Q→1. Let x and $y \geq 0$ be two real number
Prove $|x| \leq y$ iff $-y \leq x \leq y$.

Given x and $y \geq 0$ be two real number

SHOW: 1) If $|x| \leq y$ then we have $-y \leq x \leq y$
2) If $-y \leq x \leq y$ then $|x| \leq y$

Proof 1) Let $|x| \leq y$; x and $y \geq 0$ be two real number

SHOW: $-y \leq x \leq y$

$\boxed{\text{Case-1 If } x > 0}$ $\Rightarrow |x| \leq y$

$\in$ $+x \leq y$; $x > 0$

(Since $y > 0$)
$\Rightarrow -y < 0$

Hence $-y < x \leq y$

$\boxed{\text{Case-2 If } x < 0}$ $-x = |x| \leq y$

$-x \leq y$

$-y \leq x$

Since x is negative and $y > 0 \Rightarrow x \leq y$

Hence $-y \leq x \leq y$

2) Let $-y \le x \le y$; x and $y \ge 0$ be two real number

SHOW: $|x| \le y$

Given: $-y \le x \le y$

$\Rightarrow$ $-y \le x$ and $x \le y$

$\Rightarrow$ $-x \le y$ and $x \le y$

Hence $\pm x \le y$

Hence $|x| \le y$

Q→2. If $r \in \mathbb{Q}$ s.t $r \neq 0$

Let x be irrational number

Prove ① $x + r$ is irrational number

② xr is also irrational number

① Given Let $r \in \mathbb{Q}$ s.t. $r \neq 0$

$\Rightarrow$ $r = \frac{p}{q}$ where $q \neq 0$, $p, q \in \mathbb{Z}$

$\gcd(p, q) = 1$

Given x is irrational number

SHOW: ① $x + r$ is irrational number.

Let if possible $x + r$ is not irrational no.

$\Rightarrow$ $x + r$ is rational number

say $x + r = \frac{a}{b}$; $a, b \in \mathbb{Z}$
$b \neq 0$

$$x = \frac{a}{b} - r$$

$$= \frac{a}{b} - \frac{p}{q}$$ (Since $r = \frac{p}{q}$; $q \neq 0$, $p, q \in \mathbb{Z}$)

$$= \frac{aq - bp}{bq}$$ where $aq - bp \in \mathbb{Z}$
$bq \in \mathbb{Z}$; $bq \neq 0$

Hence $x \in \mathbb{Q}$

which is contradiction

Hence our Supposition is wrong

Hence $x + r$ is irrational number

(2) This is Homework. for students.

Q → 3. Let S and T be non-empty bounded subset of $\mathbb{R}$

Prove: if $S \subseteq T$ then $\inf(T) \leq \inf(S) \leq \operatorname{Sup}(S) \leq \operatorname{Sup}(T)$

Given: S and T are two bounded nonempty subset of $\mathbb{R}$

$\Rightarrow$ $\operatorname{Sup}(S)$ and $\operatorname{Sup}(T)$ exist

Given:- $S \subseteq T$

Show:- $\inf(T) \leq \inf(S) \leq \operatorname{Sup}(S) \leq \operatorname{Sup}(T)$

① We first show $\Rightarrow$ $\operatorname{Sup}(S) \leq \operatorname{Sup}(T)$

Let $a \in S$ $\Rightarrow$ $a \in T$ $\quad$ ($\because S \subseteq T$)

$\Rightarrow$ $a \leq \operatorname{Sup}(T)$

(Since a is any arbitrary element of S
and $a \leq \operatorname{Sup}(T)$)

Hence $a \leq \operatorname{Sup}(T)$; for any $a \in S$

$\Rightarrow$ $\operatorname{Sup}(T)$ is u.b. of S

but we know $\operatorname{Sup}(S)$ is l.u.b

$\Rightarrow$ $\operatorname{Sup} S \leq \operatorname{Sup} T$

(2) Now we show:→ $\inf(S) \leq \sup(S)$

Let $\inf(S) = a$ and Let $\sup(S) = b$

$\Rightarrow a \leq s \; ; \forall s \in S$ $\Rightarrow s \leq b \quad \forall s \in S$

