THE RIEMANN INTEGRAL

1. Introduction

The birth of integral calculus occured more than 2000 years ago when the greeks attempted to determine areas by a process which they called the method of Exhaustion. The essential ideas of this method are very simple and can be described briefly as follows: Given a region whose area is to be determined, we inscribe in it a polygonal region which approximates the given region and whose area we can easily compute. Then we choose another polygonal region which gives a better approximation, and we continue the process, taking polygons with more and more sides in an attempt to exhaust the given region. This method was successfully used by Archimedes (287-212 B. C.) to find the exact formulas for the area of a circle and a few other special figures. The development of the method of Exhaustion. beyond the point to which Archimedes carried it had to wait nearly eighteen centuries until the use of algebraic symbols and techniques became a standard part of mathematics.
Let us start with a simple problem : Find the area A of the region enclosed by a circle of radius r units.
For an arbitrary n, consider the n equal inscribed and superscribed triangles as shown in Figure (drawn in class) Since A is between the total areas of the inscribed and superscribed triangles, we have

$$n\frac{1}{2}r^2\sin(2\pi/n) \le A \le n(r^2 tan(\pi/n))$$
$$\frac{n}{2\pi}\sin(2\pi/n) \le \frac{A}{\pi r^2} \le \frac{n}{\pi}\tan(\pi/n)$$
$$\frac{\sin(2\pi/n)}{2\pi/n} \le \frac{A}{\pi r^2} \le \frac{\sin(\pi/n)}{\pi/n}\frac{1}{\cos(\pi/n)}$$

Taking limit $n \to \infty$ and applying sandwich theorem, $A = \pi r^2$. We will use this idea to define and evaluate the area of the region under a graph of a function.
Suppose f is a non-negative function defined on the interval $[a, b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of f between the areas of the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure (drawn in class). If the total areas of the inscribed and superscribed rectangles converge to the same limit as we make the partition of $[a, b]$ finer and finer then the area of the region under the graph of f can be defined as this limit and f is said to be integrable.

2. What is Riemann Integration?

2.1. **Partitions.** We say that two intervals are almost disjoint if they are disjoint or intersect only at a common endpoint. For example, the intervals $[0, 1]$ and $[1, 3]$ are almost disjoint, whereas the intervals $[0, 2]$ and $[1, 3]$ are not.

Definition 2.1. Partitions Let I be a nonempty, compact interval. A partition of I is a finite collection $\{I_1, I_2, ..., I_n\}$ of almost disjoint, nonempty, compact (closed and bounded) subintervals whose union is I.

A partition of $[a, b]$ with subintervals $I_k = [x_{k-1}, x_k]$ (for $k = 1, 2, ..., n$) is determined by the set of endpoints of the intervals

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Abusing notation, we will denote a partition P either by its intervals

$$P = \{I_1, I_2, ..., I_n\}$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$$

We'll adopt either notation as convenient; the context should make it clear which one is being used. There is always one more endpoint than interval.

Example 2.1. *The set of intervals* $\{[0, 1/5], [1/5, 1/4], [1/4, 1/3], [1/3, 1/2], [1/2, 1]\}$ *is a partition of* $[0, 1]$. *The corresponding set of endpoints is*

$$\{0, 1/5, 1/4, 1/3, 1/2, 1\}.$$

Definition 2.2. Length of an Interval We denote the length of an interval $I = [c, d]$ by $|I| = d-c$.

We note that the sum of the lengths $|I_k| = x_k - x_{k-1}$ of the almost disjoint sub intervals in a partition $\{I_1, I_2, ..., I_n\}$ of an interval I is equal to length of the whole interval. This is obvious geometrically; algebraically, it follows from the telescoping series

$$\sum_{k=1} I_k = x_n - x_{n-1} + x_{n-1} - x_{n-2} + \ + x_2 - x_1 + x_1 - x_0 = x_n - x_0 = |I|.$$

2.2. **Upper and Lower Riemann Sum.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with

$$M = Sup\{f(x) : x \in [a, b]\} \quad m = Inf\{f(x) : x \in [a, b]\}$$

If P is a partition of $[a, b]$. For the partition P of [a, b], we define

$$M_i = Sup\{f(x) : x_{i-1} \leq x \leq x_i\} \quad m_i = Inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

These supremum and infimum are well-defined, finite real numbers since f is bounded (So f is bounded in these subintervals $I_k = [x_{k-1}, x_k]$ for $k = 1, 2, ..., n$. Moreover,

$$m \leq m_k \leq M_k \leq M. \tag{1}$$

Remark 2.1. If f is continuous on the interval I, then it is bounded and attains its maximum and minimum values on each subinterval, but a bounded discontinuous function need not attain its supremum or infimum.

Definition 2.3. Lower Riemann sum : The Lower sum, denoted with $L(P, f)$ of f with respect to the partition P is given by

$$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Definition 2.4. Upper Riemann sum : The Upper sum, denoted with $U(P, f)$ of f with respect to the partition P is given by

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Remark 2.2. Since f is bounded, there exist real numbers m and M such that $m \leq f(x) \leq M$, $\forall x \in [a, b]$. Thus for any partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$, of $[a, b]$, we define

$$M_i = Sup\{f(x) : x_{i-1} \leq x \leq x_i\} \quad m_i = Inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

These supremum and infimum are well-defined, finite real numbers since f is bounded (So f is bounded in these subintervals $I_k = [x_{k-1}, x_k]$ for $k = 1, 2, ..., n$. Moreover, we have

$$m \leq m_k \leq M_k \leq M.$$

which implies

$$m \sum_{k=1}^{n} |I_j| \leq \sum_{k=1}^{n} |I_k| m_k \leq \sum_{k=1}^{n} |I_k| M_k \leq M \sum_{j=1}^{k} |I_j|$$

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \tag{2}$$

$$U(P, f) - L(P, f) \geq 0 \tag{3}$$

Example 2.2. *Consider the function* $f : [0, 1] \to \mathbb{R}$ *defined by* $f(x) = x$ *and take* $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. *The subintervals are* $I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, $I_3 = [\frac{2}{3}, 1]$.
And $M_1 = \frac{1}{3}$, $m_1 = 0$; $M_2 = \frac{2}{3}$, $m_2 = \frac{1}{3}$ *and* $M_3 = 1$ *and* $m_3 = \frac{2}{3}$, *and each subinterval is of length* $\frac{1}{3}$, *then*

$$U(p, f) = \frac{1}{3}.\frac{1}{3} + \frac{2}{3}\frac{1}{3} + 1\frac{1}{3} = \frac{2}{3}$$

and

$$L(p, f) = 0.\frac{1}{3} + \frac{1}{3}\frac{1}{3} + \frac{2}{3}\frac{1}{3} = \frac{1}{3}$$

Definition 2.5. Refinement of a Partition: A partition Q is called a refinement of the partition P if $P \subseteq Q$.

