RA - I Mid-Sem Exam

Q5.(b) $\{a_n\}$ & $\{b_n\}$ are cauchy

$\Rightarrow$ $\{a_n\}$ & $\{b_n\}$ are convergent.

Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

Let $\{c_n\} = \{a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots\}$

where $c_{2n-1} = a_n \ \forall\ n \geq 1$

and $c_{2n} = b_n \ \forall\ n \geq 1$

TP $\{c_n\}$ is cauchy iff $A = B$.

($\Rightarrow$) Suppose $\{c_n\}$ is cauchy. Let $\varepsilon > 0$ be arbitrary

Then for $\varepsilon/3 > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that

$|c_n - c_m| < \varepsilon/3 \ \forall\ n, m \geq N_\varepsilon$

Clearly $2n > 2n-1 \geq n \ \forall\ n \in \mathbb{N}$

& $2m > 2m-1 \geq m \ \forall\ m \in \mathbb{N}$

$\Rightarrow |a_n - b_m| < \varepsilon/3 \ \forall\ n, m \geq N_\varepsilon$ — (i) (½ mark)

($\because a_n = c_{2n-1}$ and $b_m = c_{2m}$)

Now, consider $|A - B|$

$\leq |A - a_n| + |a_n - b_m| + |b_m - B|$ — (1)

(By Triangle's inequality)

$\because \lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$

$\therefore$ By def^n, for $\varepsilon/3 > 0$, $\exists\ K_1, K_2 \in \mathbb{N}$ such that

$|a_n - A| < \varepsilon/3 \ \forall\ n \geq K_1$ — (ii)

and $|b_m - B| < \varepsilon/3 \ \forall\ m \geq K_2$ — (iii) (½ mark)

Let $K = \max\{K_1, K_2, N_\varepsilon\}$

Then, $\forall\ n, m \geq K$

$|A-B| < \epsilon/3 + \epsilon/3 + \epsilon/3$

(Using (i), (ii) & (iii) in eq^n (1))

$\Rightarrow |A-B| < \epsilon$ ($\because$ A & B are independent of n & m)

$\Rightarrow 0 \leq |A-B| < \epsilon\ \forall\ \epsilon > 0$ (As $\epsilon > 0$ was arbitrary)

$\Rightarrow |A-B| = 0 \Rightarrow A - B = 0.$

$\Rightarrow A = B$ (1 mark)

($\Leftarrow$) Suppose $A = B$.

for $\epsilon/2 > 0$, $\exists\ K_1, K_2 \in \mathbb{N}$ such that

$|a_n - A| < \epsilon/2\ \forall\ n \geq K_1$

and $|b_m - B| < \epsilon/2\ \forall\ m \geq K_2$

Let $K = \max\{K_1, K_2\}$

Then $|a_n - b_m| \leq |a_n - A| + |A - b_m|$

$= |a_n - A| + |B - b_m|$

$< \epsilon/2 + \epsilon/2\ \ \forall\ n, m \geq k$

$\Rightarrow |a_n - b_m| < \epsilon\ \forall\ n, m \geq k$ — (2)

$\therefore$ for every $\epsilon > 0$, $\exists\ k \in \mathbb{N}$ s.t.

$|a_n - b_m| < \epsilon\ \forall\ n, m \geq k$ (0.75 marks)

~~Moreover, whenever $n, m \geq K \geq K_1, K_2$~~

Moreover, $\because \{a_n\}$ & $\{b_n\}$ are cauchy

$\Rightarrow$ for $\epsilon > 0$, $\exists\ K_1' \in \mathbb{N}$ and $K_2' \in \mathbb{N}$ such that

$|a_n - a_m| < \epsilon\ \forall\ n, m \geq K_1'$ — (3)

$|b_n - b_m| < \epsilon\ \forall\ n, m \geq K_2'$ — (4)

Let $N_0 = (\max\{K, K_1', K_2'\}) \times 2$

Then, by eqn (2), (3) and (4)

$|c_n - c_m| < \epsilon \ \forall\ n, m \geq N_0$

$\therefore$ for every $\epsilon > 0$, $\exists\ N_0 \in \mathbb{N}$ such that $|c_n - c_m| < \epsilon$
$\forall\ n, m \geq N_0$

By definition, $\{c_n\}$ is Cauchy. (0.75 marks)

Q6.(b) Given $\{x_n\}$ is an unbounded sequence.

