Ques 1. Let $\varepsilon > 0$, be any given positive real no.

RTP : For this given $\varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$ s.t $\left| b/n^2 - 0 \right| < \varepsilon$ $\forall\, n \geq N_\varepsilon$.

(i.e we have to identify that $N_\varepsilon \in \mathbb{N}$, for which

$\left| b/n^2 - 0 \right| < \varepsilon \quad \forall\, n \geq N_\varepsilon$)

By ARCHIMEDEAN PROPERTY
$(x = \varepsilon > 0,\ y = |b|)$ $\exists\, N_\varepsilon \in \mathbb{N}$ s.t $N_\varepsilon \varepsilon > |b|$

$$\Rightarrow \frac{|b|}{N_\varepsilon} < \varepsilon \quad \text{—— (2)}$$

Hence, from (2) we can write,

$$\left| \frac{b}{n^2} - 0 \right| = \frac{|b|}{n^2} < \frac{|b|}{n} < \varepsilon \quad \forall\, n \geq N_\varepsilon$$

$$\left[n^2 \geq n \Rightarrow \frac{1}{n^2} \leq \frac{1}{n} \right]$$

$$\left[\because n \geq N_\varepsilon \Rightarrow \frac{1}{n} \leq \frac{1}{N_\varepsilon} \right]$$

So, we have identified N_ε, corresponding to given $\varepsilon > 0$, by Arch. property s.t $\left| \frac{b}{n^2} - 0 \right| < \varepsilon \quad \forall\, n \geq N_\varepsilon$

As $\varepsilon > 0$ is arbitrary positive real no. so, $\lim_{n \to \infty} b/n^2 = 0$

[by definition]

Ques 2. TP: $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Rationalise $(\sqrt{n+1} - \sqrt{n})$

so, $\dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \underbrace{\dfrac{1}{(\sqrt{n} + \sqrt{n+1})} \leq \dfrac{1}{\sqrt{n}}}_{\boxed{}\text{ (1)}}$

Let $\varepsilon > 0$, be any given positive real no.

RTP: For this given $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t $|(\sqrt{n+1} - \sqrt{n}) - 0| < \varepsilon$ $\forall n \geq N_\varepsilon$

(i.e we have to identify that $N_\varepsilon \in \mathbb{N}$, for which $|(\sqrt{n+1} - \sqrt{n}) - 0| < \varepsilon \ \forall n \geq N_\varepsilon$)

By ARCHIMEDEAN PROPERTY $(x = \varepsilon^2 > 0,\ y = 1)$ $\exists N_\varepsilon \in \mathbb{N}$ s.t $N_\varepsilon \varepsilon^2 > 1$

$\Rightarrow \varepsilon^2 > \dfrac{1}{N_\varepsilon} \Rightarrow \varepsilon > \dfrac{1}{\sqrt{N_\varepsilon}}$ —— (2)

Hence from (2) we can write,

$|(\sqrt{n+1} - \sqrt{n}) - 0| = \underbrace{\sqrt{n+1} - \sqrt{n} \leq \dfrac{1}{\sqrt{n}}}_{\text{by (1)}} \underbrace{\leq}_{} \underbrace{\dfrac{1}{\sqrt{N_\varepsilon}} < \varepsilon}_{\text{by (2)}}$

(as $n \geq N_\varepsilon$
so, $\frac{1}{n} \leq 1/N_\varepsilon$
so, $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N_\varepsilon}}$).

So, we have identified N_ε, corresponding to given $\varepsilon > 0$ by Arch. property s.t $\left[\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right] < \varepsilon \quad \forall \; n \geq N_\varepsilon$

As $\varepsilon > 0$ is arbitrary positive real no. so, $\lim_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

[by definition]

Ques 3. As we know $\lim_{n\to\infty} x_n = x > 0$

So, By Defn. it is known that for every $\varepsilon > 0$ $\exists$ $N_\varepsilon \in \mathbb{N}$ s.t $|x_n - x| < \varepsilon$ $\forall$ $n \geq N_\varepsilon$.

In particular take $\varepsilon = x/2 > 0$; so for this particular $\varepsilon = x/2 > 0$ $\exists$ $N_{x/2} \in \mathbb{N}$ s.t

$$|x_n - x| < x/2 \quad \forall \; n \geq N_{x/2}$$

$$\Rightarrow \quad -x/2 < x_n - x < x/2$$

$$\Rightarrow \quad x/2 < x_n < 3x/2 < 2x$$

So, we can conclude by choosing $K = N_\varepsilon$, we get that for every $n \geq K$;

$$x/2 < x_n < 2x.$$

Ques. 4 Let $\varepsilon > 0$, be any given positive real no.

RTP: For this given $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t

$$\left|\left(\frac{n}{n+1}\right) - 1\right| < \varepsilon \quad \forall \; n \geq N_\varepsilon \quad \text{or}$$

$$\left|\frac{1}{n+1}\right| < \varepsilon \quad \forall \; n \geq N_\varepsilon$$

(i.e we have to identify that $N_\varepsilon \in \mathbb{N}$, for which $\left|\frac{1}{n+1}\right| < \varepsilon \;\forall\; n \geq N_\varepsilon$)

By ARCH. PROPERTY,

$(x = \varepsilon, y = 1)$ $\exists \; N_\varepsilon \in \mathbb{N}$ s.t $N_\varepsilon \varepsilon > 1$

$$\Rightarrow \varepsilon > 1/N_\varepsilon \quad \text{— (1)}$$

From (1) we can get,

$$\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{N_\varepsilon} < \varepsilon \quad \forall \; n \geq N_\varepsilon$$

so, we have identified N_ε, corresponding to given $\varepsilon > 0$, by Arch. property s.t $\left|\left(\frac{n}{n+1}\right) - 1\right| < \varepsilon \;\forall\; n \geq N_\varepsilon$

As $\varepsilon > 0$ is arbitrary positive real no. so,

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

(by def.)

Q5)

$x_n = (-1)^n n^2$; $n \in \mathbb{N}$

Suppose this sequence converges to L.

Then for every $\varepsilon>0$, $\exists$ $N_\varepsilon \in \mathbb{N}$ s.t $|x_n - L| < \varepsilon$ $\forall$ $n \geq N_\varepsilon$

In particular, take $\varepsilon = 1$.

so, for this $\varepsilon = 1 > 0$ $\exists$ a $N_1 \in \mathbb{N}$ s.t

$$|x_n - L| < 1 \quad \forall \; n \geq N_1$$

$$\Rightarrow \; |(-1)^n n^2 - L| < 1 \quad \forall \; n \geq N_1 \quad \text{—— (1)}$$

so, $\forall$ $n \geq N_1$

$$n^2 = |(-1)^n n^2| = |(-1)^n n^2 - L + L| \leq |(-1)^n n^2 - L| + |L|$$

$$< 1 + |L| \quad (\text{by (1)})$$

Hence, $\forall$ $n \geq N_1$ we have $n^2 < 1 + |L|$, so

this ~~appears~~ implies the set $\{n^2 \in \mathbb{N} \, ; \, n \geq N_1\}$ is

bonded from above which is ABSURD.

So, our assumption $\{x_n\}$ converges to L is wrong

—— (a)

H.W

This sequence doesn't converge to $+\infty$ or $-\infty$.

[Try to prove this statement on your own] —— b

Hence; by a & b we get that the sequence neither converges nor diverges.

Hence it is OSCILLATORY.