If any answer contains any of the following, they will be given **0 marks** for that section of the answer:

1. Pictorial representation or graph of a function on $\mathbb{R}$ or $\mathbb{R}^n$ $(n \in \mathbb{N})$
2. Arithmetic operations on ∞ or $-\infty$ or similar undefined numbers on $\mathbb{R}$ or $\mathbb{R}^n$ $(n \in \mathbb{N})$
3. Converse of known theorem, that is not true
4. Incorrect definitions used
5. Inadequate or non-existent steps in calculations
6. Integration or properties of Riemannn integrable functions

Q.3) a) What is equivalent criterion for differentiability for $z = f(x, y)$ at any point in its domain? (1.25 marks)

Ans: Any one of the following definitions are valid:

Definition 1: A function $f(x, y) : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$, is differentiable at a point (x_0, y_0) of D if $\exists$ a point $\alpha = (\alpha_1, \alpha_2)$ and functions $\varepsilon_1(h, k)$, $\varepsilon_2(h, k)$ on $\mathbb{R}^2$, such that $f(x_0 + h, y_0 + k) - f(x_0, y_0) = h\alpha_1 + k\alpha_2 + h\varepsilon_1(h, k) + k\varepsilon_2(h, k)$ where $\varepsilon_1(h, k), \varepsilon_2(h, k) \to 0$ as $(h, k) \to (0, 0)$

Definition 2: A function $f(x, y) : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$, is differentiable at a point (x_0, y_0) of D if and only if $\lim\limits_{\rho \to 0} \dfrac{\triangle f(x_0, y_0) - df(x_0, y_0)}{\rho} = 0$, where:

1. $\triangle f(x_0, y_0) = f(x_0 + h, y_0 + k) - f(x_0, y_0)$
2. $df(x_0, y_0) = hf_x(x_0, y_0) + kf_y(x_0, y_0)$
3. $\rho = \sqrt{h^2 + k^2}$

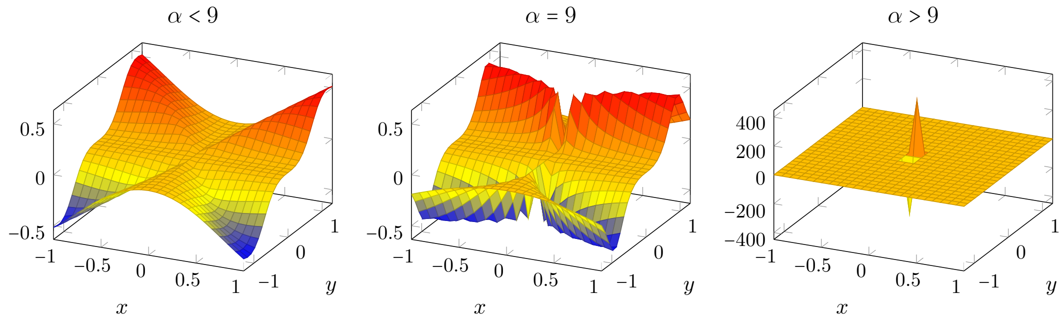
Definition 3: A function $f(x, y) : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$, is differentiable at a point (x_0, y_0) of D if both $f_x(x, y)$ and $f_y(x, y)$ exist in an open neighbourhood containing (x_0, y_0) and at least one of them is continuous at (x_0, y_0)

+1.25 marks Exact same or re-worded definition

0 marks Otherwise

Q.3) b) Determine all values of $\alpha > 0$ for which $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 y^3}{|x|^3 + |y|^\alpha}$ exists? (2.5 marks)

Ans: The set of all possible values for α in $(0, \infty)$ is $(0, 9)$ **+0.5 marks**



Claim 1: If $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3}{|x|^3 + |y|^\alpha}$ exists, $\alpha < 9$

Proof: If the limit exists, let it be ξ. An intuitive observation shows that:

$$\xi = \lim_{\substack{(x,y)\to(0,0) \\ \text{along the curve } x=y^3}} \frac{x^2y^3}{|x|^3 + |y|^\alpha} = \lim_{y\to 0} \frac{(y^3)^2y^3}{|y^3|^3 + |y|^\alpha} = \lim_{y\to 0} \frac{y|y|^8}{|y|^9 + |y|^\alpha} = \lim_{y\to 0} \frac{y}{|y|} \frac{1}{1 + |y|^{\alpha-9}}$$

