NATURAL NUMBERS

1. Introduction

The invention of the axiomatic method goes back to the Greeks: Euclid tried to build his geometry on just five postulates, which the Greeks viewed as self- evident truths. It was realized only in the 19th century that these truths were not self evident at all, but rather a collection of axioms describing Euclidean geometry and distinguishing it from other geometries satisfying other axioms. This insight made mathematicians wonder whether other areas of mathematics could also be described using the axiomatic method.

The axiomatisation of modern mathematics was a process that started at the end of the 19th century. In working with Galois groups and, more generally, with permutation groups, mathematicians began to realise that many theorems (Lagrange's result that the order of an element divides the order of the group, Cauchy's theorem that if a prime p divides the order of a group, then the group has an element of order p, Sylow's theorem, or the decomposition of abelian groups like the class group of binary quadratic forms into cyclic subgroups) could be stated and proved for abstract groups. Finding the right axioms of abstract groups was a problem occupying numerous mathematicians from Kronecker to Weber. At the same time, Guiseppe Peano (1858-1932) found axiomatic descriptions of the natural numbers and of vector spaces, and Moore came up with the field axioms. Although it was immediately realized that group theory could be built on just the axioms, it took a while until Steinitz did something similar for fields, and the general theory of rings had to wait for Fraenkel and Emmy Noether.

In this chapter we will first present the way to define natural numbers in terms of Peano axioms, and then develop the basic properties of the natural numbers from the Peano axioms; the construction of negative integers and rationals will then be built upon the set of natural numbers. Before we proceed let us define the set of natural number informally

2. NATURAL NUMBERS

The natural numbers $\{1, 2, 3, 4, ...\}$ are the most primitive of the number systems, but they are used to build the integers, which in turn are used to build the rationals. Furthermore the rationals are used to build the real numbers.

Definition 2.1. Natural Numbers A natural number is any element of the set

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

which is the set of all the numbers created by starting with 1 and then counting forward indefinitely. We call $\mathbb{N}$ the set of natural numbers.

3. Mathematical Induction

I like to think of mathematical induction via an analogy. How can I convince you that I can climb a ladder? Well, first I show you that I can climb onto the first rung, which is obviously important. Then I convince you that for any rung, if I can get to that rung, then I can get to the next one.

Are you convinced? Well, let's see. I showed you I can get to the first rung, and then, by the second property, since I can get to the first rung, I can get to the second. Then, since I can get to the second rung, by the second property, I can get to the third, and so on and so forth. Thus I can get to any rung.

Suppose we wish to find the formula for the sum of positive integers $1, 2, 3, \cdots, n$, that is, a formula which will give the value of $1+2+3$ when $n = 3$, the value $1+2+3+4$, when $n = 4$ and so on and suppose that in some manner we are led to believe that the formula $1+2+3+\cdots+n = n(n+1)/2$ is the correct one. How can this formula actually be proved? We can, of course, verify the statement for as many positive integral values of n as we like, but this process will not prove the formula for all values of n. What is needed is some kind of chain reaction which will

The cool thing about induction (we will henceforth drop the formality of "mathematical induction") is that it allows us to prove infinitely many statements. How does it do this? Suppose we have a proposition $P(n)$ for each natural number $n \in \mathbb{N}$ and we want to prove that for all $n \in \mathbb{N}$ the statement $P(n)$ is true.

(1) **Basis Step**: Verify that $P(1)$ is true.
(2) **Inductive Step** : Show that if $P(k)$ is true for some integer k (greater than or equal to 1) then $P(k+1)$ is also true. i.e., truth of $P(k)$ implies the truth of $P(k+1)$.

Then, $P(n)$ is true for all natural numbers n.

Example 3.1. *Use mathematical induction to show that for all $n \in \mathbb{N}$.*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

holds or the formula is true.

Consider : $P(n): \quad 1+2+3+\cdots+n=\frac{n(n+1)}{2}$ *and* $S=\{m\in\mathbb{N}, P(m) \text{ is true}\}$.

