ASSIGNMENT - 8
SOLUTIONS

1)

$f(x,y) = 1 \quad$ if $x=0$ or $y=0$

$f(x,y) = 0 \quad$ otherwise

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

If we consider (x,y) approaching to $(0,0)$ along the line $y=x$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} f(x,y) = 0$$

and we know $f(0,0) = 1$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$$

Hence the function is discontinous.

2) $f(x,y) = |x|+|y| \qquad g(x,y) = |xy|$

a) $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

This limit does not exist as for $\lim_{x\to 0} \frac{|x|}{x}$ we have

$$\lim_{x\to 0^+} \frac{x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{-x}{x} = -1$$

RHL $\neq$ LHL

$\therefore$ Limit does not exist

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

and this limit does not exist

$$g_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|0+h.0| - |0.0|}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$g_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{|0.0+h| - |0.0|}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

b) $x_0 \neq 0$

$$g_y(x_0,0) = \lim_{h\to 0} \frac{g(x_0,0+h) - g(x_0,0)}{h} = \lim_{h\to 0} \frac{|x_0 h| - |x_0.0|}{h}$$

$$= \lim_{h\to 0} \frac{|x_0 h|}{h}$$ and this limit does not exist

$y_0 \neq 0$

$$g_x(0,y_0) = \lim_{h\to 0} \frac{g(0+h,y_0) - g(0,y_0)}{h} = \lim_{h\to 0} \frac{|h y_0| - |0.y_0|}{h}$$

$$= \lim_{h\to 0} \frac{|h y_0|}{h}$$ and this limit does not exist

4) $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\frac{2h^2.0}{h^2+0^2} - 0}{h} \right) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\frac{2(0^2).h}{0^2+h^2} - 0}{h} \right) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Partial derivatives exist at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$ so $df(0,0) = 0$

Hence

$$\Delta f(0,0) = f(h,k) - f(0,0) = \frac{2h^2k}{h^2+k^2} = \frac{2h^2k}{\rho^2}$$

If we take $h = \rho\cos\theta$ and $k = \rho\sin\theta$ we get

$$\frac{\Delta f - df}{\rho} = \frac{2\rho^2\cos^2\theta\,\rho\sin\theta}{\rho 3} = 2\cos^2\theta\sin\theta$$

The limit does not exist.

Therefore f is not differentiable at $(0,0)$.

5) $f(x,y) = \begin{cases} \dfrac{x^2y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

The partial derivatives $f_x(0,0) = f_y(0,0) = 0$ and so $df = 0$ (can be done as shown in previous question)

$$\Delta f\,(0,0) = f(h,k) - f(0,0) = \frac{h^2k}{\sqrt{h^2+k^2}} = \frac{h^2k}{\rho}$$

If we take $h = \rho\cos\theta$ and $k = \rho\sin\theta$ we get

$$\frac{\Delta f - df}{\rho} = \frac{\rho^2\cos^2\theta\,\rho\sin\theta}{\rho^2} = \rho\cos^2\theta\sin\theta$$

$\rho\cos^2\theta\sin\theta \to 0$ as $\rho \to 0$

Hence $\displaystyle\lim_{\rho\to 0} \frac{\Delta f(0,0) - df(0,0)}{\rho} = 0$

Therefore f is differentiable at $(0,0)$

7) $z = f(x,y)$, $x = r\cos\theta$ and $y = r\sin\theta$

a) $$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dr}\right) + \left(\frac{\partial z}{\partial y}\cdot\frac{dy}{dr}\right)$$

$$= \left(\frac{\partial f}{\partial x}\cdot\frac{d\,(r\cos\theta)}{dr}\right) + \left(\frac{\partial f}{\partial y}\cdot\frac{d\,(r\sin\theta)}{dr}\right)$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

b) $$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\cdot\frac{dx}{d\theta}\right) + \left(\frac{\partial z}{\partial y}\cdot\frac{dy}{d\theta}\right)$$

$$= \left(\frac{\partial z}{\partial x}\cdot\frac{d\,(r\cos\theta)}{d\theta}\right) + \left(\frac{\partial z}{\partial y} + \frac{d\,(r\sin\theta)}{d\theta}\right)$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$$

