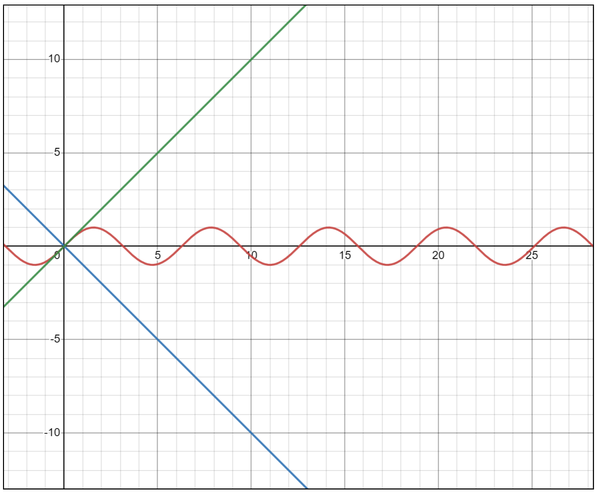
If any answer contains any of the following, they will be given **0 marks** for that section of the answer:

1. Pictorial representation or graph of a function on $\mathbb{R}$ or $\mathbb{R}^n$ $(n \in \mathbb{N})$
2. Arithmetic operations on ∞ or $-\infty$ or similar undefined numbers on $\mathbb{R}$ or $\mathbb{R}^n$ $(n \in \mathbb{N})$
3. Converse of known theorem, that is not true
4. Incorrect definitions used
5. Integration or properties of Riemannn integrable functions

Q.3) b) Can you prove $-x \leq \sin x \leq x$ for $x \geq 0$? (1.5 marks)

Ans:



Desmos : Graphs of $f(x) = -x$, $g(x) = \sin x$ and $h(x) = x$

Proof 1: Let $f(y) = \sin y \ \forall \ y \in \mathbb{R}$,

For $x = 0$, $f(0) = 0$ and thus, $-2(0) \leq f(0) \leq -1(0)$ **+0.25 marks**

For $x \neq 0$, $f(y)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. **+0.5 marks**

Thus by Lagrange's Mean Value Theorem,

$\exists \ \eta_x \in (0, x)$ such that $f'(\eta_x) = \dfrac{f(x) - f(0)}{x - 0}$ **+0.5 marks**

$$\cos(\eta_x) = f'(\eta_x) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x}$$

$\implies -1 \leq \dfrac{\sin x}{x} \leq 1$ (Range of $\cos x$ is $[-1, 1]$) **+0.25 marks**

$\implies -x \leq \sin x \leq x$

Proof 2: Let $g(x) = x - \sin x \ \forall \ x \in \mathbb{R}$ and $h(x) = x + \sin x \ \forall \ x \in \mathbb{R}$, Both $g(x)$ and $h(x)$ are continuous and differentiable on $\mathbb{R}$ **+0.5 marks**

$g'(x) = 1 - \cos x \geq 0 \implies g(x)$ is an increasing function
Thus, $x \geq 0 \implies g(x) \geq g(0) = 0 - \sin 0 = 0 \implies x \geq \sin x$ **+0.5 marks**

$h'(x) = 1 + \cos x \geq 0 \implies h(x)$ is an increasing function
Thus, $x \geq 0 \implies h(x) \geq h(0) = 0 + \sin 0 = 0 \implies \sin x \geq -x$ **+0.5 marks**

Proof 3: Let $f(x) = \sin x \ \forall \ x \in \mathbb{R}$,

For $x = 0$, $f(0) = 0$ and thus, $-2(0) \leq f(0) \leq -1(0)$ **+0.25 marks**

For $x \neq 0$, since $f(x)$ is continuous and infinitely differentiable on $\mathbb{R}$ with each of its m^{th} derivatives continuous on $\mathbb{R}$ ($m \in \mathbb{N}$), we can then apply Taylor's Theorem in a neighbourhood around the point $x = 0$. **+0.5 marks**

$f(x) = \sin x \implies f(0) = 0$
$f'(x) = \cos x$ **+0.25 marks**

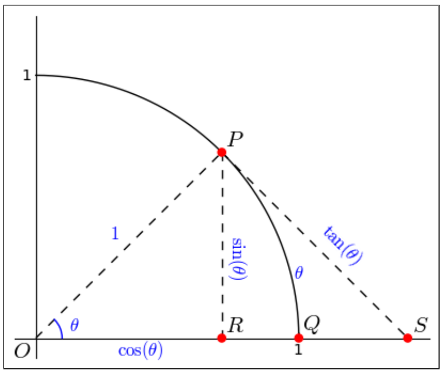
By Taylor's Theorem, $\exists \ \xi_x \in (0, x)$ where $f(x) = f(0) + f'(\xi_x)\dfrac{(x-0)}{1!}$ **+0.25 marks**
$\implies f(x) = \cos(\xi_x)x \implies -x \leq \sin x \leq x$ (Range of $\cos x$ is $[-1, 1]$) **+0.25 marks**

