Assignment 1

August 22, 2023

1. Prove that $2n - 3 \leq 2^{n-2}$ for all $n \geq 5$, $n \in \mathbb{N}$.

2. Show that $2^n \leq (n+1)!$ for all $n \in \mathbb{N}$.

3. Let x and $y \geq 0$ be two real numbers. Prove that $|x| \leq y$ iff $-y \leq x \leq y$.

4. If r is rational ($r \neq 0$) and x is irrational. Prove that $r + x$ and rx are irrational.

5. Let S and T be nonempty bounded subsets of $\mathbb{R}$.
   Prove if $S \subseteq T$, then $infT \leq infS \leq supS \leq supT$.

6. Let U and V be nonempty bounded subsets of $\mathbb{R}$. Prove $sup(U \cup V) = max\{supU, supV\}$. (What about $inf(U \cup V)$?)

7. Let S be a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$ be a real number and let $aS = \{as; s \in S\}$. Prove that

   $$inf(aS) = a.\inf(S) \quad sup(aS) = a.sup(S).$$

   (b) Let $b < 0$ be a real number and let $bS = \{bs; s \in S\}$. Prove that

   $$inf(bS) = b.sup(S) \quad sup(bS) = b.inf(S).$$

   In particular, $b = -1$, imply

   $$inf(-S) = -sup(S) \quad sup(-S) = -inf(S).$$

   (Now try to prove that the two statements in Completeness axioms are equivalent).

8. $x \in \mathbb{R}$. Then $|x| < \varepsilon$ for every $\varepsilon > 0$ iff $x = 0$.

9. Prove whether the following sets are bounded from below or above (or both) and then find Supremum or Infimum (or both) with justification.
   a) $\{y = 1 - \frac{1}{n}, n \in \mathbb{N}\}$.
   b) $\{y = x + x^{-1};\ x > 0\}$.
   c) $\{y = 2^x + 2^{\frac{1}{x}};\ x > 0\}$.
   d) $\{x \in \mathbb{R};\ x^2 - 3x + 2 < 0\}$.

10. Let S be nonempty subset of $\mathbb{R}$. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - \frac{1}{n}$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + \frac{1}{n}$ is an upper bound of S, then $u = SupS$.