Series

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1 Introduction

Now we will investigate what may happen when we add all terms of a sequence together to form what will be called an infinite series. The old Greeks already wondered about this, and actually did not have the tools to quite understand it. This is illustrated by the old tale of Achilles and the Tortoise.

1.1 Zeno's Paradox (Achilles and the Tortoise)

Achilles, a fast runner, was asked to race against a tortoise. Achilles can run 10 yards per second, the tortoise only 5 yards per second. The track is 100 yards long. Achilles, being a fair sportsman, gives the tortoise 10 yard advantage. Who will win? Both start running, with the tortoise being 10 yards ahead. After one second, Achilles has reached the spot where the tortoise started. The tortoise, in turn, has run 5 yards. Achilles runs again and reaches the spot the tortoise has just been. The tortoise, in turn, has run 2.5 yards. Achilles runs again to the spot where the tortoise has just been. The tortoise, in turn, has run another 1.25 yards ahead.

This continuous for a while, but whenever Achilles manages to reach the spot where the tortoise has just been a second ago, the tortoise has again covered a little bit of distance, and is still ahead of Achilles. Hence, as hard as he tries, Achilles only manages to cut the remaining distance in half each time, implying, of course, that Achilles can actually never reach the tortoise. So, the tortoise wins the race, which does not make Achilles very happy at all. What is wrong with this line of thinking? Let us look at the difference between Achilles and the tortoise:

Time	Difference
t = 0	10 yards
t = 1	5 = 10/2 yards
t = 1 + 1/2	2.5 =10/4 yards
t = 1 + 1/2+1/4	1.25 = 10/8 yards
t = 1 + 1/2 + 1/4 +1/8	0.625 = 10/16 yards

and so on. In general we have:

Time	Difference
$t = 1 + 1/2 + 1/4 + 1/8 + ...1/2^n$	$10/2^n$ yards

Now we want to take the limit as n goes to infinity to find out when the distance between Achilles and the tortoise is zero. But that involves adding infinitely many numbers in the above expression for the time, and we (the Greeks and Zeno) don't know how to do that. However, if we define

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... \frac{1}{2^n}$$

then, dividing by 2 and subtracting the two expressions:

$$S_n - \frac{1}{2} S_n = \left(1 - \frac{1}{2^n}\right)$$

or equivalently, solving for s_n:

$$S_n = 2\left(1 - \frac{1}{2^n}\right)$$

Now $\{S_n\}$ is a simple sequence, for which we know how to take limits. In fact, from the last expression it is clear that $\lim S_n = 2$ as n approaches infinity. Hence, we have - mathematically correctly - computed that Achilles reaches the tortoise after exactly 2 seconds, and then, of course passes it and wins the race. A much simpler calculation not involving infinitely many numbers gives the same result:

- Achilles runs 10 yards per second, so he covers 20 yards in 2 seconds.
- The tortoise runs 5 yards per second, and has an advantage of 10 yards. So, it also reaches the 20 yard mark after 2 seconds.
- Therefore, both are even after 2 seconds.

Of course, Achilles will finish the race after 10 seconds, while the tortoise needs 18 seconds to finish, and Achilles will clearly win. The problem with Zeno's paradox is that Zeno was uncomfortable with adding infinitely many numbers. In fact, his basic argument was that if you add infinitely many numbers, then — no matter what those numbers are — you must get infinity. If that was true, it would take Achilles infinitely long to reach the tortoise, and he would loose the race. However, reducing the infinite addition to the limit of a sequence, we have seen that this argument is false. One reason for looking so carefully at sequences is that it allows us to to quickly obtain the properties of infinite series.

We know (at least theoretically) how to deal with finite sums of real numbers.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$$

More interest in mathematics though tends to lie in the area of infinite series $\sum_{k=1}^{\infty} a_k$:

2 Infinite Series

Definition 2.1. Let be $\{a_n\}$ be a sequence of real numbers. An expression of the form

$$a_1 + a_2 + ... + a_n + ...$$

is called an infinite series. It is often convenient to denote the above infinite sum as

$$\sum_{n=1}^{\infty} a_n$$

1. The number a_n is called as the n-th term of the series.
2. The sequence $\{S_n\}$, defined by $S_n = \sum_{k=1}^{n} a_k$ is called the sequence of partial sums of the series.
3. If the sequence of partial sums $\{S_n\}$ **converges to a limit** L**, we say that the series** $\sum_{n=1}^{\infty} a_n$ **converges and its sum is** L**.**
4. If the sequence of partial sums $\{S_n\}$ does not converge, we say that the series diverges.
5. We say that **a series** $\sum_{n=1}^{\infty} a_n$ **satisfies Cauchy criterion if its sequence of partial sums** $\{S_n\}$ **is a Cauchy sequence.**
6. We say $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\{S_n\}$ is monotonically increasing.

