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Assignment-3 Solutions

(1) Let $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} x_n + y_n = M$ (as it is given that $\{x_n\}$ and $\{x_n+y_n\}$ are convergent sequence)

Let $\varepsilon > 0$ be any given real number

As $\lim_{n\to\infty} x_n = L$, so for this given $\frac{\varepsilon}{2} > 0$, $\exists$ $N_{\varepsilon/2} \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon/2 \quad \forall\, n \geq N_{\varepsilon/2}$

and as $\lim_{n\to\infty} x_n + y_n = M$, for this given $\frac{\varepsilon}{2} > 0$, $\exists$ $M_{\varepsilon/2} \in \mathbb{N}$ s.t. $|(x_n+y_n) - M| < \varepsilon/2 \ \forall\, n \geq M_{\varepsilon/2}$

(we obtain the above statements
(By definition of limit of a sequence)

Let $N'_{\varepsilon/2} = \max(N_{\varepsilon/2}, M_{\varepsilon/2})$

So for all $n \geq N'_{\varepsilon/2}$

$$|y_n - (M-L)| = |(x_n + y_n - M) - (x_n - L)|$$

$$|(x_n+y_n-M) - (x_n - L)| \leq |x_n + y_n - M| + |x_n - L|$$

(Triangle Inequality $|a-b| \leq |a| + |b|$)

$$|(x_n+y_n-M) - (x_n-L)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So corresponding to an $\varepsilon > 0$ we have identified $N'_{\varepsilon/2}$ s.t. $(y_n - (M-L)) < \varepsilon \quad \forall\, n \geq N'_{\varepsilon/2}$

and as choice of ε is arbitrary we have $\lim_{n\to\infty} y_n = M-L$

Hence y_n converges.

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(2) Let $\lim_{n\to\infty} x_n y_n = M$ and $\lim_{n\to\infty} x_n = x \neq 0$

(as it is given $x_n y_n$ converges and x_n converges to $x \neq 0$)

By Algebra of Limits (Theorem 2.5 property (v))

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{x_n y_n}{x_n}\right) = \frac{\lim_{n\to\infty} x_n y_n}{\lim_{n\to\infty} x_n} = \frac{M}{x}$$

where $x \neq 0$

Hence y_n also converges.

(3) Let $\lim_{n\to\infty} (-1)^n x_n = L$ and ε be any given real number

As $\lim_{n\to\infty} (-1)^n x_n = L$, for this given $\varepsilon > 0$, $\exists$ $N_\varepsilon \in \mathbb{N}$ s.t. $|(-1)^n x_n - L| < \varepsilon$ $\forall$ $n \geq N_\varepsilon$

(Definition of limit of a sequence has been used above)

So for all $n \geq N_\varepsilon$

$$\begin{aligned} |x_n - |L|| &= ||x_n| - |L|| \\ &\leq |(-1)^n x_n - L| \\ &< \varepsilon \end{aligned}$$

(as $x_n \geq 0$ is given so $|x_n| = x_n$)

(Triangle Inequality $||a|-|b|| \leq |a-b|$)

- So for an $\varepsilon > 0$ we have $N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - |L|| < \varepsilon$ $\forall$ $n \geq N_\varepsilon$ and as choice of ε is arbitrary we have

$\lim_{n\to\infty} x_n = |L|$ and x_n converges.

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(4) a) $x_n = \frac{(-1)^n n}{n+1}$

when n is even, $x_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$

when n is odd, $x_n = \frac{-n}{n+1} = -1 + \frac{1}{n+1}$

As $n \to \infty$, $x_n \to 1$ for even values of n
and as $n \to \infty$, $x_n \to -1$ for odd values of n.

By the definition of oscillatory sequences (Definition 2.4) we know that an oscillatory sequence doesn't converge, hence $x_n = \frac{(-1)^n n}{n+1}$ is not convergent if we prove it is an oscillatory sequence

Q4. (a) $x_n = \frac{(-1)^n n}{n+1} = \begin{cases} -1 + \frac{1}{n+1}, & n \text{ odd} \\ 1 - 1/_{n+1}, & n \text{ even} \end{cases}$

$\because n < n+1 \quad \forall \; n \geq 1$

$\Rightarrow \frac{n}{n+1} < 1 \quad \forall \; n \geq 1$

$\Rightarrow |x_n| < 1 \quad \forall \; n \geq 1$

$\Rightarrow \{x_n\}$ is Bounded

Hence, not dgt.