Example 2.3. *Consider the partitions of* $[0, 1]$ *with endpoints*

$$P = \{0, 1/2, 1\}, \quad Q = \{0, 1/3, 2/3, 1\}, \quad R = \{0, 1/4, 1/2, 3/4, 1\}.$$

Thus, P , Q, *and* R *partition* $[0, 1]$ *into intervals of equal length* $1/2$, $1/3$, *and* $1/4$, *respectively. Then* Q *is not a refinement of* P *but* R *is not a refinement of* P.

The following is a simple observation.

Lemma 2.1. *If* P_1 *is any partition on* I. Q_1 *is any refinement of* P_1, *then*

$$L(P_1, f) \leq L(Q_1, f) \tag{4}$$

Proof. Let $P_1 = \{x_0, x_1, x_2, ..., x_{k-1}, x_k, ..., x_n\}$ and without loss of generality we may assume $Q_1 = \{x_0, x_1, x_2, ..., x_{k-1}, z, x_k, ..., x_n\}$ that is that Q_1 contains just one more point than P.

Let

$$m'_k = Inf\{f(x) : x \in [x_{k-1}, z]\}$$
$$m''_k = Inf\{f(x) : x \in [z, x_k]\}$$

Then $m_k \leq m'_k$ and $m_k \leq m''_k$ where $m_k = Inf\{f(x) : x \in [x_{k-1}, x_k]\}$

$$\begin{aligned} L(P_1, f) &= m_0(x_1 - x_0) + ... + m_k(x_k - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &= m_0(x_1 - x_0) + ... + m_k(x_k - z + z - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &\leq m_0(x_1 - x_0) + ... + m'_k(z - x_{k-1}) + m''_k(x_k - z) + ... + m_{n-1}(x_n - x_{n-1}) \\ &= L(Q_1, f) \end{aligned}$$

□

Similarly, one can prove

Lemma 2.2. *If P_2 is any partition on I. If Q_2 is any refinement of P_2, then*

$$U(Q_2, f) \leq U(P_2, f) \tag{5}$$

Proof. Proof is similar to the previous one. $\square$

Lemma 2.3. *If Q is a refinement of P, then*

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f) \tag{6}$$

Proof. Putting $P_1 = P_2 = P$ and $Q_1 = Q_2 = Q$ in (4) and (5). $\square$

Lemma 2.4. *If P_1 and P_2 be any two partitions, then*

$$L(P_1, f) \leq U(P_2, f) \tag{7}$$

Proof. Let $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. So by (6), $L(P_1, f) \leq L(Q, f) \leq U(Q, f) \leq U(P_2, f)$. $\square$

Definition 2.6. Upper integral Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$. Then the upper integral of f is

$$\overline{\int_a^b} f dx = Inf\{U(P, f) : P \in \mathcal{P}\} \tag{8}$$

Definition 2.7. Lower integral Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$. Then the lower integral of f is

$$\underline{\int_a^b} f dx = Sup\{L(P, f) : P \in \mathcal{P}\} \tag{9}$$

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Lemma 2.5. *We have*

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx \tag{10}$$

Proof. In (7) we fix P_2 and take supremum over all partitions P_1, then we get $\underline{\int_a^b} f dx \leq U(P_2, f)$. Now we take infimum over all partitions P_2 to get the desired result. $\square$

Definition 2.8. Riemann integrability: $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx \tag{11}$$

and the value of the limit is denoted by $\int_a^b f dx$. We say $f \in \mathcal{R}[a, b]$.

Remark 2.3. Suppose $f \in \mathcal{R}[a,b]$. Then by equation (2) of Remark 2.2 and (8) and (9) and (10) and (11)

$$m(b-a) \leq L(P,f) \leq \int_a^b f dx \leq U(P,f) \leq M(b-a)$$

where P is any partition of $[a,b]$.

Example 2.4. *The constant function $f(x) = 1$ on $[0,1]$ is Riemann integrable, and*

$$\int_0^1 1dx = 1$$

To show this, let $P = \{I_1, I_2, ..., I_n\}$ be any partition of $[0,1]$ with endpoints $\{x_0 = 0, x_1, x_2, ..., x_{n-1}, x_n = 1\}$.
Since f is constant,

$$M_k = Sup\{f(x) : x_{k-1} \leq x \leq x_k\} = 1 \qquad m_k = Inf\{f(x) : x_{k-1} \leq x \leq x_k\} = 1$$

for $k = 1, ..., n$. Therefore

$$U(P,f) = L(P,f) = \sum_{k=1}^{n}(x_k - x_{k-1}) = x_n - x_0 = 1$$

Geometrically, this equation is the obvious fact that the sum of the areas of the rectangles over (or, equivalently, under) the graph of a constant function is exactly equal to the area under the graph. Thus, every upper and lower sum of f on $[0,1]$ is equal to 1, which implies that the upper and lower integrals

$$\overline{\int_0^1} dx = Inf\{U(P,f) : P \in \mathcal{P}\} = 1 \qquad \underline{\int_0^1} dx = Sup\{L(P,f) : P \in \mathcal{P}\} = 1$$

are equal, and the integral of f is 1.

More generally, the same argument shows that every constant function $f(x) = c$ is integrable and

$$\int_a^b c\ dx = c(b-a)$$

The following is an example of a discontinuous function that is Riemann integrable.

Example 2.5. *The function*

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

Show that f is Riemann integrable, and

$$\int_0^1 f(x)\ x = 0$$

.

To show this, let $P = \{I_1, I_2, ..., I_n\}$ be a partition of $[0,1]$ the set of endpoints of the intervals

$$P = \{x_0 = 0, x_1, x_2, ..., x_{n-1}, x_n = 1\}.$$

The first interval in the partition is $I_1 = [0, x_1]$, where $0 < x_1 \leq 1$, and $M_1 = 1$, $m_1 = 0$, since $f(0) = 1$ and $f(x) = 0$ for $0 < x \leq x_1$.

Then, since $f(x) = 0$ for $x > 0$, for other partitions, we have

$$M_k = Sup\{f(x) : x_{k-1} \leq x \leq x_k\} = 0 \ \ and \ \ m_k = Inf\{f(x) : x_{k-1} \leq x \leq x_k\} = 0 \qquad k = 2, 3, ..., n.$$

and

$$M_1 = Sup\{f(x) : x_0 \le x \le x_1\} = 1 \;\; and \;\; m_1 = Inf\{f(x) : x_0 \le x \le x_1\} = 0$$

It follows that $U(P,f) = 1.(x_1 - x_0) + 0(x_2 - x_1) + \cdots 0(x_n - x_{n-1}) = x_1$.
And $L(P,f) = 0(x_1 - x_0) + 0(x_2 - x_1) + \cdots 0(x_n - x_{n-1}) = 0$. *As* P *is any arbitary partition of* $[a, b,$ *so taking infimum over all partitions, we have* $\overline{\int_0^1} f(x)dx = Inf\{U(P,f) : P \in \mathcal{P}\} = 0$.
Also $\underline{\int_0^1} f(x)dx = Sup\{L(P,f) : P \in \mathcal{P}\} = 0$.
Hence $\underline{\int_0^1} f(x)dx = \overline{\int_0^1} f(x)dx = 0$. *In this example, the infimum of the upper Riemann sums is not attained and* $U(P,f) > \overline{\int_0^1} f(x)dx = 0$ *for every partition* P.

