Q→1 Let $h : [a,b] \longrightarrow \mathbb{R}$ be a function

Let $h \in \mathcal{R}[a,b]$ exist

then $h^2 \in \mathcal{R}[a,b]$

Pf $\because h \in \mathcal{R}[a,b]$ so h is bounded on $[a,b]$

$\exists$ a $k \in \mathbb{R}^+ : |h(x)| \le k \quad \forall x \in [a,b]$

$|h^2(x)| \le k^2 \quad \forall x \in [a,b]$

This show $f^2(x)$ is bounded on $[a,b]$

Let us choose $\varepsilon > 0$

$\because h$ is integrable on $[a,b]$

$\exists$ a partition P of $[a,b]$: $U(P,f) - L(P,h) < \frac{\varepsilon}{2k}$ — ⊛

Let $P = \{x_0, x_1, \ldots, x_n\}$; $x_0 = a$
$x_n = b$
& $x_0 < x_1 < x_2 \ldots < x_n$

Let $M_r = \sup\limits_{[x_{r-1}, x_r]} h(x)$ | $M_r' = \sup\limits_{[x_{r-1}, x_r]} h^2(x)$

Let $m_r = \inf\limits_{[x_{r-1}, x_r]} h(x)$ | $m_r' = \inf\limits_{[x_{r-1}, x_r]} h^2(x)$

$\forall r = 1$ to n

For any two points α & β in $[x_{r-1}, x_r]$, we have

$$|h^2(\alpha) - h^2(\beta)| = |h(\alpha) + h(\beta)| \cdot |h(\alpha) - h(\beta)|$$
$$\le 2k\, |h(\alpha) - h(\beta)|$$

— 1

$\because$ h is bounded on $[h_{r-1}, h_r]$

with $\sup\limits_{x\in[x_{r-1},x_r]} h(x) = M_r$

& $\inf\limits_{x\in[x_{r-1},x_r]} h(x) = m_r$

So Sup of Set $\left\{ |h(\alpha)-h(\beta)| \;;\; \begin{matrix}\alpha\in[x_{r-1},x_r] \\ \beta\in[x_{r-1},x_r]\end{matrix} \right\} = M_r - m_r$ — (2)

$\because$ h^2 is bounded on $[x_{r-1}, x_r]$

with $\inf\limits_{x\in[x_{r-1},x_r]} h(x) = m_r'$

$\sup\limits_{x\in[x_{r-1},x_r]} h(x) = m_r'$

So Sup of Set $\{ |h^2(\alpha) - h^2(\beta)| \;;\; \alpha,\beta\in[x_{r-1},x_r] \} = M_r' - m_r'$ — (3)

(By (1), (2), (3))

$\therefore$ $M_r' - m_r' \le 2k(M_r - m_r)$ $\quad \forall r = 1$ to n

Consider $U(P,f^2) - L(P,f^2) = \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1})$

$\le 2k \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$

$= 2k \; U(P,f) - L(P,f)$

$< \varepsilon$. (by ★)

Q→2. If $f, g \in R[a,b]$ So is fg

Since $f \in R[a,b] \Rightarrow f$ is bounded

Since $g \in R[a,b] \Rightarrow g$ is bounded

Hence fg is bounded

(Pg→12)

$\because f, g \in R[a,b]$ $\quad f+g \in R[a,b]$ ✓ [by Prop→2]

~~by Prop→2 on Pg→12~~

~~By Q→1~~

$(f+g)^2 \in R[a,b]$ ✓ [by Q→1 earlier]

& $f^2 \in R[a,b]$

& $\frac{f^2}{2} \in R[a,b]$ [by Prop→1]

$\frac{g^2}{2} \in R[a,b]$

By Prop→4 of Notes $|f| \in R[a,b]$ if $f \in R[a,b]$

Use: $fg = \frac{(f+g)^2}{2} - \frac{f^2}{2} - \frac{g^2}{2}$

Hence $fg \in R[a,b]$

Q-3. Let $\int_a^b f(x)dx$ exist

Show: $\lim_{t\to b^-}\int_a^t f(x)dx = \int_a^b f(x)dx$

$\int_a^b f(x)dx$ exist $\Rightarrow f\in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

Consider $\left|\int_a^b f(x)dx - \int_a^t f(x)dx\right| = \left|\int_a^b f + \int_t^a f(x)dx\right|$

$$= \left|\int_t^b f(x)\,dx\right|$$

$$\leq \int_t^b |f(x)|dx \quad (\because f\in R[a,b] \Rightarrow |f|\in R[a,b])$$

$$\leq M(b-t) \quad \left(\text{by Remark-2.3}\right)$$

$\to 0$ as $t\to b^-$

Hence $\lim_{t\to b^-}\int_a^t f(x)dx = \int_a^b f(x)dx$

$\boxed{\text{Remark-2.3}}$

$\boxed{\text{If } g\in R[a,b] \text{ then } g \text{ is bdd}}$

Q→ (c) SHOW $f(x)=\frac{1}{x^2}$ is Riemann Integrable

Result :- Let $f:[a,b]\to\mathbb{R}$ be integrable.
If $\exists\ k>0 : f(x)\geq k\ \ \forall x\in[a,b]$

then $\frac{1}{f}\in R[a,b]$

Pf $\left|\frac{1}{f(x)}\right|\leq\frac{1}{k}\ \ \forall x\in[a,b]\Rightarrow\frac{1}{f}$ is bdd

Let us choose $\varepsilon>0$

$\because f$ is Riemann integ.

