Real Analysis - I

Assignment - 6

Q1. To Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$

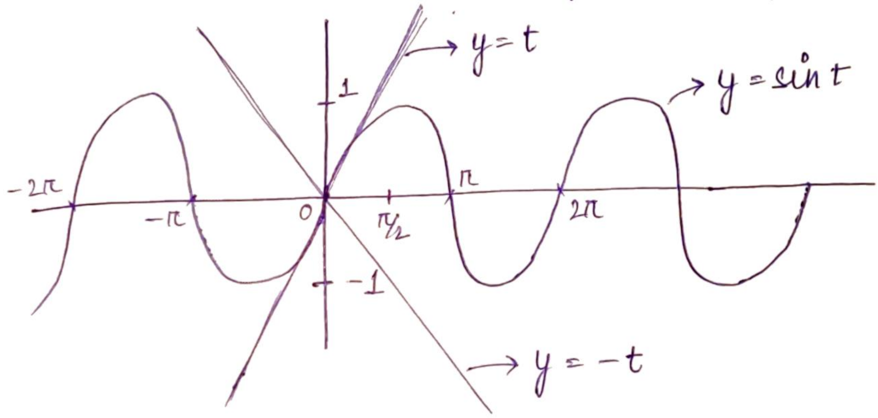
Pf We know that $\left.\begin{array}{l} -1 \le \sin\theta \le 1 \\ -1 \le \cos\theta \le 1 \end{array}\right] \forall\ \theta \in \mathbb{R}$ —— ①
—— ②

Integrating ② with respect to θ from 0 to t where $t \geq 0$.

$$\Rightarrow \int_0^t -1\, d\theta \le \int_0^t \cos\theta \le \int_0^t 1\, d\theta$$

$$\Rightarrow -t \le \sin t \le t \quad ,(t \geq 0) \quad \text{—— ③}$$

This can also be observed graphically



Graph of sint always lies b/w the lines $y=t$ & $y=-t$.

~~Similarly, integrate ② w.r to θ from t to 0 where t<0~~

~~$\Rightarrow \int_t^0 -1\, d\theta \le \int_0$~~

where $t<0$

If we substitute t by $(-t)$, in eq^n ③, the inequality still remains the same

$$\Rightarrow -t \le \sin t \le t \quad \forall\ t \in \mathbb{R}$$

Again integrating both sides w.r. to t from 0 to x where $x \geq 0$

$$\Rightarrow \int_0^x -t\, dt \le \int_0^x \sin t\, dt \le \int_0^x t\, dt$$

$$\Rightarrow -\frac{x^2}{2} \le 1-\cos x \le \frac{x^2}{2}, \quad (x \ge 0)$$

If $x<0$, replace x by $(-x)$

$$\Rightarrow -\frac{x^2}{2} \le \underset{\underset{1-\cos x}{\parallel}}{1-\cos(-x)} \le \frac{x^2}{2}, \quad (x<0) \qquad (\because \cos(-x) = \cos x)$$

$$\therefore -\frac{x^2}{2} \le 1-\cos x \le \frac{x^2}{2} \quad \forall\ x \in \mathbb{R}$$

$$\Rightarrow 1-\frac{x^2}{2} \le \cos x \le 1+\frac{x^2}{2} \quad \forall\ x \in \mathbb{R} \text{ --- } (4)$$

from (2) & (4), we get

$$1-\frac{x^2}{2} \le \cos x \le 1 \quad \forall\ x \in \mathbb{R}$$

Again integrating w.r to x from 0 to t where $t \ge 0$.