$\Rightarrow$ either $\{x_n\}$ is unbounded from above or below or both.

~~let us assume that $\{x_n\}$ is unbounded from above.~~

$\Rightarrow \nexists\ M \in \mathbb{N}$ such that $|x_n| \leq M \ \forall\ n \in \mathbb{N}$

$\Rightarrow$ for every $K \in \mathbb{N}$, $\exists\ x_{n_k'} \in \{x_n\}$ such that

$|x_{n_k'}| > K$ (0.5 marks)

for $k = 1$, $\exists\ x_{n_1'} \in \{x_n\}$ with $|x_{n_1'}| > 1$

let $n_1 = n_1'$

for $k = 2$, $\exists\ x_{n_2'} \in \{x_n\}$ with $|x_{n_2'}| > 2$

Choose $n_2' > n_1$

[Such a choice is always possible. For if $|x_n| \leq 2 \ \forall\ n > n_1$

Let $L = \max\{|x_1|, |x_2|, \ldots |x_{n_1 - 1}|, 2\}$

$\Rightarrow |x_n| \leq L \ \forall\ n \in \mathbb{N}$

$\Rightarrow \{x_n\}$ is bounded which contradicts the hypothesis]

Let $n_2' = n_2$

Similarly, choose $x_{n_k} \ \forall\ k$ such that $n_k > n_{k-1}$

and $|x_{n_k}| > k \ \forall \ k \in \mathbb{N}$ (0.5 marks)

$$\Rightarrow \frac{1}{|x_{n_k}|} < 1/k \ \forall \ k \quad \text{—} (1)$$

Observe that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ as $\{n_k\}$ is an increasing sequence of natural nos. by choice. Let $\varepsilon > 0$ be arbitrary

Consider $\left|\frac{1}{x_{n_k}} - 0\right|$

$$= \left|\frac{1}{|x_{n_k}|}\right| < \left|1/k\right| \quad (\text{By } (1))$$

$$\Rightarrow \left|\frac{1}{x_{n_k}} - 0\right| < 1/k$$ (0.5 marks)

By archimedean property, $\exists \ N_0 \in \mathbb{N}$ such that

$$1 < \varepsilon N_0$$

$$\Rightarrow 1/N_0 < \varepsilon$$

$\therefore$ whenever $k \geq N_0$

$$\left|\frac{1}{x_{n_k}} - 0\right| < 1/k \leq 1/N_0 < \varepsilon$$

$\Rightarrow$ for every $\varepsilon > 0$, $\exists \ N_0 \in \mathbb{N}$ such that

$$\left|\frac{1}{x_{n_k}} - 0\right| < \varepsilon \ \forall \ k \geq N_0$$

$$\Rightarrow \lim_{k \to \infty} \frac{1}{x_{n_k}} = 0$$ (0.5 marks)

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Q6. (a) $\{x_n\}$ defined by $x_1 = 1$

$$x_{n+1} = x_n\left(1 + \frac{\sin n}{2^n}\right) \ \forall\ n \geq 1$$

For any $k \in \mathbb{N}$ (fixed),

$$x_{k+1} = x_k\left(1 + \frac{\sin k}{2^k}\right)$$

$$= x_{k-1}\left(1 + \frac{\sin(k-1)}{2^{k-1}}\right)\left(1 + \frac{\sin k}{2^k}\right)$$

$$= x_{k-2}\left(1 + \frac{\sin(k-2)}{2^{k-2}}\right)\left(1 + \frac{\sin(k-1)}{2^{k-1}}\right)\left(1 + \frac{\sin k}{2^k}\right)$$

$$\vdots$$

$$= x_1\left(1 + \frac{\sin 1}{2}\right)\left(1 + \frac{\sin 2}{2^2}\right)\cdots\left(1 + \frac{\sin k}{2^k}\right)$$