So $\exists$ a Partition P of $[a,b]$: $U(P,f)-L(P,f)<k^2\varepsilon$ ★

Let $P=\{x_0,\dots,x_n\}$; $x_0=a<x_1<x_2\dots<x_{n-1}<x_n=b$

Let $M_r=\sup\{f(x) ; x\in[x_{r-1},x_r]$
$m_r\ \inf\{f(x) ; x\in[x_{r-1},x_r]$

$M_r'=\sup\left\{\frac{1}{f(x)} ; x\in[x_{r-1},x_r]\right\}$
$m_r'=\inf\left\{\frac{1}{f(x)} ; x\in[x_{r-1},x_r]\right\}\ \ \forall r=1$ to n

For any two points α,β in $[x_{r-1},x_r]$

$$\left|\frac{1}{f(\alpha)}-\frac{1}{f(\beta)}\right|=\frac{|f(\alpha)-f(\beta)|}{|f(\alpha)|\cdot|f(\beta)|}\leq\frac{|f(\alpha)-f(\beta)|}{k^2}\quad(1)$$

~~$\because f$ is bdd $\Rightarrow$ Sup of Set $\{|f(\alpha)-f(\beta)| : \alpha,\beta\in[x_{r-1},x_r]\}=M_r-m_r$~~

~~$\because\frac{1}{f}$ is bdd $\Rightarrow$ Sup of Set $\{|\frac{1}{f(\alpha)}-\frac{1}{f(\beta)}| : \alpha,\beta\in[x_{r-1},x_r]\}=M_r'-m_r'$~~

$\because$ f is bdd

$\Rightarrow$ Sup of Set $\{ |f(\alpha) - f(\beta)| : \alpha, \beta \in [x_{r-1}, x_r] \} =$

$= M_r - m_r$

$\because \frac{1}{f}$ is bdd

So Sup of Set $\{ |\frac{1}{f(\alpha)} - \frac{1}{f(\beta)}| ; \alpha, \beta \in [x_{r-1}, x_r] \}$

$= M_r' - m_r'$.

By ① $\frac{M_r - m_r}{k^2}$ is u.b. of Set $\left\{ \left|\frac{1}{f(\alpha)} - \frac{1}{f(\beta)}\right| : \alpha, \beta \in [x_{r-1}, x_r] \right\}$

$\therefore M_r' - m_r' \leq \frac{1}{k^2}(M_r - m_r) \quad \forall r = 1 \text{ to } n$

Consider $U(P, \frac{1}{f}) - L(P, \frac{1}{f}) = \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1})$

$\leq \frac{1}{k^2} \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$

$= \frac{1}{k^2}(U(P, f) - L(P, f))$

$< \varepsilon$ (by ✩)

H.P.

Q → 4 (c)

One can show $g(x) = x^2 \in R[1,2]$

& $g(x) \geq 1 \quad \forall x \in [1,2]$

Hence $\frac{1}{g(x)} \in R[1,2]$ (Using Result)

i.e $f(x) \in R[1,2]$

Q→4 (a) $f(x) = x$; $[0,1]$
(b) $f(x) = x^3$; $[3,7]$ SHOW: $f \in \mathcal{R}$
(c) $f(x) = \frac{1}{x^2}$; $[1,2]$

(a) Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\}$

$$\Delta x_i = (x_i - x_{i-1}) = \frac{1}{n} \quad \forall i = 1 \text{ to } n$$

$$U(P_n, f) - L(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{i-1}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n^2} \longrightarrow 0 \text{ as } n \to \infty$$

(b) Let $P_n = \Big\{ 3, 3+\frac{1}{n}, \dots, 3+\frac{n-1}{n}, 3+\frac{n}{n},$
$3+\frac{n+1}{n}, \dots, 3+\frac{2n-1}{n}, 3+\frac{2n}{n} = 5,$
$3+\frac{2n+1}{n}, \dots, 3+\frac{3n-1}{n}, 3+\frac{3n}{n} = 6,$
$3+\frac{3n+1}{n}, \dots, 3+\frac{4n-1}{n}, 3+\frac{4n}{n} = 7 \Big\}$

