Assignment 4

September 29, 2022

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x \to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.

2. Consider two functions f, g.
(a) Show that if both $\lim_{x \to c} f(x)$ and $\lim_{x \to c}(f(x) + g(x))$ exist, then $\lim_{x \to c} g(x)$ exists.
(b) If both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} f(x)g(x)$ exist,, does it follow that $\lim_{x \to c} g(x)$ exists?

3. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

4. Show that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

5. Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x \to \infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x \to \infty} f(x) = 0$.

6. Let $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ for $x \neq 0$. Find the limit of $\lim_{x \to 0} f(x)$.