Basis Step*: Thus* $P(1)$ *asserts that* $1=\frac{1}{2}1(1+1)=1$. *So* $P(1)$ *is a true assertion which serves as our basis for induction. That is* $1\in S$.

Inductive hypothesis *: For the induction step, suppose* $P(k)$ *is true that is* $k\in S$. *That is, we suppose*

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}.$$

is true.

Inductive Step *: Since we wish to prove* $P(k+1)$ *is true from this that is* $k+1\in S$. *We add* $k+1$ *to both sides to obtain*

$$\begin{aligned}1+2+3+\cdots+k+(k+1)&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}\\&=\frac{(k+1)((k+1)+1)}{2}\end{aligned}$$

Thus $P(k+1)$ *holds if* $P(k)$ *holds, that is* $k+1\in S$ *if* $k\in S$. *By the principle of mathematical induction, we conclude that is* $P(n)$ *is true for all* n.

Example 3.2. *Use induction to show that*

$$1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}=\frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

Consider : $P(n): \quad 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}=\frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$

Basic Step *:Thus* $P(1)$ *asserts that* $1+\frac{1}{3}=\frac{4}{3}=\frac{3}{2}(1-\frac{1}{3^2})=\frac{3}{2}\frac{8}{9}=\frac{4}{3}$. *So* $P(1)$ *is a true assertion which serves as our basis for induction.*

Inductive hypothesis *: For the induction step, suppose* $P(k)$ *is true that is* $k\in S$. *That is, we suppose*

$$1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}=\frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right), \tag{1}$$

is true.

Inductive Step *: Since we wish to prove $P(k+1)$ is true. We add $\frac{1}{3^{k+1}}$ to both sides of* (1) *to obtain*

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}} &= \frac{3}{2}\left(1-\frac{1}{3^{k+1}}\right)+\frac{1}{3^{k+1}} \\ &= \frac{3}{2}-\frac{3}{2}\frac{1}{3^{k+1}}+\frac{1}{3^{k+1}} \\ &= \frac{3}{2}-\frac{1}{2}\frac{1}{3^{k+1}} \\ &= \frac{3}{2}-\frac{3}{2}\frac{1}{3^{k+2}} \\ &= \frac{3}{2}\left(1-\frac{1}{3^{(k+1)+1}}\right) \end{aligned}$$

Thus $P(k+1)$ holds if $P(k)$ holds. By the principle of mathematical induction, we conclude that $P(n)$ is true for all n.

Example 3.3. *For every positive integer n, $n(n+1)$ is even (that is divisible by 2).*

Home Work.

Example 3.4. *Using the inductive hypothesis prove that $n(n+1)$ is even.*

3.1. **A variant of Induction.** To prove $P(n)$ is true for all natural numbers greater than n_0:

(1) **Basis Step**: Verify that $P(n_0)$ is true, and
(2) **Inductive Step** : Show that if $P(k)$ is true, then $P(k+1)$ is also true for any natural numbers greater k than or equal to n_0.

then we conclude that that is $P(n)$ is true for all n greater than or equal to n_0.

Example 3.5. *For every positive integer n greater than or equal to 2, n^3-n is a multiple of 6.*
Consider $P(n)$: n^3-n is a multiple of 6 for n greater than or equal to 2 and

Basic Step *: Now for $n=2$, $n^3-n=2^3-2=6$ is a multiple of 6. So $P(2)$ is a true assertion.*

Inductive hypothesis *: For the induction step, suppose $P(k)$ (where k greater than or equal to 2) is true where k is greater than 2. That is, we suppose k^3-k is a multiple of 6 for some integer m, that is $k^3-k=6r$ for some $r\in\mathbb{N}$.*

Inductive Step : *We wish to prove $P(k+1)$ is true from this, that is $(k+1)^3-(k+1)$ is a multiple of* 6. *Now*

$$\begin{aligned}(k+1)^3-(k+1) &= k^3+3k^2+3k+1-k-1\\ &= (k^3-k)+3k(k+1)\\ &= 6r+3.2l \quad \textit{as } n(n+1) \textit{ is even}\\ &= 6(r+l)\end{aligned}$$