Now, let us assume on contrary that
$\lim_{n\to\infty} x_n = L$

Let $\epsilon = 1/2 > 0$. So $\exists \; N_{1/2} \in \mathbb{N}$ s.t.

$$\left| \frac{(-1)^n n}{n+1} - L \right| < 1/2 \quad \forall \; n \geq N_{1/2}$$

$$\Rightarrow L - 1/2 < \frac{(-1)^n n}{n+1} < L + 1/2 \quad \forall \; n \geq N_{1/2}$$

$$\Rightarrow L - 1/2 < -1 + \frac{1}{2n+2} < L + 1/2 \quad \forall \; n \geq N_{1/2} \quad \text{—— (1)}$$

and $L - 1/2 < 1 - \frac{1}{2n+1} < L + 1/2 \quad \forall \; n \geq N_{1/2}$

$$\Rightarrow -L - 1/2 < -1 + \frac{1}{2n+1} < -L + 1/2 \quad \forall \; n \geq N_{1/2} \quad \text{—— (2)}$$

Adding (1) & (2)

$\Rightarrow \quad -1 < -2 + \dfrac{1}{2n+2} + \dfrac{1}{2n+1} < 1 \quad \forall\, n \geq N_{1/2}$

Taking limit $n \to \infty$

$\Rightarrow \quad -1 \leq -2 \leq 1 \quad$ ⨳

$\therefore \nexists\, L \in \mathbb{R}$ such that $\lim_{n \to \infty} x_n = L$

$\therefore \{x_n\}$ is oscillatory.

(4) b) $x_n = 2^n$

Method 1 – Let $G > 0$ be an arbitrary large real number.

we need to prove that for this given $G > 0$, $\exists\ N_G \in \mathbb{N}$ s.t.

$x_n = 2^n > G \quad \forall\ n \geq N_G$, i.e.,

$n \log 2 > \log G \quad \forall\ n \geq N_G$, i.e.,

$$n > \frac{\log G}{\log 2} \quad \forall\ n \geq N_G .$$

Let us take $N_G = \left[\frac{\log G}{\log 2}\right] + 1$

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For all $n \geq N_G = \left[\dfrac{\log G}{\log 2}\right] + 1$

$$x_n = 2^n > G \quad \forall\, n \geq N_G$$

As G is arbitrarily large, $x_n = 2^n$ diverges to $+\infty$.

Method 2 – $x_n = 2^n$ & $y_n = n$ $\forall\, n \geq 1$

$$y_n = n < x_n = 2^n \quad \forall\, n \geq 1$$

(This can be proved using PMI)

Now we know $\lim_{n \to \infty} y_n = +\infty$ or $\{y_n\}$ diverges to $+\infty$. So by corollarly 2.8

$\lim_{n \to \infty} x_n = +\infty$ or $\{x_n\}$ also diverges to $+\infty$

- Proving $n < 2^n$ $\forall\, n \geq 1$ using PMI

Base case: For $n = 1$ we get

$1 < 2$ which is always true

Inductive Hypothesis: Let the inequality be true for n, we have

$$n < 2^n$$

Inductive Proof: Now we need to prove the above inequality is true for $n+1$

or $n + 1 < 2^{n+1}$

we know $n < 2^n$

Multiplying 2 on both sides

$$2n < 2 \cdot 2^n$$

$$n + n < 2^{n+1} \quad (\text{as } n \text{ is } \geq 1)$$

$$\Rightarrow n + 1 < 2^{n+1}$$

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(5) we are given $a \in \mathbb{R}$

If $a \in \mathbb{Q}$, then $x_n = a + \frac{1}{n}$ is a monotonically decreasing sequence which converges to a, i.e.,

$$\lim_{n\to\infty} a + \frac{1}{n} = a + \lim_{n\to\infty} \frac{1}{n} = a + 0 = a$$