Proof 4: Let $f(x) = -x$, $g(x) = \sin x$ and $h(x) = x \ \forall \ x \in \mathbb{R}$,

Since $f(x)$, $g(x)$ and $h(x)$ are differentiable on $\mathbb{R}$, **+0.5 marks**

$f(0) = g(0) = h(0) = 0$
$f'(x) = -1$, $g'(x) = \cos x$ and $h'(x) = 1$
$f'(x) \leq g'(x) \leq h'(x)$ **+0.5 marks**
$\implies f(x) = -x \leq g(x) = \sin x \leq h(x) = x \ \forall \ x \geq 0$ **+0.5 marks**

Proof 5:



+0.5 marks for correct figure

From figure, area of $\triangle\, OPQ \leq$ area of $\triangledown\, OPQ \leq$ area of $\triangle\, OPS$
$\implies \dfrac{1}{2}\sin\theta \leq \dfrac{1}{2}\theta \leq \dfrac{1}{2}\tan\theta$ $\left(\theta \in \left[0, \dfrac{\pi}{2}\right]\right)$
$\implies -\theta \leq 0 \leq \sin\theta \leq \theta \ \forall \ \theta \in \left[0, \dfrac{\pi}{2}\right]$ **+0.5 marks**

For $\theta \geq \dfrac{\pi}{2}$, $-\theta \leq -1 \leq \sin\theta \leq 1 \leq \theta$ **+0.5 marks**

Therefore, $\forall \ \theta \geq 0$, $-\theta \leq \sin\theta \leq \theta$

Q.4) a) What is the level curve of $z = f(x, y)$? (1 mark)

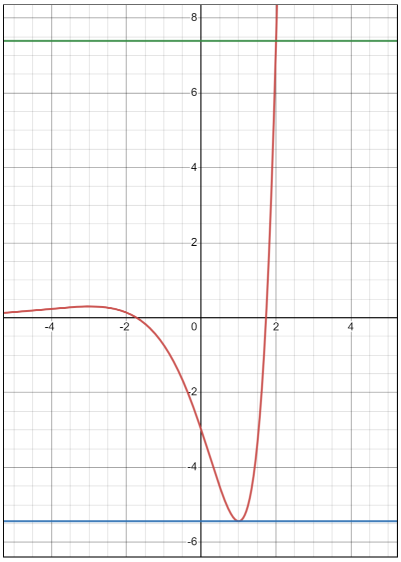
Ans: Given a function $z = f(x, y)$ and a real number c in the range of f, the set of points in the plane where a function $f(x, y)$ has the constant value $f(x, y) = c$ is called a level curve of f.

+1 mark Exact same or re-worded definition

0 marks Otherwise

Q.4) b) Find the maximum or the minimum values of $f(x) = (x^2 - 3)e^x$ on the interval $[-2, 2]$ (2 marks)

Ans:



Desmos : Graph of $f(x) = (x^2 - 3)e^x$

The final answer is:
Maximum value is e^2 at $x = 2$ **+0.25 marks**
Minimum value is $-2e$ at $x = 1$ **+0.25 marks**

Since $f(x)$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$,
$\exists\ a, b \in [-2, 2]$ such that $f(a) = \sup\{f(x) \,|\, x \in [-2, 2]\}$ and $f(b) = \inf\{f(x) \,|\, x \in [-2, 2]\}$
Thus, both maximum and minimum values exist in $[-2, 2]$ **+0.25 marks**

$f'(x) = (2x - 3 + x^2)e^x$, i.e. exist $\forall\ x \in (-2, 2)$

$f'(2) = \lim\limits_{h\to 0-} \dfrac{f(2 + h) - f(2)}{h}$ exists and $f(-2) = \lim\limits_{h\to 0+} \dfrac{f(-2 + h) - f(-2)}{h}$ exists
$\implies$ There are no critical points where the derivative is not defined **+0.25 marks**

If local maxima or minima exists at $x = c \in [-2, 2]$, $f'(c) = 0$
Finding the roots of $f'(x) = (2x - 3 + x^2)e^x = 0$,
$\implies 2x - 3 + x^2 = 0$ ($e^x \neq 0 \ \forall \ x \in [-2, 2]$)
$\implies x = 1$ or $x = -3$ ($x = -3$ is neglected as $-3 \notin [-2, 2]$)
$\implies x = 1$, i.e. $x = 1$ is a root of $f'(x) = 0$ **+0.25 marks**

Therefore, the only critical point is $x = 1$.