Example 2.1. *The series converges as* $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{(k+1)} = 1 - \frac{1}{n+1} \to 1$ *as* $n \to \infty$*.*

Example 2.2. *The series* $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ *diverges as* $S_n = \log(n+1) \to \infty$ *as* $n \to \infty$*.*

Example 2.3. *The series* $\sum_{n=1}^{\infty} 1$ *diverges as the sequence of partial sums* $S_n = \sum_{k=1}^{\infty} 1 = n$ *diverges.*

Example 2.4. *The series* $\sum_{n=1}^{\infty} n$ *diverges as the sequence of partial sums* $S_n = \sum_{k=1}^{\infty} k = \frac{n(n+1)}{2}$ *diverges.*

Remark 2.1. If $a_n \geq 0$ for all $n \geq n_0$, then the sequence $\{S_n\}$ is monotonically increasing.

Theorem 2.1. Cauchy Criterion for Convergence *A series converges iff it satisfies Cauchy criterion.*

Proof. A series converges iff the sequence of partial sums converges i.e iff the sequence of partial sums is a Cauchy sequence. $\square$

Example 2.5. *A series of the form* $\sum_{n=0}^{\infty} ar^n$ *for constants* a *and* r *is called a geometric series. For* $r \neq 1$ *the partial sums are given by*

$$\sum_{n=0}^{N} ar^n = a\frac{1 - r^{N+1}}{1 - r}$$

Taking the limit as N *goes to infinity, gives us that*

$$= \sum_{n=0}^{\infty} ar^n \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ +\infty & \text{if } a \neq 0 \text{ and } |r| > 1 \end{cases}$$

Example 2.6. $\sum_{n=1}^{\infty} 1/n$ *diverges.*
Consider the sequence of partial sums $\{S_n\}$*, where* $\{S_n\} = \sum_{k=1}^{n} \frac{1}{k}$*. This series diverges to* $+\infty$*. To prove this we need to estimate the* n*th term in the sequence of partial sums.*

$$S_1 = 1,\ S_2 = \frac{3}{2},\ S_3 = \frac{11}{6},\ S_4 = \frac{25}{12}, \ldots\ldots..$$

Now consider the following subsequence extracted from the sequence of partial sums:

$$\begin{aligned}
&S_1 = 1, \\
&S_2 = \frac{3}{2} \\
&S_4 = 1 + 1/2 + 1/3 + 1/4 > 1 + 1/2 + (1/4 + 1/4) = 1 + 1/2 + 1/2 = 2 \\
&S_8 = 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) \\
&\quad > 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) > 1 + 1/2 + 1/2 + 1/2 = 1 + 3/2
\end{aligned}$$

In general, by induction we have that that

$$S_{2^k} > 1 + k/2$$

for every $k \geq 1$*. Hence, the subsequence* $\{S_{2^k}\}$ *extracted from the sequence of partial sums* $\{S_n\}$ *is unbounded. But then the sequence* $\{S_n\}$ *cannot converge either, and must, in fact, diverge to infinity.*

Example 2.7. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*
$S_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *is monotonically increasing sequence (as all the terms are positive). If we can show that there exists a subsequence of* $\{S_n\}$ *which is bounded (which means it is convergent) then we are done. Because by Example 4.6 of the previous note "Sequence" that will imply* $\{S_n\}$ *is convergent.*
Consider the sequence

$$y_n = S_{2^n - 1}.$$

By induction, we will prove that $y_n \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$*.*
For $n = 1$*,* $y_1 = 1$ *so it is true for* $n = 1$*.*
Suppose it is true for $n = k$*. Required to prove it is true for* $n = k + 1$

$$\begin{aligned}
S_{2^{k+1}-1} &= S_{2^k - 1} + \frac{1}{(2^k)^2} + \cdots + \frac{1}{(2^{k+1} - 1)^2} \\
&<< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}} + \cdots + \frac{1}{(2^{k+1} - 1)^2}
\end{aligned}$$

And we know

$$\frac{1}{(2^k)^2}+\cdots+\frac{1}{(2^{k+1}-1)^2}\leq\frac{1}{(2^k)^2}+\cdots+\frac{1}{(2^k)^2}=\frac{2^{k+1}-2^k}{(2^k)^2}=\frac{1}{2^k}$$

Hence $y_{k+1}=S_{2^{k+1}-1}\leq 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{k-1}}+\frac{1}{2^k}$

Hence by induction, $y_n\leq 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$ for all $n\in\mathbb{N}$
So

$$y_n=2(1-\frac{1}{2^n})\leq 2\quad\forall\ n\in\mathbb{N}$$

So $\{y_n\}$ is bounded and it is monotonically increasing. Hence it converges. As $\{y_n\}$ is a subsequence of $\{S_n\}$ (and $\{S_n\}$ is monotonically increasing) so $\{S_n\}$ converges by Example 4.6 of previous chapter.

2.1 k-tail of a series

Let $\sum_{n=1}^{\infty}a_n$ be an infinite series and $k\in\mathbb{N}$. Then the series $\sum_{n=k}^{\infty}a_n$ is k-tail of the series $\sum_{n=1}^{\infty}a_n$.