$$\cancel{\Rightarrow x - \frac{x^3}{6} \le \sin x}$$

$$\Rightarrow t-\frac{t^3}{6} \le \sin t \le t, \quad t \ge 0 \text{ --- } (5)$$

$$\cancel{\Rightarrow -t \le \sin t \le \frac{t^3}{6} - t}$$

Replacing t by $(-t)$ when $t<0$

$$\Rightarrow -t+\frac{t^3}{6} \le -\sin t \le -t \qquad (\because \sin(-t) = -\sin t)$$

$$\Rightarrow t \le \sin t \le t-\frac{t^3}{6}, \quad t<0 \text{ --- } (6)$$

Hence, we get

$$1-\frac{t^2}{6} \le \frac{\sin t}{t} \le 1 \quad , \text{ when } t>0 \quad \text{—— (5)}$$

and $$1 \le \frac{\sin t}{t} \le 1-\frac{t^2}{6} \quad , \text{ when } t<0 \quad \text{—— (6)}$$

Using squeeze theorem in (5) & (6), we get

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \quad \left(\text{as } \lim_{t\to 0} 1 = 1 \text{ and } \lim_{t\to 0}\left(1-\frac{t^2}{6}\right)=1\right)$$

Q3. Given $f: \mathbb{R} \to \mathbb{R}$ is continuous

$$\lim_{|x|\to\infty} f(x) = 0 .$$

If f is bounded on $\mathbb{R}$ and attains either an absolute max^m or an absolute min^m on $\mathbb{R}$.

Let $g(x) = |f(x)| \ \forall \ x \in \mathbb{R}$

Case 1 $g(x) \equiv 0$

$\Rightarrow |f(x)| \equiv 0$

$\Rightarrow |f(x)| = 0 \ \forall \ x \in \mathbb{R} \Rightarrow f(x) = 0 \ \forall \ x \in \mathbb{R}$

$\therefore$ f is clearly bounded by 0 on $\mathbb{R}$ and attains both absolute max^m & absolute min^m on $\mathbb{R}$.

Case 2 Suppose $g(x) \not\equiv 0$

$\Rightarrow \exists \ c \in \mathbb{R}$ such that $g(c) \neq 0$

Also, $g(c) = |f(c)| > 0$.

As $\lim_{|x|\to\infty} f(x) = 0$, by definition, for every $\epsilon>0$, $\exists$ a real no. $G_\epsilon > 0$ such that $|f(x)| < \epsilon \ \forall \ |x| \geq G_\epsilon$

$\Rightarrow g(x) < \epsilon \ \forall \ |x| > G_\epsilon$

Let $\epsilon = g(c) > 0 \Rightarrow \exists\ \delta_{g(c)} > 0$ such that

$$g(x) < g(c) \quad \forall\ |x| > \delta_{g(c)}$$

$$\Rightarrow g(x) < g(c) \quad \forall\ x > \delta_{g(c)} \text{ and } x < -\delta_{g(c)}$$

Note that $c \in [-\delta_{g(c)}, \delta_{g(c)}]$

$\left(\because \text{If } c < -\delta_{g(c)} \text{ or } c > \delta_{g(c)} \Rightarrow g(c) < g(c) \text{ ✗} \right)$

$\because f(x)$ is continuous on $\mathbb{R} \Rightarrow |f(x)|$ is continuous on $\mathbb{R}$

$\Rightarrow g(x)$ is continuous on $\mathbb{R}$

$\Rightarrow g : [-\delta_{g(c)}, \delta_{g(c)}] \longrightarrow \mathbb{R}$ is continuous

$\therefore$ g being continuous on a closed and bounded interval of $\mathbb{R}$, attains its maximum & minimum on $[-\delta_{g(c)}, \delta_{g(c)}]$

$\Rightarrow \exists\ a, b \in [-\delta_{g(c)}, \delta_{g(c)}]$ such that

$$g(a) \le g(x) \le g(b) \quad \forall\ x \in [-\delta_{g(c)}, \delta_{g(c)}]$$

Note that $g(c) \le g(b)$

$\therefore\ g(x) < g(c) \le g(b) \quad \forall\ |x| > \delta_{g(c)}$

and $g(x) \le g(b) \quad \forall\ x \in [-\delta_{g(c)}, \delta_{g(c)}]$ (i.e. $\forall\ |x| < \delta_{g(c)}$)

$\Rightarrow g(x) \le g(b) \quad \forall\ x \in \mathbb{R}$

$\Rightarrow |f(x)| \le |f(b)| \quad \forall\ x \in \mathbb{R}$

$\therefore$ f is bounded on $\mathbb{R}$ by $M = |f(b)|$

Also, $|f|$ attains maximum at $x = b$

$\therefore$ f attains either absolute maximum or absolute minimum at $x = b$.

