Q2 ①

a) Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. The f^n is continuous & partial derivatives exist.

$f_x(x,y) = 2x - 2$ & $f_y(x,y) = 2y - 6$

so $f_x(x,y) = f_y(x,y) = 0$ if $x = 1$ & $y = 3$.

so critical point occurs at $(1,3)$ → ①

Let us check whether it is a local maxima / minima or saddle. Now $f(1,3) = 4$.

Method 1

$f(x,y) = x^2 - 2x + 1 + y^2 - 2 \cdot 3y + 9 + 4$

$= (x-1)^2 + (y-3)^2 + 4 \geq 4 = f(1,3)$

so $f(1,3) \leq f(x,y)$ $\forall$ $(x,y) \in D$ domain of f

so it is the pt of local minima — 1.5

Method 2

Or $f_{xx}(x,y) = 2$ so $f_{xx}(1,3) = 2 > 0$

$f_{yy}(x,y) = 2$ so $f_{yy}(1,3) = 2$

& $f_{xy}(x,y) = f_{yx}(x,y) = 0$ so $f_{xy}(1,3) = 0$

$D = f_{xx}(1,3) f_{yy}(1,3) - f_{xy}^2(1,3) = 4 > 0$

so by 2nd derivative test, $(1,3)$ is the local minima. — 1.5

2) b) Equivalent condⁿ of differentiability (2)

f is differentiable at (x_0, y_0) iff

$$\lim_{P \to 0} \frac{\Delta f(x_0, y_0) - df(x_0, y_0)}{P} = 0 \quad (0.25)$$

where $(h, k) \in \mathbb{R}^2$ & $\Delta f(x_0, y_0) = f(x_0+h, y_0+k) - f(x_0, y_0)$

& $df(x_0, y_0) = h f_x(x_0, y_0) + k f_y(x_0, y_0)$

& $P = \sqrt{h^2+k^2}$ (0.05)

[Now $(x,y) \neq (0,0)$ & $\varepsilon > 0$ be any arbitrary +ve real no.

$|f(x,y) - f(0,0)| = |f(x,y) - 0|$

$$= \frac{|x|\sqrt{|y|}}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{|y|}}{\sqrt{x^2+y^2}} \quad [\because |x| \leq \sqrt{x^2+y^2}]$$

$$\leq \sqrt{|y|} \leq \sqrt{x^2+y^2} \quad (0.75)$$

So $\forall (x,y) \in \mathbb{R}^2$ with $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta_\varepsilon = \varepsilon/2$

will imply $|f(x,y) - f(0,0)| \leq \sqrt{x^2+y^2} < \frac{\varepsilon}{2} < \varepsilon$ (0.75)

So f is continuous at $(x,y) = (0,0)$ (ans)

If f is diff at $(0,0)$ then by equivalent condition

$$\lim_{P \to 0} \frac{\Delta f(x_0, y_0) - df(x_0, y_0)}{P} = 0. \longrightarrow (*)$$

Let us evaluate $f_x(0,0)$ & $f_y(0,0)$.

$f(x,0) = \frac{x\sqrt{0}}{\sqrt{x^2}} = 0 \;\&\; f(0,y) = \frac{0.\sqrt{|y|}}{\sqrt{y^2}} = 0.$

So $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$ .5

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$ .5

so $df(0,0) = h f_x(0,0) + k f_y(0,0) = 0$

So it remains to prove $\lim_{P\to 0} \frac{\Delta f(0,0)}{P} = 0$ (by $\times$) $\xrightarrow{(-)}$

where $\Delta f(0,0) = f(h, k+0) - f(0,0) = \frac{h\sqrt{|k|}}{\sqrt{h^2+k^2}} - 0$

so we need to prove by $(-)$ $\quad (P = \sqrt{h^2+k^2}$

$$\lim_{P\to 0} \frac{h\sqrt{|k|}}{\sqrt{h^2+k^2}\cdot\sqrt{h^2+k^2}} = \lim_{P\to 0} \frac{h\sqrt{|k|}}{(h^2+k^2)}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h\sqrt{|k|}}{(h^2+k^2)} \to 1$$

If we take (h,k) approaches $(0,0)$ along $h=k$, then $\lim_{h\to 0} \frac{h\sqrt{|h|}}{2h^2} = \lim \frac{\sqrt{|h|}}{h}$

This limit does not exist

so the f^n is not diff at $(0,0)$ since .5

$\lim_{P\to 0} \frac{\Delta f(0,0) - df(0,0)}{P}$ does not exist.

