Q-4 (a) When can you say a function $f: D \to \mathbb{R}$ is Unif. cts?

(i) for given $\varepsilon > 0$, $\exists\ \delta > 0$ (depend on ε)

s.t. $\forall\ x, y \in D$ with $0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ — 1 marks

Hence δ depends only on ε, not on points x and y.

(b) Using the definition

Show: $f: [1,2] \to \mathbb{R}$ defd by $f(x) = x^{3/2}$ is Unif. cts

Given $\varepsilon > 0$, $\forall\ x, y \in [1,2]$ with $0 < |x-y| < \delta$

where δ we will choose later

Consider $|f(x) - f(y)| = |x^{3/2} - y^{3/2}|$

$= |x^{1/2} - y^{1/2}| \cdot |x + \sqrt{xy} + y|$ — 1

$\leq |x^{1/2} - y^{1/2}| \cdot (|x| + |\sqrt{xy}| + |y|)$ (By Δgle ineq.)

$\leq |x^{1/2} - y^{1/2}|\ (2 + \sqrt{2 \cdot 2} + 2)$ $(\because x, y \leq 2)$

$\leq 6\,|x^{1/2} - y^{1/2}|$

$\leq 6\ \dfrac{|x-y|}{(\sqrt{x} + \sqrt{y})}$

$\leq \dfrac{6}{2}\,|x-y|$ $\left(\begin{array}{l} \because x, y \geq 1 \\ \sqrt{x} + \sqrt{y} \geq 1 + 1 \geq 2 \\ \dfrac{1}{\sqrt{x}+\sqrt{y}} \leq \dfrac{1}{2} \end{array}\right.$

$\leq 3|x-y|$

$\leq 3 \cdot \delta$ choose $\delta < \dfrac{\varepsilon}{3}$

$\leq \varepsilon$ — 1

For writing exact value of δ is of 0.5 marks.

Method-2

(b) Given $\varepsilon > 0$,

$|f(x) - f(y)| =$

$=$

$= |x$

$\leq |x$

$\leq$

$\leq$

$\leq \varepsilon$

For findin

(M-3)

$|f(x) - f(y)| = |x$

Method-2

(b) Given $\varepsilon>0$, $\forall x,y\in[1,2]$ with $0<|x-y|<\delta$
where δ we will find later

$$|f(x)-f(y)| = |x^{3/2}-y^{3/2}|$$
$$= \left|\frac{x^3-y^3}{x^{3/2}+y^{3/2}}\right|$$

Use $a^3-b^3=(a-b)(a^2+b^2+ab)$

$$= |x-y|\cdot\left|\frac{x^2+y^2+xy}{x^{3/2}+y^{3/2}}\right| \quad \text{— } (1)$$

$$\leq |x-y|\cdot\frac{12}{2} \longrightarrow$$

$\because$ $x,y\in[1,2]$
$x^2+y^2+xy\leq 4+4+4$
≤ 12

& $x,y\geq 1$
$x^{3/2}+y^{3/2}\geq 2$
$\frac{1}{x^{3/2}+y^{3/2}}\leq\frac{1}{2}$

$$\leq 6|x-y|$$
$$\leq 6\delta$$
$$\leq \varepsilon \quad \text{choose } \delta=\frac{\varepsilon}{6}.$$

(1)

For finding exact value of δ is of (0.5) marks

(M-3) $|f(x)-f(y)| = |x^{3/2}-y^{3/2}| = |x^{3/2}-y^{3/2}+xy^{1/2}-yx^{1/2}|$
$$= |(x-y)(x^{1/2}+y^{1/2})|$$
$$\leq |x-y|\cdot|2\sqrt{2}|. \quad \text{choose } \delta=\frac{\varepsilon}{2\sqrt{2}}$$

Q→4 (c) Is $f : (0, \infty) \longrightarrow \mathbb{R}$ defd by $f(x) = \frac{1}{x^3}$ is U.C

let $\langle x_n \rangle$ and $\langle y_n \rangle$ be 2 seq. in $(0, \infty)$ defd by

$$\left.\begin{aligned} x_n &= \frac{1}{n} \\ y_n &= \frac{1}{n+1} \end{aligned}\right] \text{— 0.5} \quad \text{Both belong to } (0, \infty)$$

Although $|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$

$$\lim_{n \to \infty} |x_n - y_n| = 0 \quad \text{— 0.5}$$

But $|f(x_n) - f(y_n)| = |n^3 - (n+1)^3|$

$$= 3n^2 + 3n + 1$$

$$\not\longrightarrow 0 \quad \text{as } n \to \infty \quad \text{— 0.5}$$

By Sequential defn of U.C., $f(x) = \frac{1}{x^3}$ is NOT U.C.

Q→5 ⓐ If function has bdd. deriv.
then it is U.C.