b) Using the second result obtained in part (a)

$$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$$

$$f(x,y) = x^2 + 2xy$$

$$\frac{\partial f}{\partial x} = 2x + 2y = 2(x+y)$$

$$\frac{\partial f}{\partial y} = 2x$$

$$\frac{\partial z}{\partial \theta} = r\left(-2(x+y)\sin\theta + 2x\cos\theta\right)$$

we know $x = r\cos\theta \Rightarrow \cos\theta = x/r$

and $y = r\sin\theta \Rightarrow \sin\theta = y/r$

$$\frac{\partial z}{\partial \theta} = r\left(\frac{-2(x+y)y}{r} + \frac{(2x)x}{r}\right)$$

$$\frac{\partial z}{\partial \theta} = -2xy - 2y^2 + 2x^2$$

$$\frac{\partial z}{\partial \theta} = 2(x^2 - xy - y^2)$$

8) $z(u,v) = z(x(u,v), y(u,v))$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}$$

$z = 4e^{x}\ln y$, $x = \ln(u\cos v)$, $y = u\sin v$

$(u,v) = \left(2, \frac{\pi}{4}\right)$

$$\frac{\partial z}{\partial x} = 4e^{x}\ln y \quad , \frac{\partial z}{\partial y} = \frac{4e^{x}}{y}$$

$$\frac{dx}{du} = \frac{\cos v}{u\cos v} = \frac{1}{u} \qquad \frac{\partial x}{\partial v} = \frac{-u\sin v}{u\cos v} = -\tan v$$

$$\frac{\partial y}{\partial u} = \sin v \qquad \frac{\partial y}{\partial v} = u\cos v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= 4e^{x}\ln y \cdot \frac{1}{u} + \frac{4e^{x}}{y}\sin v$$

$$= 4e^{\ln(u\cos v)}\ln(u\sin v)\frac{1}{u} + 4e^{\ln(u\cos v)}\frac{1}{u\sin v}\sin v$$

$$= 4e\, u\cos v \ln(u\sin v)\frac{1}{u} + 4u\cos v\frac{1}{u}$$

$$= 4\cos v\left(\ln(u\sin v) + 1\right)$$

$$= 4\frac{1}{\sqrt{2}}\left(\ln\left(2\times\frac{1}{\sqrt{2}}\right) + 1\right)$$

$$= \frac{4\times 1.34}{\sqrt{2}} \approx 3.8$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= -4e^{x}\ln y \tan v + \frac{4e^{x}}{y}\, u\cos v$$

$$= -4e^{\ln(u\cos v)}\ln(u\sin v)\tan v + 4e^{\ln(u\cos v)}\frac{1\cdot u\cos v}{u\sin v}$$

$$= -4u\cos v\ln(u\sin v)\frac{\sin v}{\cos v} + 4u\cos v\frac{1}{u\sin v}u\cos v$$

$$= -4u\ln(u\sin v)\sin v + \frac{4u\cos^2 v}{\sin v}$$

$$= -4\times\cancel{2}^{\sqrt{2}}\times 0.34\times\frac{1}{\cancel{\sqrt{2}}} + 4\times\cancel{2}\times\frac{1}{\cancel{2}}\times\sqrt{2}$$

$$= 4\sqrt{2}(-0.34+1)$$

$$\approx 3.7$$

$\boxed{\text{OR}}$

we can directly write $z(u,v) = z(x(u,v), y(u,v))$ and take partial derivatives –

$$z = 4e^{\ln(u\cos v)}\ln(u\sin v) = 4u\cos v\ln(u\sin v)$$

$$\frac{\partial z}{\partial u} = 4\cos v\ln(u\sin v) + 4u\cos v\frac{\sin v}{u\sin v}$$

$$= 4\cos v\ln(u\sin v) + 4\cos v$$

$$\frac{\partial z}{\partial y} = -4u\sin v\ln(u\sin v) + 4u\cos v\frac{u\cos v}{u\sin v}$$

$$= -4u\sin v\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$$