Theorem 2.2. *Let $\sum_{n=1}^{\infty}a_n$ be an infinite series. Denote by $A_k:=\sum_{n=k}^{\infty}a_n$. Then the series $\sum_{n=1}^{\infty}a_n$ converges iff sequence of k-tails $\{A_k\}$ goes to 0 as $k\to\infty$ i.e $\lim_{k\to\infty}A_k=0$.*

Proof. Let $\{A_k\}$ converges to 0 i.e $\lim_{k\to\infty}A_k=0$, so $\{A_k\}$ is a Cauchy sequence.. Required to prove $\{S_n\}$ is a Cauchy sequence. For any $m>k$

$$|A_m-A_k|=|\sum_{j=k+1}^{m}a_j|=|S_m-S_k|$$

The l.h.s is a Cauchy sequence, so r.h.s is a Cauchy sequence. Now $\{S_n\}$ is a Cauchy sequence implies it converges.

Conversely, let $S_n=\sum_{j=1}^{n}a_j$, then $\sum_{n=1}^{\infty}a_n=S_{k-1}+A_k$. Then As $\sum_{n=1}^{\infty}a_n=$ converges. Let $\sum_{n=1}^{\infty}a_n=L$. Also $S_{k-1}\to L$ as k goes to ∞ or $\lim_{k\to\infty}S_{k-1}=0$. Hence $\lim_{k\to\infty}A_k=0$. □

2.2 Algebra of Series

Lemma 2.3. *I) If $\sum_{n=1}^{\infty}a_n$ converges to L and $\sum_{n=1}^{\infty}b_n$ converges to M, then $\sum_{n=1}^{\infty}(a_n\pm b_n)$ converges to $L\pm M$.*
II) If $\sum_{n=1}^{\infty}a_n$ converges to L and $c\in\mathbb{R}$, then the series $\sum_{n=1}^{\infty}ca_n$ converges to cL.

Remark 2.2. The converse of the first one is not true that is $\sum_{n=1}^{\infty}(a_n+b_n)$ converges does not imply $\sum_{n=1}^{\infty}a_n$ converges and $\sum_{n=1}^{\infty}b_n$ converges. Take $a_n=1$ and $b_n=-1$. Then $(a_n+b_n)=0$ so $\sum_{n=1}^{\infty}(a_n+b_n)=0$.
What about the product??

Lemma 2.4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n = L$. Then the sequence of partial sums. $\{S_n\}$ also converges to L. Now

$$a_n = (S_n - S_{n-1}) \to (L - L) = 0 \ as \ n \to \infty.$$

Hence the proof.

Or Given $\varepsilon > 0$ arbitrarily fixed. As $\{S_n\}$ converges it is a Cauchy sequence. For this given $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$|S_n - S_m| < \varepsilon \ \ \forall \ m, n \geq N_\varepsilon$$

Take $m = n + 1$, then

$$|a_{n+1}| = |S_{n+1} - S_n| < \varepsilon \ \forall \ n \geq N_\varepsilon$$

As $\varepsilon > 0$ is arbitrary then the above implies $\lim_{n\to\infty} a_n = 0$. (As $\lim_{n\to\infty} |x_n| = 0$ implies $\lim_{n\to\infty} x_n = 0$. Remember!) □

Example 2.8. *Let us consider some examples:*

1. $\sum_{n=1}^{\infty} \sin n$ *diverges because* $\lim_{n\to\infty} \sin n \neq 0$.
2. *If* $|x| \geq 1$ *then* $\sum_{n=1}^{\infty} x^n$ *diverges as* $\lim_{n\to\infty} x^n \neq 0$.

Remark 2.3. But the converse of Lemma 2.4 is not true as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, although $\lim_{n\to\infty} \frac{1}{n} = 0$.

2.3 Absolute Convergence of a Series

If the terms $\{a_n\}$ of an infinite series $\sum_{n=1}^{\infty} a_n$ are all non negative, then the partial sums $\{S_n\}$ form a non decreasing sequence, So by Theorem of monotone sequence, $\{S_n\}$ either converges or diverges to ∞. In particular, $\sum_{n=1}^{\infty} |a_n|$ is meaningful for any sequences $\{a_n\}$ whatsoever.

Definition 2.2. Absolute Convergence The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 2.9. *The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ *converge absolutely.*

Example 2.10. *The series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *converge absolutely,*

Theorem 2.5. $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} |a_k|$. As the $\sum_{n=1}^{\infty} |a_n|$ converges so $\{T_n\}$ is a Cauchy sequence. for a Given $\varepsilon > 0$ there exists a $n \in \mathbb{N}$ such that

$$|T_m - T_n| < \varepsilon \ \ \forall \ \ m \geq n \geq N$$

So we have for all $m \geq n \geq N$

$$|S_m - S_n| = |\sum_{k=n+1}^{m} a_k| \leq \sum_{k=n+1}^{m} |a_k| = |T_m - T_n| < \varepsilon.$$

Hence the proof. $\square$

Remark 2.4. The Converse of the above theorem is not true. For example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} |\frac{(-1)^{n+1}}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Remark 2.5. For a series of non negative terms convergence is same as absolute convergence. For example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (absolutely).