Example 2.6. *Let* $g(x)$ *be Dirichlet's function* $g : [0,1] \to \mathbb{R}$

$$g(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Let P *be any partition of* $[0,1]$. *Then* $m_k = 0$ *and* $M_k = 1$ *for all* $k = 1, \cdots, n$. *Thus, for every partition* P, $L(g,P) = 0 < U(g,P) = 1$ *which implies* $\underline{\int_0^1} g(x)dx = 0 < 1 = \overline{\int_0^1} g(x)dx$. *Thus, the nowhere continuous function* g *is not integrable on* $[0,1]$.

2.3. Riemann's criterion for integrability.

Theorem 2.6. (Riemann's criterion for integrability)1: *Then* f *is integrable on* $[a,b]$ $\iff$ *for every* $\varepsilon > 0$ *there exists a partition* P_ε *such that*

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon. \tag{12}$$

Proof. Suppose f is integrable that is $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx = \int_a^b f dx$. Let $\varepsilon > 0$ be given. Then by the definition (8 and 9) there exist partitions P_1 and P_2 such that

$$U(P_2, f) - \int_a^b f dx < \varepsilon/2 \text{ and } \int_a^b f dx - L(P_1, f) < \varepsilon/2$$

Now let $P_\varepsilon = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then

$$U(P_\varepsilon, f) < U(P_2, f) \text{ and } L(P_1, f) < L(P_\varepsilon, f)$$

Hence we have

$$U(P_\varepsilon, f) - \int_a^b f dx < U(P_2, f) - \int_a^b f dx < \varepsilon/2 \text{ and } \int_a^b f dx - L(P_\varepsilon, f) < \int_a^b f dx - L(P_1, f) < \varepsilon/2$$

So adding the two inequalities $U(P_\varepsilon, f) - \int_a^b f dx < \varepsilon/2$ and $\int_a^b f dx - L(P_\varepsilon, f) < \varepsilon/2$, we get, $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$.

Conversely, for every $\varepsilon > 0$ there exists a partition P_ε such that

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon. \tag{13}$$

By definition (8 and 9) for any partition P

$$U(P, f) \ge \overline{\int_a^b} f dx \text{ and } L(P, f) \le \underline{\int_a^b} f dx \tag{14}$$

In particular, we take $P = P_\varepsilon$, then by (13) and(14), we have

$$\overline{\int_a^b} f dx - \underline{\int_a^b} f dx \le U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$$

Since ε is arbitary, we can say $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$. So f is integrable in $[a, b]$. $\square$

Example 2.7. *Consider the following function $f : [0, 1] \to \mathbb{R}$*

$$f(x) = \left\{ \begin{array}{ll} 1 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{array} \right.$$

Clearly $U(P, f) = 1$ for any partition P. We notice that $L(P, f)$ will be less than 1. We can try to isolate the point $\frac{1}{2}$ in a sub-interval of small length. Consider the partition $P_\varepsilon = \{0, \frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, 1\}$. Then

$$L(P_\varepsilon, f) = 1.(\frac{1}{2} - \frac{\varepsilon}{2}) + 0.(\frac{1}{2} + \frac{\varepsilon}{2} - \frac{1}{2} + \frac{\varepsilon}{2}) + 1.(1 - \frac{1}{2} - \frac{\varepsilon}{2}) = 1 - \varepsilon$$

Therefore, for given $\varepsilon > 0$, there exists a partition $P_\varepsilon = \{0, \frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, 1\}$ such that

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

. Hence f is integrable and $\int_0^1 f(x)dx = 1$.

The proof of the following corollary is immediate from the previous theorem.

Corollary 2.7. (Riemann's criterion for integrability) 2: *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that*

$$\lim_{n\to\infty} (U(P_n, f) - L(P_n, f)) = 0. \tag{15}$$

and we have

$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \int_a^b f(x)dx$$

Proof. Suppose f is integrable. We choose $\varepsilon = \frac{1}{n}$, then by previous the Theorem (2.6), for this choice of ε we get a partition P_n such that

$$0 \leq U(P_n, f) - L(P_n, f) < \frac{1}{n}. \tag{16}$$

Taking limit on both sides of the inequality (16) we get the desired result (15).

Conversely, suppose there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} (U(P_n, f) - L(P_n, f)) = 0. \tag{17}$$

Then by the definition for any $n \in \mathbb{N}$

$$\overline{\int_a^b} f dx \leq U(P_n, f) \quad and \quad L(P_n, f) \leq \underline{\int_a^b} f dx$$

Also by (10) we have, $0 \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx$. So we can write

$$0 \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx \leq U(P_n, f) - L(P_n, f).$$

Then taking limit $n \to \infty$ we get our required result. $\square$

Example 2.8. *Consider $f(x) = x^2$ in $[0, 1]$.*

P_n be the partition of $[0, 1]$ into n-intervals of equal length $1/n$ with endpoints $x_k = k/n$ for $k = 0, 1, 2, ..., n$. If $I_k = [(k-1)/n, k/n]$ is the $k-th$ interval, then

$$M_k = Sup\{f(x) : x_{k-1} \leq x \leq x_k\} = x_k^2 = \left(\frac{k}{n}\right)^2$$

$$\text{and} \quad m_k = Inf\{f(x) : x_{k-1} \leq x \leq x_k\} = x_{k-1}^2 = \left(\frac{k-1}{n}\right)^2 \qquad k = 1, 2, 3, ..., n.$$

since f is increasing. Using the formula for the sum of squares we get

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\begin{aligned} U(P_n, f) &= \sum_{k=1}^{n} x_k^2 \,(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k^2 \frac{1}{n} \\ &= \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{6n^2}(n+1)(2n+1) = \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n}) \end{aligned}$$

$$\begin{aligned} L(P_n, f) &= \sum_{k=1}^{n} x_{k-1}^2 \,(x_k - x_{k-1}) = \sum_{k=1}^{n} x_{k-1}^2 \frac{1}{n} \\ &= \frac{1}{n^3}\sum_{k=1}^{n} (k-1)^2 = \frac{1}{6n^2}(n-1)(2n-1) = \frac{1}{6}(1-\frac{1}{n})(2-\frac{1}{n}) \end{aligned}$$

So

$$\lim_{n\to\infty} (U(P_n, f) - L(P_n, f)) = 0.$$

So we have

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \frac{1}{3}$$

Hence f is integrable and $\int_0^1 x^2 dx = \frac{1}{3}$.