2) c). $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$, $f_x(x,y) = x$, $f_y(x,y) = y$

$\vec{\nabla} f(1,1) = 1\hat{i} + 1\hat{j}$

so it increases the most (when $D_{\vec{u}} f(1,1) = \vec{\nabla} f(1,1) \cdot \vec{u}$

$= |\vec{\nabla} f(1,1)| \cdot |\vec{u}| \cos\theta = |\vec{\nabla} f(1,1)| \cos\theta = |\vec{\nabla} f(1,1)|$

that is $\cos\theta = 1$ implies $\theta = 0$) so $\vec{u_1}$ & $\vec{\nabla} f(1,1)$ have same direction & that mean) $\vec{u_1} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{2}$ 1

(ii) decreases the most when $\cos\theta = -1$ that is $\theta = \pi$, so. $\vec{u_2}$ & $\vec{\nabla} f(1,1)$ are in opposite direction to each other) so $\vec{u_2} = -\vec{u_1} = -\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$ 1

(iii) no change if $\cos\theta = 0$ that is $\theta = \pi/2$ so $\vec{u_3}$ & $\vec{\nabla} f(1,1)$ must be orthogonal to each other, hence $\vec{u_3} = -\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$ or $\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$ 1

Q3) a) Given $f(x)$ & $g(x)$ is are continuous at $x = 0$.

R.T.P. $g(x) = xf(x)$ is diff at $x=0$.

g is a continuous f^n at $x=0$, since $f(x)$ & $h(x) = x$ are continuous f^ns at $x=0$. & $g(0) = 0f(0) = 0$. 0.5

$$\lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{hf(h) - 0}{h} = \lim_{h\to 0} f(h) = f(0).$$ 1

so limit exists. & $g'(0) = f(0)$.

3)b) ~~If~~ Roll's Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b) & $f(a)=0=f(b)$. Then there exists at least a $c\in(a,b)$ s.t $f'(c)=0$. 0.75

3)c) Consider $h(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\cdots+\frac{a_{n-1}x^2}{2}+\frac{a_nx}{1}$

Now $h(0)=0$ & $h(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_{n-1}}{2}+a_n=0$ (Given in Quest) 1

h is continuous on $[0,1]$ & differentiable on $(0,1)$ 0.5 with $h(1)=h(0)=0$.

So by Roll's theorem there exists at least 1 $c\in(0,1)$ s.t. $h'(c)=0$. Now $h'(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$

So the given polynomial $p(x)=h'(x)$. & $p(c)=0$ for $c\in(0,1)$ already proved. 1

3)d) R.T. find $\lim_{x\to+\infty} x^{1/x}$. Let $y(x)=x^{1/x}\Rightarrow \log y=\frac{1}{x}\log x$

$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log x}{x}=\left[\frac{+\infty}{\infty}\right]$ form as. $f_1(x)=\log x$

& $g_1(x)=x$ & $\lim_{x\to\infty}f_1(x)=\lim_{x\to\infty}g_1(x)=+\infty$

Now. $f_1'(x)=\frac{1}{x}$ & $g_1'(x)=1$.

3)d) R.T. find $\lim_{x\to0}\frac{e^{2x}-\cos x}{x}$. $\left[\frac{0}{0}\right]$ form. Let $f_1(x)=e^{2x}-\cos x$

& $g_1(x)=x$ & f_1, g_1 are continuous at $x=0$. & $f_1(0)=0=g_1(0)$

so $\lim_{x\to 0} \frac{e^{2x}-\cos x}{x} = \left[\frac{0}{0}\right]$ form.

Both of them are differentiable at $x=0$. 6

& $g_1'(x) = 1 \neq 0$ in a nbdhd of 0.) → .75

$$\lim_{x\to 0} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0} \frac{2e^{2x}+\sin x}{1} = \frac{\lim_{x\to 0}(2e^{2x}+\sin x)}{\lim_{x\to 0} 1} \quad \{\text{Alg. of limit}$$

$$= \frac{2}{1} = 2. \} \to .75$$

So by L'Hospital's rule stated below, $\lim_{x\to 0} \frac{f(x)}{g(x)}$ exists.

$$\lim_{x\to 0} \frac{f_1(x)}{g_1(x)} = \lim_{x\to 0} \frac{f_1'(x)}{g_1'(x)} = 2. \longrightarrow .5$$

L'Hospital's Rule: (f & $g : [a,b] \to \mathbb{R}$ be 2 continuous f^ns on $[a,b]$. & differentiable at every point in (a,b) except possibly at $x_0 \in (a,b)$.) → .25 {Suppose $f(x_0) = 0 = g(x_0)$ & $g'(x) \neq 0$ in a deleted nbdhd of x_0. & $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists} .25 Then

$\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists & $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ } → .75

Q3) e) i) True, (ii) False. (Check my notes).

