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Q1 Suppose that f & g are continuous on $[a,b]$ differentiable on (a,b), that $c \in [a,b]$ & that $g(x) \neq 0$ ~~from~~ for $x \neq c$ & $x \in [a,b]$. Let $A = \lim_{x \to c} f(x)$ & $B = \lim_{x \to c} g(x)$. If $B = 0$ & $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists. Then $A = 0$.
(If $c = a$ then $\lim_{x \to a+}$ & $c = b$ then $\lim_{x \to b-}$)

Pf: Given $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists say $\lim_{x \to c} \frac{f(x)}{g(x)} = M$ (say).

$A = \lim_{x \to c} f(x)$ & $B = \lim_{x \to c} g(x) = 0$ (Given).

When $x \neq c$.

$f(x) = \frac{f(x)}{g(x)} \cdot g(x)$ [As $g(x) \neq 0$ for $x \neq c$ (given)]

$$A = \lim_{x \to c} f(x) = \lim_{x \to c} \left\{ \frac{f(x)}{g(x)} \cdot g(x) \right\}$$

$$= \lim_{x \to c} \frac{f(x)}{g(x)} \lim_{x \to c} g(x) \quad \text{[By Algebra of limits]}$$

$$= M \cdot 0 = 0.$$

Q2. Suppose that f & g are continuous on $[a,b]$ & differentiable on (a,b). Let $A = \lim_{x \to c} f(x)$ & $B = \lim_{x \to c} g(x)$. If $g(x) > 0$ for all $x \in [a,b]$

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but $x \neq c$. If $A > 0$ & $B = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = +\infty$.
If $A < 0$ & $B = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = -\infty$.

Pf! $\lim_{x \to c} f(x) = A > 0$ (Given). So there exists corresponding to $\varepsilon = \frac{A}{2}$ $\delta > 0$ s.t $|f(x) - A| < \frac{A}{2}$ for all $x \in (c-\delta, c+\delta) \setminus \{c\}$
[If $c = a$, then $\forall x \in (c, c+\delta)$ & if $c = b$ then $\forall x \in (c-\delta, c)$].

So $-\frac{A}{2} < f(x) - A < A/2$ $\forall x \in (c-\delta, c+\delta) \setminus \{c\}$

hence $\frac{A}{2} < f(x) < \frac{3A}{2}$ $\forall x \in (c-\delta, c+\delta) \setminus \{c\}$.

so $f(x) > \frac{A}{2} > 0$ $\forall x \in (c-\delta, c+\delta) \setminus \{c\}$ (*)

Hence in a deleted nbdhd of c. the function $\frac{1}{f}$ is defined. So $\frac{g(x)}{f(x)}$ is defined in a deleted nbdhd of c.

Hence $\lim_{x \to c} \frac{g(x)}{f(x)} = \frac{\lim_{x \to c} g(x)}{\lim_{x \to c} f(x)} = \frac{0}{A} = 0$

By Algebra of limits as $A > 0$

Let $G > 0$ be any real no (however large).

As $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$. So corresponding to this given $\varepsilon = 1/G$, there exists. $\delta_1 > 0$. s.t

$$\left|\frac{g(x)}{f(x)} - 0\right| < \frac{1}{G} \quad \forall\, x \in (c-\delta_1, c+\delta_1)\setminus\{c\}$$
$$\subseteq (c-\delta, c+\delta).$$

Now $f(x) > 0$ by $(*)$ in $(c-\delta, c+\delta)\setminus\{c\}$ & $g(x) > 0 \ \because \forall\, x \in [a,b]$ & $x \neq c$ (Given)

so $0 < \dfrac{g(x)}{f(x)} < \dfrac{1}{G} \quad \forall\, x \in (c-\delta_1, c+\delta_1)\setminus\{c\}$.

so $0 < G < \dfrac{f(x)}{g(x)} \quad \forall\, x \in (c-\delta_1, c+\delta_1)\setminus\{c\}$.

Hence for given $G > 0$, we have found δ_1.

So $\displaystyle \lim_{x\to c} \frac{f(x)}{g(x)} = +\infty$.

2nd part will follow similar way.

Q3 Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 1 \end{cases}$

& $g(x) = x^2 \ \forall\, x \in [0,1]$. Then f & g are diff on $(0,1)$, & $g(x) > 0$ for $x \in [0,1]$.

A : Show that $\displaystyle \lim_{x\to 0+} f(x) = 0 = \lim_{x\to 0+} g(x)$.

& $\displaystyle \lim_{x\to 0+} \frac{f(x)}{g(x)}$ does not exist.

So It is easy to prove $A = \displaystyle \lim_{x\to 0+} f(x) = 0 =$

$\lim_{x \to 0+} g(x) = B$. (4)

So $A = 0$, hence we can not apply previous result.

Now $\frac{f(x)}{g(x)} = \frac{x^2 \sin\frac{1}{x}}{x^2}$ when $x \neq 0$

$= \frac{\sin\frac{1}{x}}{1} = \frac{f_1(x)}{g_1(x)}$ where $f_1(x) = \sin\frac{1}{x}$, $g_1(x) = 1$, $x \neq 0$.