Example 2.9. *Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$.*

P_n be the partition of $[0, 1]$ into n-intervals of equal length $1/n$ with endpoints $x_k = k/n$ for $k = 0, 1, 2, ..., n$. If $I_k = [(k-1)/n, k/n]$ is the $k-th$ interval, then

$$M_k = Sup\{f(x) : x_{k-1} \leq x \leq x_k\} = x_k = \left(\frac{k}{n}\right)$$

$$\text{and} \quad m_k = Inf\{f(x) : x_{k-1} \leq x \leq x_k\} = x_{k-1} = \left(\frac{k-1}{n}\right) \qquad k = 1, 2, 3, ..., n.$$

Hence

$$\begin{aligned} U(P_n, f) &= \sum_{k=1}^{n} x_k \,(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k \frac{1}{n} \\ &= \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2n^2}n(n+1) = \frac{1}{2}(1+\frac{1}{n}) \end{aligned}$$

$$
\begin{aligned}
L(P_n, f) &= \sum_{k=1}^{n} x_{k-1}^2 \, (x_k - x_{k-1}) = \sum_{k=1}^{n} x_{k-1}^2 \frac{1}{n} \\
&= \frac{1}{n^2} \sum_{k=1}^{n} (k-1) = \frac{1}{2n^2} n(n-1) = \frac{1}{2}(1 - \frac{1}{n})
\end{aligned}
$$

So

$$\lim_{n \to \infty} (U(P_n, f) - L(P_n, f)) = 0.$$

Hence f is integrable and $\int_0^1 x dx = \frac{1}{2}$.

Example 2.10. Integrability of Continuous functions *The Cauchy criterion leads to the following fundamental result that :*

Theorem 2.8. *Every continuous function is Riemann integrable.*

Proof. To prove this, we use the fact that a continuous function oscillates by an arbitrarily small amount on every interval of a sufficiently refined partition. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on a compact interval. A continuous function on a compact set is bounded, Let $\varepsilon > 0$ be given. A continuous function on a compact set is uniformly continuous, so there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \qquad \textit{for all } x, y \in [a, b] \textit{ such that } |x - y| < \delta \tag{18}$$

Choose a partition $P_\varepsilon = \{I_1, I_2, ..., I_r\}$ of $[a, b]$ such that $|I_k| < \delta$ for every k; for example, we can take r intervals of equal length $(b - a)/r$ with $r > (b - a)/\delta$. Since f is continuous, it attains its maximum and minimum values M_k and m_k on the compact interval I_k at points x_k and y_k in I_k. These points satisfy $|x_k - y_k| < \delta$, so $M_k - m_k = f(x_k) - f(y_k) < \frac{\varepsilon}{b-a}$ by (18).

The upper and lower sums of f therefore satisfy

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) = \sum_{k=1}^{r} (M_k - m_k)|I_k| = \sum_{k=1}^{r} (f(x_k) - f(y_k))|I_k| < \sum_{k=1}^{r} \frac{\varepsilon}{b-a}|I_k| = \frac{\varepsilon}{b-a} \sum_{k=1}^{r} |I_k| = \varepsilon$$

and Theorem 2.6 implies that f is integrable. □

The function $f(x) = x^2$ on $[0, 1]$ considered is integrable since it is continuous.

Theorem 2.9. *Suppose $f : [a, b] \to \mathbb{R}$ be a bounded function which has finitely many discontinuities. Then $f \in \mathcal{R}[a, b]$.*

Proof follows by constructing suitable partition with sub-intervals of sufficiently small length around the discontinuities as observed in the above example.

Example 2.11. *Example 2.7*

Example 2.12. Integrability of Monotone functions *The Cauchy criterion leads to the following fundamental result that :*

Theorem 2.10. *Every monotone function is Riemann integrable.*

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is monotonically increasing (the proof is similar in the other case.) Choose a partition $P_n = \{I_1, I_2, ..., I_n\}$ of $[a, b]$ such that $|I_k| = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for $i = 1, 2, ..., n$. Therefore

$$\begin{aligned} 0 \le U(P_n, f) - L(P_n, f) = \sum_{k=1}^{n}(M_k - m_k)|I_k| = \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))|I_k| &= \sum_{k=1}^{n}\frac{b-a}{n}(f(x_k) - f(x_{k-1})) \\ &= \frac{b-a}{n}(f(b) - f(a)) \end{aligned}$$

Now $\frac{b-a}{n}(f(b) - f(a)) \to 0$ as $n \to \infty$, so we have $\lim_{n\to\infty}(U(P_n, f) - L(P_n, f)) = 0$. Hence f is integrable by Corollary 2.7. ☐

Example 2.13. *Consider $f : [1, b] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.*
Divide the interval in geometric progression and compute $U(P_n, f)$ and $L(P_n, f)$ to show that $f \in \mathcal{R}[1, b]$.
Let $P_n = \{1, r, r^2, \cdots, r^n = b\}$ be a partition on $[1, b]$. Then

$$\begin{aligned} U(P_n, f) &= f(1)(r-1) + f(r)(r^2 - r) + \cdots + f(r^{n-1})(r^n - r^{n-1}) \\ &= (r-1) + \frac{1}{r}(r^2 - r) + \cdots + \frac{1}{r^{n-1}}(r^n - r^{n-1}) \\ &= n(r-1) \\ &= n(b^{\frac{1}{n}} - 1) \end{aligned}$$

Then $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{b^{\frac{1}{n}} - 1}{\frac{1}{n}} = \log b$.
Similarly,

$$\begin{aligned} L(P_n, f) &= f(r)(r-1) + f(r^2)(r^2 - r) + \cdots + f(r^n)(r^n - r^{n-1}) \\ &= \frac{1}{r}(r-1) + \frac{1}{r^2}(r^2 - r) + \cdots + \frac{1}{r^n}(r^n - r^{n-1}) \\ &= \frac{n}{r}(r-1) \\ &= \frac{(b^{\frac{1}{n}} - 1)}{b^{\frac{1}{n}}\frac{1}{n}} \end{aligned}$$

Then $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} \frac{b^{\frac{1}{n}} - 1}{b^{\frac{1}{n}}\frac{1}{n}} = \log b$.
So we have $\lim_{n\to\infty}(U(P_n, f) - L(P_n, f)) = 0$. *Hence f is integrable by Corollary 2.7.*

2.4. **Riemann Sum.**

Definition 2.9. Riemann sum: Let $P = \{x_1, x_2, ..., x_n\}$ be a partition of $[a, b]$. Choose $\xi_k \in I_k$ for each k. The set $\{\xi_k : 1 \le k \le n\}$ is called a selection from P. The expression $S(P, f, \xi)$ is defined as

$$S(P, f, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \tag{19}$$

is the Riemann sum for f with respect to the partition P and selection ξ.

Then it is easy to show the following

Lemma 2.11. *If $M = Sup\{f(x) : x \in [a,b]\}$ and $m = Inf\{f(x) : x \in [a,b]\}$, then*

$$m(b-a) \le L(P,f) \le S(P,f,\xi) \le U(P,f) \le M(b-a).$$

for any selection ξ from P.

