Ques 1. Let $\varepsilon > 0$ be any arbitrary real no.

Now, we know $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

that implies for $\varepsilon = 1/2$, $\exists\ \delta_{(1/2)} > 0$ s.t $0 < |\theta - 0| < \delta_{1/2}$ implies $\left| \frac{\sin\theta}{\theta} - 1 \right| < 1/2$

$$\Rightarrow \left| \frac{\sin\theta}{\theta} \right| = \left| \frac{\sin\theta}{\theta} - 1 + 1 \right| \leq \left| \frac{\sin\theta}{\theta} - 1 \right| + 1 < 1/2 + 1 = 3/2$$

$$\Rightarrow |\sin\theta| \leq \frac{3}{2} |\theta|$$

so, $0 < |\theta| < \delta_{1/2}$ implies $\theta/2 < \sin\theta < \frac{3\theta}{2}$ —— (1)

Let $c \in [0, \pi/2]$

$$|f(x) - f(c)| = |\sin x - \sin c|$$

$$= \left| \sin \frac{((x+c) + (x-c))}{2} - \sin \frac{((x+c) - (x-c))}{2} \right|$$

$$= \left| 2 \cos \frac{(x+c)}{2} \sin\left(\frac{x-c}{2}\right) \right| \leq 2 \left| \sin\left(\frac{x-c}{2}\right) \right|$$

—— (2) (as $\cos\alpha \leq 1$)

Take $\theta = \left(\frac{x-c}{2}\right)$ in (1), so whenever $0 < \left|\frac{x-c}{2}\right| < \delta_{1/2}$

then by (1) $\left| \sin\left(\frac{x-c}{2}\right) \right| < \frac{3}{4} |x-c|$ —— (3)

so, $0 < |x-c| < \delta_0 \Rightarrow \min\left\{ 2\delta_{1/2}, \frac{2\varepsilon}{3} \right\}$

then we have,

$$|f(x) - f(c)| \leq 2 \left| \frac{\sin(x-c)}{2} \right| \quad (\text{by } (2))$$

$$\leq 2 \left(\frac{3}{4} |x-c| \right) \quad (\text{by } (3))$$

$f(x) - f(c) \leq \frac{3}{2}|x-c| < \varepsilon$

So, corresponding to $\varepsilon > 0$, we have found $\delta_0(\varepsilon) = \min\left\{ 2\delta_{1/2}, \frac{2\varepsilon}{3} \right\}$ s.t $0 < |x-c| < \delta_0(\varepsilon)$ implies $|\sin x - \sin c| < \varepsilon$.

Ans 2. A function 'f' such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y, where 'C' is a constant independent of x and y, is called a LIPSCHITZ FUNCTION.

Yes, every Lipschitz fn. is continous.

(By $\varepsilon - \delta$ defn. of cont. fn. at c)
choose $\delta = \varepsilon / C$

then, $|f(x) - f(y)| < \varepsilon$ whenever,
$|x-y| < \delta$

Ques 3. Suppose $f : [0,1] \rightarrow \mathbb{R}$ is continous fn.

consider $g : [0,1] \rightarrow \mathbb{R}$ is continous fn.

Let $g(x) = f(x) - x$.

when $x \in \mathbb{Q}$; $f(x) \notin \mathbb{Q}$; so, $g(x) \notin \mathbb{Q}$

If $x \notin \mathbb{Q}$ then $f(x) \in \mathbb{Q}$; but $g(x) \notin \mathbb{Q}$

So, $g(x) \notin \mathbb{Q} \ \forall \ x \in [0,1]$

Suppose $x_1, x_2 \in [0,1]$ s.t $g(x_1) \neq g(x_2)$. Then $g(x_1)$ and $g(x_2)$ are 2 irrational numbers and between any 2 irrational no. $\exists$ a rational no., so $\exists$ $\mu \in \mathbb{Q}$ s.t $g(x_1) < \mu < g(x_2)$.

But $\exists$ no $x \in [0,1]$ s.t $g(x) = \mu$. So g fails to satisfy IVP. Hence, g can't be continous and so is 'f'.

