Assignment - 5

Q→1 Let $f: [a,b] \to \mathbb{R}$ is cts

If $c \in (a,b)$: $f(c) > 0$

If $0 < \beta < f(c)$ then show $\exists\, \delta > 0 : f(x) > \beta \; ; \forall x \in (c-\delta, c+\delta) \subseteq [a,b]$

Let $g : [a,b] \to \mathbb{R}$

$g(x) = f(x) - \beta$ It is cts

$g(c) = f(c) - \beta$

$> 0 \quad [\because 0 < \beta < f(c)]$

Th^m 2.3 Let f be a real-valued fn : $\lim_{x \to c} f(x) = L \; ; L > 0$

then $\exists$ open interval $(c-\delta, c+\delta)$ containing c : $f(x) > 0 \; ; \forall x \in (c-\delta, c+\delta) \setminus c$

$\exists\, \delta > 0 : \quad g(x) > 0 \quad ; \forall x \in (c-\delta, c+\delta)$

$f(x) - \beta > 0$

$f(x) > \beta \quad ; \forall x \in (c-\delta, c+\delta)$

H.P.

Q→2. Let $f : \mathbb{R} \to \mathbb{R}$ is cts : $\lim_{|x| \to \infty} f(x) = 0$

Prove 1) f is bdd on $\mathbb{R}$

2) f attains either abs. maximum or an absolute minimum

Let $g(x) = |f(x)|$

~~cts f~~

Q→2. Let $f: \mathbb{R} \to \mathbb{R}$ is cts : $\lim_{|x| \to \infty} f(x) = 0$

Prove 1) f is bdd on $\mathbb{R}$

2) f attains either abs. maximum or an abs minimum.

Let $g(x) = |f(x)|$

$\exists\ x_0 \in \mathbb{R}$: $g(x) \le g(x_0)$; $\forall x \in \mathbb{R}$

$\exists\ y_0 \in \mathbb{R}$: $g(x) \ge g(y_0)$; $\forall x \in \mathbb{R}$

~~Case 1 If $g(x) \not\equiv 0$; $\forall x$~~

wlg $g(x) \not\equiv 0$; $\forall x \in \mathbb{R}$

$g(x) \ge 0$ and $g(x) \not\equiv 0$

means $\exists$ $c \in \mathbb{R}$: $g(c) \ne 0$, $g(c) > 0$

Given :- $\lim_{|x| \to \infty} f(x) = 0 \Rightarrow \lim_{|x| \to \infty} g(x) = 0$

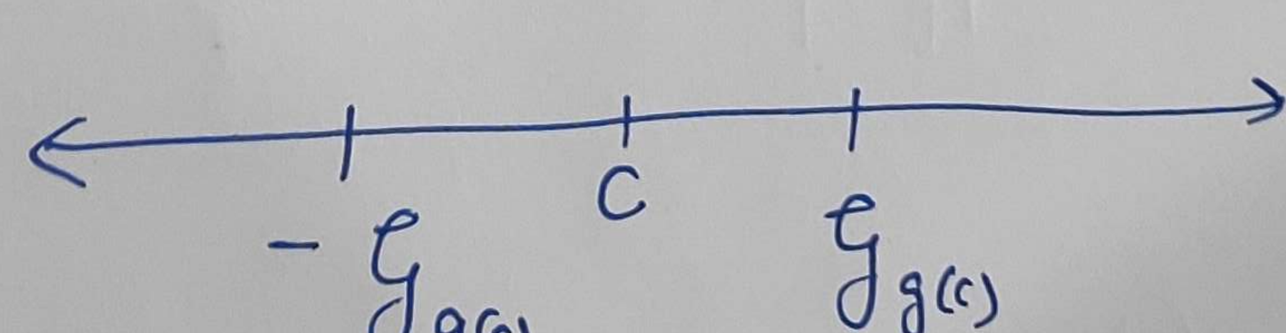
means for each $\varepsilon > 0$, $\exists$ a real no. $\xi > 0$: $|f(x)| < \varepsilon$; $\forall |x| > \xi$

: $g(x) < \varepsilon$; $\forall |x| > \xi$

In particular for $\varepsilon = g(c)$

$\exists$ $\xi_{g(c)}$: $|g(x)| < g(c)$; $\forall |x| > \xi_{g(c)}$.

that means $g(x) < g(c)$; $\forall x > \xi_{g(c)}$ & $x > -\xi_{g(c)}$.



