Assignment-1

Q1. $2n-3 \le 2^{n-2} \ \forall\ n \geqslant 5,\ n \in \mathbb{N}$

By using variant of PMI,

for $n=5$ LHS $2(5)-3 = 7$

RHS $2^{5-2} = 2^3 = 8$

as $7 \le 8 \Rightarrow$ true for $n=5$

let us assume that it is true for $n=k$ $(k \geqslant 5)$

i.e. $2k-3 \le 2^{k-2}$ —— (*)

Now, we prove it for $n=k+1$

Consider $2(k+1)-3$

$= 2k-3+2$

$\le 2^{k-2}+2$ (using (*))

$= 2(2^{k-3}+1)$

$\le 2(2^{k-3} \cdot 2)$ (as $k-3 \geqslant 2 \Rightarrow 2^{k-3} \geqslant 4$)

$= 2^{k-1}$

$= 2^{(k+1)-2}$

Hence proved.

Q2 $2^n \le (n+1)! \ \forall\ n$

for $n=1$ LHS $2^1 = 2$] as LHS = RHS

RHS $(1+1)! = 2$] $\Rightarrow$ true for $n=1$.

Let us assume it is true for $n=k$ $(k \geq 1)$, i.e.

$2^k \leq (k+1)!$ ——— $(*)$

Now, we prove it for $n=k+1$

Consider 2^{k+1}

$= 2^k \cdot 2$

$\leq (k+1)! \cdot 2$ (Using $(*)$)

$\leq (k+2) \cdot (k+1)!$ (as $k+2 \geq 3 > 2$)

$= ((k+1)+1)!$

Hence proved (by PMI)

Q3. Let $x, y \in \mathbb{R}$, $y \geq 0$

TP $|x| \leq y \Leftrightarrow -y \leq x \leq y$

$|x| \leq y \Leftrightarrow x \leq y$ and $-x \leq y$

$\Leftrightarrow x \leq y$ and $x \geq -y$

$\Leftrightarrow -y \leq x \leq y$

Q4. Let $r \in \mathbb{Q}$, $(r \neq 0)$ and $x \in \mathbb{Q}^c$ (Irrational)

Then $r = p/q$, $p, q \in \mathbb{Z}$, $(q \neq 0)$ $(p \neq 0)$

TP $r + x \notin \mathbb{Q}$ and $rx \notin \mathbb{Q}$

Suppose $r + x \in \mathbb{Q}$

$\Rightarrow \exists$ integers c, d $(d \neq 0)$ such that $r + x = c/d$

$\Rightarrow x = c/d - r = c/d - p/q$

$= \dfrac{cq - pd}{dq} \in \mathbb{Q}$ $(dq \neq 0)$

$\Rightarrow x \in \mathbb{Q}$ which is a contradiction as $x \notin \mathbb{Q}$

$\therefore r + x \notin \mathbb{Q}$.

Let $rx \in \mathbb{Q}$. $\exists$ $a, b \in \mathbb{Z}$, $(b \neq 0)$ such that

$rx = a/b \Rightarrow x = \dfrac{a}{br}$ (as $r \neq 0$)

$\Rightarrow x = \dfrac{aq}{bp} \in \mathbb{Q}$ $(bp \neq 0)$

$\Rightarrow x \in \mathbb{Q}$ again a contradiction

$\Rightarrow rx \in \mathbb{Q}$.

Q5. Let S & T be nonempty bdd. subsets of $\mathbb{R}$

TP If $S \subseteq T$, $\inf T \leq \inf S \leq \sup S \leq \sup T$

As S & T are bounded subsets of $\mathbb{R}$ $\Rightarrow$ they are bounded from above as well as below.

By completeness axiom of R, supremum & ~~infimu~~ infimum of both S & T exist.

