Q→1 Show directly from definition of cauchy Sequence that
If $\langle x_n \rangle$ & $\langle y_n \rangle$ are cauchy seq.
then $\langle x_n y_n \rangle$ is a chy seq

Pf

Use fact: If a seq. is cauchy
then it must be bounded

$\Rightarrow \exists \; M_1 > 0, M_2 > 0 \; : \; |x_n| \le M_1 \quad \forall n \in \mathbb{N}$ — (+)

$|y_n| \le M_2, \quad \forall n \in \mathbb{N}$ — (++)

— 1

Let $\varepsilon > 0$ be any given real no.

then for $\frac{\varepsilon}{2M_2} > 0$, $\exists \; N_1 \in \mathbb{N}_{\frac{\varepsilon}{2M_2}}$:

s.t. $|x_n - x_m| < \frac{\varepsilon}{2M_2}$; $\forall n, m \ge N_1$

— 0.5

— (1)

||ly for $\frac{\varepsilon}{2M_1} > 0$, $\exists \; N_2 \in \mathbb{N}_{\frac{\varepsilon}{2M_1}}$

s.t. $|y_n - y_m| < \frac{\varepsilon}{2M_1}$; $\forall n, m \ge N_2$

— 0.5

— (2)

Taking $N = \max\{N_1, N_2\}$ ——— 1

Consider $|x_n y_n - x_m y_m|$

$$= |x_n y_n - x_m y_n + x_m y_n - x_m y_m|$$

$$\leq |x_n - x_m| \cdot |y_n| + |y_n - y_m| \cdot |x_m|$$

(by Δgle inequality)

$$\leq |x_n - x_m| M_2 + |y_n - y_m| \cdot M_1 \quad \text{(by ⊕, ++)}$$

$$\leq \frac{\varepsilon}{2M_2} \cdot M_2 + \frac{\varepsilon}{2M_1} \cdot M_1 \quad \text{(by (1), (2))}$$

$$< \varepsilon \quad ; \quad \forall m, n \geq N$$

1.5

Since $\varepsilon > 0$ is arb.

So $\langle x_n y_n \rangle$ is a cgy Sequence

Q→2: Let $x_n = \sqrt{n}$; $\forall n \in \mathbb{N}$

1) SHOW: $\langle x_n \rangle$ satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.

2) SHOW: $\langle x_n \rangle$ is not a cauchy sequence.

Pf $0 \le |x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}$ — 1 marks

We know $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Apply Sandwich Thm

taking $a_n = 0$; $\forall n$

$y_n = |x_{n+1} - x_n|$

$z_n = \frac{1}{\sqrt{n}}$

We have $a_n \le y_n \le z_n$

$\downarrow$ $\downarrow$

0 0

Hence $\lim_{n\to\infty} y_n = 0$

$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ ⟶ 1 marks

Use: If $\langle x_n \rangle$ is cvg seq, ⟶ 0.5
then $\langle x_n \rangle$ is cgt seq.

Let if possible $\langle x_n \rangle$ is chy

for any $G > 0$, (However large)

$\exists \; n_G = ([G]+1)^2$ s.t. $x_n = \sqrt{n} > G$; ~~$\forall n \ge n_G (> G^2)$~~

$\forall n \ge n_G (> G^2)$

So $\langle x_n \rangle$ is divergent.

which is contradiction to our assumption

that $\langle x_n \rangle$ is a chy seq.

⟶ 1.5

Q-3. Let $S = \left\{ \frac{(n+1)^2}{2^n} \; ; \; n \in \mathbb{N} \right\}$

Method 1 -

$\{$ $S = \{x_n \; ; \; n \in \mathbb{N}\}$ where $x_n = \frac{(n+1)^2}{2^n}$

We first prove that $\{x_n\}$ is a monotonically decreasing seq^n for all $n \geq n_0$.

Consider,

$$x_n - x_{n+1} = \frac{(n+1)^2}{2^n} - \frac{(n+2)^2}{2^{n+1}}$$

$$= \frac{1}{2^{n+1}} \left\{ 2(n+1)^2 - (n+2)^2 \right\}$$

$$= \frac{1}{2^{n+1}} \left\{ 2n^2 + 4n + 2 - n^2 - 4n - 4 \right\}$$

$$= \frac{1}{2^{n+1}} (n^2 - 2)$$

So, $x_n - x_{n+1} > 0$ for all $n \geq 2$

$\Rightarrow$ $x_n > x_{n+1}$ for all $n \geq 2$

Hence $\{x_n\}$ is a decreasing seq for all $n \geq 2$ $\}$

(2 marks)

$\{$ which implies

$$\cdots \leq x_{n+1} \leq x_n \leq \cdots \leq x_2 \quad \text{for any } n \geq 2$$

Hence $x_n \leq x_2 = \frac{9}{4}$ $\forall$ $n \geq 2$

$x_1 = 2 < x_2 = \frac{9}{4}$

Hence $\quad x_n \le x_2 = \frac{9}{4} \quad \forall\ n \ge 1 \ \}$ (1 mark)

$\{$ So $x_2 = \frac{9}{4}$ is an upper bound of S
also it belongs to S ($x_2 \in S$)
So $\sup S = \frac{9}{4} \ \}$ (1 mark)

Method-2

$$S = \{x_n \ ; \ n \in \mathbb{N}\}$$

where $\quad x_n = \frac{(n+1)^2}{2^n} \quad ; \quad n \ge 1$

$x_1 = x_3 = 2 > 1 \quad , \ x_2 = \frac{9}{4} > 1$

$x_4 = \frac{25}{16} > 1 \quad , \quad x_5 = \frac{9}{8} > 1$

$\{$ We first show that $x_n < 1 \quad \forall\ n \ge 6$, by induction.