Remark 2.6. To see whether a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (that is $\sum_{n=1}^{\infty} |a_n|$ converges): If we are given that the sequence of partial sums $T_n = \sum_{k=1}^{n} |a_k|$ is bounded above then we are done. Because $\{T_n\}$ is obviously monotonically increasing sequence ($T_{n+1} = \sum_{k=1}^{n+1} |a_k| = T_n + |a_{n+1}|$ hence $T_{n+1} - T_n = |a_{n+1}| \geq 0$ for all $n \in \mathbb{N}$). And we all know a monotonically increasing sequence bounded above converges.
Or you can show that there exists a convergent subsequence of $\{T_n\}$, then apply the result of Example 4.6 of the previous chapter.

3 Conditional Convergence

Definition 3.1. Conditional Convergence A series converges conditionally, if it converges, but not absolutely.

Example 3.1. *Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge absolutely, conditionally, or not at all (this series is called alternating harmonic series)?*

Definition 3.2. Conditional Convergence A series converges conditionally, if it converges, but not absolutely.

Theorem 3.1. Algebra on Absolutely Convergent Series:
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series. Then
(i) The sum $\sum_{n=1}^{\infty} (a_n + b_n)$ is absolutely convergent.
(ii) The difference $\sum_{n=1}^{\infty} (a_n - b_n)$ of the two series is again absolutely convergent. (iii) The product of the two series is again absolutely convergent. Its limit is the product of the limit of the two series.

4 Convergence Tests

Theorem 4.1. (Comparison Test) *Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \geq 0$ for all $n \geq N_0$.*
(i) If $\sum_{n=1}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all $n \geq N_1$, then $\sum_{n=1}^{\infty} |b_n|$ (also $\sum_{n=1}^{\infty} b_n$) converges.
(ii) If $\sum_{n=1}^{\infty} a_n = +\infty$ and $c_n \geq a_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} c_n$ diverges.

Proof. (i)Given $\varepsilon > 0$ arbitrarily fixed. Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$ and $T_n' = \sum_{k=1}^{n} |b_k|$. For $m \geq n \geq N_2 = max(N_0, N_1)$ we have

(1)
$$|T_m - T_n| = |\sum_{k=n+1}^{m} b_k| \leq \sum_{k=n+1}^{m} |b_k| = |T_m' - T_n'| \leq \sum_{k=n+1}^{m} a_k = S_m - S_n = |S_m - S_n|$$

where the first inequality follows from the Triangle Inequality. Since $\sum_{n=1}^{\infty} a_n$ converges, it satisfies the Cauchy criterion. For the given $\varepsilon > 0$, there exists a natural number $N_\varepsilon \in \mathbb{N}$ such that

(2)
$$|S_m - S_n| < \varepsilon \quad \forall\ m, n \geq N_\varepsilon$$

We choose $N = max(N_2, N_\varepsilon)$, then combining (1) and (2), we have

(3)
$$|T_m - T_n| \leq |T_m' - T_n'| \leq |S_m - S_n| < \varepsilon \quad \forall\ m \geq n \geq N$$

As $\varepsilon > 0$ is arbitary. It then follows from the above that the series $\sum_{n=1}^{\infty} |b_n|$ as well as $\sum_{n=1}^{\infty} b_n$ also satisfies the Cauchy criterion and hence it also converges.

(ii) Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} c_k$. Since $c_n \geq a_n$ for all n, we have $T_n \geq S_n$ for all n. So by Corollary 2.7 of the previous chapter, we can conclude that $\lim_{n\to\infty} T_n = +\infty$ as $\lim_{n\to\infty} S_n = +\infty$. So $\sum_{n=1}^{\infty} c_n$ diverges. □

Remark 4.1. Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \geq 0$ for all n.
(i) If $\sum_{n=1}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all $n \geq n_0$, then $\sum_{n=1}^{\infty} b_n$ converges.
(ii) If $\sum_{n=1}^{\infty} a_n = +\infty$ and $c_n \geq a_n$ for all $n \in \geq n_0$, then $\sum_{n=1}^{\infty} c_n$ diverges.

Example 4.1. *By Comparison Test the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Example 4.2. *By Comparison Test the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges as $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges*

Example 4.3. *By Comparison Test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p < 1$. As $n^p < n$ for $0 < p < 1$. Then $\frac{1}{n} < \frac{1}{n^p}$ for all $n \in \mathbb{N}$. As $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.*

Theorem 4.2. (Limit Comparison Test) *Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two infinite series. Suppose also that $r = \lim_{n\to\infty} |\frac{x_n}{y_n}|$ exists, and $0 < r < +\infty$. Then $\sum_{n=1}^{\infty} x_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} y_n$ converges absolutely.*

Proof. Since $r = \lim_{n\to\infty} |\frac{x_n}{y_n}|$ exists, and r is between 0 and $+\infty$. Choose $\varepsilon = \frac{r}{2}$ as $r > 0$, then for this choice of $\varepsilon > 0$, there exists a $N_{r/2} \in \mathbb{N}$ such that

$$\left| |\frac{x_n}{y_n}| - r \right| < \varepsilon \quad \forall\ n \geq N_{r/2}$$

$$0 < \frac{r}{2} = r - \frac{r}{2} < |\frac{x_n}{y_n}| < r + \frac{r}{2} = \frac{3}{2}r \quad \forall\ n \geq N_{r/2} \text{ (Remember: Problem 1 Assignment 1)}$$