Although $\lim_{x \to 0+} g_1(x) = 1$ but $\lim_{x \to 0+} f_1(x)$ that $\lim_{x \to 0+} \sin\frac{1}{x}$ does not exists. So $\lim_{x \to 0+} \frac{f_1(x)}{g_1(x)}$ does not exist so $\lim_{x \to 0+} \frac{f(x)}{g(x)}$ does not exist.

Another way:

$\lim_{x \to 0+} f(x) = 0 = \lim_{x \to 0+} g(x)$. So $\lim_{x \to 0+} \frac{f(x)}{g(x)}$ is $\left[\frac{0}{0}\right]$ indeterminate form. & $g'(x) \neq 0$ for $x \neq 0$. But $\lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{\cos\frac{1}{x} + 2x\sin\frac{1}{x}}{2x}$

~~But~~ Here $\lim_{x \to 0+} g'(x) = 0$ but $\lim_{x \to 0+} f'(x)$ does not exists as $\lim_{x \to 0+} \cos\frac{1}{x}$ does not exists.

So. We can not say $\lim_{x \to 0+} \frac{f'(x)}{g'(x)}$ exists ~~or~~ & also it is not in indetermenate form. So we can not apply L' Hospital.

Q5) Let f be differentiable on $(0,\infty)$. & suppose that $\lim_{x\to\infty}(f(x)+f'(x)) = L$. Show that $\lim_{x\to\infty} f(x) = L$ & $\lim_{x\to\infty} f'(x) = 0$.

Pf) Consider $f_1(x) = f(x)e^x$ & $g_1(x) = e^x$

So Now $\lim_{x\to\infty}(f(x)+f'(x)) = L$ implies $\lim_{x\to\infty} f(x)$ exists? (check) so $\exists M>0$ s.t $|f(x)| \le M$ for all $x > x_0$

So $\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} f(x)e^x$

$= \lim_{x\to\infty} f(x)\, e^x = \pm\infty$ (not by Algebra of Limits)

($+\infty$ if $\lim_{x\to+\infty} f(x)$ +ve) & $-\infty$ if $\lim_{x\to+\infty} f(x)$ is -ve

Take $\lim_{x\to+\infty} f_1(x) = +\infty$. ($-\infty$ will have to consider

Then $\lim_{x\to\infty} g_1(x) = +\infty = \lim_{x\to\infty} f_1(x)$.

& $g_1'(x) = e^x \neq 0$ for all $x \in (0,+\infty)$.

Let us look at $\lim_{x\to+\infty} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to+\infty} \frac{\cancel{e^x}[f(x)+f'(x)]}{\cancel{e^x}}$

$= \lim_{x\to\infty}(f(x)+f'(x)) = L$ (Given)

So L'Hospital's rule 6 (in note)

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$$\lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty}\frac{f_1'(x)}{g_1'(x)} = L \rightarrow (+)$$

but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{f(x)e^x}{e^x} = \lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} \rightarrow (-)$

Combining (+) & (−), $\lim_{x\to\infty} f(x) = L. \longrightarrow (x)$

Hence $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}[f(x)+f'(x)-f(x)]$

$$= \lim_{x\to\infty}[f(x)+f'(x)] - \lim_{x\to\infty} f(x)$$

$$= L - L \quad \text{by } (x) \text{ \& Given hypothesis}$$

$$= 0.$$

Q6 Evaluate the limit, $\lim_{x\to\infty} x^{\sin\left(\frac{1}{x}\right)}$

Let $y(x) = x^{\sin\left(\frac{1}{x}\right)}$ $\quad \lim y(x) = [\infty^0]$ form

then $h(x) = \log y(x) = \sin\frac{1}{x}\log x = \dfrac{\log x}{\frac{1}{\sin\left(\frac{1}{x}\right)}} \rightarrow (1).$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\frac{\log x}{\left(\sin\frac{1}{x}\right)^{-1}} = ??$$

Take $f(x) = \log x$ & $g(x) = \left(\sin\frac{1}{x}\right)^{-1}$

& $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty}\left(\sin\frac{1}{x}\right)^{-1}$

$$f'(x) = \frac{1}{x} \quad \& \quad g'(x) = \frac{1}{x^2}\left(\cos\frac{1}{x}\right)\left(\sin\frac{1}{x}\right)^{-2}$$

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$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{\frac{1}{x}\left(\sin\frac{1}{x}\right)^2 x^2}{\cos\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(\sin\frac{1}{x}\right)^2}{\left(\cos\frac{1}{x}\right)\frac{1}{x}}$$

→ (a) is $\left[\frac{0}{0}\right]$ fo

Now consider $f_1(x) = \left(\sin\frac{1}{x}\right)^2$ & $g_1(x) = \left(\cos\frac{1}{x}\right)\frac{1}{x}$ so $\lim_{x\to 0} f_1(x) = 0 = \lim g_1(x)$.