$P(n) = x_n < 1$

RTP - $P(n)$ is true $\forall\ n \ge 6$

when $n = 6$, $x_6 = \frac{49}{64} < 1$

So $P(6)$ is true

Let us assume $P(k)$ is true for any $k \ge 6$, that is

$$x_k = \frac{(k+1)^2}{2^k} < 1$$

Now, $$x_{k+1} = \frac{(k+2)^2}{2^{k+1}} = \frac{(k+1)^2}{2^k} \cdot \frac{(k+2)^2}{2(k+1)^2} < \frac{(k+2)^2}{2(k+1)^2}$$

as $\frac{(k+1)^2}{2^k} < 1$ ($P(k)$ is true)

Now, $$\frac{(k+2)^2}{2(k+1)^2} = \frac{k^2+4k+4}{2k^2+4k+2}$$

$$= \frac{k^2+4k+4}{k^2+4k+4+k^2-2} = \frac{(k+2)^2}{(k+2)^2+k^2-2} < 1$$

for $k \geq 6$, (as $k^2-2 > 0$ for $k \geq 6$)

Hence $P(k+1)$ is true.

So by Principle of strong mathematical induction

$P(n)$ is true $\forall\ n \geq 6$.

So $x_n < 1 \quad \forall\ n \geq 6$ } (2 marks)

{ But x_1, x_2, x_3, x_4, x_5 all are > 1

and Max of them is $x_2 = \frac{9}{4}$ & $x_n < 1 \quad \forall\ n \geq 6$

So $x_n \leq x_2 = \frac{9}{4} \quad \forall\ n \geq 1$ } (1 mark)

{ As $x_2 \in S$. so x_2 is an upper bound for S

and $x_2 \in S$ so $x_2 = \frac{9}{4}$ is the supremum }

(1 mark)

Q-4.

Method 1 - $\{a_n\}$ be a seq^n such that

$$\lim_{n\to\infty} \frac{a_n}{n} = L \quad \& \quad L>0$$

RTP - $\{a_n\}$ diverges.

{ For every $\varepsilon>0$, $\exists$ $N_\varepsilon \in \mathbb{N}$ (depending on $\varepsilon>0$) such that

$$\left|\frac{a_n}{n} - L\right| < \varepsilon \quad \forall \; n \geq N_\varepsilon$$

In particular, for $\varepsilon = \frac{L}{2}$, there exists $N_{\frac{L}{2}} \in \mathbb{N}$ s.t.

$$\left|\frac{a_n}{n} - L\right| < \frac{L}{2} \quad \forall \; n \geq N_{\frac{L}{2}}$$ } (1 mark)

{ which implies $-\frac{L}{2} < \frac{a_n}{n} - L < \frac{L}{2} \quad \forall \; n \geq N_{\frac{L}{2}}$

So $\frac{L}{2} < \frac{a_n}{n} < \frac{3L}{2} \quad \forall \; n \geq N_{L/2}$

So $n \cdot \frac{L}{2} < a_n \quad \forall \; n \geq N_{L/2}$ }

(1 mark)

Now $b_n = \frac{L}{2} \cdot n > 0 \quad \forall \; n \geq 1$

{ & $b_n < a_n \quad \forall \; n \geq N_{L/2}$

We know $\{b_n\}$ diverges

So by ordered properties of limits $\{a_n\}$ diverges }

(1 mark)

Method 2 - $\{a_n\}$ be a seq^n such that

$$\lim_{n\to\infty} \frac{a_n}{n} = L \quad \& \quad L>0$$

RTP - $\{a_n\}$ diverges

We will prove by contradiction.

Suppose $\{a_n\}$ converges that is

$$\lim_{n\to\infty} a_n = l \quad \text{—} (*) \quad (l \text{ is finite})$$ (1 mark)

Consider,

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \left(\frac{1}{n} \times a_n\right)$$

$$= \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} a_n\right) \quad \text{— (1)}$$ by Algebra of limit

As we know $\lim_{n\to\infty} \frac{1}{n} = 0$ — (2)

By (1), (2) & (*), we have

$$\lim_{n\to\infty} \frac{a_n}{n} = 0 \times l = 0$$ (1 mark)

which is a contradiction, since we have to given that $\lim_{n\to\infty} \frac{a_n}{n} = L > 0$

Hence, $\{a_n\}$ diverges. (1 mark)