Ques-4 let us assume $f'(c) = 0$

Let $\varepsilon > 0$, since $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists & equal to

$f'(c) = 0$

So, corr. to $\varepsilon > 0$ $\exists$ $\delta(\varepsilon) > 0$ s.t $0 < |h-0| < \delta(\varepsilon)$

implies $\left| \frac{f(c+h) - f(c)}{f(h)} - f'(c) \right| < \varepsilon$

that is, $\left| \frac{f(c+h)}{f(h)} \right| < \varepsilon$ (as $f(c) = 0$) —— (1)

Now, let $0 < |h-0| < \delta(\varepsilon)$ then,

$$\left| \frac{g(c+h) - g(c)}{h} - 0 \right| = \left| \frac{|f(c+h)|}{h} \right| = \left| \frac{f(c+h)}{h} \right| < \varepsilon$$

(by (1))

$\{ \because g(c) = |f(c)| = 0 \}$

hence, corr. to $\varepsilon > 0$, we have found,

$\delta(\varepsilon) > 0$ s.t $\left| \frac{g(c+h) - g(c)}{h} - 0 \right| < \varepsilon$ when $0 < |h| < \delta(\varepsilon)$

so, $\lim_{h\to 0} \frac{g(c+h) - g(c)}{h}$ exists and equal to 0.

so, $g'(c) = 0$

conversely, let $g(x) = |f(x)|$ is differentiable at $x = c$.

that means $\lim_{h\to 0} \frac{|f(c+h)| - |f(c)|}{h}$ exists

i.e. $\lim_{h\to 0^+} \frac{|f(c+h)|}{h} = \lim_{h\to 0^-} \left| \frac{f(c+h)}{h} \right|$ —— (2)

as $h \to 0^+$, $h>0$ so $\frac{|f(c+h)|}{h} > 0$

so, by ordered prop.

$$\lim_{h\to 0^+} \frac{|f(c+h)|}{h} \geq 0 \quad \text{——} \ (3)$$

similarly, $\lim_{h\to 0^-} \frac{|f(c+h)|}{h} \leq 0$ (as $h\to 0^-$ implies $h<0$ & $\frac{|f(c+h)|}{h} < 0$)

—— (4)

so, by (2), (3), (4)

$$\lim_{h\to 0} \frac{|f(c+h)|}{h} = \lim_{h\to 0^+} \frac{|f(c+h)|}{h} = \lim_{h\to 0^-} \frac{|f(c+h)|}{h} = 0$$

$\Rightarrow$ Let $\varepsilon > 0$ be any real no. so, $\exists\ \delta(\varepsilon) > 0$ s.t

$$\left| \frac{|f(c+h)|}{h} - 0 \right| < \varepsilon \text{ when } 0 < |h| < \delta(\varepsilon)$$

—— (5)

so,
$$\left| \frac{f(c+h) - f(c)}{h} - 0 \right| = \left| \frac{f(c+h)}{h} \right|$$

$$= \left| \frac{|f(c+h)|}{h} \right| < \varepsilon \quad (\text{by } (5))$$

when $0 < |h| < \delta(\varepsilon)$

hence, $f'(c) = 0$

Ques. 5 (a) $f(x) = |x| + |x+1|$

$$f(x) = \begin{cases} -2x-1 & x \leq -1 \\ 1 & -1 < x \leq 0 \\ 2x+1 & x \geq 0 \end{cases}$$

At $x = -1$

$$\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0$$

$$\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^-} \frac{-2(-1+h)-1-1}{h}$$

$$= \lim_{h \to 0^-} \frac{2-2h-2}{h} = -2$$

So, LHL $\neq$ RHL. Hence limit doesn't exist.
fn. isn't differentiable

Similarly at $x=0$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(2h+1) - 1}{h} = 2$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{1-1}{h} = 0$$

LHL $\neq$ RHL. limit doesn't exist

Therefore, fn. $f(x)$ isn't differentiable

(b) $g(x) = 2x + |x|$

$$g(x) = \begin{cases} 3x & x > 0 \\ x & x \leq 0 \end{cases}$$

At $x = 0$

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \frac{3h - 0}{h} = 3$$

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \frac{h - 0}{h} = 1$$

LHL $\neq$ RHL. Hence limit doesn't exist at $x=0$.
So, function $g(x)$ isn't differentiable.

(c) Homework.

(d) Homework.

Ques 6.

Given : $f: [0,1] \longrightarrow \mathbb{R}$

f: differentiable fn.

$f(0) = 0$ & $f(1) = 1$

$f'(x) = 2x$

Let $g(x) = f(x) - x^2$

(as $f(x)$ = differentiable fn. and x^2 = continous fn. on $\mathbb{R}$, hence $g(x)$ is also a continous fn.)

(similarly $g(x)$ is also differentiable) [algebra on diff. fns.]

so, $g(0) = 0$, $g(1) = 0$

so, according to Mean Value Theorem

$$g'(c) = \frac{g(1) - g(0)}{1-0} = 0$$

so, $g'(c) = f'(c) - 2c = 0$

$f'(c) = 2c$

Hence, this shows that $\exists$ a point c in $(0,1)$ s.t

$f'(c) = 2c$.