Assignment 4 Answers

Farhan Ali

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1. Test the convergence or divergence of the series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)! + 4^{n+1}}{(3n+1)!}$.

Ans: Let $S_k = \displaystyle\sum_{n=1}^{k} c_n$ and $c_n = \dfrac{(3n)! + 4^{n+1}}{(3n+1)!} = a_n + b_n$ where $a_n = \dfrac{(3n)!}{(3n+1)!}$ and $b_n = \dfrac{4^{n+1}}{(3n+1)!}$.

Now $c_n = a_n + b_n \geq a_n \quad \forall n \geq \mathbb{N}$

It is easy to see $a_n = \dfrac{(3n)!}{(3n+1)!} = \dfrac{1}{3n+1} > \dfrac{1}{3n+3} = \dfrac{1}{3}\dfrac{1}{n+1}$

where we have used the fact $3n+3 > 3n+1 \quad \forall n \geq 1$.

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ is a divergent series so by Comparison test $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges and again by comparison test $\displaystyle\sum_{n=1}^{\infty} c_n$ diverges as $c_n \geq a_n \quad \forall n \geq \mathbb{N}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$.

Ans: As $\displaystyle\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ will diverge

(Contrapositive of $\displaystyle\sum_{i=1}^{n} a_i$ is convergent $\implies \displaystyle\lim_{n\to\infty} a_n = 0$)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{\frac{1}{n}}+1}$.

Ans: As $\displaystyle\lim_{n\to\infty} \frac{5}{2^{\frac{1}{n}}+1} = \frac{5}{2} \neq 0$, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{\frac{1}{n}}+1}$ will diverge

(Contrapositive of $\displaystyle\sum_{i=1}^{n} a_i$ is convergent $\implies \displaystyle\lim_{n\to\infty} a_n = 0$)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$.

Ans: We can decompose the fraction $\dfrac{2}{n^2+2n}$ as $\dfrac{2}{n^2+2n} = \dfrac{1}{n} - \dfrac{1}{n+2}$

The partial sum $S_n = \sum_{j=1}^{n} \frac{2}{j^2+2j} = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+2}\right)$. We have a telescoping series and in each partial sum, most of the terms cancel and we obtain the formula $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}, \quad \text{so} \sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{2}.$$

(e) $\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$.

Ans: Since $2n^2+3n-5 \geq 2n^2 \ \forall\ n \geq 2$, $\frac{1}{2n^2} \geq \frac{1}{2n^2+3n-5} \ \forall\ n \geq 2$

$\sum_{n=2}^{k} \frac{1}{2n^2} \geq \sum_{n=2}^{k} \frac{1}{2n^2+3n-5} \ \forall\ k \geq 2$, hence $\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$ converges by Comparison test

2. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of non negative real numbers, such that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n$ both converge, then prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Ans: Let $S_k = \sum_{n=1}^{k} b_n$ and $T_k = \sum_{n=1}^{k} a_n b_n$.

Given $\sum_{n=1}^{\infty} b_n$ converges i.e sequence of partial sums $\{S_k\}$ is convergent, hence bounded above (by S say i.e $S_k \leq S \quad \forall k \geq 1$).

As $\sum_{n=1}^{\infty} a_n^2$ converges, so $\lim_{n\to\infty} a_n^2 = 0$. As $a_n \geq 0 \quad \forall n \in \mathbb{N}$, $\lim_{n\to\infty} \sqrt{a_n^2} = |a_n| = a_n = 0$. So the sequence $\{a_n\}$ is bounded as it is convergent $\Rightarrow \exists\ M > 0$ such that $a_n = |a_n| \leq M \quad \forall n \in \mathbb{N}$. Now

$$T_k = \sum_{n=1}^{k} a_n b_n \leq M \sum_{n=1}^{k} b_n = M S_k \leq MS$$

So $\{T_k\}$ is a bounded above sequence. And $T_{k+1} - T_k = \sum_{n=1}^{k+1} a_n b_n - \sum_{n=1}^{k} a_n b_n = a_k b_k \geq 0$. So $\{T_k\}$ are monotonically increasing sequence. Now $\{T_k\}$ is monotonically increasing and bounded sequence, therefore it is convergent which implies that $\sum_{n=1}^{\infty} a_n b_n$ converges.

3. Can you give an example of a convergent series $\sum_{n=1}^{\infty} x_n$ and a divergent series $\sum_{n=1}^{\infty} y_n$. such that $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent? Explain.