Also, given $S \subseteq T$

Let $s \in S \Rightarrow s \in T$ (as $S \subseteq T$)

$\Rightarrow \inf T \leq s$ (By prop. of infimum)

As s is an arbitrary element of S

$\Rightarrow \inf T \leq s \ \forall s \in S$

$\Rightarrow \inf T$ is a lower bound for S

$\Rightarrow \inf T \leq \inf S$ (as $\inf S$ is the greatest lower bound for S) —— (1)

By property of sup & inf,

$\inf S \leq s \leq \sup S \ \forall s \in S$

$\Rightarrow \inf S \leq \sup S$ —— (2)

Now, for any $s \in S$, $s \in T \Rightarrow s \leq \sup T$ (By prop. of supremum)

$\Rightarrow s \leq \sup T \ \forall s \in S$

$\Rightarrow \sup S \leq \sup T$ (as $\sup S$ is ~~the~~ least upper bound of S) —— (3)

By compiling (1), (2) & (3) we get

$\inf T \leq \inf S \leq \sup S \leq \sup T$.

Q6. Let U & V be nonempty bounded subsets of $\mathbb{R}$

TP $\sup(U \cup V) = \max\{\sup U, \sup V\}$

Pf U & V bounded $\Rightarrow$ Bounded from both sides.

By Completeness axiom, sup & inf exist for both U & V.

As $U \subseteq U \cup V$ & $V \subseteq U \cup V$

for any $x \in U \cup V \Rightarrow x \in U$ or $x \in V$

$\Rightarrow x \leq M$ for some $M \in \mathbb{R}$ (as both U & V are bounded)

$\Rightarrow U \cup V$ is bounded above & hence $\sup(U \cup V)$ exists

By previous result,

$\sup U \leq \sup(U \cup V)$ —— (1) (Taking $S = U$ & $T = U \cup V$)

Similarly, $\sup V \leq \sup(U \cup V)$ —— (2)

From (1) & (2), $\max(\sup U, \sup V) \leq \sup(U \cup V)$ —— (3)

Moreover, if $x \in U \cup V \Rightarrow x \in U$ or $x \in V$

$\Rightarrow x \leq \sup U$ or $x \leq \sup V$

$\Rightarrow x \leq \max\{\sup U, \sup V\} \ \forall x \in U \cup V$

$\Rightarrow \max\{\sup U, \sup V\}$ is an upper bnd. of $U \cup V$

$\Rightarrow \sup(U \cup V) \leq \max\{\sup U, \sup V\}$ —— (4)

(3) & (4) $\Rightarrow \sup(U \cup V) = \max\{\sup U, \sup V\}$.

In the same way, try proving that
$\inf(U \cup V) = \min\{\inf U, \inf V\}$.

Q7. Let S be a nonempty bdd. subset of $\mathbb{R}$

(a) Let $a \in \mathbb{R}$, $a > 0$

$aS = \{as ; s \in S\}$

Pf As S is a bounded set $\Rightarrow M_1 \le s \le M_2 \ \forall s \in S$

$\Rightarrow aM_1 \le as \le aM_2 \ \forall s \in S$

$\Rightarrow aS$ is a bounded set

So, sup & inf of aS exist in $\mathbb{R}$.
(By completeness axiom).

$\inf(aS) \le as \ \forall s \in S$ or $\forall as \in aS$

$\Rightarrow a^{-1}\inf(aS) \le s \ \forall s \in S$ $\quad (a^{-1} = 1/a$ as $a > 0)$

$\Rightarrow a^{-1}\inf(aS)$ is a lower bound for S

Now, if we can show that for every $\epsilon > 0$, $\exists s_\epsilon \in S$
such that $a^{-1}\inf(aS) \le s_\epsilon < a^{-1}\inf(aS) + \epsilon$,
then we get $a^{-1}\inf(aS) = \inf S$. (By Corollary 3.2)

As ~~as~~ $\inf(aS)$ is the g.l.b. of aS
$\Rightarrow$ for every $a\epsilon > 0$, $\exists$ an element $as_\epsilon \in aS$
(depending on $\epsilon > 0$)
s.t. $\inf(aS) \le as_\epsilon < \inf(aS) + a\epsilon$

Dividing by $a (>0)$

$\Rightarrow a^{-1}\inf(aS) \le s_\epsilon < a^{-1}\inf(aS) + \epsilon$ ~~$\forall \epsilon > 0$~~
($\epsilon > 0$ was arbitrary)

Hence, $a^{-1}\inf(aS) = \inf S$

$\Rightarrow \inf(aS) = a \inf S$.

