Assignment 6

October 17, 2023

1. Prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous such that given any two points $x_1 < x_2$, there exists a point x_3 such that $x_1 < x_3 < x_2$ and $f(x_3) = g(x_3)$. Show that $f(x) = g(x)$ for all x.

3. Suppose f is a function from $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{|x|\to\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either an absolute maximum or an absolute minimum!

4. There does not exist a continuous function f from $[0, 1]$ onto $\mathbb{R}$ – Why?

5. Find a continuous function f from $(0, 1)$ onto $\mathbb{R}$.

6. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that for each $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove there exists a point c in $[a, b]$ such that $f(c) = 0$.

7. Let $f : (0, 1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors,} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

(a) Show that f is continuous at every irrational.
(c) Show that f is discontinuous at every rational.