If $a \notin \mathbb{Q}$, then by density property of $\mathbb{R}$, $\exists$ a rational number b_1 s.t. $a < b_1 < a+1$

- Case 1 – If $a + \frac{1}{2} < b_1$, then by density property

  $\exists\ b_2' \in \mathbb{Q}$ s.t. $a < b_2 < a + 1/2 < b_1$
  (Here $a = a$ and $b = a + 1/2$ in accordance with Theorem 4.2 given in notes)

- Case 2 – If $b_1 < a + 1/2$, then by density property
  $\exists\ b_2'' \in \mathbb{Q}$ s.t. $a < b_2'' < b_1 < a + 1/2$

$$\text{Take } b_2 = \begin{cases} b_2' & \text{if } a + 1/2 < b_1 \\ b_2'' & \text{if } b_1 < a + 1/2 \end{cases}$$

(b_1 cannot be equal to $a + 1/2$ as $a + 1/2 \notin \mathbb{Q}$ and $b_1 \in \mathbb{Q}$)

From this we observe that in any case $b_2 < b_1$ and $b_2 < a + 1/2$. Similarly using the above method we can ~~find~~ choose $b_3, b_4, \ldots$ & so on all rational no's s.t. $b_{n+1} < b_n$ & $b_{n+1} < a + \frac{1}{n+1}$ $\forall\ n \geq 1$.

so $\{b_n\}$ is monotonically decreasing &

$a < b_n < a + \frac{1}{n} \quad \forall n \geq 1$, so by sandwich

lemma $\quad x_n = a, \; y_n = b_n$.

$\quad z_n = a + 1/n$

& $\lim x_n = \lim z_n = a$ so $\lim b_n = a$ & $b_n \in \mathbb{Q}$

$\forall n \geq 1$

So $\{b_n\}$ is a sequence of rational numbers &
it is monotonically decreasing and converges to a.

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(7) Let $\{|x_n|\}$ converge and let x_n be a monotonically increasing sequence. Now we need to prove $\lim x_n$ exists.

As $\{|x_n|\}$ converges, so the sequence $|x_n|$ must be bounded from both above and below. Suppose it is bounded from above by L, say, i.e.,

$$|x_n| \leq L \quad \forall n \geq 1$$

Hence $x_n \leq |x_n| \leq L \quad \forall n \geq 1$

So x_n is also bounded above and L is an upper bound. So by completeness axiom Sup T exists. As $\{x_n\}$ is monotonically increasing and Sup T exists. So by Theorem 3.1 $\{x_n\}$ converges to its supremum.

We can similarly prove when x_n is a monotonically decreasing sequence.

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(6) Let $\{x_n\}$ be a bounded sequence and ~~sup $\{x_n : n \in \mathbb{N}\}$~~

$s = \sup\{x_n : n \in \mathbb{N}\}$ & $s \notin \{x_n : n \in \mathbb{N}\}$

We need to prove that there exists a subsequence of $\{x_n\}$ which converges to s.

As $s-1 < s$, so by the property of supremum $\exists\ n_1 \in \mathbb{N}$ s.t. $x_{n_1} \in S = \{x_n : n \in \mathbb{N}\}$ &

$$s-1 < x_{n_1} < s \quad (\text{as } s \notin S \text{ so } x_{n_1} \neq s,\ \text{so } x_{n_1} < s),$$

As $s-1/2 < s$, so by property of supremum $\exists$ $n_2 \in \mathbb{N}$ s.t. $x_{n_2} \in S$ &

$$s-1/2 < x_{n_2} < s$$

We claim that $n_2 > n_1$. If there were no $n_2 > n_1$ such that $s-1/2 < x_{n_2}$ so this would imply $x_n \leq s-1/2\ \forall\ n > n_1$. So for every $n \in \mathbb{N}$ we would obtain $x_n \leq \max\{s-1/2, x_1, x_2, \ldots x_n\} < s$

So sup S would be less than s which is a contradiction. Hence $\exists\ n_2 > n_1$ s.t. $s-1/2 < x_{n_2} < s$

Now suppose that

$n_1, n_2, \ldots n_k$ have been found s.t.