Similarly, prove that $\sup(aS) = a \cdot \sup(S)$

(b) Let $b \in \mathbb{R}$, $b < 0$

$bS = \{bs ; s \in S\}$

T.P. $\inf(bS) = b \cdot \sup(S)$, $\sup(bS) = b \cdot \inf(S)$.

Pf As S is a bdd set $\Rightarrow M_1 \le s \le M_2 \ \forall s \in S$

$\Rightarrow bM_2 \le bs \le bM_1 \ \forall s \in S$

$\Rightarrow bS$ is a bdd. set

So, sup & inf for bS exist in $\mathbb{R}$

$\inf(bS) \le bs \ \forall s \in S$ or $\forall bs \in bS$

$\Rightarrow \frac{1}{b}\inf(bS) \ge s \ \forall s \in S$ $\quad$ (as $1/b < 0$)

$\Rightarrow \frac{1}{b}\inf(bS)$ is an upper bnd. for S

Now, if we show that for every $\epsilon>0$, $\exists\, s_\epsilon \in S$ such that $b^{-1}\inf(bS)-\epsilon < s_\epsilon \le b^{-1}\inf(bS)$, then we get

$$b^{-1}\inf(bS) = \sup(S)$$

As $\inf(bS)$ is the greatest lower bound for bS

$\Rightarrow$ for every $-b\epsilon>0$, $\exists$ an element $bs_\epsilon \in bS$ s.t.

$$\inf(bS) \le bs_\epsilon < \inf(bS) - b\epsilon$$

Dividing by $b\,(<0)$

$\Rightarrow b^{-1}\inf(bS)-\epsilon < s_\epsilon \le b^{-1}\inf(bS)$

$\because \epsilon>0$ was arbitrary

$\Rightarrow b^{-1}\inf(bS) = \sup(S)$

$\Rightarrow \inf(bS) = b\sup(S)$

Similarly, prove that $\sup(bS) = b\cdot\inf(S)$

In particular, $b=-1 \Rightarrow \left.\begin{array}{l}\inf(-S) = -\sup(S)\\ \sup(-S) = -\inf(S)\end{array}\right]$ (*)

To prove the following two statements of Completeness axiom are equivalent:-

1) If S is a nonempty subset of $\mathbb{R}$ which is bounded above, then $\sup S$ exists.

2) If S is a nonempty subset of $\mathbb{R}$ which is bounded below, then $\inf S$ exists.

Pf Assume that (1) is true.

We need to prove (2).

Given S is a nonempty subset of $\mathbb{R}$ which is bdd. below

$\Rightarrow M \le s \;\forall\, s\in S$

$\Rightarrow -s \le -M \;\forall\, s\in S$ or $\forall\,(-s)\in -S$

$\Rightarrow -S$ is a nonempty subset of $\mathbb{R}$ which is bdd. above

$\Rightarrow \sup(-S)$ exists (by (1))

$\Rightarrow \inf S = -\sup(-S)$ exists (by (*))

Hence (2) is also true.

Similarly, we can prove (1) assuming (2) is true.

So (1) & (2) are equivalent.

Q8. $x\in\mathbb{R}$

TP $|x|<\epsilon \;\forall\, \epsilon>0 \Leftrightarrow x=0$

($\Leftarrow$) If $x=0 \Rightarrow |x| = 0 < \epsilon \;\forall\, \epsilon>0$

$\Rightarrow$ $\exists$ $|x| < \epsilon$ $\forall$ $\epsilon > 0$

T.P. $x = 0$

Suppose $x \neq 0$, let $\epsilon = \frac{|x|}{2} > 0$, then by given condition, $|x| < \epsilon = \frac{|x|}{2}$

$\Rightarrow 1 < \frac{1}{2}$ (as $|x| \neq 0$)

which is a contradiction. Hence our assumption $x \neq 0$ is wrong. So, $x = 0$.