Let us assume for the sake of contradiction that $\alpha \geq 9$,

$$\implies \lim_{y\to 0} 1 + |y|^{\alpha-9} = \begin{cases} 1 & \text{if } \alpha > 9 \\ 2 & \text{if } \alpha = 9 \end{cases}$$

$$\implies \lim_{y\to 0} \frac{y}{|y|} \text{ exists as } \lim_{y\to 0} \frac{y}{|y|} = \lim_{y\to 0} \frac{y}{|y|} \frac{1 + |y|^{\alpha-9}}{1 + |y|^{\alpha-9}} \text{ (Product of limits)}$$

$$\implies \lim_{y\to 0} \frac{y}{|y|} = \begin{cases} \xi & \text{if } \alpha > 9 \\ 2\xi & \text{if } \alpha = 9 \end{cases} \implies \xi = 0$$

since $\lim_{y\to 0} \frac{y}{|y|}$ is independent of α and $\xi = 2\xi \Leftrightarrow \xi = 0$

$$\implies \lim_{y\to 0} \frac{y}{|y|} = 0 \textbf{ (Contradiction)}$$

Therefore, $\alpha < 9$ **+1 mark**

Claim 2: If $0 < \alpha < 9$, $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3}{|x|^2 + |y|^\alpha}$ exists and is 0

Proof: If $\alpha < 9$, $3 - \dfrac{\alpha}{3} > 0$

$$\implies 0 \leq \left|\frac{x^2y^3}{|x|^3 + |y|^\alpha}\right| = |y|^{3-\frac{\alpha}{3}} \frac{|x|^2|y|^{\frac{2\alpha}{3}}}{|x|^3 + |y|^\alpha} = |y|^{3-\frac{\alpha}{3}} \frac{\left(\frac{|x|}{|y|^{\frac{\alpha}{3}}}\right)^2}{\left(\frac{|x|}{|y|^{\frac{\alpha}{3}}}\right)^3 + 1}$$

For any arbitrary real number $\gamma > 0$, $\begin{cases} \text{if } \gamma \geq 1,\ 1 + \gamma^3 \geq \gamma^3 \geq \gamma^2 \implies \dfrac{\gamma^2}{1 + \gamma^3} \leq 1 \\ \text{if } \gamma < 1,\ \gamma^2 < 1 \text{ and } 1 + \gamma^3 > 1 \implies \dfrac{\gamma^2}{1 + \gamma^3} \leq 1 \end{cases}$

and clearly for $(x, y) \neq (0, 0)$, $\dfrac{|x|}{|y|^{\frac{\alpha}{3}}} > 0$

$$\implies 0 \leq \left|\frac{x^2y^3}{|x|^3 + |y|^\alpha}\right| \leq |y|^{3-\frac{\alpha}{3}} \text{ and since } 3 - \frac{\alpha}{3} > 0,\ \lim_{y\to 0} |y|^{3-\frac{\alpha}{3}} = 0$$

$$\implies \lim_{y\to 0}\left|\frac{x^2y^3}{|x|^3+|y|^\alpha}\right| = 0 \text{ (Sandwich Theorem)}$$

$$\implies \lim_{y\to 0}\frac{x^2y^3}{|x|^3+|y|^\alpha} = 0 \text{ (Sandwich Theorem again)}$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^3}{|x|^2+|y|^\alpha} = 0$ for $0<\alpha<9$ **+1 mark**

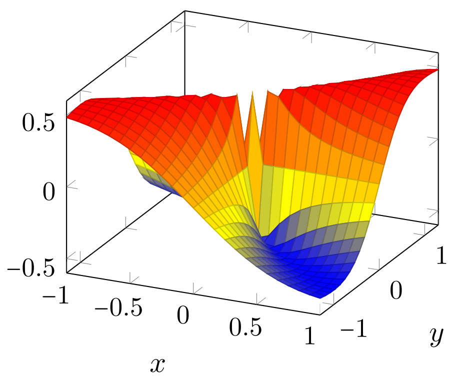
Note: This question has been designed in such a way that this is the only logical solution. As far as I know, there are **no other alternate solutions** barring minor details.