In fact, one has the following Darboux theorem:

Definition 2.10. The norm (or mesh) of the partition $P = \{x_1, x_2, ..., x_n\}$ denoted by $||P||$ is the length of the longest of these sub-intervals i.e $||P|| = max\{|x_i - x_{i-1}| : i = 1, ..., n\}$.

Theorem 2.12. *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function if and only if there exists a real number $\mathcal{R}(f)$ satisfying the following: for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P with $||P|| := max|x_i - x_{i-1}| < \delta$ we have*

$$|S(P,f,\xi) - \mathcal{R}(f)| < \varepsilon \tag{20}$$

for any selection ξ from P and in that case we have $\int_a^b f(x)dx = \mathcal{R}(f)$.

Corollary 2.13. *If f is Riemann integrable if and only if for every sequence of partitions $\{P_n\}$ with $||P_n|| \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} L(P_n,f) = \lim_{n\to\infty} U(P_n,f) = \int_a^b f(x)dx$. Also we have, $\lim_{n\to\infty} S(P_n,f) = \int_a^b f(x)dx$,*

Proof. The proof follows from Theorem 2.12 and Lemma 2.11. □

Problem 2.1. Let us evaluate the sum $\lim_{n\to\infty} x_n$ where $x_n = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n}$ using the above theorem. Basically we have to write x_n as a Riemann sum of some function on some interval. Note that

$$x_n = \frac{1}{n}\left(1 + \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + ... + \frac{1}{1+\frac{n}{n}}\right) = S(P_n, f, \xi)$$

where $f(x) = \frac{1}{x}$ for $x \in [1,2]$ and it is Riemann integrable as it is a continuous function and $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, ...1+\frac{n}{n}\}$ with $\xi_k = 1+\frac{k}{n}$ for $k = 1, 2, ...n$. By previous theorem $\lim_{n\to\infty} x_n = \lim_{n\to\infty} S(P_n, f, \xi) = \int_1^2 \frac{1}{x}dx = \log 2$.

Remark 2.4. From the above theorem, we note that for any sequence of partitions $\{P_n\}$ such that $||P_n|| \to 0$ as $n \to \infty$ but $U(P_n,f) - L(P_n,f) \nrightarrow 0$ as $n \to \infty$ then f is not integrable.

Example 2.14. *Show that the function $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) = \{ \begin{array}{ll} 1+x & x \in \mathbb{Q} \\ 1-x & x \notin \mathbb{Q} \end{array}$$

is non integrable.

Consider the sequence of partitions $\{P_n\}$ defined by $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$. Then for any j $(1, 2, ..., n)$ supremum of f in $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ is $(1 + \frac{j}{n})$ and the infimum of f in $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ is $(1 - \frac{j-1}{n})$ (although not achieved). So $||P_n|| \to 0$ as $n \to \infty$ but

$$\begin{aligned} U(P_n,f) &= \left(1+\frac{1}{n}\right)\frac{1}{n} + \left(1+\frac{2}{n}\right)\frac{1}{n} + ... + \left(1+\frac{n}{n}\right)\frac{1}{n} \\ &= 1 + \frac{1}{n^2}\frac{(n(n+1))}{2} \\ &= 1 + \frac{1}{2}\left(1+\frac{1}{n}\right) \\ &= \frac{3}{2} + \frac{1}{n} \ \to \frac{3}{2} \ \ n \to \infty \end{aligned}$$

$$\begin{aligned}
L(P_n, f) &= \left(1 - \frac{1}{n}\right)\frac{1}{n} + \left(1 - \frac{2}{n}\right)\frac{1}{n} + \ldots\ldots\ldots\ldots + \left(1 - \frac{n}{n}\right)\frac{1}{n} \\
&= 1 - \frac{1}{n^2}\frac{(n(n+1))}{2} \\
&= 1 - \frac{1}{2}\left(1 + \frac{1}{n}\right) \\
&= \frac{1}{2} - \frac{1}{n} \quad \to \frac{1}{2} \quad n \to \infty
\end{aligned}$$

is not integrable by Corollary 2.13.

2.5. **Properties of Definite Integral:** In this section we prove some properties of definite integral.

Property 1: For a constant $c \in \mathbb{R}$ and $f \in \mathcal{R}[a, b]$, then $cf \in \mathcal{R}[a, b]$ and we have $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Proof. Let $\varepsilon > 0$ be given. If $c = 0$, it is clear cf is integrable. So, assume $c \neq 0$. Given $\varepsilon > 0$. As $f \in \mathcal{R}[a, b]$, so by Theorem 2.12, for given $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P with $||P|| := max|x_i - x_{i-1}| < \delta$ we have

$$|S(P, f, \xi) - \int_a^b f(x)dx| < \frac{\varepsilon}{|c|} \tag{21}$$

for any selection x_i from P. Now

$$\begin{aligned}
|S(P, cf, \xi) - c\int_a^b f(x)dx| &= |\sum_{i=1}^{n} cf(\xi_i)(x_i - x_{i-1}) - c\int_a^b f(x)dx| \\
&= |c\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - c\int_a^b f(x)dx| \\
&= |c||\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - c\int_a^b f(x)dx| \\
&< |c|\frac{\varepsilon}{|c|} \quad \text{by (21)} \\
&= \varepsilon
\end{aligned}$$

Hence for given $\varepsilon > 0$ there exists $\delta > 0$ such that for any partition P with $||P|| := max|x_i - x_{i-1}| < \delta$ we have

$$|S(P, cf, \xi) - c\int_a^b f(x)dx| < \varepsilon$$

for any selection ξ from P. By Theorem 2.12, we have $cf \in \mathcal{R}[a, b]$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$. □

Property 2: For a constant $c_1, c_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{R}[a, b]$. Then $c_1f_1 + c_2f_2 \in \mathcal{R}[a, b]$.

$$\int_a^b (c_1f_1(x) + c_2f_2(x))dx = c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx$$

Proof. Let $\varepsilon > 0$ be given. By Property 1 it is clear c_1f_1 is integrable. As $c_1f_1 \in \mathcal{R}[a, b]$, so by Theorem 2.12, for given $\varepsilon > 0$ there exists $\delta_1 > 0$ (depending on ε) such that for any partition P

with $||P|| := max|x_i - x_{i-1}| < \delta_1$ we have

$$|S(P, c_1f_1, \xi) - \int_a^b c_1f_1(x)dx| = |S(P, c_1f_1, \xi) - c_1\int_a^b f_1(x)dx| < \frac{\varepsilon}{2} \tag{22}$$

for any selection ξ from P.
Also By Property 1, c_2f_2 is integrable. As $\int_a^b c_2f_2(x)dx = c_2\int_a^b f_2(x)dx$. As $c_2f_2 \in \mathcal{R}[a,b]$, so by Theorem 2.12, for given $\varepsilon > 0$ there exists $\delta_2 > 0$ (depending on ε) such that for any partition P with $||P|| := max|x_i - x_{i-1}| < \delta_2$ we have