We know $g(x)$ is cts on $[-\xi_{g(c)}, \xi_{g(c)}] \longrightarrow \mathbb{R}$

In closed and bdd interval $g(x)$ attain Sup and Inf.

$\exists$ $a, b \in [-\xi_{g(c)}, \xi_{g(c)}]$: $g(a) \le g(x) \le g(b)$; $\forall x \in [\xi_{g(c)}, \xi_{g(c)}]$

In particular $g(c) \le g(b)$.

Ans→2 continued

we have

~~$g(c)$~~ $g(x) < g(c)$; $\forall |x| > \xi_c$

&

$g(a) \le g(x) \le g(b)$; $\forall x \in [-\xi_c, \xi_c]$

& $g(c) \le g(b)$ ∵ $c \in [\xi_c, \xi_c]$.

~~choose $\max\{g(b), g(c)\}$~~

~~Case-1 If $\max\{g(b), g(c)\} = g(b)$~~
~~Then $g(x)$ is bdd on $\mathbb{R}$~~

~~Case-2 If $\max\{g(b), g(c)\} = g(c)$~~
~~Then $g(x)$ is bdd on $\mathbb{R}$.~~

Hence $g(x) \le g(b)$ $\forall x \in \mathbb{R}$.

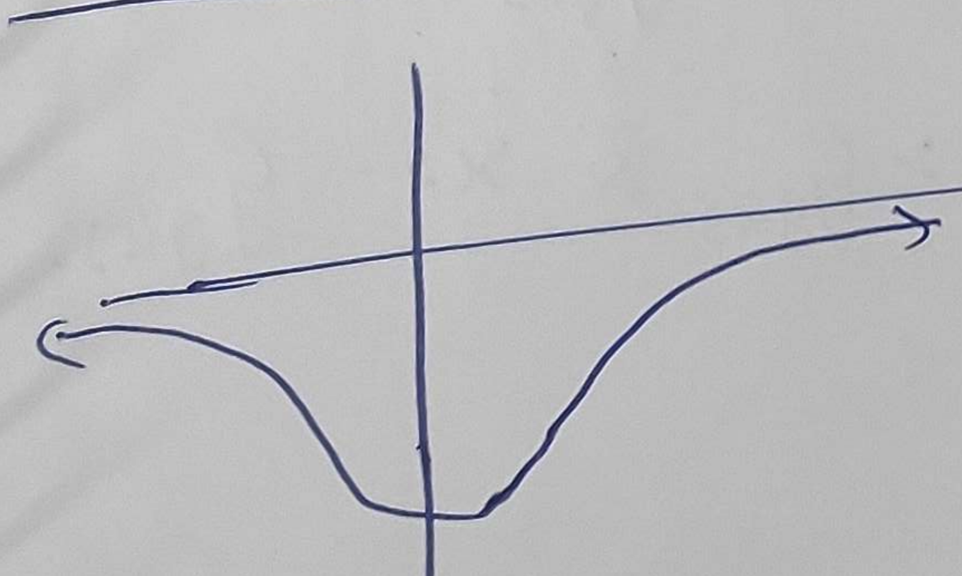
⇒ g attain Max at $x = b$.

∵ $g = |f(x)|$

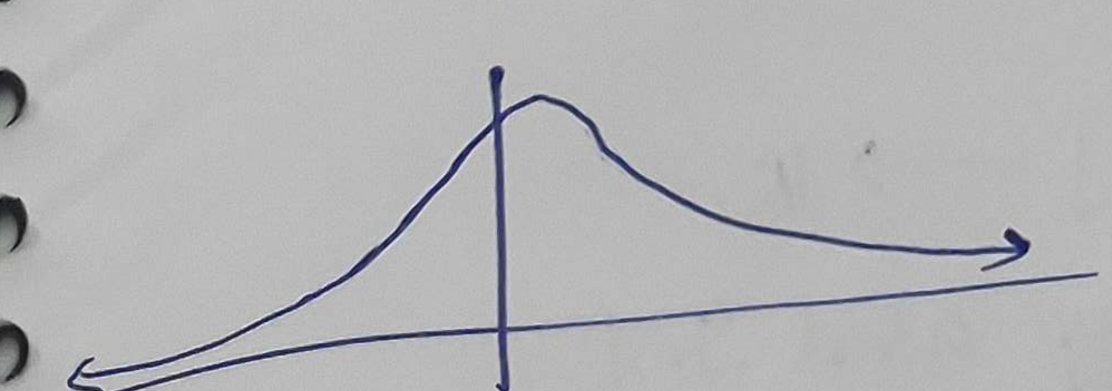
Hence $|f(x)|$ attain Max. at $x = b$

Hence $f(x)$ attains either maximum or minimum.