0.5 0.5

Q4) a) Given f be a continuous f^n on $[a,b]$. s.t $\int_a^b g(x)f(x) = 0$ for every continuous g on $[a,b]$. R.T.P $f(x) = 0$ for all $x \in [a,b]$.

~~Take~~ Since $\int_a^b g(x)f(x) = 0$ for every continuous g on $[a,b]$. In particular take $g(x) = f(x)$. so that will imply by $(+)$, $\int_a^b f^2(x)\,dx = 0$. $\longrightarrow (-)$ 7

Now. $f^2(x) \geq 0 \ \forall x \in [a,b]$. hence $\int_a^b f^2(x)\,dx \geq 0$. $\longrightarrow$ 1

Suppose $f \not\equiv 0$. So $\exists\, x_0 \in [a,b]$ & a nbdhd of x_0. s.t. $f^2(x) \neq 0 \ \forall x \in (x_0-\delta, x_0+\delta) \subseteq [a,b]$, $\delta > 0$. as f is continuous.

As $f^2(x) \geq 0 \ \forall x \in [a,b] \Rightarrow f^2(x) > 0 \ \forall x \in (x_0-\delta, x_0+\delta)$. $\longrightarrow$ .75

$$0 = \int_a^b f^2(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} f^2(x)\,dx = f^2(c_1)\,2\delta > 0 \text{ (Integral mean value theorem)}$$

by $(-)$ additive property.

so $0 < 0$. which is a contradiction

Hence $f^2(x) \equiv 0 \ \forall x \in [a,b] \Rightarrow f \equiv 0 \ \forall x \in [a,b]$ $\longrightarrow$ .75

Q4) b) $f : [0,1] \to \mathbb{R}$ s.t. $f(x)$ is differentiable & $f'(x)$ is continuous.

$$S_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(t)\,dt$$

$$= n\left\{\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(t)\,dt\right\}$$

$$= n\left\{\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \sum_{k=0}^{n-1}\int_{k/n}^{\frac{k+1}{n}} f(t)\,dt\right\}$$

Q4) b) Let f be integrable on $[0,1]$. as f is continuous there, so is $g(x) = x-1$. Hence the product $f(x)g(x)$ is integrable on $[0,1]$. 8

By Integration by parts

$$\int_0^1 f(x)\, g'(x)\,dx + \int_0^1 f'(x)\, g(x)\,dx = f(1)g(1) - f(0)g(0)$$

$$= 0 - (-1) f(0) = f(0) \quad (*)$$

$$\text{L.H.S of } (*) = \int_0^1 f(x)\,dx + \int_0^1 (x-1) f'(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 (1-x) f'(x)\,dx$$

As f' is continuous we can apply integral mean value th. theorem, $\exists\, \theta \in (0,1)$

Now by Integral mean value $(1-x \geq 0 \;\forall\, x \in [0,1])$

$$\text{s.t } \int_0^1 (1-x) f'(x)\,dx = f'(\theta) \int_0^1 (1-x)\,dx = \frac{f'(\theta)}{2} \quad (x)$$

Combining $(*)$ & (x).

$$\int_0^1 f(x)\,dx - \frac{f'(\theta)}{2} = f(0)$$ hence the result.

Q4) c) Second Fundamental Theorem of Calculus

If f be integrable on $[a,b]$. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)\,dt$. Then F is continuous. 0.5

& if f is continuous at $x = x_0 \in (a,b)$, then 0.5

F is diff. at $x = x_0$ & $F'(x_0) = f(x_0)$. 0.5

(9)

Integral Mean Value Theorem

Suppose $f, g \in [a,b] \longrightarrow \mathbb{R}$ s.t

a) $g(x) \geq 0 \ \forall x \in [a,b]$

b) f is bnded & $m \leq f(x) \leq M \ \forall x \in [a,b]$.

c) $f(x)$ & $f(x)g(x)$ are Riemann integ.

