Practice Problem with Solutions (Chapter -Series)

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1 Problems

1. Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(3n)!+4^{n+1}}{(3n+1)!}$.

2. Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$.

3. Test the convergence or divergence of the series $\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$.

4. Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}+3}{n^2-n+1}$.

5. Prove if $\{a_n\}$ is a decreasing sequence of real numbers and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

6. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non negative numbers and $p > 1$, then $\sum_{n=1}^{\infty} a_n^p$ converges.

7. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of non negative real numbers, such that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n$ both converge, then prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

2 Solutions

Problem 1. Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(3n)!+4^{n+1}}{(3n+1)!}$.

Let $S_k = \sum_{n=1}^{k} c_n$ and $c_n = \frac{(3n)!+4^{n+1}}{(3n+1)!} = a_n + b_n$ where $a_n = \frac{(3n)!}{(3n+1)!}$ and $b_n = \frac{4^{n+1}}{(3n+1)!}$ Now

$$c_n = a_n + b_n \geq a_n \quad \forall\ n \geq \mathbb{N} \tag{1}$$

It is easy to see

$$a_n = \frac{(3n)!}{(3n+1)!} = \frac{1}{3n+1} > \frac{1}{3n+3} = \frac{1}{3}\frac{1}{n+1}$$

where we have used the fact $3n+3 > 3n+1\ \forall\ n \geq 1$. Now $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is a divergent series so by Comparison test $\sum_{n=1}^{\infty} a_n$ diverges. And again by comparison test $\sum_{n=1}^{\infty} c_n$ diverges as by (1) $c_n \geq a_n$.

Problem 2. Consider $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$.

We can decompose the fraction $\frac{2}{n^2+2n}$ as

$$\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}.$$

The partial sum $S_n = \sum_{j=1}^{n} \frac{2}{j^2+2j} = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+2}\right)$

We have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}, \quad \text{so} \sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{2}.$$

Problem 3. Consider $\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$.

Let us observe that

$$\begin{aligned} \frac{1}{(\frac{1}{2}n-5)^3} &= \frac{2^3}{(n-10)^3} \quad \forall\ n \geq 11 \\ &= \frac{2^3}{m^3} \quad \forall\ m \geq 1 \end{aligned}$$

where in the last step we have taken $n - 10 = m$ and $n \geq 11$ implies $m \geq 1$. Now $8\sum_{m=1}^{\infty} \frac{1}{m^3}$ converges so $\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$ converges by $p = 3$ series test.

Problem 4. Determine the convergence of $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}+3}{n^2-n+1}$.

Let us take $a_n = \dfrac{\sqrt{n}+3}{n^2-n+1}$ and $b_n = \frac{1}{n^{\frac{3}{2}}}$ and look at

$$\begin{aligned} r = \lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} &= \lim_{n\to\infty}\frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1} \\ &= \lim_{n\to\infty}\frac{(1+\frac{3}{\sqrt{n}})}{1-\frac{1}{n}+\frac{1}{n^2}} = 1 \end{aligned}$$

By Limit Comparison test the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}+3}{n^2-n+1}$ converges.
If we have taken $b_n = \frac{1}{n^2}$ we would get $r = \infty$ so in that case we can not make any conclusion from Limit comparison test (as to apply Limit comparison test we need $0 < r < \infty$).

Problem 5. Prove if $\{a_n\}$ is a decreasing sequence of real numbers and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

Let $S_k = \sum_{n=1}^{k} a_n$. As $\sum_{n=1}^{\infty} a_n$ converges, so $\lim_{n\to\infty}|a_n| = 0$ by Lemma 2.4 and $\{S_k\}$ is a Cauchy sequence. Suppose $\varepsilon > 0$ be any small real number. Then there exists $N_0 \in \mathbb{N}$ (depending on ε) such that

$$|S_m - S_k| < \varepsilon \quad \forall\ m, k \geq N_0$$

In particular we put $k = N_0$, then

$$\begin{aligned} &|S_m - S_{N_0}| < \varepsilon \quad \forall m \ \geq N_0 \\ \Rightarrow\ &|a_m + a_{m-1} + \ldots\ldots.. + a_{N_0+1} + a_{N_0}| < \varepsilon \quad \forall m \ \geq N_0 \end{aligned} \tag{2}$$

