Q.3) c) Let $a_n \geq 0$, then show that both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converge and diverge together (3 marks)

Ans: We are done if we can show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges ($a_n \geq 0$, both must either converge or diverge, i.e. cannot be oscillating) **+0.5 marks**

Claim 1: If $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges

Since $0 \leq \frac{a_n}{1+a_n} \leq a_n$, so $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges by Comparison Test **+1 mark**

Alternate Proof of Claim 1:

Since $\left| \frac{a_n}{\frac{a_n}{1+a_n}} \right| = |1+a_n|$ and $\lim_{n\to\infty} |1+a_n| = \lim_{n\to\infty} 1+a_n = 1 > 0$, so $\sum_{n=1}^{\infty} \frac{|a_n|}{|1+a_n|} = \sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges by Limit Comparision Test **+1 mark**

Claim 2: If $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, $\sum_{n=1}^{\infty} a_n$ converges

Since $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$

$\implies \exists N_1 \in \mathbb{N}$ such that $\frac{a_n}{1+a_n} < \frac{1}{2}\ \forall n \geq N_1$

$\implies \frac{a_n}{1+a_n} < \frac{1}{2}\ \forall n \geq N_1$

$\implies 0 \leq a_n < 1\ \forall n \geq N_1$

$\implies \frac{1}{1+1} \leq \frac{1}{1+a_n} < \frac{1}{1+0}\ \forall n \geq N_1$

$\implies \frac{a_n}{2} \leq \frac{a_n}{1+a_n}\ \forall n \geq N_1$

Thus, $\sum_{n=1}^{\infty} a_n$ converges by Comparison Test **+1.5 marks**

Alternate Proof of Claim 2:

Since $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$

$\implies \lim_{n\to\infty} 1 - \frac{1}{1+a_n} = 0$

$\implies \lim_{n\to\infty} \frac{1}{1+a_n} = 1$

$\implies \lim_{n\to\infty} 1 + a_n = 1$
$\implies \lim_{n\to\infty} a_n = 0$

Hence $\{a_n\}$ is bounded above by $M \geq 0 \in \mathbb{R}$ but:
$a_n = \frac{a_n}{1+a_n}(1+a_n) = \frac{a_n}{1+a_n} + \frac{a_n^2}{1+a_n} \leq \frac{(1+M)a_n}{1+a_n}$

Hence $\sum_{n=1}^{\infty} a_n$ converges by Comparison Test **+1.5 marks**

Alternate Proof of Claim 2:

Since $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$
$\implies \lim_{n\to\infty} 1 - \frac{1}{1+a_n} = 0$
$\implies \lim_{n\to\infty} \frac{1}{1+a_n} = 1$
$\implies \lim_{n\to\infty} 1 + a_n = 1$
$\implies \lim_{n\to\infty} a_n = 0$

Since $\left|\frac{a_n}{\frac{a_n}{1+a_n}}\right| = |1+a_n|$ and $\lim_{n\to\infty} |1+a_n| = \lim_{n\to\infty} 1 + a_n = 1 > 0$, so $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$ converges by Limit Comparision Test **+1.5 marks**

Q.4) a) What is the completeness axiom of $\mathbb{R}$? (1 mark)

Ans: Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$, i.e. $\sup A$ exists and $\sup A \in \mathbb{R}$.

Similarly, every nonempty subset A of $\mathbb{R}$ that is bounded below has a greatest lower bound in $\mathbb{R}$, i.e. $\inf A$ exists and $\inf A \in \mathbb{R}$.

+0.5 marks Exact same or re-worded definition of first statement
+0.5 marks Exact same or re-worded definition of second statement
+0.5 marks Only considers sets that are bounded both above and below
0 marks Otherwise

Q.4) b) What is the infimum of a set? Let A be a nonempty bounded subset of strictly positive numbers. Let $\frac{1}{A} = \{\frac{1}{x} \mid x \in A\}$. Let $\inf A > 0$. What is $\sup \frac{1}{A}$? (1.5 + 2.5 marks)

Ans: Definition: A lower bound y_0 of a non-empty set $A \subseteq \mathbb{R}$ is said to be infimum of A , i.e. $\inf A$ if whenever v is a lower bound of A, $v \leq y_0$.

Alternate Definition: For a non empty set $A \subseteq \mathbb{R}$ which is bounded from below by y_0, y_0 is said to be the infimum of A iff $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in A$ (depending on ε) such that $y_0 \leq x_\varepsilon < y_0 + \varepsilon$.

+1.5 marks Exact same or re-worded definition

0 marks Otherwise

Proof: Given non-empty bounded subset A of strictly positive numbers with $\inf A > 0$, we have to show that for $\dfrac{1}{A} = \{\dfrac{1}{x} \,|\, x \in A\}$, $\sup \dfrac{1}{A} = \dfrac{1}{\inf A}$ **+0.5 marks**

Claim 1: $\dfrac{1}{A}$ is bounded above by $\dfrac{1}{\inf A}$

$\forall\, x \in A$, $0 < \inf A \leq x$ ($\inf A$ is a lower bound)

$\implies \forall\, x \in A, \dfrac{1}{x} \leq \dfrac{1}{\inf A}$

$\implies \forall\, y \in \dfrac{1}{A}, y \leq \dfrac{1}{\inf A}$

$\implies \dfrac{1}{A}$ is bounded above by $\dfrac{1}{\inf A}$ **+0.25 marks**

By completeness axiom of $\mathbb{R}$, $\sup \dfrac{1}{A}$ exists and belongs in $\mathbb{R}$ **+0.25 marks**

