Q2 a) $y = \dfrac{x^2}{x+2}$

$(-3, -9)$

$x = -3$

$x + h = -3 + h$

(0.5) $f(x+h) = f(-3+h) = \dfrac{(-3+h)^2}{(-3+h+2)}$

(0.5) $f(-3) = \dfrac{(-3)^2}{-3+2} = \dfrac{9}{-1} = -9$

(0.5) $\lim\limits_{h \to 0} \dfrac{f(-3+h) - f(-3)}{-3+h-(-3)} = \dfrac{\dfrac{9+h^2-6h}{h-1} + 9}{h}$

$= \dfrac{9+h^2-6h+9h-9}{h} = \dfrac{h^2+3h}{h}$

$= \dfrac{h(h+3)}{h} = 3$

Q2 b) Let $\lim_{x \to c} \frac{f(x)}{g(x)}$ exist and be $= L$

$$f(x) = \frac{f(x)}{g(x)} g(x)$$ (0.5)

Applying limit on both sides

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x)}{g(x)} g(x) \right)$$

$$= \lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) \times \lim_{x \to c} g(x)$$

$$= L \times 0 = 0$$ (0.5)

$$\lim_{x \to c} f(x) = 0$$

but we are given $\lim_{x \to c} f(x) \neq 0$
(contradiction)

$\Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)}$ does not exist

Hence Proved (0.5)

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2c) $y = x^3$ $(2, 4)$

(0.5) $\frac{dy}{dx} = 3x^2$

$3x^2$ is equal to the slope of the line passing through the points $P(2, 4)$ and $P(x, x^3)$

A great man's courage to fulfill his vision comes from passion

$$3x^2 = \frac{x^3-4}{x-2}$$

$$3x^2(x-2) = x^3-4$$

$$3x^3 - 6x^2 = x^3 - 4$$

$$2x^3 - 6x^2 + 4 = 0$$

(0.5) $$(x-1)(2x^2-4x-4) = 0$$

$$2(x-1)(x^2-2x-2) = 0$$

sowing this we get $1 \pm \sqrt{3}$

Finding slopes of tangent lines -

$$f'(1-\sqrt{3}) = 3(1-\sqrt{3})^2$$

(0.5) $$= 3(1-2\sqrt{3}+3) = 12-6\sqrt{3}$$

$$f'(1+\sqrt{3}) = 3(1+\sqrt{3})^2 = 3(1+2\sqrt{3}+3)$$

$$= 12+6\sqrt{3}$$

$$f'(1) = 3(1)^2 = 3$$

Equations of lines -

(0.5) $$y-4 = (12-6\sqrt{3})(x-2)$$

$$y-4 = 3(x-2)$$

$$y-4 = (12+6\sqrt{3})(x-2)$$

Q6 b)

Definition 9.1. Let $\tilde{u} = u_1 i + u_2 j$ be any unit vector. Then the **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of $\tilde{u}$ is

$$D_{\tilde{u}}(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

We can replace $\tilde{u} = u_1 \tilde{i} + u_2 \tilde{j}$ by $\tilde{u} = \cos\theta i + \sin\theta j$, and obtain

$$D_{\tilde{u}}(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0 + s\cos\theta, y_0 + s\sin\theta) - f(x_0, y_0)}{s}$$

Q6 c)

Geometrical Interpretation The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\tilde{u}$ intersects S in a curve C (Figure 14.28). The rate of change of f in the direction of $\tilde{u}$ is the slope of the tangent to C at P in the right-handed system formed by the vectors $\tilde{u}$ and k.

1 mark

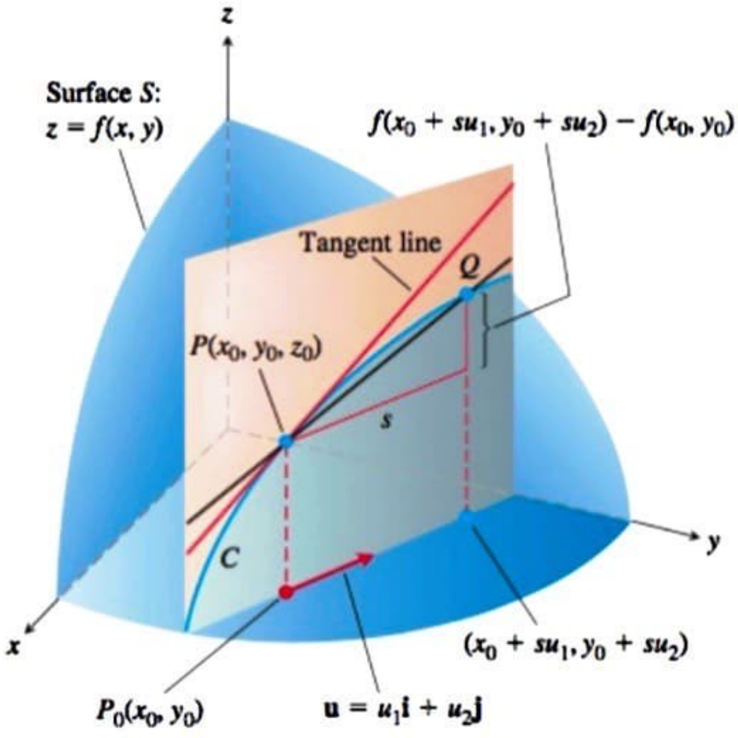

FIGURE 14.28 The slope of the trace curve C at P_0 is $\lim_{Q \to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = D_{\mathbf{u}}f|_{P_0}.$$

When $\tilde{u} = i$, the directional derivative at P_0 is $\frac{\partial f}{\partial x}$ evaluated at (x_0, y_0). When $\tilde{u} = j$, the directional derivative at P_0 is $\frac{\partial f}{\partial y}$ evaluated at (x_0, y_0).

0.5 mark

Diagram was not required

Consider the function $g : [a,b] \to \mathbb{R}$
defined by $g(x) = e^{-x} f(x)$ — 0.5

then g is continuous on $[a,b]$ and
g is differentiable on (a,b)

$g(c_1) = e^{-c_1} f(c_1) = 0$ as $f(c_1)=0$
$g(c_2) = e^{-c_2} f(c_2) = 0$ as $f(c_2)=0$
$g(c_3) = e^{-c_3} f(c_3) = 0$ as $f(c_3)=0$

Applying Rolle's theorem on (c_1, c_2)
& (c_2, c_3) separately,
$\exists\ d_1 \in (c_1, c_2)$ and $d_2 \in (c_2, c_3)$
s.t. $g'(d_1) = 0$ & $g'(d_2) = 0$

$[c_1 < d_1 < c_2 < d_2 < c_3]$

Now, $g'(x) = -e^{-x} f(x) + e^{-x} f'(x)$

$$g''(x) = \left[f(x) + f''(x) - 2f'(x)\right] e^{-x}$$

$\Rightarrow g'$ is continuous on $[d_1, d_2]$
as f' and f and e^{-x} are
continuous on $[d_1, d_2]$ and g'
differentiable on (d_1, d_2) as
f' and e^{-x} differentiable on (d_1, d_2)

1 mark

$g'(d_1) = g'(d_2) = 0$

so, using Rolle's theorem
$\exists\ e_1 \in (d_1, d_2)$ s.t

$$g''(e_1) = 0$$

so, $\boxed{f(e_1) + f''(e_1) - 2f'(e_1) = 0}$

0.5 mark for this