Ans: Let $S_n = \sum_{i=1}^{n} x_i$ and $S_n^{'} = \sum_{i=1}^{n} y_i$, Since $\sum_{n=1}^{\infty} x_n$ is a convergent series, $\lim_{n\to\infty} S_n = L$ (for some $L \in \mathbb{R}$) and since $\sum_{n=1}^{\infty} y_n$ is a divergent series, $S_n^{'}$ diverges to $+\infty$ or $-\infty$,

$S_n + S_n^{'} = \sum_{i=1}^{n} x_i + y_i$ diverges and thus, $\sum_{n=1}^{\infty} (x_n + y_n)$ is always divergent

4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of non negative numbers and $p > 1$, then $\sum_{n=1}^{\infty} a_n^p$ converges.

Ans: Let $S_k = \sum_{n=1}^{k} a_n$ and $T_k = \sum_{n=1}^{k} a_n^p$. As $\sum_{n=1}^{\infty} a_n$ is a convergent series, so $\lim_{n\to\infty} a_n = 0$.
So for $\varepsilon = 1$, $\exists\ N_1 \in \mathbb{N}$ such that $a_n < 1 \quad \forall n \geq N_1$
$\implies a_n^p < a_n < 1 \quad \forall n \geq N_1$

As $\sum_{n=1}^{\infty} a_n$ is a convergent series, so $\{S_k\}$ is a Cauchy sequence. So for any arbitrary $\varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that

$$S_k - S_m = |S_k - S_m| < \varepsilon \quad \forall k \geq m \geq N_\varepsilon$$

We choose $N = \max(N_1, N_\varepsilon)$. So $\forall k \geq m \geq N$

$$|T_k - T_m| = T_k - T_m = \sum_{n=m+1}^{k} a_n^p < \sum_{n=m+1}^{k} a_n = S_k - S_m = |S_k - S_m| < \varepsilon$$

Hence $\{T_n\}$ is a Cauchy sequence, so it converges.

5. If $\sum_{n=1}^{\infty} a_n$ converges with $a_n > 0$ then is always $\sum_{n=1}^{\infty} \sqrt{a_n}$ convergent? Either prove it or give a counterexample.

Ans: Taking $a_n = \frac{1}{n^2}$ is enough as $a_n > 0\ \forall\ n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series but $\sum_{n=1}^{\infty} \sqrt{a_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series

6. If $\sum_{n=1}^{\infty} a_n$ converges with $a_n > 0$ then is always $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ convergent? Either prove it or give a counterexample.

Ans: Let $S_n = \sum_{i=1}^{n} a_i$ and $T_n = \sum_{i=1}^{n} \sqrt{a_i a_{i+1}}$,
By AM-GM inequality, $\frac{a_n + a_{n+1}}{2} \geq \sqrt{a_n a_{n+1}}\ \forall\ n \in \mathbb{N}$
$\implies \sum_{i=1}^{n} \frac{a_i + a_{i+1}}{2} \geq \sum_{i=1}^{n} \sqrt{a_i a_{i+1}}\ \forall\ n \in \mathbb{N}$

$$\implies \sum_{i=1}^{n} \frac{a_i + a_i}{2} + \frac{a_{n+1} - a_1}{2} \geq T_n \ \forall \ n \in \mathbb{N}$$
$$\implies \sum_{i=1}^{n} \frac{a_i + a_i}{2} + \frac{a_{n+1} - a_1}{2} + \frac{a_{n+1} + a_1}{2} \geq \sum_{i=1}^{n} \frac{a_i + a_i}{2} + \frac{a_{n+1} - a_1}{2} \geq T_n \ \forall \ n \in \mathbb{N}$$
$$\implies S_{n+1} = \sum_{i=1}^{n+1} \frac{a_i + a_i}{2} \geq T_n \ \forall \ n \in \mathbb{N}$$

Since $\sum_{n=1}^{\infty} a_n$ converges, $\{S_n\}$ is convergent and $\{S_{n+1}\}$ is a subsequence of a convergent sequence, hence it is convergent itself.

Since $\{T_n\}$ is bounded above by a convergent sequence, $\{T_n\}$ is convergent itself and since the partial sum sequence $\{T_n\}$ converges, $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is a convergent series always

7. If $\sum_{n=1}^{\infty} a_n$ converges with $a_n > 0$ then $\sum_{n=1}^{\infty} b_n$ where $b_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ always divergent?

Ans:

Example: Taking $a_n = (\frac{1}{2})^n$ is enough as $a_n > 0 \ \forall \ n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (\frac{1}{2})^n$ is a convergent series

But $b_n \geq \dfrac{1}{2n} \ \forall \ n \in \mathbb{N}$ so,

$\sum_{n=1}^{\infty} \dfrac{1}{2n}$ is a divergent series and thus by Comparison test $\sum_{n=1}^{\infty} b_n$ diverges

General Proof: For any given a_n, $b_n \geq \dfrac{a_1}{n}$ $(a_1 > 0) \ \forall \ n \in \mathbb{N}$ so,

$\sum_{n=1}^{\infty} \dfrac{a_1}{n}$ is a divergent series and thus by Comparison test $\sum_{n=1}^{\infty} b_n$ diverges always