There exist constants $c = \frac{r}{2}$ and $C = \frac{3r}{2}$ with $0 < c < C < +\infty$ such that

$$c < \left|\frac{x_n}{y_n}\right| < C \quad \forall\ n \geq N_{r/2}$$

Assume that $\sum_{n=1}^{\infty} x_n$ converges absolutely i.e $\sum_{n=1}^{\infty} |x_n|$ converges. We have that $c|y_n| < |x_n| \quad \forall\ n \geq N_{r/2}$. Therefore, by the Comparison Test $\sum_{n=1}^{\infty} c|y_n| = c\sum_{n=1}^{\infty} |y_n|$ converges. As $c \neq 0$, that will imply $\sum_{n=1}^{\infty} |y_n|$ converges.

Now Assume that $\sum_{n=1}^{\infty} y_n$ converges absolutely i.e $\sum_{n=1}^{\infty} |y_n|$ converges. We have that $C|y_n| > |x_n| \quad \forall\ n \geq N_{r/2}$. As $\sum_{n=1}^{\infty} |y_n|$ converges so by Algebra of limits $\sum_{n=1}^{\infty} C|y_n| = C\sum_{n=1}^{\infty} |y_n|$ converges ($C > 0$). Therefore, by the Comparison Test $\sum_{n=1}^{\infty} |x_n|$ converges.

□

Remark 4.2. If $r = 0$, then $\sum_{n=1}^{\infty} y_n$ converges absolutely implies $\sum_{n=1}^{\infty} x_n$ converges absolutely . But $\sum_{n=1}^{\infty} x_n$ converges absolutely does not mean that $\sum_{n=1}^{\infty} y_n$ converges absolutely. Take $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$, then $r = \lim_{n\to\infty} |\frac{x_n}{y_n}| = \lim_{n\to\infty} \frac{1}{n} = 0$

Example 4.4. *To test the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$*, let*

$$a_n = \frac{1}{2^n - n} \textit{ and } b_n = \frac{1}{2^n}$$

Then

$$\lim_{n\to\infty} |\frac{a_n}{b_n}| = \lim_{n\to\infty} \frac{2^n}{2^n - n} = \lim_{n\to\infty} \frac{1}{1 - \frac{n}{2^n}} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$*, so* $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ *converges (absolutely).*

Example 4.5. *To test the series* $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ *for convergence, let*

$$a_n = \frac{1}{n\sqrt[n]{n}} \textit{ and } b_n = \frac{1}{n}$$

Then $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*. But* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges so we can not conclude anything about the convergence of* $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ *by Comparison test. We will use Limit Comparison test*

$$\lim_{n\to\infty} |\frac{a_n}{b_n}| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ *diverges as* $\sum_{n=1}^{\infty} \frac{1}{n}$ *by Limit Comparison Test.*

Theorem 4.3. (Cauchy Condensation Test) *Suppose $\{a_n\}$ is a decreasing sequence of positive terms. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.*

Proof. Assume that $\sum_{n=1}^{\infty} a_n$ converges. Since $\{a_n\}$ is a decreasing sequence, we have that

$$\begin{aligned} 2^{k-1}a_{2^k} &= a_{2^k} + a_{2^k} + a_{2^k} + \dots\dots + a_{2^k} \\ &\le a_{2^{k-1}+1} + a_{2^{k-1}+2} + \dots\dots a_{2^{k-1}+2^{k-2}} + \dots\dots + a_{2^k} \end{aligned}$$

Therefore, we have that

$$\sum_{k=1}^{N} 2^k a_{2^k} = 2\sum_{k=1}^{N} 2^{k-1}a_{2^k} \le 2\sum_{k=1}^{N}\sum_{j=2^{k-1}+1}^{2^k} a_j = 2\sum_{m=2}^{2^N} a_m.$$

Now the partial sums on the right are bounded, by assumption (since $\{a_n\}$ is a sequence of positive terms). Hence the partial sums on the left are also bounded. Since all terms (of the sequence $\{2^k a_{2^k}\}$) are positive, the partial sums now form an increasing sequence that is bounded above, hence it must converge. Hence the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Now, assume that Now, assume that $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges. Then we have that

$$\begin{aligned} \sum_{j=2^{k-1}+1}^{2^k} a_j &= a_{2^{k-1}+1} + a_{2^{k-1}+2} + \dots\dots + a_{2^{k-1}+2^{k-2}} + a_{2^k} \\ &\le a_{2^{k-1}} + a_{2^{k-1}} + \dots\dots\dots\dots + a_{2^{k-1}} + \dots\dots\dots + a_{2^{k-1}} \\ &= 2^{k-1}a_{2^{k-1}} \end{aligned}$$