$$g_1'(x) = -\frac{1}{x^2}\cos\frac{1}{x} - \frac{1}{x^3}\sin\frac{1}{x} \neq 0 \ \forall x > 0.$$

$$f_1'(x) = -\frac{1}{x^2} 2\sin\frac{1}{x}\cos\frac{1}{x}$$

$$\lim_{x\to\infty}\frac{f_1'(x)}{g_1'(x)} = \lim_{x\to\infty}\frac{\frac{1}{x^2}2\sin\frac{1}{x}\cos\frac{1}{x}}{\frac{1}{x^2}\cos\frac{1}{x} + \frac{1}{x^3}\sin\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{2\sin\frac{1}{x}\cos\frac{1}{x}}{\cos\frac{1}{x} + \frac{\sin 1/x}{x}}$$

$$= \frac{\lim_{x\to\infty} 2\sin\frac{1}{x}\cos\frac{1}{x}}{\lim_{x\to\infty}\left(\cos\frac{1}{x} + \frac{\sin 1/x}{x}\right)}$$ [By Algebra of limits]

$$= \frac{0}{1} = 0.$$

By L'Hospital's rule applied to f_1 & g_1,

$$\lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} = \lim_{x\to\infty}\frac{f_1'(x)}{g_1'(x)} = \frac{0}{1} = 0.$$

Now $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} = 0$ by (a).

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where

$$k = -\sqrt{\frac{e_3 - e_2}{e_1 - e_2}}.$$

Thus if one defines

$$\operatorname{sn} u = S((e_1 - e_2)^{-\frac{1}{2}}u), \quad \operatorname{cn} u = C((e_1 - e_2)^{-\frac{1}{2}}u), \quad \operatorname{dn} u = D((e_1 - e_2)^{-\frac{1}{2}}u),$$

then sn, cn and dn are elliptic functions which satisfy the identities

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \quad k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1,$$

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \quad \frac{d}{du}\operatorname{cn} u = -\operatorname{dn} u \operatorname{sn} u, \quad \frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

These functions are the elliptic functions of Jacobi.

Hence by L'Hospital's rules applied to f & g.

($\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists & equal to 0) gives.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0.$$

so by (1) $\lim_{x\to\infty} \log y(x) = 0$

$$\lim_{x\to\infty} x^{\sin\frac{1}{x}} = \lim_{x\to\infty} e^{\log y(x)} = e^0 = 1.$$

as exp. f^n is cont

X

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Q7 Show that $1+\frac{x}{2}-\frac{x^2}{8} \leq \sqrt{1+x} \leq 1+\frac{x}{2}$ $(x>0)$

Pf: ~~$f(x)=\sqrt{1+x}$~~ Consider, $f:(-1,\infty) \to \mathbb{R}$
defined by $f(x)=\sqrt{1+x}$.
Apply Taylor's theorem on $(0,x)$. So
$\exists$ $c_x \in (0,x)$ s.t.

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(c_x)+\cancel{\frac{x^3}{3!}f'''(c_x)}$

Now $f'(t)=\frac{1}{2}(1+x)^{-1/2}$, $f''(t)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$

So $f'(0)=\frac{1}{2}$ & $f''(0)=-\frac{1}{4}$ & $f(0)=1$

$f(x)=1+\frac{x}{2}-\underbrace{\frac{x^2}{8}(1+c_x)^{-\frac{3}{2}}}_{\geq 0} \leq 1+\frac{x}{2}$ $\forall x>0$ $\longrightarrow (+)$

$\because (1+c_x)^{-3/2} \leq 1$, so $-\frac{x^2}{8} \leq -\frac{x^2}{8}(1+c_x)^{-\frac{3}{2}}$

Adding $1+\frac{x}{2}$ both sides of the inequality

$1+\frac{x}{2}-\frac{x^2}{8} \leq 1+\frac{x}{2}-\frac{x^2}{8(1+c_x)^{3/2}}=f(x)=\sqrt{1+x}$ $(x>0)$ $\longrightarrow (-)$

Combining (+) & (−), we get the result.

Q8 Show that for $x \in \mathbb{R}$ with $|x|^5 < \frac{51}{10^4}$, we can replace $\sin x$ by $x-\frac{x^3}{6}$ with an error of magnitude less than or equal to 10^{-4}.

Pf: Let $f:(0,\infty) \to \mathbb{R}$ defined by $f(x)=\sin x$

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Apply Taylor's theorem on $(0, x)$.

$\exists\ c_x \in (0, x)$ s.t

$$\sin x = f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \frac{x^5}{5!} f^{(5)}(c_x).$$

$$= x - \frac{x^3}{3!} + (\cos c_x)\frac{x^5}{5!}$$

Hence $\sin x - \left(x - \frac{x^3}{3!}\right) = (\cos c_x)\frac{x^5}{5!}$

$$\left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| = \underbrace{|\cos c_x|}_{|\cos c_x| \le 1}\left|\frac{x^5}{5!}\right| \le \frac{|x|^5}{5!} = \frac{5!}{10^4\, 5!}$$

$|x|^5 \le \frac{5!}{10^4}$ (Given)

so $\left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| \le \frac{1}{10^4}$ $(x > 0)$

Q9. Given an example of a f^n f s.t f' exists but $g = f'$ is not continuous.