$\implies \sup \dfrac{1}{A} \leq \dfrac{1}{\inf A}$

Claim 2: $\sup \dfrac{1}{A} = \dfrac{1}{\inf A}$

$\forall\, x \in \dfrac{1}{A}, 0 < x \leq \sup \dfrac{1}{A}$ ($\sup \dfrac{1}{A}$ is an upper bound)

$\implies \forall\, x \in \dfrac{1}{A}, \dfrac{1}{\sup \frac{1}{A}} \leq \dfrac{1}{x}$

$\implies \forall\, x \in A, \dfrac{1}{\sup \frac{1}{A}} \leq x$

$\implies \dfrac{1}{\sup \frac{1}{A}} \leq \inf A$ ($\dfrac{1}{\sup \frac{1}{A}}$ is a lower bound)

$\implies \dfrac{1}{\inf A} \leq \sup \dfrac{1}{A}$

Since $\dfrac{1}{\inf A} \leq \sup \dfrac{1}{A}$ and $\dfrac{1}{\inf A} \geq \sup \dfrac{1}{A}$, $\sup \dfrac{1}{A} = \dfrac{1}{\inf A}$ **+1.5 marks**

Alternate Proof of Claim 2:

By definition of $\inf A$, $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in A$ such that $0 < \inf A \leq x_\varepsilon < \inf A + \varepsilon$

$\implies \dfrac{1}{\inf A + \varepsilon} < \dfrac{1}{x_\varepsilon} \leq \sup \dfrac{1}{A}$

Taking $\varepsilon = \dfrac{1}{n}\ \forall\, n \in \mathbb{N}$, We construct three sequences $a_n = \dfrac{1}{\inf A + \frac{1}{n}}$, $b_n = \dfrac{1}{x_{\frac{1}{n}}}$ and $c_n = \sup \dfrac{1}{A}$ $\forall\ n \in \mathbb{N}$

$\{\inf A + \dfrac{1}{n}\}$ converges to $\inf A > 0$ (by addition of convergent sequences)

$\{a_n\} = \{\dfrac{1}{\inf A + \frac{1}{n}}\}$ converges to $\dfrac{1}{\inf A}$ (by division of convergent sequences)

Since $a_n < b_n \leq c_n \implies a_n \leq c_n\ \forall\ n \in \mathbb{N}$

$\implies \dfrac{1}{\inf A} = \lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} c_n = \sup \dfrac{1}{A}$

Since $\dfrac{1}{\inf A} \leq \sup \dfrac{1}{A}$ and $\dfrac{1}{\inf A} \geq \sup \dfrac{1}{A}$, $\sup \dfrac{1}{A} = \dfrac{1}{\inf A}$ **+1.5 marks**

Alternate Proof of Claim 2:

By definition of $\inf A$, $\forall\ \varepsilon > 0$, $\exists\ y_\varepsilon \in A$ such that $0 < \inf A \leq y_\varepsilon < \inf A + \varepsilon$

$\implies \dfrac{1}{\inf A + \varepsilon} < \dfrac{1}{y_\varepsilon} \leq \dfrac{1}{\inf A}$

By definition of $\sup \dfrac{1}{A}$, $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in \dfrac{1}{A}$ such that $\sup \dfrac{1}{A} - \varepsilon < x_\varepsilon \leq \sup \dfrac{1}{A}$

If we show that $\dfrac{1}{\inf A}$ satisfies the above property, we are done as supremum is unique

Since all elements of $\dfrac{1}{A}$ are strictly positive, we can assume WLOG that $\dfrac{1}{\inf A} - \varepsilon > 0$ (otherwise x_ε can be any element in A since we have already shown that $\dfrac{1}{\inf A}$ is an upper bound of A)

And since $\dfrac{1}{\inf A} - \varepsilon > 0 \implies \dfrac{1}{\inf A} > \varepsilon > 0$, we take our $x_\varepsilon = \dfrac{1}{y_\delta}$

(where $y_\delta \in A$ and $\delta = \dfrac{\inf A}{1 - \inf A\varepsilon} - \inf A > 0$)

Since $\dfrac{1}{\inf A + \delta} < \dfrac{1}{y_\delta} \leq \dfrac{1}{\inf A}$ (by definition of $\inf A$)

$\implies \dfrac{1}{\inf A - \inf A + \frac{\inf A}{1 - \inf A\varepsilon}} < \dfrac{1}{y_\delta} \leq \dfrac{1}{\inf A}$

$\implies \dfrac{1 - \inf A\varepsilon}{\inf A} < \dfrac{1}{y_\delta} \leq \dfrac{1}{\inf A}$

$\implies \dfrac{1}{\inf A} - \varepsilon < \dfrac{1}{y_\delta} \leq \dfrac{1}{\inf A}$

Thus $\forall\ \varepsilon > 0$, $\exists\ x_\varepsilon \in \dfrac{1}{A}$ such that $\dfrac{1}{\inf A} - \varepsilon < x_\varepsilon \leq \dfrac{1}{\inf A}$ **+1.5 marks**

Q.5) a) What is the rational zeros theorem? (1.5 marks)

Ans: Suppose $c_0, c_1, \ldots, c_n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ such that r is a solution of the polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$.
If $r = \dfrac{c}{d}$ where c, d are integers such that $\gcd(c, d) = 1$ and $d \neq 0$, then $c \mid c_0$ and $d \mid c_n$.

+1.5 marks Exact same or re-worded definition

0 marks Otherwise