Let $f: D \to \mathbb{R}$ be a cts f^n and diff.ble

Let $x, y \in D$

Then $f_0 : [x,y] \to \mathbb{R}$ defd by $f_0(t) = f(t)$ $\forall t \in [x,y]$

It is again cts f.
& $f_0(t)$ is diffble on (x,y). — 0.5

By Lag. Mean Value Thm $\exists$ $C_{x,y} \in (x,y)$: $\frac{f_0(x) - f_0(y)}{x-y} = f_0'(C_{xy})$ — 0.5

$$\left|\frac{f_0(x) - f_0(y)}{x-y}\right| = |f_0'(C_{xy})|$$

$$\left|\frac{f(x) - f(y)}{x-y}\right| = |f'(C_{x,y})| \le M$$

— 0.5

Since deriv. of f is bdd,
So $\exists M > 0$:
$|f'(t)| \le M$ $\forall t \in [x,y]$
— 0.5

So $|f(x) - f(y)| \le M\,|x-y|$

Since x and y are arb. points in D

For any given $\varepsilon > 0$, $\exists$ $\delta_\varepsilon = \frac{\varepsilon}{M}$

s.t. $\forall x, y \in D$: $0 < |x-y| < \delta$ implies $|f(x) - f(y)| \le M|x-y|$
$\le M \cdot \delta_\varepsilon$
$\le M \cdot \frac{\varepsilon}{M}$
$\le \varepsilon$

So f is U.C. — 0.5

Q→5 (b) $f:(0,1)\longrightarrow\mathbb{R}$ defd by $f(x)=\sqrt{1+x^2}$ is U.C.

$$f'(x)=\frac{x}{\sqrt{1+x^2}} \quad \text{— (1)}$$

$$\le 1 \qquad \forall x\in(0,1)$$

$\because 0<x<1$

$\Rightarrow x>0$ | $x<1$

$1+x^2>1$

$\sqrt{1+x^2}>1$

$\frac{1}{\sqrt{1+x^2}}<1$

So f' is bdd fn on $(0,1)$ — (0.5)

By Previous Q→5 (a) → (0.5)

f is Uniform cts → (0.5) marks.

(Since it has bdd. derivatives).

Method-2) for given $\varepsilon>0$,

$\forall x,y\in(0,1)$ with $0<|x-y|<\delta$ where δ we will find later

we have

$$|f(x)-f(y)| = |\sqrt{1+x^2}-\sqrt{1+y^2}|$$

$$= \left|\frac{x^2-y^2}{\sqrt{1+x^2}+\sqrt{1+y^2}}\right|$$

$$= |x-y|\cdot\left|\frac{x+y}{\sqrt{x^2+1}+\sqrt{1+y^2}}\right| \quad \text{— (1)}$$

$\because x,y\in(0,1)$

$$\le |x-y|\cdot\left|\frac{x}{\sqrt{1+x^2}}+\frac{y}{\sqrt{1+y^2}}\right|$$

$$\le |x-y|\cdot 2$$

$$\le \varepsilon \quad \text{choose } \delta=\frac{\varepsilon}{2}.$$

(1)

To find exact value of δ is of 0.5 marks

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(6a) $$S_n = \sum_{k=1}^{n} \frac{k}{(2k-1)^2(2k+1)^2}$$

$$S_n = \frac{1}{8} \sum_{k=1}^{n} \frac{8k}{(2k-1)^2(2k+1)^2}$$

$$= \frac{1}{8}\left[\sum_{k=1}^{n} \frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2}\right]$$

$$= \frac{1}{8}\left[\left(1 + \cancel{\frac{1}{3^2}} + \cancel{\frac{1}{5^2}} + \cdots \frac{1}{(2n-1)^2}\right) - \left(\cancel{\frac{1}{3^2}} + \cancel{\frac{1}{5^2}} + \cdots + \cancel{\frac{1}{(2n-1)^2}} + \frac{1}{(2n+1)^2}\right)\right]$$

2^{n-1}

$$= \frac{1}{8}\left[1 - \frac{1}{(2n+1)^2}\right]$$ — 1

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{8}\left(1 - \frac{1}{(2n+1)^2}\right)$$

$$= \frac{1}{8} - \lim \frac{1}{(2n+1)^2} = \frac{1}{8}$$

(By algebra of limits of seq)

Hence $\boxed{S_n = \frac{1}{8}}$ — 0.5

(6b) Consider the function $g : [a, b] \rightarrow \mathbb{R}$ defined by $g(x) = e^{-x} f(x)$ — 0.5

then g is continuous on $[a, b]$ and g is differentiable on (a, b)

$g(c_1) = e^{-c_1} f(c_1) = 0$ as $f(c_1) = 0$
$g(c_2) = e^{-c_2} f(c_2) = 0$ as $f(c_2) = 0$
$g(c_3) = e^{-c_3} f(c_3) = 0$ as $f(c_3) = 0$

Applying Rolle's theorem on (c_1, c_2) & (c_2, c_3) separately,
$\exists$ $d_1 \in (c_1, c_2)$ and $d_2 \in (c_2, c_3)$
s.t. $g'(d_1) = 0$ & $g'(d_2) = 0$

(2)

$[c_1 < d_1 < c_2 < d_2 < c_3]$

Now, $g'(x) = -e^{-x} f(x) + e^{-x} f'(x)$

$$g''(x) = \left[f(x) + f''(x) - 2f'(x)\right] e^{-x}$$

$\Rightarrow$ g' is continuous on $[d_1, d_2]$ as f' and f and e^{-x} are continuous on $[d_1, d_2]$ and g' differentiable on (d_1, d_2) as f' and e^{-x} differentiable on (d_1, d_2)

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$g'(d_1) = g'(d_2) = 0$

So, using Rolle's theorem
$\exists\ e_1 \in (d_1, d_2)$ s.t

$g''(e_1) = 0$

so, $\boxed{f(e_1) + f''(e_1) - 2f'(e_1) = 0}$