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Q2 a)

Show that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$.

Proof. Given $\lim_{x\to\infty} xf(x) = L$. Let $\{x_n\}$ be any sequence diverges to ∞. Then $\lim_{n\to\infty} x_n f(x_n) = L$ as $\lim_{x\to\infty} xf(x) = L$. Let $X_n = x_n f(x_n)$ and $Y_n = \frac{1}{x_n}$. Now $\lim_{n\to\infty} X_n = L$ and $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \frac{1}{x_n} = 0$. So by Algebra of limits $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} X_n Y_n = \lim_{n\to\infty} X_n \times \lim_{n\to\infty} Y_n = L \times 0 = 0$. So for any arbitrary sequence $\{x_n\}$ converging to ∞, $\lim_{n\to\infty} f(x_n) = 0$ concluding $\lim_{x\to\infty} f(x) = 0$. $\square$

Q2 b) Example of any such continuous function could be given (domain to be specified)

For Example -

- $\lim_{x\to\infty} \frac{1}{x} = 0$ (but $x \neq 0$ or $x \in R\setminus\{0\}$)

- $f(x) = a$ (a is constant, $x \in R$)

  $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} a = a$