So $P(k+1)$ is true whenever $P(k)$ (where k greater than or equal to 2*)for $k \geq 2$. Hence we can conclude that is $P(n)$ is true for all n greater than or equal to* 2 *that for every positive integer n greater than or equal to* 2*, n^3-n is a multiple of* 6*.*

3.2. **Strong Mathematical Induction. Principle of Mathematical Induction (PSMI)**:

The Principle of Mathematical Induction states Suppose we have a proposition $P(n)$ for each natural number $n \in \mathbb{N}$ and we want to prove that for all $n \in \mathbb{N}$ the statement $P(n)$ is true.

(1) **Basis Step**: $P(1)$ is true.
(2) **Inductive Step** : For every integer $k \in \mathbb{N}$ (with k greater than or equal to 1), if $P(k')$ is true for all $k' \in \{1, 2, \cdots, k\}$, then $P(k+1)$ is also true.

We conclude that that is $P(n)$ is true for all n.

Example 3.6. *Any positive integer greater than* 1 *has a prime factorization.*

Proof. Here, the predicate we want to prove is $P(n) =$ "n has a prime factorization."

Base case: Note that the base case is $n = 2$, because the claim does not apply when $n = 1$. Since 2 itself is prime, we can conclude $P(2)$ is true.

Inductive hypothesis : Every integer between 2 and k (including 2 and k) has a prime factorization. **Inductive Step** : Let $k+1$ be any positive integer greater than 1. If $k+1$ is prime, then $k+1$ itself is a prime factorization of $k+1$. Otherwise, $k+1 = ab$ where $a, b \in \{2, \cdots, k-1\}$. Since a and b are between 2 and k, the inductive hypothesis tells us that a and b each have a prime factorization, so they can be expressed as the product of prime numbers. Since $k+1$ is the product of a and b, this implies $k+1$ can also be expressed as the product of prime numbers, so $P(k+1)$ is true By induction, we conclude that is $P(n)$ is true for all n. ∎

4. N as a well-ordered set

We start by defining the relevant concept. For $x, y \in \mathbb{N}$ we say that

$$x < y \text{ if there is an } n \in \mathbb{N} \text{ such that } x + n = y. \tag{2}$$

and $x \leq y$ implies either $x = y$ or $x < y$.

Theorem 4.1. *Every nonempty subset $S \subseteq \mathbb{N}$ has a smallest element.*

Proof. Let S be a subset of the positive integers with no least element. Clearly, $1 \notin S$ since it would be the least element if it were. Let T be the complement of S; so $1 \in T$. Now suppose every positive integer $\leq n$ is in T. Then if $n + 1 \in S$, it would be the least element of S, since every integer smaller than $n + 1$ is in the complement of S. This is not possible, so $n + 1 \in T$ instead.

The above paragraph implies that every positive integer is in T, by strong induction. So S is the empty set. This completes the proof. So our assumption is wrong (that is S be a subset of the positive integers with no least element is wrong). Hence every nonempty subset $S \subseteq \mathbb{N}$ has a smallest element. □

Second, here is a proof of the principle of induction using the well-ordering principle:

5. The set of Integers $\mathbb{Z}$ and Rational Numbers $\mathbb{Q}$

Small children first learn to add and to multiply positive integers. After subtraction is introduced, the need to expand the number system to include 0 and negative integers becomes apparent. At this point the world of numbers is enlarged to include the set $\mathbb{Z}$ of all integers. Thus we have

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\}$$

Soon the space $\mathbb{Z}$ also becomes inadequate when division is introduced. The solution is to enlarge the world of numbers to include all fractions. Accordingly, we study the space $\mathbb{Q}$ of all rational numbers, i.e., numbers of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $\mathbb{Q}$ contains all terminating decimals such as $1.23 = \frac{123}{100}$.