Hence $a \leq s \leq b$; for any $s \in S$

Hence $\inf(S) \leq \sup(S)$

(3) Now we show:→ $\inf(T) \leq \inf(S)$

This is Homework for Students

Let S and T be two bounded subset of $\mathbb{R}$ ⑥

P→4 Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Given: Let S and T be two bounded subset of $\mathbb{R}$

$\Rightarrow \sup S$ and $\sup T$ exist

Show: $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Without loss of generality

Take $\max\{\sup S, \sup T\} = \sup S$ —①

Let a be any arbitrary element of $S \cup T$

$\Rightarrow a \in S \cup T$

Either $a \in S$ or $a \in T$

$\Rightarrow a \leq \sup(S)$

$\Rightarrow a \leq \sup T \leq \sup S$ (by ①)

Hence $a \leq \sup(S) = \max\{\sup S, \sup T\}$

$\Rightarrow a \leq \max\{\sup S, \sup T\}$

Since a is arbitrary element of $S \cup T$

$\Rightarrow \max\{\sup S, \sup T\}$ is u.b. for set $(S \cup T)$

But $\sup(S \cup T)$ is l.u.b.

Hence $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$ —②

In Q→3 we have seen that

If $A \subseteq B$; A and B be two bounded non-empty subset of $\mathbb{R}$

then $\sup A \leq \sup B$

Take $A = S$ $B = S \cup T$

~~and~~ $A \subseteq B$

(by Q-3), we can say $\sup A \leq \sup B$

i.e. $\sup(S) \leq \sup(S \cup T)$

(by ①) $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$

③

By ②, ③, we have

$\sup(S \cup T) \leq \max\{\sup S, \sup T\} \leq \sup(S \cup T)$

Hence $\sup(S \cup T) = \max\{\sup S, \sup T\}$

Q→6 Let $x \in \mathbb{R}$

then $|x| < \varepsilon$ for every $\varepsilon > 0$ iff $x = 0$

→

Let $x = 0$

then $|x| < \varepsilon$ for every $\varepsilon > 0$

This is obvious

← Let $x \in \mathbb{R}$ s.t. $|x| < \varepsilon$ for every $\varepsilon > 0$

Show: $x = 0$

Let if possible $x \neq 0$

In particular chose $\varepsilon = \frac{|x|}{2}$

then we have $|x| < \frac{|x|}{2}$

$\Rightarrow \frac{|x|}{2} < 0$

$\Rightarrow |x| < 0$

i.e. absurd

Hence our Supposition is wrong

Hence $x = 0$

⑧

Q→5. Let S be a non-empty bounded set in $\mathbb{R}$

(a) Let $a>0$ be a real number

Let $aS = \{as ; s \in S\}$

Prove 1) $\inf(aS) = a \cdot \inf(S)$

2) $\sup(aS) = a \cdot \sup(S)$

NOTE → In this question, we will use Result of Q→6.

Proof (a)

Let S be non-empty bounded subset of $\mathbb{R}$

$\Rightarrow$ $\sup S$ ~~and $\sup T$~~ exist.

SHOW 1) $\inf(aS) = a \inf(S)$

Let $u \in aS \Rightarrow u = as$; f.s. $s \in S$

We also know that $\inf(S) \leq s$; $\forall s \in S$

$a \cdot \inf(S) \leq as$; $\forall s \in S$

$\Rightarrow a \cdot \inf(S) \leq u$

Since u is arbitrary element of S

$\Rightarrow a \cdot \inf(S) \leq u$; $\forall u \in S$

$\Rightarrow a \cdot \inf(S)$ is l.b. for Set $\{aS\}$.

but $\inf(aS)$ is greatest lower bound

$\Rightarrow a \cdot \inf(S) \leq \inf(aS)$. —— ①

Let $\varepsilon > 0$

(By Th^m-3.1) $\exists$ $s_\varepsilon \in S$ s.t. $Inf(S) \leq s_\varepsilon < Inf(S) + \frac{\varepsilon}{a}$