Therefore, similar to the above we get

$$\sum_{m=2}^{2^N} a_m = \sum_{k=1}^{N}\sum_{m=2^{k-1}+1}^{2^k} a_m \le \sum_{k=1}^{N} 2^{k-1}a_{2^{k-1}}.$$

Now the sequence of partial sums on the right is bounded, by assumption. There- fore, the left side forms an increasing sequence that is bounded above, and therefore must converge. $\square$

Example 4.6. *(p-series) For fixed $p \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series and converges if and only if $p > 1$. The special case when $p = 1$ is the harmonic series.*

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{2^{kp}} = \sum_{k=1}^{\infty} 2^{k(1-p)}$$

is a geometric series with ratio 2^{1-p} so it converges only when $2^{1-p} < 1$ (see Example 2.1). Since 2^{1-p} only when $p > 1$, it follows from the Cauchy Condensation Test that the p-series converges when $p > 1$ and diverges when $p \le 1$.

Example 4.7. *We will show* $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges if and only if* $p > 1$ *by applying Abel's test.*
Let $a_n = \frac{1}{n(\log n)^p}$ *so* $2^k a_{2^k} = \frac{2^k}{2^k(\log 2^k)^p} = \frac{1}{(k \log 2)^p} = \frac{1}{k^p(\log 2)^p}$ *So by Cauchy Condesation test* $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges if and only if* $\frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges. And by previous example or again by Cauchy Condesation test, the series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *is called a p-series and converges if and only if* $p > 1$.

Theorem 4.4. (Ratio Test) *A series* $\sum_{n=1}^{\infty} a_n$ *of nonzero terms*
(i) converges absolutely if $\alpha = \lim sup|\frac{a_{n+1}}{a_n}| < 1$,
(ii) diverges if $\beta = \lim inf|\frac{a_{n+1}}{a_n}| > 1$.
(iii) Otherwise, $\beta = \lim inf|\frac{a_{n+1}}{a_n}| \leq 1 \leq \lim sup|\frac{a_{n+1}}{a_n}| = \alpha$ *and the test gives no information.*

Proof. (i) Given $\alpha := \lim sup|\frac{a_{n+1}}{a_n}| < 1$. Then choose an $\varepsilon_0 > 0$ so that $\alpha + \varepsilon_0 < 1$. So by (Theorem 7.1 of the previous chapter) there is a natural number N such that

$$|\frac{a_{n+1}}{a_n}| < (\alpha + \varepsilon_0) \quad \forall\, n \geq N$$

Multiplying both sides by $|a_n|$,

$$|a_{n+1}| < |a_n|(\alpha + \varepsilon_0) \quad \forall\, n \geq N$$

In particular we choose $n = N$ to get

$$|a_{N+1}| < |a_N|(\alpha + \varepsilon_0)$$

Therefore we also have

$$|a_{N+2}| < |a_{N+1}|(\alpha + \varepsilon_0) < |a_N|(\alpha + \varepsilon_0)^2$$

Repeating this procedure, we get that

$$|a_k| = |a_{N+(k-N)}| < |a_N|(\alpha + \varepsilon_0)^{k-N} = \frac{|a_N|}{(\alpha + \varepsilon_0)^N}(\alpha + \varepsilon_0)^k \quad \forall k > N$$

Since $\alpha + \varepsilon_0 < 1$, the geometric series $\sum_{n=1}^{\infty}(\alpha + \varepsilon_0)^n$ converges. Thus, the Comparison Test shows that $\sum_{n=1}^{\infty} a_n$ also converges.

The other two parts are proven in much the same fashion as the previous theorem. □

Example 4.8. *Consider* $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. *Then*

$$\alpha = \beta = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n}.$$

Since $(n+1)! = (n+1)n!$, *we have*

$$\alpha = \beta = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

So by Comparison test, the series converges.

Example 4.9. *Consider* $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.
We can see that

$$\alpha = \beta = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \lim_{n\to\infty} (\frac{n+1}{n})^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e > 1.$$

So by Comparison test, the series diverges.

Example 4.10. *Consider* $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$.
Since

$$\left|\frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!}\right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \frac{(2n)!}{n!n!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

We see that

$$\alpha = \beta = \lim_{n\to} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4} < 1.$$

So by Comparison test, the series converges.

Example 4.11. *When* $\beta = \lim inf|\frac{a_{n+1}}{a_n}| = 1 = \lim sup|\frac{a_{n+1}}{a_n}| = \lim|\frac{a_{n+1}}{a_n}| = \alpha$*:*
Take $a_n = \frac{1}{n}$*, then* $\lim inf|\frac{a_{n+1}}{a_n}| = 1 = \lim sup|\frac{a_{n+1}}{a_n}|$ *but* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*
Take $a_n = \frac{1}{n^2}$*, then* $\lim inf|\frac{a_{n+1}}{a_n}| = 1 = \lim sup|\frac{a_{n+1}}{a_n}|$ *but* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Example 4.12. *When* $\beta = \lim inf|\frac{a_{n+1}}{a_n}| = 1 < \lim sup|\frac{a_{n+1}}{a_n}| = \alpha$*: Consider the series* $2+2+8+8+\cdots$

$$a_n = \begin{cases} 2^n & n \textit{ is odd} \\ 2^{n-1} & n \textit{ is even} \end{cases}$$

Then $2 \leq a_n$ *for all* $n \in \mathbb{N}$*. So by Comparison test* $\sum_{n=1}^{\infty} a_n$ *diverges. But* n *is odd* $\frac{a_{n+1}}{a_n} = 1$ *and* n *is even* $\frac{a_{n+1}}{a_n} = 4$*, so* $\beta = \lim inf|\frac{a_{n+1}}{a_n}| = 1 < \lim sup|\frac{a_{n+1}}{a_n}| = 4$.