Q4. TP $\nexists$ a continuous onto function f from $[0,1]$ to $\mathbb{R}$

Let us assume on contrary that $\exists$ $f:[0,1] \to \mathbb{R}$ such that f is continuous ~~and~~ onto $\mathbb{R}$.

$\because f$ is continuous on a closed & bounded interval $[0,1]$

$\Rightarrow f$ is bounded on $[0,1]$ and attains its supremum & infimum on $[0,1]$.

$\Rightarrow \exists$ $a, b \in [0,1]$ such that $f(a) \le f(x) \le f(b)$ $\forall$ $x \in [0,1]$ —— (1)

As f is onto function, let $c > f(b)$ $(\in \mathbb{R})$

Then $\exists$ $y \in [0,1]$ such that $f(y) = c > f(b)$

$\Rightarrow$ $y \in [0,1]$ such that $f(y) > f(b)$

But this contradicts (1).

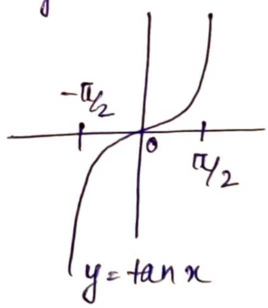
$\therefore$ No such continuous function exists from $[0,1]$ to $\mathbb{R}$.

Q5. We know that ~~f(x)~~ $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ defined by $f(x) = \tan x$ is continuous and onto. (Infact bijective)

Also, define $g: (0,1) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$g(x) = (1-t)\left(-\frac{\pi}{2}\right) + t\frac{\pi}{2}$$

$$= (2t-1)\frac{\pi}{2}$$



g being a linear map is continuous

Also, g is bijective (Exercise)

~~gof : (~~ $\therefore$ $f \circ g : (0,1) \longrightarrow \mathbb{R}$ is a continuous onto function

($\because$ Composition of two continuous functions is continuous
Composition of two bijective functions is bijective)

Q6. Given $f:[a,b] \longrightarrow \mathbb{R}$ is continuous.

For each $x \in [a,b]$, $\exists\ y \in [a,b]$ such that

$$|f(y)| \leq \tfrac{1}{2} |f(x)|.$$

TP $\exists$ a point $c \in [a,b]$ such that $f(c) = 0$.

Pf $\because f:[a,b] \longrightarrow \mathbb{R}$ is continuous

$\Rightarrow |f|$ is also continuous on $[a,b]$ and hence is bounded on $[a,b]$

(Every continuous function on a closed & bdd. interval $[a,b]$ of $\mathbb{R}$ is bounded & attains its supremum & infimum in ~~$\mathbb{R}$~~ $[a,b]$.)

$\Rightarrow \exists\ c_1, c_2 \in [a,b]$ such that

$$|f(c_1)| \leq |f(x)| \leq |f(c_2)| \quad \forall\ x \in [a,b]$$

Also, as $c_1 \in [a,b] \Rightarrow \exists\ y_1 \in [a,b]$ such that

$$|f(y_1)| \leq \frac{1}{2} |f(c_1)|$$

$$\Rightarrow |f(c_1)| \leq |f(y_1)| \leq \frac{1}{2} |f(c_1)|$$

$$\Rightarrow \tfrac{1}{2} |f(c_1)| \leq 0$$

$$\Rightarrow |f(c_1)| = 0 \Rightarrow f(c_1) = 0$$

$\therefore \exists\ c_1 \in [a,b]$ such that $f(c_1) = 0$.