$n_1 < n_2 < \ldots < n_k$ &

$$s - \frac{1}{j} < x_{n_j} < s \quad \forall\ j = 1, 2, 3, \ldots k$$

Then as we proceed from n_1 to n_2 we can find a natural no. $n_{k+1} > n_k$ s.t.

$$s - \frac{1}{k+1} < x_{n_{k+1}} < s$$

In this way we obtain $\{n_1 < n_2 < n_3 < \ldots\}$ a subsequence $\{x_{n_k}\}$ s.t. $s - \frac{1}{k} < x_{n_k} < s$

Now by Sandwich theorem

$\lim x_{n_k} = s$ as

$s - \frac{1}{k} < x_{n_k} < s$ and $\lim_{k\to\infty} s = s$ and

$\lim_{k\to\infty} s - \frac{1}{k} = s$

(8) $a_n = \left(1+\frac{1}{n}\right)^n$

a) Take $x_1 = 1, x_2 = x_3 = \dots = x_{n+1} = \left(1+\frac{1}{n}\right)$

$\forall n \in \mathbb{N}$

Applying AM > GM we get

$$\frac{1+n\left(1+\frac{1}{n}\right)}{1+n} \geq \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow \frac{\frac{n(1+n)}{n}+1}{1+n} \geq \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$\left(\frac{1}{1+n}+1\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

so $a_{n+1} > a_n \quad \forall \; n \geq 1$

Hence $\{a_n\}$ is strictly monotonically increasing

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b) $b_n = \left(1+\frac{1}{n}\right)^{n+1}$

Applying HM < GM with $Y_1 = 1$, $Y_2 = Y_3 = \cdots = Y_{n+1} = 1 + \frac{1}{n-1}$

$$\frac{n+1}{1+n\left(\frac{1}{1+\frac{1}{n-1}}\right)} < \sqrt[n+1]{\left(1+\frac{1}{n-1}\right)^n}$$

$$\Rightarrow \frac{1+n}{1+\cancel{n}\left(\frac{n-1}{\cancel{n}}\right)} < \sqrt[n+1]{\left(1+\frac{1}{n-1}\right)^n}$$

$$\Rightarrow 1+\frac{1}{n} < \sqrt[n+1]{\left(1+\frac{1}{n-1}\right)^n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n+1} < \left(1+\frac{1}{n-1}\right)^n$$

$\Rightarrow b_n < b_{n-1} \quad \forall\ n \geq 2$

so $\{b_n\}$ is strictly montonically decreasing

c) we know $a_n < b_n \quad \forall\ n \geq 1$

& $a_1 \leq a_n < b_n < b_1 \quad \forall\ n \geq 1$

(as $\{a_n\}$ is increasing & $\{b_n\}$ is decreasing)

$a_n < b_1 \quad \forall\ n \geq 1$, so $\{a_n\}$ is monotonically increasing and is bounded above by b_1. So $\{a_n\}$ converges.

Now $a_1 < b_n \quad \forall\ n \geq 1$, so b_n is monotonically decreasing and is bounded below by a_1, so $\{b_n\}$ converges.

So $\{a_n\}$ and $\{b_n\}$ both converge to its supremum and infimum respectively.

Morever $b_n = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)a_n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty} a_n \quad \left(\text{Algebra of limits}\right)$$

$\lim b_n = 1 . \lim a_n$

$\lim b_n = \lim a_n$ and this limit is defined to be e

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(9) Example of an unbounded sequence that has a convergent subsequence –

$$x_n = \begin{cases} n & n \text{ is even} \\ 1/n & n \text{ is odd} \end{cases}$$

$$x_n = \left\{ 1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, 8, \frac{1}{9}, \ldots \right\}$$

$\{x_n\}$ is not bounded above

But if we take the subsequence $\{x_{2i+1}\}$, i.e., $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$ we see that it converges to 0.