$$|S(P, c_2f_2, \xi) - \int_a^b c_2f_2(x)dx| = |S(P, c_2f_2, \xi) - c_2\int_a^b f_2(x)dx| < \frac{\varepsilon}{2} \tag{23}$$

for any selection ξ from P. Now

Let $\delta_\varepsilon = min(\delta_1, \delta_2)$. So for given $\varepsilon > 0$, there exists this δ_ε such that for any partition P with $||P|| := max|x_i - x_{i-1}| < \delta_\varepsilon$ we have

$$\begin{aligned}
&|S(P, c_1f_1 + c_2f_2, \xi) - (c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx)| \\
&= |\sum_{i=1}^n (c_1f_1(\xi_i) + c_2f_2(\xi_i))(x_i - x_{i-1}) - (c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx)| \\
&= |\sum_{i=1}^n c_1f_1(\xi_i)(x_i - x_{i-1}) + \sum_{i=1}^n c_2f_2(\xi_i)(x_i - x_{i-1}) - (c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx)| \\
&\leq |S(P, c_1f_1, \xi) - c_1\int_a^b f_1(x)dx| + |S(P, c_2f_2, \xi) - c_2\int_a^b f_2(x)dx| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for any selection ξ from P. Hence $c_1f_1 + c_2f_2$ is integrable by Theorem 2.12 and $\int_a^b (c_1f_1 + c_2f_2)dx = c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx$. □

Example 2.15. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $f(x) \geq 0$ for every $x \in [a,b]$. If f is integrable, show that $\int_a^b f dx \geq 0$. (See Problem 5, Assignment 9).*

Property 3: Let $h, g \in \mathcal{R}[a,b]$. And $h(x) \leq g(x)$ for all $x \in [a,b]$. Then

$$\int_a^b h(x)dx \leq \int_a^b g(x)dx$$

Proof. Let $f \in \mathcal{R}[a,b]$ with $0 \leq f(x)$ for all $x \in [a,b]$. Then by Remark 2.2, we have

$$m(b-a) \leq \int_a^b f dx \leq M(b-a)$$

Since $0 \leq f(x)$ for all $x \in [a,b]$, we have $m \geq 0$ which in turn implies

$$0 \leq m(b-a) \leq \int_a^b f dx$$

We put $f = g - h$. Then $f(x) = g(x) - h(x) \geq 0$ for all $x \in [a,b]$. Then we can write by Property 2,

$$0 \leq \int_a^b f(x)dx = \int_a^b (g(x) - h(x))dx = \int_a^b g(x)dx - \int_a^b h(x)dx \quad \Rightarrow \quad \int_a^b h(x)dx \leq \int_a^b g(x)dx$$

□

Property 4: If $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and

$$|\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx$$

Proof. For the partition P of [a, b], the notation is

$$M_i = Sup\{f(x) : x_{i-1} \leq x \leq x_i\} \quad m_i = Inf\{f(x) : x_{i-1} \leq x \leq x_i\} \text{ for } i = 0, 1, ...n.$$

□

Let us denote by

$$M_i' = Sup\{|f|(x) : x_{i-1} \leq x \leq x_i\} \quad m_i' = Inf\{|f|(x) : x_{i-1} \leq x \leq x_i\} \text{ for } i = 0, 1, ...n.$$

We claim that

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

Now $x, y \in [x_{i-1}, x_i]$, we have $m_i \leq f(x) \leq M_i$ and $-M_i \leq -f(x) \leq -m_i$, then

$$f(x) - f(y) \leq M_i(f) - m_i(f).$$
$$\text{or } f(y) - f(x) \leq M_i(f) - m_i(f)$$
$$\text{so } |f(x) - f(y)| \leq M_i(f) - m_i(f)$$

Note that for $x, y \in [x_{i-1}, x_i]$,

$$|f|(x) - |f|(y) \leq |f(x) - f(y)| \leq M_i(f) - m_i(f).$$

Now take supremum over x (fixing y) and then taking infimum over y, to conclude the claim. Now since f is integrable, there exists partitions $\{P_n\}$ such that $\lim_{n\to\infty}(U(P_n, f) - L(P_n, f)) = 0$ which implies

$$\lim_{n\to\infty}(U(P_n, |f|) - L(P_n, |f|)) = 0$$

So $|f| \in \mathcal{R}[a,b]$.

Now

$$-|f|(x) \leq f(x) \leq |f|(x)$$

Then by Property 3,

$$-\int_a^b |f|(x)dx \leq \int_a^b f(x)dx \leq \int_a^b |f|(x)dx$$

Hence

$$|\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx.$$

Example 2.16. *Suppose $h : [a,b] \to \mathbb{R}$ be a function. If $h \in \mathcal{R}[a,b]$ exists then so is $h^2 \in \mathcal{R}[a,b]$. Using the above prove that if $f, g \in \mathcal{R}[a,b]$, so is fg.*
Refer to Assignment 9.

3. Additive Property

Domain decomposition property

Theorem 3.1. *Let f be bounded on $[a,b]$ and let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$. In this cases*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof. Let f be integrable on $[a,b]$. For $\varepsilon > 0$, there exists partition

$$U(P,f) - L(P,f) < \varepsilon \tag{24}$$

With out loss of generality we can assume that P contain c. (otherwise we can refine P by adding c and the difference will be closer than before). Let $P_1 = P \cap [a,c]$ and $P_2 = P \cap [c,b]$. Then P_1 and P_2 are partitions on $[a,c]$ and $[c,b]$ respectively. Also by (24),

$$U(P_1,f) - L(P_1,f) < \varepsilon$$

and

$$U(P_2,f) - L(P_2,f) < \varepsilon$$

. This implies f is integrable on $[a,c]$ and $[c,b]$.
Conversely, suppose f be is integrable on $[a,c]$ and $[c,b]$. Then for $\varepsilon > 0$, there exists partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(P_1,f) - L(P_1,f) < \varepsilon/2$$

and

$$U(P_2,f) - L(P_2,f) < \varepsilon/2$$

. Now take $P = P_1 \cup P_2$. Then

$$U(P,f) - L(P,f) \leq U(P_1,f) - L(P_1,f) + U(P_2,f) - L(P_2,f) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So f is Riemann integrable. As f is Riemann integrable on $[a,b]$, for every sequence of partition $\{P_n\}$ in $[a,b]$ with $||P_n|| \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} S(P_n,f) = \int_a^b f(x)dx$ by Corollary 2.13.

$$\begin{aligned}
\int_a^b f(x)dx = \lim_{n\to\infty} S(P_n,f) &= \lim_{n\to\infty} \sum_{P_n} f(\zeta_k)(x_k - x_{k-1}) \\
&= \lim_{n\to\infty} \Big(\sum_{P_n \cap [a,c]} f(\zeta_k)(x_k - x_{k-1}) + \sum_{P_n \cap [c,b]} f(\zeta_k)(x_k - x_{k-1}) \Big) \\
&= \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}$$