Q10. Let $\epsilon > 0$ be any arbitrary no.

As S is bounded above nonempty subset of $\mathbb{R}$

$\Rightarrow$ sup S exists say x (By completeness axiom)

T.P. $x = u$.

As $\sup S = x$ & by property (i),

$u - 1/n$ is not an u.b. for every n

$\Rightarrow u - 1/n < x$ $\forall$ $n \in \mathbb{N}$ (as x is an u.b. of S) — ①

As $x = \sup S$, by property (ii),

$u + 1/n$ is an u.b. of S $\forall$ n.

$\Rightarrow x \leq u + 1/n$ $\forall$ n (as x is the l.u.b.) — ②

So, by ① & ②,

$u - 1/n < x \leq u + 1/n$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow |x - u| \leq 1/n$ $\forall$ $n \in \mathbb{N}$ — ③

for every $\epsilon > 0$, $\exists$ a natural no. $n_\epsilon > 1/\epsilon$

(Take $x = \epsilon$ & $y = 1$ in archimedian property)

$\Rightarrow |x - u| \leq 1/n_\epsilon$ (By ③)

$\Rightarrow |x - u| < \epsilon$

As $\epsilon > 0$ was arbitrary

$\Rightarrow |x - u| < \epsilon$ $\forall$ $\epsilon > 0$

$\Rightarrow x - u = 0$ (By Problem 8)

$\Rightarrow x = u$

Q9. (a) $\{y = 1 - 1/n,\ n \in \mathbb{N}\}$

As $n \geq 1$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow 1/n \leq 1$ $\forall$ n

Also, $1/n > 0$ $\forall$ n

$\Rightarrow 0 < 1/n \leq 1$ $\forall$ n $\Rightarrow -1 \leq -1/n < 0$ $\forall$ n

$\Rightarrow 0 \leq 1 - 1/n < 1$ $\forall$ n

$\Rightarrow$ The set is bounded
$\Rightarrow$ sup & inf exist.
Now, if $n=1$, $1-1/n = 0 \in \{y = 1-1/n, n\in \mathbb{N}\} = S$.

So, 0 is a lower bound belonging to the set
$\Rightarrow 0 = \inf S$. (By Remark 3.5)

TP $1 = \sup S$.
Clearly 1 is an u.b. of S
Let $\varepsilon > 0$ be arbitrary
Then $\exists$ $\cancel{n_\varepsilon}$ $1/n_\varepsilon < \varepsilon$ (By archimedian property)

$$\Rightarrow -\varepsilon < -1/n_\varepsilon$$

$$\Rightarrow 1-\varepsilon < 1 - 1/n_\varepsilon < 1$$

So, $\exists$ $1-1/n_\varepsilon \in S$ greater than $1-\varepsilon$

$\Rightarrow$ $1-\varepsilon$ is not an u.b. of S for every $\varepsilon > 0$
$\Rightarrow 1 = \sup S$. (as 1 is the least u.b.)

(b) $\{y = x + x^{-1} ; x > 0\} = S$

$$y = x + 1/x + 2 - 2 = \frac{x^2 - 2x + 1}{x} + 2$$

$$= \frac{1}{x}(x-1)^2 + 2 \geq 2 \quad \left(\begin{array}{l}\text{as } (x-1)^2 \geq 0 \\ \& \ x > 0\end{array}\right)$$

As y is an arbitrary element of S
so, $y \geq 2 \ \forall \ y \in S$
$\Rightarrow 2$ is a l.b. for S
Also, for $x = 1$, $x + 1/x = 2 \Rightarrow 2 \in S$
so $2 = \inf S$ (By Remark 3.5)

Next, assume that S is bdd. above
Then $\sup S$ exists (by Completeness axiom)
say $\sup S = M$
$\because 0 < y = x + 1/x \leq M \ \forall \ x > 0$
$\Rightarrow M > 0$

If $x = M \Rightarrow y_M = M + 1/M \leq M$ (as M is sup S)

But $M + 1/M > M$ (as $1/M > 0$)

So, sup S cannot exist i.e. S is not bounded above.