$\Rightarrow$ $a s_\varepsilon < a\, Inf(S) + a\frac{\varepsilon}{a}$

(Since $a s_\varepsilon \in aS \Rightarrow Inf(aS) < a s_\varepsilon$) $\cancel{< a\, Inf(S) + \varepsilon}$

Hence we get $Inf(aS) < a s_\varepsilon < a\, Inf(S) + a\frac{\varepsilon}{a}$

$\Rightarrow Inf(aS) < a\, Inf(S) + \varepsilon$

$\Rightarrow Inf(aS) - a\, Inf(S) < \varepsilon$

by (1), we have $Inf(aS) - a\, Inf(S) > 0$

$\Rightarrow |Inf(aS) - a\, Inf(S)| < \varepsilon$

Take $x = Inf(aS) - a\, Inf(S)$

Now we use Result of Q→b

i.e. $|x| < \varepsilon$; for any $\varepsilon > 0 \Rightarrow x = 0$

as $\varepsilon > 0$ is arb.

$\Rightarrow Inf(aS) - a\, Inf\, S = 0$

$\Rightarrow Inf(aS) = a\, Inf(S)$

H.P.

Q→5

SHOW: 2) $\sup(aS) = a\sup(S)$

Let $v \in aS$

$\Rightarrow v = as$; f.s. $s \in S$

We know $s \leq \sup(S)$; $\forall s \in S$

$as \leq a\sup(S)$; $\forall s \in S$

$v \leq a\sup(S)$

Since v is arb. elt of aS

Hence $v \leq a\sup(S)$; $\forall v \in aS$

But $a\sup(S)$ is u.b. for set aS

Hence $\sup(aS) \leq a\sup(S)$ — (2)

Use Th^m $u = \sup(S)$ iff for every $\varepsilon > 0$

$\exists\, x_\varepsilon \in S : u - \varepsilon < x_\varepsilon \leq u$

Let $\varepsilon > 0$, $\exists\, s_\varepsilon \in S : \sup(S) - \frac{\varepsilon}{a} < s_\varepsilon \leq \sup(S)$

$\Rightarrow a\sup(S) - \varepsilon < as_\varepsilon$

Since $as_\varepsilon \in aS$ which implies $as_\varepsilon \leq \sup(aS)$

By combining above inequality, we get

$a\sup(S) - \varepsilon < \sup(aS)$

$a\sup(S) - \sup(aS) < \varepsilon$, where LHS ≥ 0 (by (2))

Now, we apply Result of Q→6
by taking $x = a\,\mathrm{Sup}(S) - \mathrm{Sup}(aS)$

As $\epsilon > 0$ is arb.
Hence $a\,\mathrm{Sup}(S) - \mathrm{Sup}(aS) = 0$
Hence $\mathrm{Sup}(aS) = a\,\mathrm{Sup}(S)$

(b) Let $b < 0$ be a real no.
Let $bS = \{bs \,;\, s \in S\}$

<u>Prove</u> 1) $\mathrm{Inf}(bS) = b\,\mathrm{Sup}(S)$
2) $\mathrm{Sup}(bS) = b\,\mathrm{Inf}(S)$

Hw. for students

<u>Remark</u>:-
Take $b = -1$
we get
$$\boxed{\begin{array}{l} \mathrm{Inf}(-S) = -\mathrm{Sup}(S) \\ \mathrm{Sup}(-S) = -\mathrm{Inf}(S) \end{array}}$$

Prove whether foll. sets are bdd from above | below
Then find Sup, Inf

(12)

Q→7

(a) $\{y = 1 - \frac{1}{n} \; ; n \in \mathbb{N}\}$

(b) $\{y = x + \frac{1}{x} \; ; x > 0\}$

(c) $\{y = 2^{x} + 2^{1/x} \; ; x > 0\}$

(a) This is HW.

Proof (b) $S = \{y = x + \frac{1}{x} \; ; x > 0\}$

For $x > 0$, we have $x + \frac{1}{x} > 0$. Hence set is bdd from below

Use: AM $\geq$ GM

$$\frac{a+b}{2} \geq (ab)^{1/2} \quad ; a, b \in \mathbb{R}^{+}$$

Taking $a = x$, $b = \frac{1}{x}$, we get

$$y = x + \frac{1}{x} \geq 2\left(x \cdot \frac{1}{x}\right)^{1/2}$$

$$y = x + \frac{1}{x} \geq 2$$ Since y is arb. elt of S.