Now

$$|a_m|(m - N_0) < |a_m + a_{m-1} + \ldots\ldots.. + a_{N_0+1} + a_{N_0}| \quad \forall m \ \geq N_0 \tag{3}$$

as $\{a_n\}$ is a decreasing sequence, so $a_m \leq a_{m-1} \leq a_{m-2} \leq \cdots a_2 \leq a_1$ for all $m \in \mathbb{N}$. So from (2) and (3),

$$|a_m|(m - N_0) < |a_m + a_{m-1} + \ldots\ldots.. + a_{N_0+1} + a_{N_0}| < \varepsilon \quad \forall m \geq N_0$$

Taking limit $m \to \infty$

$$\begin{aligned} &0 \leq \lim_{m\to\infty} |a_m|(m - N_0) \leq \varepsilon \\ &0 \leq \lim_{m\to\infty} |ma_m| - N_0 \lim_{m\to\infty} |a_m| \leq \varepsilon \\ &0 \leq \lim_{m\to\infty} |ma_m| \leq \varepsilon \quad \text{as} \lim_{m\to\infty} |a_m| = 0. \end{aligned}$$

Since $\varepsilon > 0$ is arbitary. The above inequality holds for any $\varepsilon > 0$. Hence (by Problem 6, Assignment 1) $\lim_{m\to\infty}|ma_m| = 0$ implies $\lim_{m\to\infty} ma_m = 0$.($\lim_{n\to\infty}|x_n| = 0$ iff $\lim_{n\to\infty} x_n = 0$)

Problem 6. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non negative numbers and $p > 1$, then $\sum_{n=1}^{\infty} a_n^p$ converges.
Let $S_k = \sum_{n=1}^{k} a_n$ and $T_k = \sum_{n=1}^{k} a_n^p$. As $\sum_{n=1}^{\infty} a_n$ is a convergent series, so $\lim_{n\to\infty} a_n = 0$. So for $\varepsilon = 1$. There exists $N_1\mathbb{N}$ such that

$$\begin{aligned} &a_n < 1 \quad \forall n \geq N_1 \\ \Rightarrow\ &a_n^{p-1} < 1 \ \ \forall n \geq N_1 \end{aligned}$$

Given $\varepsilon > 0$. As $\sum_{n=1}^{\infty} a_n$ is a convergent series, so $\{S_k\}$ is a Cauchy sequence. So there exists $N_2 \in \mathbb{N}$ such that

$$S_k - S_m = |S_k - S_m| < \varepsilon \ \ \forall\ k, m \geq N_2$$

We choose $N = max(N_1, N_2)$. So $\forall\ k, m \geq N$

$$T_k - T_m = \sum_{n=m}^{k} a_n^p = \sum_{n=m}^{k} a_n^{p-1} a_n \leq \sum_{n=m}^{k} a_n = S_k - S_m < \varepsilon$$

Hence $\{T_n\}$ is a Cauchy sequence, so it converges.

Problem 7. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of non negative real numbers, such that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n$ both converge, then prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.
Let $S_k = \sum_{n=1}^{k} b_n$ and $T_k = \sum_{n=1}^{k} a_n b_n$. Given $\sum_{n=1}^{\infty} b_n$ converges i.e sequence of partial sums $\{S_k\}$ is convergent, hence bounded above (by S say i.e $S_k \leq S$ $\forall\ k \geq 1$).
As $\sum_{n=1}^{\infty} a_n^2$ converges. So $\lim_{n\to\infty} a_n^2 = 0$. And by Example 2.5 of previous chapter, $\lim_{n\to\infty} a_n = 0$ as $a_n \geq 0\ \forall\ n \geq 1$. So the sequence $\{a_n\}$ is bounded as it is convergent $\Rightarrow$ there exists $M > 0$ such that $a_n = |a_n| \leq M\ \forall\ n \geq 1$. Now

$$T_k = \sum_{n=1}^{k} a_n b_n \leq M \sum_{n=1}^{k} b_n = M S_k \leq M S$$

So $\{T_k\}$ is a bounded above sequence. And $T_{k+1} - T_k = \sum_{n=1}^{k+1} a_n b_n - \sum_{n=1}^{k} a_n b_n = a_k b_k \geq 0$. So $\{T_k\}$ are monotonically increasing sequence. Now $\{T_k\}$ is monotonically increasing and bounded sequence, therefore it is convergent (by Theorem 3.1 of previous chapter) which implies $\sum_{n=1}^{\infty} a_n b_n$ converges.