Q.3) a) Give the definition of Cauchy sequence (1 mark)

Ans: A sequence a_n is called a Cauchy sequence if and only if $\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ (depending on $\varepsilon > 0$) such that $|a_n - a_m| < \varepsilon\, \forall\, n, m \geq N_\varepsilon$

+1 mark Exact same or re-worded definition

0 marks Otherwise

Q.3) b) Let $\{a_n\}$ be a monotonic sequence with a convergent subsequence then the monotonic sequence $\{a_n\}$ is convergent (2 marks)

Ans: From question, we are given a monotonic sequence $\{a_n\}$ with a convergent subsequence $\{a_{n_k}\}$ and we have to prove that $\{a_n\}$ converges as well. Let $\lim\limits_{k\to\infty} a_{n_k} = L$,

WLOG we assume $\{a_n\}$ is monotonically increasing,

Claim 1: $\{a_{n_k}\}$ is monotonically increasing
Since $\{a_{n_k}\}$ is a subsequence, so $\forall\, i, j \in \mathbb{N}$ such that $j > i$, it implies $n_j \geq n_i$ and since $\{a_n\}$ is monotonic, $a_{n_j} \geq a_{n_i}$. Thus taking $j = i + 1$, $a_{n_{i+1}} \geq a_{n_i}\ \forall\ i \in \mathbb{N}$, which makes $\{a_{n_k}\}$ a monotonically increasing sequence.

$a_{n_k} \leq L\ \forall\ k \in \mathbb{N}$ (Since a monotonically increasing sequence converges if and only if it is bounded and it converges at its supremum) **+0.5 marks**

Claim 2: $\forall\ n \in \mathbb{N}$, $a_n \leq L$
Since $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, $n_k \geq k\ \forall\ k \in \mathbb{N}$
$\implies a_{n_k} \geq a_k\ \forall\ k \in \mathbb{N}$ (Since $\{a_n\}$ is a monotonically increasing sequence)
$\implies a_k \leq a_{n_k} \leq L\ \forall\ k \in \mathbb{N}$
$\implies a_n \leq L\ \forall\ n \in \mathbb{N}$ **+1 mark**

Alternate Claim 2: $\forall\ n \in \mathbb{N}$, $a_n \leq L$
Let us assume to the contrary that $\exists\ u \in \mathbb{N}$ such that $a_u > L$,
$\implies \forall\, v \in \mathbb{N}$ such that $v \geq u$, $a_v \geq a_u > L$
$\implies a_{n_k} \in S = \{a_i \,|\, i \in \mathbb{N} \text{ such that } i < u\}\ \forall\ k \in \mathbb{N}$
But, S is a finite set (**Contradiction**) **+1 mark**

Claim 3: $\{a_n\}$ is convergent
Since $\forall\ n \in \mathbb{N}$, $a_n \leq L$ and $a_n \leq a_{n+1}$, a_n converges to $\sup\{a_n\}$. **+0.5 marks**

Hence, if $\{a_n\}$ is a monotonically increasing sequence with a convergent subsequence $\{a_{n_k}\}$ then the sequence $\{a_n\}$ is convergent

Q.4) a) Show that the sequence $\{n + \frac{(-1)^n}{n}\}$ is not Cauchy (1.5 marks)

Ans: From question, we are given a sequence $\{n + \frac{(-1)^n}{n}\}$ and we have to prove that it is not Cauchy.

Proof 1: If we can show that $\{n + \frac{(-1)^n}{n}\}$ violates the definition of a Cauchy sequence, we are done

Let us assume to the contrary that $a_n = n + \frac{(-1)^n}{n}$ is Cauchy,
$\forall\, \varepsilon > 0, \exists\, N_\varepsilon \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon \,\forall\, n, m \geq N_\varepsilon$,

Since $\{\frac{1}{n}\}$ converges to 0, it is also Cauchy,
Taking $\varepsilon = \frac{1}{2}$, $\exists\, N_{\frac{1}{2}} \in \mathbb{N}$ such that $|\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{2} \;\forall\, n, m \geq N_{\frac{1}{2}}$
$\implies -|\frac{1}{n} - \frac{1}{m}| \geq -\frac{1}{2} \;\forall\, n, m \geq N_{\frac{1}{2}}$ **+0.5 marks**

Thus $\forall\, m, n \geq N_{\frac{1}{2}}$,

$$\begin{aligned}
|m + \frac{(-1)^m}{m} - n - \frac{(-1)^n}{n}| &\geq ||m-n| - |\frac{(-1)^m}{m} - \frac{(-1)^n}{n}|| \;\; (|a-b| \geq ||a| - |b||) \\
&\geq ||m-n| - ||\frac{(-1)^m}{m}| - |\frac{(-1)^n}{n}||| \;\; (|a-b| \geq ||a| - |b||) \\
&= ||m-n| - |\frac{1}{m} - \frac{1}{n}|| \\
&\geq ||m-n| - \frac{1}{2}|
\end{aligned}$$