Q9. (c) $S = \{y = 2^x + 2^{1/x} ; x > 0\}$
If $a, b \in \mathbb{R}$, $a > 0, b > 0$
then AM $\geq$ GM i.e. $\frac{a+b}{2} \geq (ab)^{1/2}$

Take $2^x = a$ & $b = 2^{1/x}$
$\quad\; >0 \qquad\quad >0$

then, $\frac{2^x + 2^{1/x}}{2} \geqslant (2^x . 2^{1/x})^{1/2}$ —— (1)

$$2^x . 2^{1/x} = 2^{x+1/x} = 2^{\frac{x^2-2x+1}{x}+2}$$
$$= 2^{\frac{(x-1)^2}{x}+2}$$
$$\geqslant 2^2 \text{ —— (2)}$$

Using (2) in (1)

$\frac{2^x + 2^{1/x}}{2} \geqslant (2^2)^{1/2} = 2$

$\Rightarrow y = 2^x + 2^{1/x} \geqslant 4$

As $y \in S$ is arbitrary $\Rightarrow y \geqslant 4 \ \forall\ y \in S$

$\therefore$ 4 is a l.b. of S

If $x = 1$, $y = 2 + 2 = 4 \in S$

By Remark 3.5, $4 = \inf S$

Suppose S is bounded above $\Rightarrow$ sup S exists (say M)

$\Rightarrow 2^x + 2^{1/x} \leq M \ \forall\ x > 0$ —— (3)

Also, $2^x + 2^{1/x} > 0 \ \forall\ x > 0 \Rightarrow M > 0$

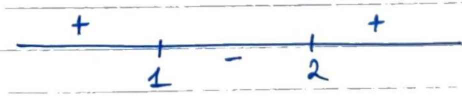
For $x = M$, $y = 2^M + 2^{1/M} > M$ (as $2^M > M$ $\forall\ M > 0$)

This contradicts (3)

Hence, S is not bounded above.

(d) $S = \{x \in \mathbb{R};\ x^2 - 3x + 2 < 0\}$

$$x^2 - 3x + 2 = x^2 - 2x - x + 2$$
$$= x(x-2) - 1(x-2)$$
$$= (x-1)(x-2)$$
$$< 0$$



So, $x^2 - 3x + 2 < 0 \Rightarrow 1 < x < 2$

[As $x^2 - 3x + 2 > 0$ if $x < 1$ $\begin{pmatrix} x-1<0 \\ x-2<0 \end{pmatrix}$

$x^2 - 3x + 2 > 0$ if $x > 2$ $\begin{pmatrix} x-2>0 \\ x-1>0 \end{pmatrix}$]

So, $S = \{x;\ 1 < x < 2\}$

Clearly, S is bounded $\Rightarrow$ sup S & inf S exist in $\mathbb{R}$

TP $1 = \inf S$

Sufficient to show that $1 + \varepsilon$ is not a l.b. of S $\forall\ \varepsilon > 0$

Let $\varepsilon > 0$ be arbitrary (such that $1+\varepsilon < 2$)

Then,

$$1 < 1+\varepsilon/2 < 1+\varepsilon$$

Also, $1 < 1+\varepsilon/2 < 2 \Rightarrow 1+\varepsilon/2 \in S$

$\therefore \exists\ s_\varepsilon = 1+\varepsilon/2$ such that $1 < s_\varepsilon < 1+\varepsilon$

As $\varepsilon > 0$ was arbitrary $\Rightarrow 1+\varepsilon$ is not a l.b. $\forall\ \varepsilon > 0$

$\Rightarrow 1$ is the g.l.b.

$\Rightarrow 1 = \inf S$

Similarly, prove that $2 = \sup S$. //