Assignment 5

October 16, 2023

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.

   Proof. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$. Then by Algebra of limits of sequences $\lim_{n\to\infty} 2x_n = 0$. As $\lim_{x\to 0} f(x) = L$, so this will imply $\lim_{n\to\infty} f(x_n) = L$ as well as $\lim_{n\to\infty} f(2x_n) = L$. But $f(2x_n) = 2f(x_n)$ by the given properties of f (Given $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. In particular, if $x = y$ that will imply $f(2x) = 2f(x)$ for all $x \in \mathbb{R}$.) Hence

   $$L = \lim_{n\to\infty} f(2x_n) = 2 \lim_{n\to\infty} f(x_n) = 2L,$$

   Hence $2L = L$ implies $L = 0$.
   Let $c \neq 0$, and let $\{x_n\}$ be any sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = c$. Then $\lim_{n\to\infty}(x_n - c) = 0$, hence $0 = \lim_{n\to\infty} f(x_n - c) = \lim_{n\to\infty}(f(x_n) - f(c))$(as $f(x-y) = f(x) - f(y)$) and so we will have $0 = \lim_{n\to\infty} f(x_n) - f(c)$ which imply $\lim_{n\to\infty} f(x_n) = f(c)$. □

2. Consider two functions f, g.
   (a) Show that if both $\lim_{x\to c} f(x)$ and $\lim_{x\to c}(f(x) + g(x))$ exist, then $\lim_{x\to c} g(x)$ exists.
   (b) If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} f(x)g(x)$ exist, does it follow that $\lim_{x\to c} g(x)$ exists?

   Proof. a) Consider $g = (f+g) - f$. So we have $\lim_{x\to c} g(x) = \lim_{x\to c}(f(x) + g(x)) - \lim_{x\to c} f(x)$ (by Algebra of limit of the functions).
   b) If $\lim_{x\to c} f(x) \neq 0$ then $\lim_{x\to c} g(x) = \frac{\lim_{x\to c} f(x)g(x)}{\lim_{x\to c} f(x)}$ (by Algebra of limit of the functions) since $\lim_{x\to c} f(x) \neq 0$. □

3. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

   Proof. Hint: Consider the function $f, g : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} -1 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ -1 \text{ if } x > 0 \end{cases}$$

   then limit of f and g do not exist at $x = 0$. But $f + g$ where $(f + g)(x) = f(x) + g(x)$ and $fg(x) = f(x)g(x)$ have limits at $x = 0$. □

4. Show that if $f:(a,\infty)\to\mathbb{R}$ is such that $\lim_{x\to\infty} xf(x)=L$ where $L\in\mathbb{R}$, then $\lim_{x\to\infty} f(x)=0$.

   Proof. Given $\lim_{x\to\infty} xf(x)=L$. Let $\{x_n\}$ be any sequence diverges to ∞. Then $\lim_{n\to\infty} x_n f(x_n)=L$ as $\lim_{x\to\infty} xf(x)=L$. Let $X_n=x_n f(x_n)$ and $Y_n=\frac{1}{x_n}$. Now $\lim_{n\to\infty} X_n=L$ and $\lim_{n\to\infty} Y_n=\lim_{n\to\infty}\frac{1}{x_n}=0$. So by Algebra of limits $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} X_nY_n=\lim_{n\to\infty} X_n\times\lim_{n\to\infty} Y_n=L\times 0=0$. So for any arbitrary sequence $\{x_n\}$ converging to ∞, $\lim_{n\to\infty} f(x_n)=0$ concluding $\lim_{x\to\infty} f(x)=0$. $\square$

5. Let $f(x)=\frac{\sqrt{1+3x^2}-1}{x^2}$ for $x\neq 0$. Find the limit of $\lim_{x\to 0} f(x)$.

   Proof. We have $f(x)=\frac{\sqrt{1+3x^2}-1}{x^2}=\frac{3x^2}{x^2(\sqrt{1+3x^2}+1)}$ for $x\neq 0$. Hence $f(x)=\frac{3}{(\sqrt{1+3x^2}+1)}$ for $x\neq 0$. Then apply the algebra of limits, $\lim_{x\to 0} f(x)=\frac{\lim_{x\to 0} 3}{\lim_{x\to 0}(\sqrt{1+3x^2}+1)}=\frac{3}{2}$. $\square$

