# Real Analysis - I

## Quiz - 3

**Q2.** (a) L' Hospital Rule I that is $\left[\frac{0}{0}\right]$ form

Let us suppose $-\infty \leq a < b \leq \infty$ and let $f, g$ are two differentiable functions on $(a, b)$ such that $g'(x) \neq 0$ $\forall\ x \in (a, b)$. Suppose that $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$

(0.5 marks)

1) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, implies $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

2) If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ implies $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

(0.5 marks)

The same result holds for

→ the left-hand limit $\lim_{x \to b^-}$.

→ the two-sided limit $\lim_{x \to x_0}$ where $x_0 \in (a, b)$ and if $f$ & $g$ are differentiable except possibly at $x_0 \in (a, b)$.

(0.5 marks)

**Q2.** (b) Define $F : (1, \infty) \longrightarrow \mathbb{R}$ by

$$F(x) = \left(1 - \frac{1}{x}\right)^x$$

Clearly, $\lim_{x \to \infty} F(x) = [1^\infty]$.

We can write $F(x) = \left(1 - \frac{1}{x}\right)^x = e^{x \log\left(1 - \frac{1}{x}\right)}$ (0.25 marks)

Let $h(x) = x \log\left(1 - \frac{1}{x}\right)$

$$\Rightarrow \lim_{x\to\infty} h(x) = \lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right)$$

$$= \lim_{y\to 0+} \frac{\log(1-y)}{y}$$ (0.25 marks)

Let $f(y) = \log(1-y)$

$g(y) = y$

Then, $\lim_{y\to 0+} f(y) = \lim_{y\to 0+} g(y) = 0$

and $g'(y) = 1 \neq 0 \;\forall\; y \in (0,1)$

and $\lim_{y\to 0+} \frac{f'(y)}{g'(y)} = \lim_{y\to 0+} \frac{-1}{(1-y)} = -1$ exists

$\therefore$ By L'Hospital Rule I,

$$\lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right) = \lim_{y\to 0+} \frac{f(y)}{g(y)} = \lim_{y\to 0+} \frac{f'(y)}{g'(y)} = -1.$$

(0.5 marks)

Let $G(y) = e^y$

$\Rightarrow G$ is continuous on $\mathbb{R}$.

$F(x) = x\log\left(1-\frac{1}{x}\right)$ is also continuous $\forall\; x>1$.

$\left(\because \frac{1}{x} < 1 \Rightarrow 1-\frac{1}{x} > 0\right)$

$\Rightarrow e^{x\log(1-\frac{1}{x})}$ is continuous on $(1,\infty)$

(Being composition of continuous functions) (0.25 marks)

$$\Rightarrow \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\log\left(1-\frac{1}{x}\right)} = e^{-1}$$ (0.25 marks)

Q3. (a) TP $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ where $f(x,y) = \dfrac{xy^2}{x^6+y^2}$

Let $\varepsilon > 0$ be any arbitrary real no.

When $(x,y) \neq (0,0)$,

$0 \leq y^2 \leq x^6 + y^2 \quad \forall\ (x,y) \neq (0,0)$

$\Rightarrow \dfrac{y^2}{x^6+y^2} \leq 1 \quad \forall\ (x,y) \neq (0,0)$ —— (1) (0.5 marks)

Let $\delta_\varepsilon = \varepsilon > 0$, then if $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta_\varepsilon = \varepsilon$

$\Rightarrow 0 < \sqrt{x^2+y^2} < \varepsilon$ —— (2) (0.25 marks)

That would imply

$$\left| f(x,y) - 0 \right| = \left| \frac{xy^2}{x^6+y^2} \right| \leq |x| \frac{y^2}{x^6+y^2} \quad \left( \because y^2 \geq 0 \ \& \ x^6+y^2 \geq 0 \right)$$

$$\leq 1 \cdot |x| \quad (\text{By eq}^n\ (1))$$

$$\leq \sqrt{x^2+y^2}$$

$$< \varepsilon \quad (\text{By eq}^n\ (2))$$ (0.5 marks)

$\therefore$ For $\varepsilon > 0$, $\exists\ \delta_\varepsilon = \varepsilon$ such that

$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta_\varepsilon = \varepsilon \Rightarrow |f(x,y) - 0| < \varepsilon$

$\because \varepsilon > 0$ was arbitrary $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$. (0.25 marks)