Hence 2 is lower bound for set (S)

But at $x = 1$, we get $y = 2 \in$ Set (S)

$\boxed{\text{Hence } \inf(S) = 2}$ (By Remark - 3.1 of Notes)

SHOW: $\text{Sup}(S)$ does not exist

Let if possible $\text{Sup}(S)$ exist say M

Consider $M + \frac{1}{M} > M = \text{Sup}(S)$; $M + \frac{1}{M} \in$ Set

Hence we get one element $M + \frac{1}{M} \in$ Set s.t.

$M + \frac{1}{M} > \text{Sup}(\text{set})$

contradict. Hence our supposition is wrong
Hence $\text{Sup}(S)$ does not exist

(c) Set $= \{ y = 2^{x} + 2^{1/x} \; ; x > 0 \}$

Again apply :- AM $\geq$ GM

we get $2^{x} + 2^{1/x} \geq 2 \left(2^{x} \cdot 2^{1/x} \right)^{1/2}$

$\geq 2 \left(2^{x + \frac{1}{x} - 2 + 2} \right)^{1/2}$

$\geq 2 \left(2^{\frac{(x-1)^2}{2x} + 1} \right)$

$\geq 2 \cdot (2^{1}) \quad \left(\because \frac{(x-1)^2}{2x} > 0 \; ; \forall x > 0 \right)$

Hence 4 is lower bound

But at $x = 1$, we get $y = 4 \in$ Set

$\boxed{\text{Hence } \inf(S) = 4}$

Show: Sup(set) does not exist

Let if possible Sup(set) exist say M

Consider $2^{M} + 2^{1/M} \geq 2^{M} > M$

So, we get one element $2^{M} + 2^{1/M} \in$ Set which is greater than Sup(set)

i.e contradiction

So, our supposition is wrong

Hence Sup(set) does not exist

Q→8. Let S be n.e. subset of $\mathbb{R}$

Prove: if a number u in $\mathbb{R}$ has properties :—

① for every $n \in \mathbb{N}$, the number $u - \frac{1}{n}$ is not u.b. of set S.

② for every $n \in \mathbb{N}$, the number $u + \frac{1}{n}$ is an u.b. of Set S. &

then $u = \text{Sup } S$.

Let $\varepsilon > 0$ be any no. (Given)

As S is bdd. above

By Completeness Axiom, Sup(S) exist say x

R.T.P $x = u$

As $\text{Sup}(S) = x$

By prop→① $u - \frac{1}{n}$ is not an u.b. ; for every $n \in \mathbb{N}$

So $u - \frac{1}{n} < x$; $\forall n \in \mathbb{N}$ —— ①

(As x is an u.b.)

Now $\text{Sup}(S) = x$

By prop ② $u + \frac{1}{n}$ is u.b. ; $\forall n \in \mathbb{N}$

But x is the smallest u.b.

So $x \le u + \frac{1}{n}$; $\forall n \in \mathbb{N}$ —— ②

By 1, 2

$u-\frac{1}{n} < x \le u+\frac{1}{n}$; $\forall n \in \mathbb{N}$

Hence $|x-u| \le \frac{1}{n}$; $\forall n \in N$ (Use Q→1 Result)

— 3

For any given $\varepsilon > 0$

$\exists$ $n_\varepsilon \in \mathbb{N}$: $n_\varepsilon > \frac{1}{\varepsilon}$ — 4

$\left\{ \because \text{we take } x = \varepsilon, y = 1 \text{ in Archimed. prop. Th}^m \right\}$

Since 3 holds $\forall n \in \mathbb{N}$

In particular, take $n = n_\varepsilon$

So $|x-u| \le \frac{1}{n_\varepsilon}$

$< \varepsilon$ (by 4)

Since $\varepsilon > 0$ is arb. element

So $|x-u| < \varepsilon$; for every $\varepsilon > 0$

Hence by Q→6 Result

$x-u = 0$

$x = u$

$\underline{\text{Sup}(S) = u}$

H.P.