For $-2 < x < 1$, $f'(x) = (x + 3)(x - 1)e^x < 0$,
and for $1 < x < 2$, $f'(x) = (x + 3)(x - 1)e^x > 0$,
By 1$^{\text{st}}$ Derivative Test, $x = 1$ is a local minimum and $f(1) = -2e$ **+0.25 marks**

Since for $-2 \leq x \leq 1$, $f'(x) = (x + 3)(x - 1)e^x \leq 0$,
$\implies f(x)$ is decreasing on $[-2, 1]$
$f(x) \leq f(-2) = e^{-2} \ \forall \ x \in [-2, 1]$ **+0.25 marks**

Since for $1 \leq x \leq 2$, $f'(x) = (x + 3)(x - 1)e^x \geq 0$,
$\implies f(x)$ is increasing on $[1, 2]$
$f(x) \leq f(2) = e^2 \ \forall \ x \in [1, 2]$ **+0.25 marks**

Therefore, the global maximum value e^2 occurs at $x = 2$ and the global minimum value $-2e$ occurs at $x = 1$

Q.5) a) What is the statement of Taylor's theorem? (1 mark)

Ans: If f and its derivative of order m are continuous and $f^{(m+1)}(x)$ exists in a neighbourhood of a. Then there exists $c_x \in (a, x)$ or $c_x \in (x, a)$ such that :

$$f(x) = f(a) + f'(a)(x - a) + f^{(2)}\frac{(x - a)^2}{2!} + \cdots + f^{(m)}\frac{(x - a)^m}{m!} + R_m(x)$$

where $R_m(x) = f^{(m+1)}(c_x)\dfrac{(x - a)^{m+1}}{(m + 1)!}$

+1 mark Exact same or re-worded definition

0 marks Otherwise

Q.5) b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function at $x = c$ and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at c iff $f'(c) = 0$ (2 marks)

Ans: $g(x)$ is continuous on $\mathbb{R}$ since it is a composition of two continuous functions on $\mathbb{R}$, i.e. $f(x)$ and $|x|$ **+0.5 marks**

We define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ to be (η is an arbitrary real number):

$$F(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & , x \neq c \\ f'(c) & , x = c \end{cases}$$

$$G(x) = \begin{cases} \dfrac{g(x) - g(c)}{x - c} & , x \neq c \\ \eta & , x = c \end{cases}$$

Claim 1: If $f'(c) = 0$, $g(x)$ is differentiable at $x = c$

Proof: By definition of differentiability, F is continuous at $x = c$

$\implies \forall\ \varepsilon > 0,\ \exists\ \delta_\varepsilon \in R^+$ such that $\forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon)$, $|F(x) - f'(c)| < \varepsilon$

$\implies \left|\dfrac{f(x) - 0}{x - c} - 0\right| = |F(x) - f'(c)| < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon) \smallsetminus \{c\}$

$\implies \left|\dfrac{|f(x)| - |f(c)|}{x - c} - 0\right| = \left|\dfrac{|f(x)|}{x - c} - 0\right| = \dfrac{|f(x)|}{|x - c|} < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon) \smallsetminus \{c\}$

$\implies \left|\dfrac{g(x) - g(c)}{x - c} - 0\right| < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon) \smallsetminus \{c\}$

$\implies G$ is continuous at $x = c$ if and only if $\eta = 0$

$\implies \lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c}$ exists and is equal to 0 **+0.5 marks**

Claim 2: If $g(x)$ is differentiable at $x = c$, $g'(c) = 0$

Proof: By definition of differentiability, G is continuous at $x = c$

$\implies \lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c} = \eta$

$\implies \lim\limits_{x \to c+} \dfrac{|f(x)|}{x - c} = \eta = \lim\limits_{x \to c-} \dfrac{|f(x)|}{x - c}$

For $x > c$, $\dfrac{|f(x)|}{x - c} \geq 0 \implies \lim\limits_{x \to c+} \dfrac{|f(x)|}{x - c} \geq 0$

For $x < c$, $\dfrac{|f(x)|}{x - c} \leq 0 \implies \lim\limits_{x \to c-} \dfrac{|f(x)|}{x - c} \leq 0$

$\implies g'(c) = \lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c} = \eta = 0$ **+0.5 marks**

Claim 3: If $g(x)$ is differentiable at $x = c$ and $g'(c) = 0$, $f'(c) = 0$

Proof: $\lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c}$ exists and is equal to 0,

$\implies \forall\ \varepsilon > 0,\ \exists\ \delta_\varepsilon \in R^+$ such that $\forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon)$, $|G(x) - g'(c)| < \varepsilon$

$\implies \left|\dfrac{|f(x)| - 0}{x - c} - 0\right| = |G(x) - g'(c)| < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon) \smallsetminus \{c\}$

$\implies \left|\dfrac{f(x) - f(c)}{x - c} - 0\right| = \left|\dfrac{|f(x)|}{x - c} - 0\right| = \dfrac{||f(x)||}{|x - c|} < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon) \smallsetminus \{c\}$

$\implies \lim\limits_{x \to c} F(x) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = 0$

$\implies f'(c) = 0$ (F is continuous at $x = c$) **+0.5 marks**

Therefore, $g(x)$ is differentiable at $x = c \Leftrightarrow f'(c) = 0$

Note: There are alternate ways of proving these claims but these claims themselves are necessary. Any valid proof for a claim gets **+0.5 marks**