Q2. Given $g, f: \mathbb{R} \longrightarrow \mathbb{R}$ are continuous

Given any two points $x_1 < x_2$, $\exists\ x_3 \in \mathbb{R}$ such that

$$x_1 < x_3 < x_2 \text{ and } f(x_3) = g(x_3)$$

TP $f(x) = g(x)\ \forall\ x$.

Let $x \in \mathbb{R}$ be arbitrary

Then $x < x + 1/n \ \forall n$.

By hypothesis, $\exists\, c_n \in \mathbb{R}$ such that $x < c_n < x + 1/n$

and $f(c_n) = g(c_n) \ \forall n \in \mathbb{N}$ —— (1)

By squeeze theorem, $\lim_{n\to\infty} x \le \lim_{n\to\infty} c_n \le \lim_{n\to\infty} (x + 1/n)$

$$\Rightarrow \lim_{n\to\infty} c_n = x$$

Now, as f & g are continuous on $\mathbb{R}$.

$\Rightarrow f$ & g are continuous at $x \in \mathbb{R}$.

$\Rightarrow \lim_{n\to\infty} f(c_n) = f(x)$ and $\lim_{n\to\infty} g(c_n) = g(x)$

(By sequential criteria for continuity)

$\Rightarrow \lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} g(c_n)$ (as $f(c_n) = g(c_n) \ \forall n$ by (1))

$\Rightarrow f(x) = g(x)$

$\because x \in \mathbb{R}$ was arbitrary $\Rightarrow f(x) = g(x) \ \forall x \in \mathbb{R}$.

Q7: Given $f : (0,1) \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \text{ where } p, q \in \mathbb{N} \ \& \ (p,q) = 1 \\ 0, & x \text{ is irrational} \end{cases}$$

(a) TS f is continuous at every irrational.

Let $b \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$ be any arbitrary irrational no.

Let $\varepsilon > 0$ be arbitrary

By Archimedean property, $\exists\, n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$.

There are only a finite no. of rationals with denominator less than n_0 in the interval $(b-1, b+1)$.

(Exercise)

$\therefore$ $\delta' > 0$ can be chosen small enough such that the neighbourhood $(b-\delta', b+\delta')$ contains no rational numbers with denominator less than n_0.

Let $\delta = \min \{\delta', |b|, |1-b|\}$

Then $(b-\delta, b+\delta) \subseteq (0,1)$ contains no rational with denominator less than n_0.

$\therefore$ $\forall\ x \in (b-\delta, b+\delta)$,

$$|h(x) - \underset{0}{\underset{\shortparallel}{h(b)}}| = |h(x)| \leq 1/n_0 < \varepsilon$$

$$h(x) = \begin{cases} 1/q_1 & \text{where } q_1 \geq n_0, \text{ if } x \text{ rational} \\ 0 & \text{, if } x \text{ irrational} \end{cases}$$

$\therefore$ for $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$|h(x) - h(b)| < \varepsilon \quad \forall \quad x \in (b-\delta, b+\delta)$
i.e. $|x-b| < \delta$

$\because$ $\varepsilon > 0$ was arbitrary

$\Rightarrow$ h is continuous at x

$\Rightarrow$ h is continuous at every irrational in $(0,1)$.

(b) TS f is discontinuous at every rational.

Let $a \in \mathbb{Q} \cap (0,1)$ be any arbitrary rational.

for every $n \in \mathbb{N}$, $\exists$ $b_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$a < b_n < a + 1/n$ (By density property).

Also, $\lim_{n\to\infty} b_n = a$ (By squeeze theorem).

suppose f is continuous at a, then by sequential criteria

$\Rightarrow$ $\lim_{n\to\infty} f(b_n) = f(a)$ $\left[\begin{array}{l} f(b_n) = 0 \ \forall\ n \text{ as } b_n \in \mathbb{R}\setminus\mathbb{Q} \\ \text{and } f(a) = 1/q' \text{ where } a = p'/q' > 0 \end{array}\right]$

$\Rightarrow$ $0 = f(a) = 1/q'$ $(\neq 0)$

which is a contradiction. $\therefore$ f is discts. at every $a \in \mathbb{Q}$.