RA - I

14/09 Quiz-1

Q4. (b) Check whether the sequence $\left\{\frac{n+1}{n\sqrt{n}}\right\}$ cgs.

Sol^n $x_n = \frac{n+1}{n\sqrt{n}} \quad \forall\ n \geq 1$

~~$\Rightarrow x_n = 1 + 1/n \quad \forall\ n \geq 1$~~

~~$\sqrt{n}$~~

$\Rightarrow x_n = 1/\sqrt{n} + 1/n\sqrt{n} \quad \forall\ n \geq 1$ —— (1) +0.25 ($\cancel{+0.5}$ marks)

Claim 1 $\lim_{n\to\infty} 1/\sqrt{n} = 0$

Let $\epsilon > 0$ be an arbitrary real number

Consider $|1/\sqrt{n} - 0| = 1/\sqrt{n}$

In case $|1/\sqrt{n} - 0| = 1/\sqrt{n} < \epsilon$

$\Rightarrow 1/\epsilon < \sqrt{n}$

$\Rightarrow 1/\epsilon^2 < n$ (squaring both sides)

Let $K_\epsilon = \lfloor 1/\epsilon^2 \rfloor + 1 \in \mathbb{N}$

$\Rightarrow$ for $n \geq K_\varepsilon \Rightarrow n \geq \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor + 1 > 1/\varepsilon^2$

$\left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon$ (+0.5 marks)

Hence, for $\varepsilon > 0$, $\exists$ $K_\varepsilon = \left\lfloor 1/\varepsilon^2 \right\rfloor + 1 \in \mathbb{N}$ such that $\left| 1/\sqrt{n} - 0 \right| < \varepsilon$ $\forall$ $n \geq K_\varepsilon$

As $\varepsilon > 0$ was arbitrary

$\Rightarrow \lim_{n \to \infty} 1/\sqrt{n} = 0.$ (+0.25 marks)

Claim 2 $\lim_{n \to \infty} 1/n\sqrt{n} = 0.$

As $\{1/n\}$ and $\{1/\sqrt{n}\}$ both are convergent

By Algebra of limits,

$$\lim_{n \to \infty} 1/n\sqrt{n} = \lim_{n \to \infty} (1/n) \cdot \lim_{n \to \infty} (1/\sqrt{n})$$

$$= 0.0$$

$$= 0$$ (+0.25 marks)

Hence, By Algebra of limits in eq^n (1),

$\lim_{n \to \infty} x_n = \lim_{n \to \infty} 1/\sqrt{n} + \lim_{n \to \infty} (1/n\sqrt{n}) = 0$ (+0.25 marks)

Q5. What is the infimum of the set
$\left\{ \frac{mn}{m+n+1}, \ m,n \in \mathbb{N} \right\}$?

Soln for $m = n = 1$, $\frac{mn}{m+n+1} = 1/3$

Let $S = \left\{ \frac{mn}{m+n+1}, \ m,n \in \mathbb{N} \right\}$

$\Rightarrow 1/3 \in S$ (+0.5 marks)

Claim $1/3$ is a lower bound of S

So, it is sufficient to show that

$$\frac{mn}{m+n+1} \geq 1/3 \quad \forall \ m,n \in \mathbb{N}$$

$$\Leftrightarrow 3mn \geq m+n+1 \quad \forall \ m,n \in \mathbb{N}$$

$$\Leftrightarrow 3mn - n \geq m+1 \quad \forall \ m,n \in \mathbb{N}$$

$$\Leftrightarrow n(3m-1) \geq m+1 \quad \forall \ m,n \in \mathbb{N}$$

$$\Leftrightarrow n \geq \frac{m+1}{3m-1} \quad \forall \ m,n \in \mathbb{N}$$ (+1 mark)

(*)

$$\Leftrightarrow n \geq \frac{3m+3}{3(3m-1)} = \frac{3m-1}{3(3m-1)} + \frac{4}{3(3m-1)}$$

Claim $m+1 \leq 3m-1 \quad \forall\ m \in \mathbb{N}$

For $m=1$, $2 \leq 2 \Rightarrow$ Inequality holds for $m=1$

Let us assume that it holds for $m=k$

$\Rightarrow \quad k+1 \leq 3k-1$ ——— ①

Now, we prove it for $m=k+1$

Consider $(k+1)+1$ ~~$\leq k+$~~

$\leq 3k-1+1$

$= 3k$

$\leq 3k+2 = 3(k+1)-1$

Hence, the inequality holds true for $m=k+1$

By Principle of Mathematical Induction,

$m+1 \leq 3m-1 \quad \forall\ m \in \mathbb{N}$ (~~+0.75~~ marks) +0.5

$\Rightarrow \dfrac{m+1}{3m-1} \leq 1 \quad \forall\ m \in \mathbb{N}$

~~By ✱, $n \geq \dfrac{m+1}{3m-1}$~~ $\because n \geq 1 \quad \forall\ n \in \mathbb{N}$

$\Rightarrow n \geq 1 \geq \dfrac{m+1}{3m-1} \quad \forall\ n \in \mathbb{N},\ m \in \mathbb{N}$

$\therefore$ ✱ holds $\forall\ m, n \in \mathbb{N}$

$\Rightarrow \dfrac{mn}{m+n+1} \geq \dfrac{1}{3} \quad \forall\ m, n \in \mathbb{N}$ (+0.5 marks)

Hence, $1/3$ is a lower bound of S & $1/3 \in S$

$$\Rightarrow 1/3 = \inf S$$

(Using the result,
If 'a' is a lower bound of a set S contained in the set S, then 'a' is the infimum of S)

(+0.5 marks)

[0.25 marks for writing statement of result]