(a) Let, $\exists$ a fn. 'f' $R \to R$ which is diff. on (a, b) s.t

$$f'(x) = h(x)$$

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$0 = f'(-1) < 1/2 < f'(0) = 0$$

By IMV of diff. fn. there must exist $c \in \mathbb{R}$ s.t $f'(c) = 1/2$

But $\exists$ no point $c \in \mathbb{R}$ s.t $f'(c) - h(c) = 1/2$.

(FALSE)

(b) $$\lim_{x \to \infty} \frac{x - \sin x}{2x + \sin x}$$

$f(x) = x - \sin x$; $g(x) = 2x + \sin x$

$f'(x) = 1 - \cos x$

$g'(x) = 2 - \cos x$

But, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x)$ doesn't exist.

(FALSE)

(c) $w = (u-s)^2 + (u+s)^2$ (True)

$= 2u^2 + 2s^2$

$w_u = 4u$

(d) $f(x) = |x|^3$

$f'(x) = 3x|x|$

$f''(x) = \begin{cases} 0 & x = 0 \\ \dfrac{6x^2}{|x|} & x \neq 0 \end{cases}$ (False)

so, $f''(0^+) = f''(0) = f''(0^-) = 0$

as $f''(0^+) \,/\, f''(0^-)$:

$f''(0^+) = \dfrac{6x^2}{x} = 6x = 0$

$f''(0^-) = \dfrac{6x^2}{-x} = -6x = 0$.

(e) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ (False)

$\cancel{f'(x) = \lim\limits_{h\to 0^-} \dfrac{\sin(x+h) - |\sin x|}{h}}$ $k\pi$.

$f(x) = |\sin x|$

for $x = k\pi$ $(k = 0, 1, \ldots)$

$\lim\limits_{h\to 0^+} \dfrac{f(k\pi + h) - f(k\pi)}{h} = \dfrac{|\sin(k\pi + h)| - |\sin k\pi|}{h} =$

$\dfrac{|\overset{0}{\cancel{\sin k\pi \cos h}} + \cos k\pi \sin h| - 0}{h} = \dfrac{|\sin h|}{h} = 1$

$$\lim_{h\to 0^-} \frac{f(k\pi+h) - f(k\pi)}{h} = \frac{|\sin h|}{h} = -1$$

as $h \to 0^-$ hence $= -1$

so, $LHL \neq RHL$;

$|\sin x|$ not differentiable at all points.

(f) (True)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^6+y^2}$$

let $x = r\cos\theta$, $y = r\sin\theta$

then $$\frac{(r\cos\theta)(\cancel{r^2}\sin^2\theta)}{r^4\,\cancel{r^6}\cos^6\theta + \cancel{r^2}\sin^2\theta} = \frac{r\cos\theta\sin^2\theta}{r^4\cos^6\theta + \sin^2\theta}$$

$$\lim_{r\to 0} \frac{r\cos\theta\sin^2\theta}{r^4\cos^6\theta+\sin^2\theta} = 0$$

(g) Let $\exists$ an onto function $f : [a,b] \to \mathbb{R}$.

Then for every $(n > 0)$ there is $x_n \in [a,b]$ s.t $f(x_n) = n$

x_n is bounded in $[a,b]$

hence x_n will have convergent subsequence, let's say x_{n_k}

Let $\lim_{k\to\infty} x_{n_k} = x \in [a,b]$

Then;

$\lim_{k\to\infty} f(x_{n_k}) = f(x)$ {by continuity}

while;

$\lim_{k\to\infty} (f(x_{n_k})) = \lim_{k\to\infty} (n_k) = \infty$ (True)

so, By contradiction; $f(x) = \infty$.

(h) $\lim_{x\to c} g(x) \cdot \lim_{x\to c} f(x) = \lim_{x\to c} g(x) f(x)$

(False)

(i) $f(x) = 0$

$g(x) = |x|$; $g(f(x)) = |f(x)| = 0$

$g(f(x))$ = differentiable fn.

$f(x)$ = differentiable fn. (False)

$g(x)$ = not differentiable at $x = 0$.

(j) Let $|f(x)| \le M$ and $|g(x)| \le N$ ($\forall$ x, y in A and some constants $M, N > 0$)

Then,

$$|f(x)\,g(x) - f(y)\,g(y)| = |f(x)(g(x) - g(y)) + f(x)g(y) - f(y)g(y)|$$

$$= f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))$$

$$\leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)|$$

$$\leq M\,|g(x) - g(y)| + N\,|f(x) - f(y)|$$

$$\leq Md\,|x-y| + Nc\,|x-y|$$

(True)

$$= C'\,|x-y|$$

Hence, f·g is Lipschitz fn. bounded on A.