Q-1 a) Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$$

If $a_n = \dfrac{n!\,n!}{(2n)!} \Rightarrow a_{n+1} = \dfrac{(n+1)!\,(n+1)!}{(2n+2)!}$

Applying ratio test ———— 1

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{(2n+2)!} \times \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)\cancel{n!}\,\cancel{n!}\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{4\left(n^2 + \frac{3n}{4} + \frac{1}{2}\right)}$$

$$= \frac{1}{4} < 1$$ → 0.5

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ———— 1

↳ 0.5

Q-1 b) Test the convergence of

$$\sum_{n=1}^{\infty} \frac{2\sqrt{n}+3}{n^3-n+1}$$

Let $a_n = \dfrac{2\sqrt{n}+3}{n^3-n+1}$ and $b_n = \dfrac{1}{n^{5/2}}$

Applying Limit Comparison Test ——— 1

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \frac{\cancel{n^{5/2}}\, \cancel{n^{1/2}} \left(2+\frac{3}{\sqrt{n}}\right)}{\cancel{n^3}\left(1-\frac{1}{n^2}+\frac{1}{n^3}\right)}$$

$$= \frac{\lim \left(2+\frac{3}{\sqrt{n}}\right)}{\lim \left(1-\frac{1}{n^2}+\frac{1}{n^3}\right)} = \frac{2+\lim \frac{3}{\sqrt{n}}}{1-\lim \frac{1}{n^2}+\lim \frac{1}{n^3}}$$

By algebra of limits

$= 2 > 0$ ——— 0.5

So $\sum_{n=1}^{\infty} a_n$ converges ——— 0.5

$\Big\}$ ——— 1

c) Example → $a_n = \dfrac{1}{n}$ ——— 1 for correct example

$\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges but $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges.

——— 0 for wrong example.

Q2. (a) ①

f & g are continuous at $x=c$

TP - $\max(f,g)$ & $\min(f,g)$ are continuous at $x=c$

(Example 3.11 of the note)

Pf [We can write

$$\max(f,g) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$$

$$\min(f,g) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$$] (0.5 marks)

[f & g are continuous at c. So by Algebra of limits $\frac{f(x)+g(x)}{2}$ & $\frac{f(x)-g(x)}{2}$ are continuous at $x=c$

Now $\frac{|f(x)-g(x)|}{2}$ is continuous at $x=c$

($h(x)$ is continuous at $x=c \Rightarrow |h(x)|$ is continuous at $x=c$)

Again by Algebra of limits $\max(f,g)$ & $\min(f,g)$ are continuous at $x=c$.] (1 mark)

Q2. (b) $I = [0, \frac{\pi}{2}]$

So, $f: I \to \mathbb{R}$, defined by

{ $f(x) = \sup\{x^2, \cos x\}$

$= \max\{x^2, \cos x\}$

x^2 & $\cos x$ both are continuous on I

So by Q2.(a) f is continuous on I. } (0.5 marks)

{ I is closed & bounded interval so image set
$S = \{f(x) : x \in I\}$ is bounded

{ $f : I \to \mathbb{R}$ continuous f^n & I is closed & bounded then image set is bounded. Also sup & inf ~~exists~~ of image set exists & it is achieved by f } } (0.5 marks)

{ So $\exists$ $x_m, x_M \in I$ s.t. $\text{Inf } S = f(x_m)$

As $f(x_m) \in S$

So $\min S = \text{Inf } S = f(x_m)$

Hence absolute minimum point exists for f on I. } (0.5 marks)

Q2. © [Suppose $f : [0,1] \to \mathbb{R}$ is continuous & the image set $S = \{f(x) : x \in [0,1]\}$ is a subset of the set of rational numbers.
& f is non-constant so S can not be a singleton set.] (0.5 marks)

[Let $p, q \in \mathbb{Q}$ & $p < q$ and $p, q \in S$

So $\exists$ $c, d \in [0,1]$ s.t. $f(c) = p$ & $f(d) = q$

So $f(c) = p < q = f(d)$

But between any 2 rational no., $\exists$ an irrational no. (by Density property)

let r is an irrational no. lying between p & q

i.e. $f(c) = p < r < q = f(d)$] (1 marks)

[Now by Intermediate value theorem of continuous function there exists $e \in (c,d)$ s.t. $f(e) = r \notin \mathbb{Q}$

but f only takes rational values.

which is a contradiction. Hence such an f can not exists.] (0.5 marks)

$|f(x) - f(c)| < \varepsilon$ and $|g(x) - g(c)| < \varepsilon$ whenever $|x-c|<\delta$

Since $f(c) = g(c)$

$\Rightarrow \quad f(c) - \varepsilon < f(x), g(x) < f(c) + \varepsilon$ whever $|x-c|<\delta$

$\Rightarrow \quad f(c) - \varepsilon < \max(f,g) < f(c) + \varepsilon$ whenever $|x-c|<\delta$

Hence, $\max(f,g)$ is continuous.] (0.5 mark)

Similarly, one can do for $\min(f,g)$.

Date
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Q-3

a) $f(x) = x|x|$

if any function is differentiable at a then,

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ $f(0) = 0$

$\therefore a = 0$

$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0} \frac{x|x|}{x} = \lim_{x \to 0} |x| = 0$ 1

So the limit exist and hence $f'(0) = 0$

b) $f: R \to R$ is continouse f^n and twice diffrentiable with $a < b$

Since $f: R - R$ is diffrentiable

$\therefore$ $f: [a,b] \to R$ is also diffrentiable

as $f(a) = 0$, $f(b) = 0$ 2

So by Rolls therorem there exists a $c \in (a,b)$

Such that $f'(c) = 0$, therefore $f'(c) = 0$ 1

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Now let a function
$g\ [a\ c] \to R$ definedby $g(x) = f'(x)$
as f is twice diffrentiable.

$\therefore$ g is continous and diffrentiable on $a, c,$

~~no~~

now $f'(a) = 0$ $\therefore$ $g(a) = 0$
$f'(c) = 0$ $\therefore$ $g(c) = 0$

(1)

$\therefore$ from rolls therorem there exist a $d \in (a, c)$ such that

$$g'(d) = 0$$

$$\therefore f''(d) = 0$$

$$d \in (a; c)$$

$$(a, c) \subseteq (a, b)$$

$$\therefore d \in (a, b)$$

(1)

$\therefore$ there exist a point x_1 in (a, b) such that

$$f''(x_1) = 0$$