If $m > n \geq N_{\frac{1}{2}}$, $m - n \geq 1$ and $|m + \frac{(-1)^m}{m} - n - \frac{(-1)^n}{n}| \geq |1 - \frac{1}{2}| \geq \frac{1}{2}$ **+1 mark**
(Contradiction)

Hence, $\{n + \frac{(-1)^n}{n}\}$ is not Cauchy

Proof 2: If $a_n = n + \frac{(-1)^n}{n}$ diverges to $+\infty$ (not convergent), we are done

We first show $\forall\, n \in \mathbb{N}$,

$$\begin{aligned}
a_{n+1} - a_n &= n + 1 + \frac{(-1)^{n+1}}{n+1} - n - \frac{(-1)^n}{n} \\
&= 1 - \frac{(-1)^n}{n} - \frac{(-1)^n}{n+1} \\
&= 1 + \frac{1}{n} + \frac{1}{n+1} \geq 0 \text{ (if } n \text{ is odd)} \\
&= 1 - \frac{1}{n} - \frac{1}{n+1} \geq 1 - \frac{1}{2} - \frac{1}{3} \text{ (if } n \text{ is even } \implies n \geq 2)
\end{aligned}$$

$a_{n+1} - a_n \geq 0 \implies a_{n+1} \geq a_n \ \forall \ n \in \mathbb{N}$ **+0.5 marks**

Let $G > 0$ be an arbitrarily large real number,
$\exists \ N = \lfloor G + 2 \rfloor \in \mathbb{N}$, such that $a_n \geq a_N \geq 2 + \lceil G \rceil - \frac{1}{\lfloor G+2 \rfloor} \geq G \ \forall \ n \geq N$
$\implies a_n = n + \dfrac{(-1)^n}{n}$ diverges to $+\infty$ **+0.5 marks**
$\implies a_n$ is not convergent
$\implies a_n$ is not Cauchy **+0.5 marks**

Proof 3: If we can show that $\{n - \dfrac{1}{n}\}$ diverges to $+\infty$, we are done
$(n + \dfrac{(-1)^n}{n} \geq n - \dfrac{1}{n}$ for $n \in \mathbb{N})$ **+0.5 marks**

Taking $a_n = n$ which diverges to $+\infty$ and $b_n = \dfrac{-1}{n}$ which converges to 0,
$a_n - b_n = n - \dfrac{1}{n}$ diverges to $+\infty$ **+0.5 marks**

Taking $a_n = n + \dfrac{(-1)^n}{n}$ and $b_n = n - \dfrac{1}{n}$ which diverges to $+\infty$ and such that $a_n \geq b_n$ $\forall \ n \in \mathbb{N}$, $a_n = n + \dfrac{(-1)^n}{n}$ diverges to $+\infty$ **+0.5 marks**
$\implies a_n$ is not convergent
$\implies a_n$ is not Cauchy

Proof 4: If we can show that $\{n + \dfrac{(-1)^n}{n}\}$ is unbounded, we are done

Let us assume to the contrary that $\{n + \dfrac{(-1)^n}{n}\}$ is bounded above by L,

Thus, $n + \dfrac{(-1)^n}{n} \leq L \ \forall \ n \in \mathbb{N}$
But for $n = \lceil L + 2 \rceil$, $a_n \geq \lceil L \rceil + 2 - \dfrac{1}{\lceil L + 2 \rceil} \geq L$ (**Contradiction**) **+1 mark**
$\implies a_n$ is not bounded above
$\implies a_n$ is not Cauchy **+0.5 marks**

Proof 5: Let us assume to the contrary that $a_n = n + \dfrac{(-1)^n}{n}$ is Cauchy,
$a_n = n + \dfrac{(-1)^n}{n}$ is Cauchy
$\implies a_n = n + \dfrac{(-1)^n}{n}$ is convergent

Assume $\lim\limits_{n \to \infty} a_n = L \ (L \in \mathbb{R})$,

Claim: $\lim\limits_{n \to \infty} \dfrac{(-1)^n}{n} = 0$
$\dfrac{1}{n}$ converges to 0

$\implies \forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ such that $|\frac{1}{n} - 0| < \varepsilon \,\forall\, n \,\geq\, N_\varepsilon$

$\implies \forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ such that $|\frac{(-1)^n}{n} - 0| < \varepsilon \,\forall\, n \,\geq\, N_\varepsilon$ (Since $|\frac{(-1)^n}{n}| = \frac{1}{n} = |\frac{1}{n}|$)

$\implies \frac{(-1)^n}{n}$ converges to 0 **+0.5 marks**

Taking $a_n = n + \frac{(-1)^n}{n}$ with $\lim\limits_{n\to\infty} a_n = L$ and $b_n = \frac{(-1)^n}{n}$ with $\lim\limits_{n\to\infty} b_n = 0$,
$a_n - b_n$ converges with limit $\lim\limits_{n\to\infty} a_n - b_n = L - 0 = L$
But, $a_n - b_n = n$ which diverges to $+\infty$ (**Contradiction**) **+0.5 marks**

Hence, $\{n + \frac{(-1)^n}{n}\}$ is not convergent and thus, not a Cauchy sequence. **+0.5 marks**