Example 4.13. *When* $\beta < 1 = \alpha$*: Consider the series* $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \cdots$

$$a_n = \begin{cases} \frac{1}{2^n} & n \textit{ is odd} \\ \frac{1}{2^{n-1}} & n \textit{ is even} \end{cases}$$

Then $0 \leq a_n \frac{2}{2^n}$ *for all* $n \in \mathbb{N}$*. So by Comparison test* $\sum_{n=1}^{\infty} a_n$ *converges. But* n *is odd* $\frac{a_{n+1}}{a_n} = 1$ *and* n *is even* $\frac{a_{n+1}}{a_n} = \frac{1}{4}$*, so* $\beta = \lim inf|\frac{a_{n+1}}{a_n}| = \frac{1}{4} < 1 = \lim sup|\frac{a_{n+1}}{a_n}| = \alpha$.

Example 4.14. *Consider the series* $\sum_{n=1}^{\infty} \frac{n}{2^n}$.
We will use the Ratio-Test. Let $a_n = \frac{n}{2^n}$*. Then*

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{2^{n+1}}.\frac{2^n}{n} = (1+\frac{1}{n})\frac{1}{2} = \frac{1}{2}.$$

Now $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2} < 1$. So (as the limit exists so lim sup is same as lim). Hence by ratio test the series converges.

Example 4.15. *Consider the series* $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.
We will use the Ratio-Test. Let $x_n = \frac{2^n n!}{n^n}$. *Then*

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2\frac{n^n}{(n+1)^n} = 2\frac{1}{(1+\frac{1}{n})^n}$$

Now $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. So

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{2}{e} < 1.$$

(as the limit exists so lim sup is same as lim). Hence by ratio test the series converges.

Theorem 4.5. (Root Test) *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with* $\alpha = \lim sup|a_n|^{1/n}$.
Then the series $\sum_{n=1}^{\infty} a_n$
(i) converges absolutely if $\alpha < 1$.
(ii) diverges if $\alpha > 1$.
(iii) Otherwise $\alpha = 1$ *and the test gives no information.*

Although this Root Test is more difficult to apply, it is better than the Ratio Test in the following sense. There are series for which the Ratio Test give no information, yet the Root Test will be conclusive. We will use the Root Test to prove the Ratio Test, but you cannot use the Ratio Test to prove the Root Test. It is important to remember that when the Root Test gives 1 as the answer for the lim sup, then no conclusion at all is possible.

The use of the lim sup rather than the regular limit has the advantage that we do not have to be concerned with the existence of a limit. On the other hand, if the regular limit exists, it is the same as the lim sup, so that we are not giving up anything using the lim sup.

Proof. (i) Suppose that $\alpha < 1$. Then choose an $\varepsilon_0 > 0$ so that $\alpha + \varepsilon_0 < 1$. Now we can use the definition of the limit superior (Theorem 7.1 or 7.2 of the previous chapter).
(We notice that

$$inf_{j\geq 1}\gamma_j = \lim sup\ |a_n|^{\frac{1}{n}} = \alpha$$

where $\gamma_j := Sup\{|a_j|^{\frac{1}{j}}, |a_{j+1}|^{\frac{1}{j+1}}, \ldots\ldots\ldots.\}$. As $inf_{j\geq 1}\gamma_j = \alpha < \alpha + \varepsilon_0$, then by the property of infimum, there exists $N = N_{\varepsilon_0} \in \mathbb{N}$ (depending on $\varepsilon > 0$) such that

$$\alpha \leq \gamma_N < \alpha + \varepsilon_0$$

which in turn implies

$$\gamma_N = Sup\{|a_N|^{\frac{1}{N}}, |a_{N+1}|^{\frac{1}{N+1}}, \ldots\ldots\ldots.\} < \alpha + \varepsilon_0$$

So we can write

$$|a_n|^{\frac{1}{n}} < \alpha + \varepsilon_0 \text{ for all } n \geq N.)$$

There is a natural number N such that

$$|a_n|^{1/n} < (\alpha + \varepsilon_0) \ \ \forall\, n \geq N$$
$$|a_n| < (\alpha + \varepsilon_0)^n$$

Since $\alpha + \varepsilon_0 < 1$, the geometric series $\sum_{n=1}^{\infty}(\alpha+\varepsilon_0)^n$ converges. So by Comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

(ii) if $\alpha > 1$, then there is a subsequence of $\{a_n\}$ whose limit is $\alpha > 1$. That means that $|a_n| > 1$ for infinitely many choices of n. In particular, the sequence $\{a_n\}$ cannot converge to 0, so the series $\sum_{n=1}^{\infty} a_n$ cannot converge.