Then $\exists$ a $c \in [m, M]$ s.t $\int_a^b f(t)g(t)dt = c\int_a^b g(t)dt$

In addition to that if f is continuous on $[a,b]$.

Then there exists $x_0 \in (a,b)$ s.t $\int_a^b f(x)g(x)dx = f(x_0)\int_a^b g(x)dx$

Method 1

Suppose $F(x) = \int_a^x (f(t) - g(t))\,dt$.

As f & g are cont. on $[a,b]$ so is $f-g$. & hence F is diff on (a,b) & $F'(x) = (f(x) - g(x))$

$= f(x) - g(x) \ \forall x \in (a,b)$. by 2nd Fundamental th.

Now $F(a) = 0$, $F(b) = \int_a^b (f(t) - g(t))\,dt$ → (1)

$= \int_a^b f(t)\,dt - \int_a^b g(t)\,dt$ (additive property)

$= 0$. → 0.5

So $F(a) = 0 = F(b)$ & F is diff on (a,b). So by Roll's theorem $\exists$ $c \in (a,b)$ s.t $F'(c) = 0$

$\Rightarrow f(c) - g(c) = 0 \Rightarrow f(c) = g(c)$. → 0.5

(10)

Method 2: Let $h(x) = f(x) - g(x)$ $\forall x \in [a,b]$.

then as. f & g are cont. on $[a,b]$ so is $h = f - g$.

Hence by Integral mean value theorem as.

h is a cont f^n. & $G(x) = 1$ is +ve & const.

f^n. so There exists an $x_0 \in (a,b)$ s.t

$$\int_a^b h(x)\,dx = \int_a^b F(x)\,G(x)\,dx = F(x_0)\int_a^b 1\,dx = F(x_0)(b-a) \quad (*)$$

Now L.H.S. $\int_a^b h(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 0$

R.H.S $= F(x_0)(b-a)$ by $(*)$ $F(x_0)(b-a) = 0$

$b \neq a$ so $F(x_0) = 0$.

Now $0 = F(x_0) = h(x_0) = f(x_0) - g(x_0)$ so $f(x_0) = g(x_0)$

Q)4d) True, since the f^n is continuous at $x = 0$.

(11) (12)

Q1)a) No. (0.5) For eg $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ & $g(x) = x$, f, g are unifo continuous, but $f(x)g(x) = x^2$ is not uniformly cont. on $\mathbb{R}$. → (1)

Q1)b) If for every seq $\{x_n\}$ in (a,b) converging to x_0, (0.5) the seq $\{f(x_n)\}$ converges to $f(x_0) = L$, then we say f is cont. at $x = x_0$. → (1)

or for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ (dependent on $\varepsilon > 0$) s.t. $\forall\, x \in (a,b)$ with (0.5) $0 < |x - x_0| < \delta_\varepsilon$ implies $|f(x) - f(x_0)| < \varepsilon$ then f is continuous at $x = x_0$. → (1)

Q1)c) $\varepsilon - \delta$ def. $\forall\, x_0 \in (0,1)$.

Let $\varepsilon > 0$ be any real no. so $\delta_\varepsilon \leq \frac{x_0}{2}$

Let $\delta_\varepsilon = \min\left\{\frac{x_0}{2}, \frac{x^{3/2}}{2}\varepsilon\right\}$ → $\forall\, \delta_\varepsilon < \frac{x_0^{3/2}\varepsilon}{2}$

(0.75) $\forall\, x \in (0,1)$ with $0 < |x - x_0| < \delta_\varepsilon \leq \frac{x_0}{2}$

will imply $-\frac{x_0}{2} < x - x_0 < \frac{x_0}{2} \Rightarrow \frac{x_0}{2} < x < \frac{3x_0}{2}$. → (1)

& this will imply (0.75)

$$|f(x) - f(x_0)| = \left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x_0}}\right| \leq \frac{|\sqrt{x} - \sqrt{x_0}|}{\sqrt{x\,x_0}}$$

(12)

$$= \frac{|x-x_0|}{\sqrt{x x_0}\,(\sqrt{x}+\sqrt{x_0})} < \sqrt{\frac{2}{x_0 x_0}}\,\frac{|x-x_0|}{\sqrt{x_0}\left(\frac{1}{\sqrt{2}}+1\right)}$$ .5

[As $\frac{x_0}{2} < x \Rightarrow \frac{1}{\sqrt{x}} < \sqrt{\frac{2}{x_0}}$ (by ⊕).

