IIITD | INDRAPRASTHA INSTITUTE of INFORMATION TECHNOLOGY DELHI

End Semester Examination

15th December 2022

Course Title : Real Analysis -1 | Time Duration : 2 hours 30 min
Course Code : MTH-240 | Total Marks : 40

Q.1)a) If f and g are uniformly continuous on $\mathbb{R}$, then is it always true $f.g$ is uniformly continuous? Justify your answer. 1.5-marks

b) What is continuity of a function $f : D(\subseteq \mathbb{R}) \to \mathbb{R}$ at a point $x = x_0$? 1.5-marks

c) Determine whether the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sqrt{x}}$ is continuous. 2.5-marks

d) State whether the following statements are **true or false**.

i) Composition of two continuous functions is a continuous function.

ii) Composition of two uniformly continuous functions may not be a uniformly continuous function.

iii) Composition of two differentiable functions is a differentiable function. 0.5+0.5+0.5=1.5-marks

Answer Either e) or f)

e) Define Limit Superior. Find the Limit Superior of the sequence $\{\frac{1}{2}, \frac{2}{3}, \frac{1}{4} \cdots\}$. 1.5+1.5=3-marks

f) State intermediate value theorem for continuous functions. Consider the function

$$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3. \end{cases}$$

It is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite. 1.5+1.5=3-marks

Q.2) a) Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find the local maxima or minima. 2.5-marks

b) State Equivalent condition of differentiability for $z = f(x,y)$ at a point (x_0, y_0) in the domain of the function ? 1-marks

$$f(x,y) = \begin{cases} \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is $f(x,y)$ continuous at $(0,0)$? Is $f(x,y)$ differentiable at $(0,0)$? 1.5+2=3.5-marks

c) Find the directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$,

i) increases most rapidly at $(1,1)$, ii) decreases most rapidly at $(1,1)$ and iii) what are the directions of zero change in f at $(1,1)$? 1+1+1=3-marks

Q.3) a) Let $f(x)$ and $g(x)$ be two functions where $f(x)$ is continuous at $x = 0$, and $g(x) = xf(x)$ for all $x \in \mathbb{R}$. Is $g(x)$ differentiable at $x = 0$? Justify your answer. 1.5-marks

b) State Roll's theorem. 1.5-marks

c) Assume that $a_0, a_1, \cdots, a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$$

Prove that the polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ has at least one root in $(0, 1)$. 2.5-marks

d) Find $\lim_{x\to 0} \frac{e^{2x}-\cos x}{x}$. Mention the L'Hospital's rule whatever you are applying to the problem. 2+1.5=3.5-marks

e) State whether the following statements are **true or false**.

i) If f is differentiable then f is always continuous.

ii) If f is differentiable then f' is always continuous. 0.5+0.5=1 mark

Q.4) a) Let f be a continuous function on $[a, b]$ and such that

$$\int_a^b g(x)f(x)dx = 0$$

for every g continuous on $[a, b]$. Show that f is identically 0. 2.5-marks

b) Let $f : [0, 1] \to \mathbb{R}$ be such that $f(x)$ is differentiable and $f'(x)$ is continuous. Then show that there exists a $\theta \in (0, 1)$ such that

$$\int_0^1 f(x)dx = f(0) + \frac{1}{2}f'(\theta).$$

3-marks

c) State Second Fundamental Theorem of Calculus or Integral mean value theorem for continuous function. Suppose f and g be two continuous functions on $[a, b]$ such that $\int_a^b f(t)dt = \int_a^b g(t)dt$. Prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$. 1.5+2=3.5-marks

d) State whether the following statement is true or false.

$$f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The function is Riemann integrable in $[-1, 1]$. 1 mark