(iii) For the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ so α turns out to be 1. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conveges, the equality $\alpha = 1$ can not guarantee either convergence or divergence of the series.

$\square$

Example 4.16. *For any $x \in \mathbb{R}$, then series $\sum_{n=1}^{\infty} \frac{|x^n|}{n!}$. To see this, we apply root test*

$$\lim sup(\frac{|x^n|}{n!})^{1/n} = \lim_{n\to\infty} (\frac{|x^n|}{n!})^{1/n} = \lim_{n\to\infty} \frac{|x|}{(n!)^{1/n}} = 0 < 1.$$

as $\lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = 0$. The Root Test shows the series converges.

Example 4.17. *Consider $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$.*
To apply the root test, we compute

$$\alpha = \lim_{n\to\infty} \sqrt[n]{(n^2+3n)^n/(4n^2+5)^n} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4} < 1.$$

Since $\alpha < 1$, the series converges absolutely.

The next lemma is used to prove Abel's Convergence Test. It is computational in nature.

Lemma 4.6. (Summation by Parts) *Consider the two sequences $\{a_n\}$ and $\{b_n\}$. Let $S_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum. Then for any $n \geq m$,*

$$\sum_{j=m}^{n} a_j b_j = (S_n b_n - S_{m-1} b_m) + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1})$$

Proof. We have

$$\begin{aligned}
\sum_{j=m}^{n} a_j b_j &= \sum_{j=m}^{n} (S_j - S_{j-1}) b_j \\
&= \sum_{j=m}^{n} S_j b_j - \sum_{j=m}^{n} S_{j-1} b_j \\
&= \sum_{j=m}^{n} S_j b_j - \sum_{j=m-1}^{n-1} S_j b_{j+1} \\
&= \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}) + (S_n b_n - S_{m-1} b_m)
\end{aligned}$$

□

Theorem 4.7. (Abel's Test) *Consider the series* $\sum_{n=1}^{\infty} a_n b_n$. *Suppose that*
(i) the partial sums $S_n = \sum_{k=1}^{n} a_k$ *form a bounded sequence,*
(ii) the sequence $\{b_n\}$ *is decreasing,*
(iii) Suppose $\lim_{n\to\infty} b_n = 0$.
Then the $\sum_{n=1}^{\infty} a_n b_n$ *series converges.*

This test is rather sophisticated. Its main application is to prove the Alternating Series test, but one can sometimes use it for other series as well, if the more obvious tests do not work.

Proof. Given $\varepsilon > 0$. First, let's assume that the partial sums S_n's are bounded by K i.e $|S_n| \leq K$ for all $n \geq 1$. Next, since the sequence $\{b_n\}$ converges to zero, we can choose an integer $N \in \mathbb{N}$ such that $|b_n| \leq \frac{\varepsilon}{2K}$. Using the Summation by Parts lemma, we then have:

$$\begin{aligned}
|\sum_{j=m}^{n} a_j b_j| &= \left| \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}) + (S_n b_n - S_{m-1} b_m) \right| \\
&\leq \left| \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}) \right| + |(S_n b_n - S_{m-1} b_m)| \\
&\leq \sum_{j=m}^{n-1} |S_j||(b_j - b_{j+1})| + |S_n||b_n| + |S_{m-1}||b_m| \\
&\leq K \sum_{j=m}^{n-1} |(b_j - b_{j+1})| + K|b_n| + K|b_m|
\end{aligned}$$

But the sequence $\{b_n\}$ is decreasing to zero, so in particular, all terms must be positive, and all absolute values inside the summation above are superfluous. But then the sum is a telescoping sum. Therefore, all that remains is the first and last term, and we have:

$$|\sum_{j=m}^{n} a_j b_j| \leq K(b_m + b_n + b_m - b_n) = 2Kb_m \leq \varepsilon \quad \forall n \geq N$$

So the sequence of partial sums of $\{a_n b_n\}$ is a Cauchy sequence, hence $\sum_{n=1}^{\infty} a_n b_n$ converges. □

Corollary 4.8. (Alternating Series Test) *If $x_1 \geq x_2 \geq ....x_n \geq ....0$ and $\{x_n\}$ converges to zero, then the alternating series $\sum_{n=1}^{\infty}(-1)^n x_n$ converges.*

This test does not prove absolute convergence. In fact, when checking for absolute convergence the term 'alternating series' is meaningless. It is important that the series truly alternates, that is each positive term is followed by a negative one, and visa versa. If that is not the case, the alternating series test does not apply (while Abel's Test may still work).

Proof. Let $a_n = (-1)^n$. Then the partial sums $S_n = \sum_{k=1}^{n} a_k$ form a bounded sequence. Then, with the given choice of $b_n = x_n$, Abel's test applies directly, showing that the series converges. □