b - a$ and $y = 1$) $n_0 \in \mathbb{N}$ be such that $n_0(b - a) > 1$. Now consider the set

$$S = \{m \in \mathbb{N} : \quad \frac{m}{n_0} > a\}.$$

Then $S$ is non-empty by Archimedean property (with $x = \frac{1}{n_0}$ and $y = a$). By well-ordering of $\mathbb{N}$, $S$ has a minimal element say $m_0$. Then $a < \frac{m_0}{n_0}$ . By the minimality of $m_0$, we see that $\frac{m_0-1}{n_0} \leq a$. Then,

$$\frac{m_0}{n_0} < a + \frac{1}{n_0} \leq a + (b-a) = b$$

Therefore,

$$a < \frac{m_0}{n_0} < b.$$

So $\beta = \frac{m_0}{n_0}$ and we are done. □

**Corollary 4.3.** *Between any two distinct real numbers there is an irrational number.*

*Proof.* Suppose $x, y \geq 0$, and $y > x > 0$. Then $\frac{1}{\sqrt{2}}x < \frac{1}{\sqrt{2}}y$. By Theorem 4.2, ( with $a = \frac{1}{\sqrt{2}}x$ and $b = \frac{1}{\sqrt{2}}y$) there exists a rational number $r$ such that $\frac{1}{\sqrt{2}}x < r < \frac{1}{\sqrt{2}}y$ which implies $x < r\sqrt{2} < y$. As $r\sqrt{2}$ is irrational, so we are done.

□

Using this **Density Property Directly**, One can derive Supremum and Infimum.

**Example 4.4.** *Consider the set $S = \{x \in \mathbb{Q} : \ x^2 < 5\}$. Note that $S = \{x \ \in \mathbb{Q} : -\sqrt{5} < x < \sqrt{5}\}$. So $\sqrt{5}$ is an upper bound of $S$.*
*Required to prove $\sqrt{5}$ is the Supremum of $S$.*
*Proof by Contradiction: Let us suppose that $\sqrt{5}$ is not the Supremum of $S$. That implies there exists a real number $M$ such that $M$ is an upper bound of $S$ and $M < \sqrt{5}$. Then there exists a rational number $r$ such that $M < r < \sqrt{5}$ (Apply Theorem 4.2 with $a = M$ and $b = \sqrt{5}$). Then $r \in S$ and $M < r$. But this contradicts the fact that $M$ is an upper bound of $S$. Thus, sup $S = \sqrt{5}$. Likewise one can show that $inf\ S = -\sqrt{5}$.*
**It follows that the set $S$ that consists of rational numbers, does not have a supremum or infimum belonging to the set $\mathbb{Q}$, that is $supD, infD \notin D$. Thus the ordered field $\mathbb{Q}$ of rational numbers does not possess the Completeness Property.**

Some more Examples: (How to derive Supremum/ Infimum)

**Example 4.5.** *The set $D = \{x \in \mathbb{R} : x^2 < 10\}$ is the open interval $(-\sqrt{10}, \sqrt{10})$ (Now we know there exists a real number $a > 0$ such that $a^2 = 10$ and that number is denoted by $\sqrt{10}$). Thus it is bounded above and below. However, $inf(D) = -\sqrt{10}$ and $sup(D) = \sqrt{10}$. Because if for any $\varepsilon > 0$, there exists $\sqrt{10} - \frac{\varepsilon}{2} \in D$ such* *that* $\sqrt{10} - \varepsilon < \sqrt{10} - \frac{\varepsilon}{2} < \sqrt{10}$ *which proves that* $sup(D) = \sqrt{10}$ *by Theorem 3.1. Similarly, for infimum we can prove this. But it has no maximum or minimum as* $-\sqrt{10}, \sqrt{10} \notin D$.

**Example 4.6.** *The set* $B = \{r \in \mathbb{Q} : r^3 \leq 7\}$ *is bounded above and* $A$ *is non-empty as* $1 \in A$. $supA = 7^{\frac{1}{3}}$ *but* $7^{\frac{1}{3}} \notin A$. *But* $7^{\frac{1}{3}}$ *is the supremum, because for any* $\varepsilon > 0$, *we can find a rational number* $r_0 \in \mathbb{Q}$ *such that* $7^{\frac{1}{3}} - \varepsilon < r_0 < 7^{\frac{1}{3}}$ *(why? use Theorem 4.2 with* $a = 7^{\frac{1}{3}} - \varepsilon$ *and* $b = 7^{\frac{1}{3}}$*). Of course* $r_0 \in B$ *as* $r_0^3 < 7$ *and* $r_0 \in \mathbb{Q}$. *As* $\varepsilon > 0$ *is arbitary, so using Theorem 3.1, we can conclude* $7^{\frac{1}{3}}$ *is the supremum of* $B$.