We have $U(P_n, f) - L(P_n, f) = \sum_{i=1}^{4n} (M_i - m_i)\Delta x_i$

$$= \sum_{i=1}^{4n} \left(\frac{i^3}{n^3} - \frac{(i-1)^3}{n^3}\right)\frac{1}{n}$$

$$a^3-b^3=(a-b)(a^2+b^2+ab)$$

$$i^3-(i-1)^3 = [i-(i-1)]\,[i^2+(i-1)^2+i(i-1)]$$
$$= 1\cdot[i^2+i^2+1-2i+i^2-i]$$
$$= 3i^2-3i+1$$

$$U(P_n,f)-L(P_n,f) = \sum_{i=1}^{4n} \frac{3i^2-3i+1}{n^4}$$

$$= \frac{3}{n^4}\sum_{i=1}^{4n} i^2 - \frac{3}{n^4}\sum_{i=1}^{4n} i + \frac{1}{n^4}\sum_{i=1}^{4n} 1$$

$$= \frac{3}{n^4}\left\{\frac{4n(4n+1)(8n+1)}{6}\right\} - \frac{3}{n^4}\left\{\frac{4n(4n+1)}{2}\right\} + \frac{4n}{n^4}$$

$\to 0$ as $n \to \infty$.

(c) Let $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots, 1+\frac{n-1}{n}, 1+\frac{n}{n}\}$

$\Delta(x_i) = x_i - x_{i-1} = \frac{1}{n}$

$U(P_n,f) - L(P_n,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta(x_i)$

$=$

Ques→5. Let g be a cts Non-negative fn on $[a,b]$

If $\int_a^b g(t)\,dt = 0$

then g is identically zero on $[a,b]$.

Let if possible g is NOT identically zero on $[a,b]$

means $\exists$ $c \in [a,b] : g(c) > 0$ — (1)

Case-1 when $a < c < b$

Let $\varepsilon = \frac{g(c)}{2} > 0$ $\quad (\because g(c) > 0)$

$\because g$ is cts at c

$\exists$ a +ve $\delta > 0$: $g(c) - \varepsilon < g(x) < g(c) + \varepsilon$; $\forall x \in [c-\delta, c+\delta]$

$0 < \frac{g(c)}{2} < g(x)$ $\quad \forall x \in [c-\delta, c+\delta]$

Hence $\int_{c-\delta}^{c+\delta} g(x)\,dx \geq \frac{g(c)}{2} \int_{c-\delta}^{c+\delta} dx \geq g(c).\delta > 0$ — (2)

Consider $\int_a^b g(x)\,dx = \int_a^{c-\delta} g(x) + \int_{c-\delta}^{c+\delta} g(x) + \int_{c+\delta}^{b} g(x)\,dx$ — (4)

$\because g(x) \geq 0$ on $[a,b] \Rightarrow$ $\left.\begin{array}{l} \int_a^{c-\delta} g(x)\,dx \geq 0 \\ \int_{c+\delta}^{b} g(x)\,dx \geq 0 \end{array}\right\}$ — (3)

By (2), (3), (4) $\int_a^b g(x)\,dx > 0$

i.e. contradiction.

Case-2 when $c = a$

Let $\varepsilon = \frac{g(a)}{2} > 0$ $\quad (\because g(a) > 0)$

$\because g$ is cts at a

$\exists\, \delta > 0 : g(a) - \varepsilon < g(x) \quad \forall x \in [a, a+\delta]$

$0 < \frac{g(a)}{2} < g(x) \quad \forall x \in [a, a+\delta]$

$$\int_a^{a+\delta} g(x) \cdot dx \geq \frac{g(a)}{2} \cdot \delta > 0 \quad \text{—— (5)}$$

$\because g(x) \geq 0$ on $[a,b] \Rightarrow \int_{a+\delta}^{b} g(x)\,dx \geq 0$ —— (6)

Consider

$$\int_a^b g(x)\,dx = \int_a^{a+\delta} g(x)\,dx + \int_{a+\delta}^{b} g(x)\,dx$$

> 0 (by (5), (6))

i.e. contradiction

Case-3 when $c = b$

Do yourself

Q->6. Let f be a cts fn on $\mathbb{R}$
Define $g = \int_0^{\sin x} f(t)\,dt$; $x \in \mathbb{R}$

Let $F(x) = \int_0^{x} f(t)\,dt$

SHOW ① $g(x)$ is diff.ble
② $g'(x)$?

As f be a cts function on $\mathbb{R}$
So it is integrable on $[-1,1]$
Hence F is diff.ble on $\mathbb{R}$
& $F'(x) = f(x)$

[by Second Fundamental Thm of Calculus]

As $g(x) = F(\sin x)$

By chain Rule $g'(x) = F'(\sin x) \cos x$
$= f(\sin x) \cos x$

Ans

Q->