The last step follows as f is Riemann integrable on $[c,b]$ and $[a,c]$, for every sequence of partition $\{P_{1,n}\}$ of $[a,c]$ with $||P_{1,n}|| \to 0$ as $n \to \infty$, and for every sequence of partition $\{P_{2,n}\}$ of $[c,b]$ with $||P_{2,n}|| \to 0$ as $n \to \infty$,we have $\lim_{n\to\infty} S(P_{1,n},f) = \int_a^c f(x)dx$ and $\lim_{n\to\infty} S(P_{2,n},f) = \int_c^b f(x)dx$ by Corollary 2.13. (Here $P_n \cap [a,c]$ is a sequence of partition of $[a,c]$ and $P_n \cap [c,b]$ is a sequence of partition of $[c,b]$.) □

Example 3.1. $f : [0,1] \to \mathbb{R}$ *defined by*

$$f(x) = \left\{ \begin{array}{cc} 0 & x \in \mathbb{Q} \\ \sin\frac{1}{x} & x \notin \mathbb{Q} \end{array} \right.$$

We will show that f is not integrable on a sub interval of $[0,1]$. *Consider the f on the sub-interval $I_1 = [\frac{2}{\pi}, 1]$. Clearly $L(P,f) = 0$ for any partition P of I_1 because $f(x) \geq 0$ in the sub interval $[\frac{2}{\pi}, 1]$. Let M_k be the Supremum of f on sub intervals $[x_{k-1}, x_k]$ of $[\frac{2}{\pi}, 1]$. Also, the minimum of M_ks is* $\sin 1$. *Therefore,*

$$U(P,f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) > (1 - \frac{2}{\pi}) \sin 1.$$

Hence $U(P,f) - L(P,f)$ can not be made less than ε for any $\varepsilon < (1 - \frac{2}{\pi}) \sin 1$. Therefore f is not Riemann integrable.

4. Fundamental Theorem of Calculus

By looking at the definitions of differentiation and integration, one may feel that these notions are totally different. Even the geometric interpretations do not give any idea that these two notions are related. In this lecture we will discuss two results, called fundamental theorems of calculus, which say that differentiation and integration are, in a sense, inverse operations.

Theorem 4.1. (First Fundamental Theorem of Calculus) *Let f be integrable on $[a,b]$. If there is a differentiable function F on $[a,b]$ such that $F'(x) = f(x)$ for all $x \in (a,b)$ then*

$$\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$. Since f is integrable we can find a partition $P := \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. By the mean value theorem there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Hence

$$\sum_{i=1}^n f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n F'(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(b) - F(a) \tag{25}$$

We know that by Lemma 2.11

$$L(P,f) \le S(P,f,c) = \sum_{i=1}^n f(c_i)(x_i - x_{i-1}) \le U(P,f)$$

$$L(P,f) \le F(b) - F(a) \le U(P,f) \text{ by (25)} \tag{26}$$

Also by Remark 2.2

$$L(P,f) \le \int_a^b f(x)dx \le U(P,f) \tag{27}$$

Hence by (26) and (27)

$$|\int_a^b f(x)dx - (F(b) - F(a))| < U(P,f) - L(P,f) < \varepsilon$$

Since $\varepsilon > 0$ is arbitary. Hence $\int_a^b f(x)dx = F(b) - F(a)$. This completes the proof.

The existence of an anti-derivative for a continuous function on $[a,b]$ follows from the second F.T.C (See Below). If an integrable function f has an anti-derivative (and if we can find it), then calculating its integral is very simple. The second F.T.C. explains this. □

Remark 4.1. It is always not true that $\int_a^b f'(x)dx = f(b) - f(a)$.? The answer is NO. For example, take $f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$, so $f'(x) = \frac{1}{2\sqrt{x}}$ for all $x \in (0,1]$ and this function is unbounded, so not Riemann integrable. So we can not write $\int_0^1 f'(x)dx = f(1) - f(0)$.

Corollary 4.2. (Integration by Parts) *If h, g is continuous on $[a,b]$ and differentiable on (a,b) and both $h'g$ and $g'h$ are integrable on $[a,b]$, then*

$$\int_a^b h'g dx + \int_a^b g'h dx = h(b)g(b) - h(a)g(a). \tag{28}$$

Proof. Let $f(x) = h'(x)g(x) + g'(x)h(x)$. By Property 2, f is integrable as $h'g + g'h$ are integrable on $[a,b]$. Let $F'(x) = f(x) = h'(x)g(x) + g'(x)h(x) = (h(x)g(x))'$. Hence by (First Fundamental Theorem of Calculus)

$$\int_a^b (h'(x)g(x) + g'(x)h(x))dx = \int_a^b f(x)dx = F(b) - F(a) = h(b)g(b) - h(a)g(a).$$

□

Theorem 4.3. (Second Fundamental Theorem of Calculus) *Let f be integrable on $[a, b]$. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$ and if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof. Suppose $M = sup\{|f(x)| : x \in [a, b]\}$. Let $a \leq x < y \leq b$. Then

$$\begin{aligned} |F(y) - F(x)| &= |\int_a^y f(t)dt - \int_a^x f(t)dt| \\ &= |\int_x^y f(t)dt| \leq \int_x^y |f(t)|dt \leq M \int_x^y 1dt = M(y - x). \end{aligned}$$

Hence F is continuous in $[a, b]$, in fact uniformly continuous. Now suppose f is continuous at x_0. Given $\varepsilon > 0$, then there exists $\delta > 0$ (depending on $\varepsilon > 0$) such that

$$|t - x_0| < \delta \ \Rightarrow \ |f(t) - f(x_0)| < \varepsilon.$$

Let x be such that $0 \leq |x - x_0| < \delta$. Then

$$\begin{aligned} |\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)| &= |\frac{1}{x - x_0} \int_x^{x_0} f(t)dt - f(x_0)| \\ &= |\frac{1}{x - x_0} \int_x^{x_0} (f(t) - f(x_0))dt| \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^{x} |f(t) - f(x_0)|dt \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^{x} \varepsilon dt = \varepsilon \end{aligned}$$

As $\varepsilon > 0$ is arbitary, the above implies that $F'(x_0)$ exists and $F'(x_0) = f(x_0)$.

□

Problem 4.1. Let f be a continuous function on $[0, \frac{\pi}{2}]$ and $\int_0^{\frac{\pi}{2}} f(t)dt = 0$. Show that there exists a $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.

Proof. As f be a continuous function on $[0, \frac{\pi}{2}]$, so it is integrable on $[0, \frac{\pi}{2}]$. Define $F : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by $F(x) = \int_0^x f(t)dt - \sin 2x$ □

Apply the second F.T.C. $F'(x) = f(x) - 2\cos 2x$ and Now $F(0) = 0$ and $F(\frac{\pi}{2}) = 0$. So We apply Rolle's theorem on F, to conclude that there exists $c \in (0, \frac{\pi}{2})$ such that $F'(c) = 0$ so $f(c) = 2\cos 2c$.