& $\frac{\sqrt{x_0}}{\sqrt{2}} < \sqrt{x}$ so $\sqrt{x_0}\left(\frac{1}{\sqrt{2}}+1\right) < \sqrt{x}+\sqrt{x_0}$

so $\frac{1}{(\sqrt{x}+\sqrt{x_0})} < \frac{1}{\sqrt{x_0}\left(\frac{1}{\sqrt{2}}+1\right)}$

.5

So $|f(x)-f(x_0)| < \frac{\sqrt{2}}{x_0\sqrt{x_0}}|x-x_0| < \frac{\sqrt{2}}{x_0^{3/2}}\delta_\varepsilon < \frac{x^{3/2}\varepsilon}{\sqrt{2}\,x^{3/2}}$

As x_0 is any arbitrary point, so f is cont at every $x \in (0,1)$.

Sequential Argument Let $x_0 \in (0,1)$.

Let $\{x_n\}$ be any seq. in $(0,1)$ converging to x_0. Let $\varepsilon > 0$ be any real no.

As $\lim_{n\to\infty} x_n = x_0$. So for $\varepsilon\sqrt{x_0}\ (=\varepsilon)$, there exists $N_{\varepsilon\sqrt{x_0}}$ s.t. $|x_n - x_0| < \varepsilon\sqrt{x_0}\ \forall\, n \geq N_{\varepsilon\sqrt{x_0}}$. → (*) .5

As $x_0, x_n \geq 0\ \forall n \in \mathbb{N}$, so $\sqrt{x_n}+\sqrt{x_0} \geq \sqrt{x_0}$ $\forall n \in \mathbb{N}$. So → (+) .5

$|\sqrt{x_n}-\sqrt{x_0}| \leq \frac{|x_n - x_0|}{\sqrt{x_n}+\sqrt{x_0}} < \frac{\varepsilon\sqrt{x_0}}{\sqrt{x_0}}$ (by (*) & (+)) for all $n \geq N_{\varepsilon\sqrt{x_0}}$ → .5

Hence $\{\sqrt{x_n}\}$ converges to $\{\sqrt{x_0}\}$. As $\{\sqrt{x_n}\}$ converges $\sqrt{x_0} \neq 0$

So by Algebra of limits of a seq $\left\{\frac{1}{\sqrt{x_n}}\right\} \to \left\{\frac{1}{\sqrt{x_0}}\right\}$ → 1

So $\{f(x_n) = \frac{1}{\sqrt{x_n}}\}$ converges to $\frac{1}{\sqrt{x_0}} = f(x_0)$

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Q1) d) (i) True, (ii) False, (iii) True.

Q1) f) Let $f(x)$ be a continuous on a domain D containing $[a,b]$ & let $f(a) < y < f(b)$. Then $\exists\, c \in (a,b)$ s.t $f(c) = y$.

$$f(x) = \begin{cases} -x^2+4 & \text{if } x \le 3 \\ 4x-8 & \text{if } x > 3. \end{cases}$$

$$\lim_{x\to 3+} f(x) = \lim_{x\to 3+} (4x-8) = 4$$

$$\lim_{x\to 3-} f(x) = \lim_{x\to 3-} (-x^2+4) = -5$$

so $\lim_{x\to 3+} f(x) \neq \lim_{x\to 3-} f(x)$. so it is a jump discon at $x=3$.

Q1) e) Let $\{a_n\}$ be a bnded seq. For each $j \geq 1$, consider the set $B_j := \{a_j, a_{j+1}, \dots\}$.

It is clear $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \supseteq B_k \supseteq B_{k+1} \supseteq \dots$

If $\beta_j = \sup B_j$ then $\beta_1 \geq \beta_2 \geq \dots \geq \beta_k \geq \beta_{k+1} \geq \dots$

Hence $\{\beta_k\}$ is a decreasing seq which is bnded from below. So $\lim \beta_k$ exists & $\lim \beta_k$ is called limit superior of $\{a_n\}$

$\{a_n\} = \{a_n = \frac{1}{n+1}$ if n odd & $a_n = \frac{n}{n+1}$ if n even.

k odd $B_k = \{\frac{1}{k+1}, \frac{k+1}{k+2}, \dots\}$ & k even $B_k = \{\frac{k}{k+1}, \frac{k}{k+2}, \dots\}$

In any case $\frac{k}{k+1} = 1 - \frac{1}{k+1} = \beta_k = \sup\{a_m ; m \geq k\} < 1$ so $\lim \beta_k = 1$