ASSIGNMENT 3

(1) Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty}((-1)^n)x_n$ exists. Show that $\{x_n\}$ converges.

(2) Let $\{x_n\}$ be a bounded sequence and let $s = Sup\{x_n;\ n \in \mathbb{N}\}$ and $s \notin \{x_n;\ n \in \mathbb{N}\}$. Show that there is a subsequence of $\{x_n\}$ that converges to s.

(3) Let $\{s_n\}$ and $\{t_n\}$ are two sequences such that $s_n \leq t_n$ for all $n \geq N_0$. Show that $lim\ inf s_n \leq lim\ inf t_n$. Similarly, $lim\ sup\ s_n \leq lim\ sup\ t_n$.

(4) Let $a_n = (1 + \frac{1}{n})^n$ and $b_n = (1 + \frac{1}{n})^{n+1}$ for $n \in \mathbb{N}$. Then
a) the sequence $\{a_n\}$ strictly increasing.
b) the sequence $\{b_n\}$ is strictly decreasing.
Show that $\{a_n\}$ and $\{b_n\}$ both have the same limit defined to be Euler's number.

(5) Let $x_n = (1 + \frac{x}{n})^n$.
a) Show that if $x > 0$, the sequence $\{x_n\}$ is bounded and strictly increasing.
b) Let $x \in \mathbb{R}$. Show that the sequence is bounded and strictly increasing for $n > -x$.
The limit of the sequence is defined by e^x.
c) Let $S_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{j=1}^{n} \frac{x^j}{j!}$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} S_n = e^x$. (That is $\{S_n\}$ is the sequence of partial sum of the series $\sum_{j=1}^{\infty} \frac{x^j}{j!}$ and $\sum_{j=1}^{\infty} \frac{x^j}{j!} = e^x$)

(6) If $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$ then

$$\lim_{n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$$

What about $\lim_{n\to\infty} \left(1 + \frac{x}{a_n}\right)^{a_n}$ for any $x \in \mathbb{R}$?

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