Problem 4.2. Consider f be a continuous function on $\mathbb{R}$ defined by

$$G(x) = \int_0^{\sin x} f(t)dt$$

Show G is differentiable on $\mathbb{R}$ and compute G'.
Let $F(x) = \int_0^x f(t)dt$. As f be a continuous function on $\mathbb{R}$, so it is integrable $[-1, 1]$. Hence F is differentiable on $\mathbb{R}$ by second FTC. Also by second FTC

$$F'(x) = f(x)$$

As $G(x) = F(\sin x)$, so by chain rule

$$G'(x) = F'(\sin x)\cos x = f(\sin x)\cos x.$$

Remark 4.2. The usual definition of the natural logarithm function depends on the Fundamental Theorem of Calculus. Recall for $x > 0$,

$$\ln x = \int_1^x \frac{dt}{t}$$

Since $1/t$ is continuous on $(0, \infty)$, So FTC shows

$$\frac{d\log(x)}{dx} = \frac{1}{x}.$$

It should also be noted that the notational convention mentioned that above equation is used to get $\ln(x) < 0$ when $0 < x < 1$. It's easy to read too much into the Fundamental Theorem of Calculus. We are tempted to start thinking of integration and differentiation as opposites of each other. But, this is far from the truth. The operations of integration and anti differentiation are different operations, that happen to sometimes be tied together by the Fundamental Theorem of Calculus. Consider the following examples

Example 4.1. *Let $f : [-1, 1] \to \mathbb{R}$.*

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It's easy to prove that f is integrable, and that $F(x) = \int_{-1}^x f(t)dt = |x| - 1$. F is not differentiable at $x = 0$ as f is discontinuous at $x = 0$.

5. Integral Mean Value Theorem

Theorem 5.1. *Suppose $f, g : [a, b] \to \mathbb{R}$ are such that*
(a) $g(x) \geq 0$ on $[a, b]$,
(b) f is bounded and $m \leq f(x) \leq M$ for all $x \in [a, b]$, and
(c) $\int_a^b f(x)dx$ and $\int_a^b f(x)g(x)dx$ both exist.
Then there is a $c \in [m, M]$ such that

$$\int_a^b f(x)g(x)dx = c\int_a^b g(x)dx. \tag{29}$$

Proof. Obviously,

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx. \tag{30}$$

As by a) $g(x) \geq 0$ on $[a, b]$, so $\int_a^b g(x)dx \geq 0$ by property 3. If $\int_a^b g(x)dx = 0$, then we are done. Otherwise $\int_a^b g(x)dx > 0$, giving

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M \tag{31}$$

Let $c = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$. Then $\int_a^b f(x)g(x)dx = c\int_a^b g(x)dx$. And $m \leq c \leq M$. □

Corollary 5.2. *Let f and g be as in Theorem 5.1, but additionally assume f is continuous. Then there is a $x_0 \in (a, b)$ such that*

$$\int_a^b f(x)g(x)dx = f(x_0)\int_a^b g(x)dx. \tag{32}$$

.

Proof. By Theorem 5.1, we have

$$\int_a^b f(x)g(x)dx = c\int_a^b g(x)dx. \tag{33}$$

As f is continuous in $[a,b]$, and $m \leq c \leq M$. So by theory of continuous function f assumes all values between m and M. So there exists $x_0 \in (a,b)$ such that $c = f(x_0)$. □

6. Change of Variable

Change of Variables Integration by substitution works side-by-side with the Fundamental Theorem of Calculus in the integration section of any calculus course. Most of the time calculus books require all functions in sight to be continuous. In that case, a substitution theorem is an easy consequence of the Fundamental Theorem and the Chain Rule. (See Exercise 8.16.) More general statements are true, but they are harder to prove.

Theorem 6.1. *Let u be a differentiable function on an open interval J such that u' is continuous, and let I be an open interval such that $u(x) \in I$ for all $x \in J$. If f is continuous on I,then $f \circ u$ is continuous on J and*

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(t)dt \tag{34}$$

Proof. Note $u(a)$ need not be less than $u(b)$, even if $a < b$.

The continuity of $f \circ u$ follows from the composition of two continuous functions. Fix c in I and let $F(u) = \int_c^u f(t)dt$. Then $F'(u) = f(u)$ for all $u \in I$ by second FTC. Let $g = F \circ u$. By the Chain Rule of differentiable functions we have $g'(x) = F'(u(x))u'(x) = f(u(x))u'(x)$, as $f(u(x))u'(x)$ is a continuous function, so integrable. Hence integrable. So by First FTC

$$\begin{aligned}\int_a^b f \circ u(x)u'(x)dx = \int_a^b g'(x)dx &= g(b) - g(a) = F(u(b)) - F(u(a)) \\ &= \int_c^{u(b)} f(t)dt - \int_c^{u(a)} f(t)dt = \int_{u(a)}^{u(b)} f(t)dt\end{aligned}$$

□

Remark 6.1. To evaluate an integral we can use change of variables in many ways. The following are examples:

Example 6.1. *Let f be a continuous function on $\mathbb{R}$. Then how to evaluate*

$$\int_a^b f(cx)dx$$

where $c \neq 0$ be any real number.
Let us take $u(x) = cx$ so $u'(x) = c$, then $f(u(x))u'(x) = f(cx)c$, hence by Change of variable (34),

$$\int_a^b f(cx)dx = \frac{1}{c}\int_a^b f(cx)cdx = \frac{1}{c}\int_a^b f \circ u(x)u'(x)dx = \frac{1}{c}\int_{u(a)}^{u(b)} f(t)dt = \frac{1}{c}\int_{ca}^{cb} f(t)dt$$

Example 6.2. *Evaluate $\int_0^1 x\sqrt{1+x^2}dx$.*
Here $f(t) = \sqrt{t}$. Taking $u(x) = 1 + x^2$, and $u'(x) = 2x$, we get $u(0) = 1, u(1) = 2$. Then by Change of variable (34)

$$\int_0^1 x\sqrt{1+x^2}dx = \int_a^b f \circ u(x)u'(x)dx = \int_{u(a)}^{u(b)} f(t)dt = \frac{1}{2}\int_1^2 \sqrt{t}dt = ?$$

Example 6.3. *Evaluate* $\int_0^1 \sqrt{1-x^2}dx$.
Consider $f(x) = \sqrt{1-x^2}$ *and* $u(x) = \sin x$ *then* $u'(x) = \cos x$ *Now* $u(0) = 0$ *and* $u(\frac{\pi}{2}) = 1$. *Then by Change of variable* (34),

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}dx = \int_{u(0)}^{u(\frac{\pi}{2})} f(t)dt &= \int_0^{\frac{\pi}{2}} f(u(x))u'(x)dx \\
&= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^x} \cos x dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\cos^2 x \; dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2x + 1)dx \\
&= \left[\frac{x}{2} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}
\end{aligned}$$