H.P.



Here $f(x)$ attains Minima



Here $f(x)$ attains maxima.

Q→3 $\nexists$ a cts fn : $[0,1]$ onto $\mathbb{R}$. Why ?

Let if possible $\exists$ a cts fn $f:[0,1]\to\mathbb{R}$ is cts & i.e. onto

By Thm 3.5

Let $S = \{f(x) ; x \in [0,1]\}$

$\operatorname{Sup}(S)$, $\operatorname{Inf}(S)$ exist.

By Thm 3.6

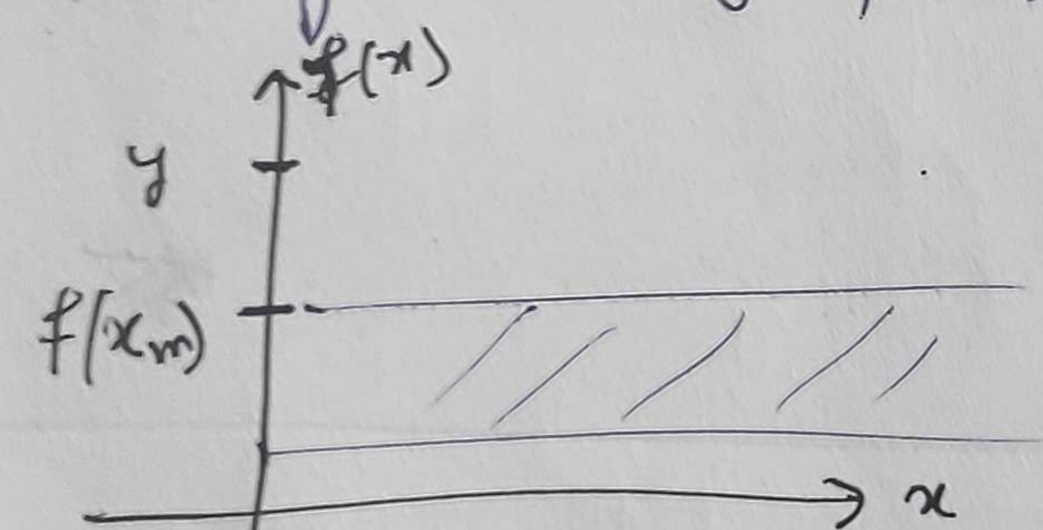
$\operatorname{Sup}(S)$ & $\operatorname{Inf}(S)$ attain by f.

Let $\operatorname{Sup} S = f(x_M)$; $x_M \in [0,1]$

& $\operatorname{Inf} S = f(x_m)$; $x_m \in [0,1]$

Hence $f(x_m) \le f(x) \le f(x_M)$

Take $y \in \mathbb{R}$: $y > f(x_M)$



$\because f$ is onto.

$\exists\ x_0 \in [0,1]$: $f(x_0) = y$

But $y > f(x_M)$

$\Rightarrow f(x_0) > f(x_M)$

$\Rightarrow f(x_0) > \operatorname{Sup} S$

$\#$

Hence our supposition is wrong

Q→4 Find a cts fn $f: (0,1) \longrightarrow \mathbb{R}$ onto.

$f: (0,1) \rightarrow \mathbb{R}$

$f(x) = \tan \frac{\pi}{2}(2x-1)$ is ~~both~~ bijection. & cts

we know $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ is /

$f_1 = \tan x$ is ~~not~~ cts & ~~not~~ bijection.

$f_2: (0,1) \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ is defd as

$f_2(x) = \frac{\pi}{2}(2x-1)$ is cts & bijection.

$f = f_1 \circ f_2 : (0,1) \rightarrow \mathbb{R}$ is cts

$f(x) = \tan \frac{\pi}{2}(2x-1)$

Use ① ∵ composition of two cts fn is continous

② Composition of two bijection map is bijection.