$(x_1 = 1)$ (0.5 marks)

$$\Rightarrow |x_{k+1}| \leq \left(1 + \frac{\sin 1}{2}\right)\left(1 + \frac{|\sin 2|}{2^2}\right)\cdots\left(1 + \frac{|\sin k|}{2^k}\right)$$

(By Triangle's Inequality)

$$\leq \left(1 + 1/2\right)\left(1 + 1/2^2\right)\cdots\left(1 + 1/2^k\right) \quad \text{—(1)}$$

$(\because |\sin\theta| \leq 1)$

Let $a_i = \left(1 + 1/2^i\right)$ $\because$ AM $\geq$ GM

$$\Rightarrow \left[\frac{\left(1+1/2\right) + \left(1+1/2^2\right) + \cdots \left(1+1/2^k\right)}{k}\right]^k \geq \left(1+1/2\right)\left(1+1/2^2\right)\cdots\left(1+1/2^k\right)$$

$$= \left(\frac{k + \sum_{j=1}^{k} 1/2^j}{k}\right)^k$$

$\therefore$ Equation ① becomes

$$|x_{k+1}| \leq \left(\frac{k + \sum_{j=1}^{k} 1/2^j}{k} \right)^k$$

$$= \left(1 + \frac{\cancel{1/2}\left(1 - 1/2^k\right)}{\cancel{1/2} \cdot k} \right)^k$$

$$\leq \left(1 + 1/k\right)^k \quad \left(\text{as } 1 - 1/2^k < 1 \ \forall k\right) \text{ — ②}$$

(0.75 marks)

We use the fact that $\lim_{k \to \infty} \left(1 + 1/k\right)^k = e$

(Euler's constant)

$\Rightarrow$ for $\varepsilon = 1$, $\exists$ $N_0 \in \mathbb{N}$ such that

$$\left| \left(1 + 1/k\right)^k - e \right| < 1 \quad \forall \ k \geq N_0$$

$$\Rightarrow \quad \left(1 + 1/k\right)^k < 1 + e \quad \forall \ k \geq N_0$$

$\therefore$ Eq^n ② becomes

$$|x_{k+1}| < 1 + e \quad \forall \ k \geq N_0 \text{ — (I)} \quad (0.5 \text{ marks})$$

Consider $|x_{n+1} - x_n| = |x_n| \left| \frac{\sin n}{2^n} \right|$

$$\leq \frac{(e+1)}{2^n} \quad \forall \ n \geq N_0 + 1$$

(By (I))

Whenever $n > m \geq N_0 + 1$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m+1} - x_m|$$

$$\leq (e+1)\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m}\right)$$

$$= (e+1)\frac{\frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)}{1/2}$$

$$\leq 2(e+1) \cdot 1/2^m \quad \text{---} \quad \text{II}$$ (0.75 marks)

Let $\varepsilon > 0$ be any arbitrary real no.
By archimedian property, $\exists\ N_1 \in \mathbb{N}$ such that $\frac{2(e+1)}{2^{N_1}} < \varepsilon$

$$\Rightarrow \frac{2(e+1)}{\varepsilon} < 2^{N_1}$$

$$\Rightarrow N_1 > \log_2\left(\frac{2(e+1)}{\varepsilon}\right)$$

Let $N = \max\{N_0+1, N_1\}$
Whenever $n > m \geq N \geq N_1$

$$\Rightarrow |x_n - x_m| \leq \frac{2(e+1)}{2^m} \leq \frac{2(e+1)}{2^N} < \varepsilon$$
(By II)

$\therefore$ for every $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that
$|x_n - x_m| < \varepsilon \ \forall\ n > m \geq N$

By definition, $\{x_n\}$ is cauchy & hence convergent.

(0.5 marks)