Q.3) c) Show by an example that the existence of partial derivatives at a given point does not imply continuity at that point.? (1.25 marks)

Ans: There are multiple such functions that are valid but the following example showcases the steps necessary for full marks:

Example: $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y) = \begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{otherwise}\end{cases}$ at $(0,0)$

$z = f(x,y)$



$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0 \implies \frac{\partial f}{\partial x}(0,0) = 0$$

$$\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0 \implies \frac{\partial f}{\partial y}(0,0) = 0$$

But if we consider the case where (x,y) approaches $(0,0)$ along the curve $y=x$, then:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the curve } y=x}} f(x,y) = \lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2} \neq 0 = f(0,0)$$

$\implies f$ is not continuous at $(0, 0)$

+0.25 marks Valid multivariate function f

+0.25 marks Valid point of discontinuity (x_0, y_0)

+0.25 marks Valid evaluation of $\dfrac{\partial f}{\partial x}(x_0, y_0)$

+0.25 marks Valid evaluation of $\dfrac{\partial f}{\partial y}(x_0, y_0)$

+0.25 marks Valid contradiction of continuity at (x_0, y_0)

0 marks None of the above

Q.7) a) Let $f(x, y) = x^3y - xy^2 + cx^2$ where c is a constant. Find c if f increases fastest at the point $P(3, 2)$ in the direction of the vector $\mathbf{v} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (2.5 marks)

Ans: For given c, $f(x, y) = x^3y - xy^2 + cx^2$

$\implies \nabla f(x, y) = (3x^2y - y^2 + 2cx)\hat{\mathbf{i}} + (x^3 - 2xy)\hat{\mathbf{j}}$

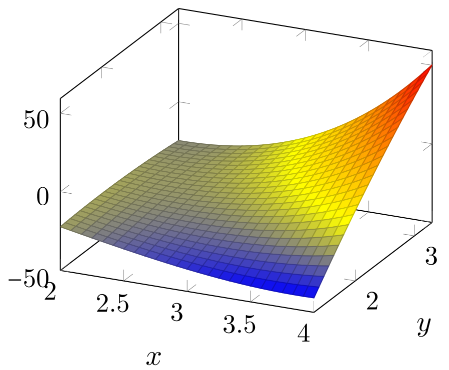
$\implies \nabla f\Big|_P = (50 + 6c)\hat{\mathbf{i}} + (15)\hat{\mathbf{j}}$

Since f increases fastest at P in the direction of $\mathbf{v}$, both $\nabla f\Big|_P$ and $\mathbf{v}$ have the same unit vector $\implies \nabla f\Big|_P$ is a positive multiple of $\mathbf{v}$ or $\exists\ \eta > 0$ such that $\nabla f\Big|_P = \eta\mathbf{v}$

$\implies \nabla f\Big|_P - \eta\mathbf{v} = (50 + 6c - 2\eta)\hat{\mathbf{i}} + (15 - 5\eta)\hat{\mathbf{j}} = \mathbf{0}$

$\implies 15 - 5\eta = 0 \implies \eta = 3 \implies 50 + 6c - 2\eta = 0 \implies c = \dfrac{-22}{3}$

$$z = x^3y - xy^2 - \frac{22}{3}x^2 \text{ at } P(3, 2)$$



Note: This question is straightforward and extremely short for **2.5 marks**, so it will have binary marking.

+2.5 marks Exact same or re-worded calculation

0 marks Otherwise

Q.7) b) Can you prove $\nabla \frac{f}{g}\Big|_{(x_0, y_0)} = \frac{g\nabla f - f\nabla g}{g^2}\Big|_{(x_0, y_0)}$? (2.5 marks)

Ans: Let $h = \frac{f}{g}$,

$\implies \frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(f \times \frac{1}{g}) = \frac{1}{g}\frac{\partial f}{\partial x} - \frac{f}{g^2}\frac{\partial g}{\partial x} = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}$ (Product/Quotient Rule)

$\implies \frac{\partial h}{\partial y} = \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}$ (Product/Quotient Rule again)

$\implies \nabla h = \frac{g\nabla f - f\nabla g}{g^2}$

Therefore, $\nabla \frac{f}{g}\Big|_{(x_0, y_0)} = \nabla h\Big|_{(x_0, y_0)} = \frac{g\nabla f - f\nabla g}{g^2}\Big|_{(x_0, y_0)}$

Note: This question is straightforward and extremely short for **2.5 marks**, so it will have binary marking.

+2.5 marks Exact same or re-worded derivation

0 marks Otherwise