Q-5. Let $f : [a,b] \to \mathbb{R}$ be a cts fn

s.t. for each $x \in [a,b]$; $\exists$ $y \in [a,b]$: $|f(y)| \leq \frac{|f(x)|}{2}$

Prove $\exists$ a $c \in [a,b]$: $f(c) = 0$.

Since f is cts $\Rightarrow$ $|f|$ is also cts

By Thm 3.5, 3.6

Thm 3.5 $\rightarrow$

$S = \{ |f(x)| ; x \in [a,b] \}$ Then $\sup(S)$, $\inf(S)$ exist

Thm 3.6 $\rightarrow$ from notes, we can say

$\sup(S)$ & $\inf(S)$ is achieved by $|f|(x)$

i.e. $|f|(x_M) = \sup(S)$; $x_M \in [a,b]$

& $|f|(x_m) = \inf(S)$; $x_m \in [a,b]$

Hence $|f|(x_m) \leq |f|(x) \leq |f|(x_M)$ — ($\star$)

Given:- For $x = x_m$; $\exists$ $y_{x_m} \in [a,b]$: $|f(y_{x_m})| \leq \frac{|f(x_m)|}{2}$ — ($\#$)

By ($\star$) $|f|(x_m) \leq |f|(y_{x_m})$

Hence $|f(x_m)| \leq |f(y_{x_m})| \leq \frac{|f(x_m)|}{2}$

Hence $|f(x_m)| = 0$

$\Rightarrow$ Hence $f(x_m) = 0$

Q→6. $f: A \to \mathbb{R}$ is U.C.

$|f(x)| \geq k > 0 \quad \forall x \in A$

Show: $\frac{1}{f(x)}$ is U.C. on A.

Given $\varepsilon > 0$

RTP: To this given $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0 : \forall x, y \in A$

$0 < |x - y| < \delta_\varepsilon \Rightarrow \left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| < \varepsilon$

We know f is U.C.

For given $\varepsilon k^2 > 0$, $\exists\, \delta_{\varepsilon k^2} > 0$:

$0 < |x - y| < \delta_{\varepsilon k^2} : |f(x) - f(y)| < \varepsilon k^2$ —— 1

Now for $\forall x, y \in A : \quad 0 < |x - y| < \delta_{\varepsilon k^2}$, consider.

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \frac{|f(x) - f(y)|}{|f(x)\, f(y)|}$$

$$\leq \frac{1}{k^2}\, |f(x) - f(y)|$$

$$\leq \frac{1}{k^2} \cdot \varepsilon k^2$$

$$\leq \varepsilon.$$

For given $\varepsilon > 0$, we have found $\delta_{\varepsilon k^2} : \forall x, y \in A$

$0 < |x - y| < \delta_{\varepsilon k^2} \Rightarrow \left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| < \varepsilon$

Hence $\frac{1}{f(x)}$ is U.C.

Q→7 $f : \mathbb{R} \to \mathbb{R}$ be a fn.

$f(x) = \frac{1}{1+x^2}$

<u>Show</u> f is U.C. on $\mathbb{R}$

for any $\varepsilon > 0$, $\exists$ $\delta_\varepsilon > 0$: $\forall x, y \in \mathbb{R}$

$0 < |x-y| < \delta_\varepsilon \Rightarrow |f(x - f(y)| < \varepsilon$

Consider $|f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \left| \frac{x^2 - y^2}{(1+x^2)(1+y^2)} \right|$

$$= \frac{|x+y| \cdot}{(1+x^2)(1+y^2)} |x-y|$$

$$\le \frac{(|x| + |y|)}{(1+x^2)(1+y^2)} \cdot |x-y|$$

$$\le \left(\frac{|x|}{1+x^2} + \frac{|y|}{y^2+1} \right) |x-y|$$

$$\le \left(\frac{1}{2} + \frac{1}{2} \right) |x-y|$$

$$\le |x-y|$$

$$< \varepsilon$$

$\left[\because \; a^2 + b^2 \ge 2|ab| \right.$
$a = 1, b = x$
$1 + x^2 \ge 2|x|$
$\left. \Rightarrow \frac{1}{2} \ge \frac{x}{1+x^2} \right]$

for given $\varepsilon > 0$, we have found $\delta_\varepsilon = \varepsilon$

; $\forall x, y \in \mathbb{R}$

$0 < |x-y| < \varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$