6. Let $\lim_{x\to 0}\frac{f(x)}{x^2}=5$, then $\lim_{x\to 0}\frac{f(x)}{x}=0$.

   Proof. Let $F(x)=\frac{f(x)}{x^2}$ when $x\neq 0$ and $G(x)=x$ which gives $F(x)G(x)=\frac{f(x)}{x}$ when $x\neq 0$. Hence $\lim_{x\to 0} F(x)=5$ (given) and $\lim_{x\to 0} G(x)=0$. So by Algebra of limits of functions $\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0} F(x)G(x)=\lim_{x\to 0} F(x)\times\lim_{x\to 0} G(x)=5\times 0=0$. $\square$

7. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ such that $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not exist.

   Proof. Hint: Example $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} 1 \text{ if } x\geq 0\\ -1 \text{ if } x<0\end{cases}$$

   Then $\lim_{x\to 0} f(x)$ does not exist. Now $x^2\geq 0$ so $f(x^2)=1$ for all $x\in\mathbb{R}$ giving $\lim_{x\to 0} f(x^2)=1$ $\square$

8. Let $f:(0,\infty)\to\mathbb{R}$. Prove that $\lim_{x\to\infty} f(x)=L$ iff $\lim_{x\to 0+} f(\frac{1}{x})=L$.

   Proof. Given $\lim_{x\to\infty} f(x)=L$. Required to prove $\lim_{x\to 0+} f(\frac{1}{x})=L$. Consider $g(x)=f(\frac{1}{x})$ Let $\{x_n\}$ be any sequence of positive real numbers converging to 0. Then the sequence $\{y_n\}$ be the sequence of positive real numbers converging to ∞ where $y_n=\frac{1}{x_n}$ for $n\geq 1$. As $\lim_{x\to\infty} f(x)=L$, so $\lim_{n\to\infty} f(y_n)=L$ which implies $\lim_{n\to\infty} f(\frac{1}{x_n})=\lim_{n\to\infty} g(x_n)=L$. Hence $\lim_{x\to 0+} f(\frac{1}{x}=\lim_{x\to 0+} g(x)=L$ as $\{x_n\}$ is any arbitrary sequence of positive real numbers converging to 0.

   Similarly, we can prove the converse. $\square$

9. Consider the function $f:[0,2]\to\mathbb{R}$ defined by $f(x)=x^{\frac{3}{2}}$. Prove that the function is continuous at $c\in[0,2]$.

Proof. Let $\varepsilon > 0$ be any given real number. When $c = 0$, we have

$$|f(x) - 0| = x^{3/2} < \varepsilon$$

whenever $0 < |x - 0| < \delta = \varepsilon^{2/3}$.
When $c \neq 0$. Then

$$\begin{aligned}
|f(x) - f(c)| &= |x^{\frac{3}{2}} - c^{\frac{3}{2}}| = |x^{\frac{1}{2}} - c^{\frac{1}{2}}||x + \sqrt{xc} + c| \\
&\leq |x^{\frac{1}{2}} - c^{\frac{1}{2}}|(2 + \sqrt{4} + 2) \quad \text{since} \quad x, c \leq 2 \\
&= 6|x^{\frac{1}{2}} - c^{\frac{1}{2}}| \\
&= \frac{6}{|x^{\frac{1}{2}} + c^{\frac{1}{2}}|}|x - c| \\
&\leq \frac{6}{c^{\frac{1}{2}}}|x - c| \quad \text{since } x \geq 0, \quad x^{\frac{1}{2}} + c^{\frac{1}{2}} \geq c^{\frac{1}{2}} \\
&< \varepsilon,
\end{aligned}$$

whenever $0 < |x - c| < \delta = \frac{\varepsilon c^{\frac{1}{2}}}{6}$. □