RA - I

Quiz - 2

Q3. (a) **Given** $f : [a,b] \to \mathbb{R}$ is continuous,
$c \in (a,b)$ such that $f(c) > 0$,
$0 < \beta < f(c)$

To prove $\exists\ \delta > 0$ such that $f(x) > \beta\ \forall\ x \in (c-\delta, c+\delta) \subseteq [a,b]$

Pf Let us assume on contrary that no such $\delta > 0$ exists

$\Rightarrow \forall\ \delta > 0,\ \exists\ x_\delta \in (c-\delta, c+\delta) \cap [a,b]$ such that $f(x_\delta) \leq \beta$

$\Rightarrow$ for $\delta = 1/n$, $\exists\ x_n \in (c - 1/n, c + 1/n) \cap [a,b]$ such that $f(x_n) \leq \beta$

$\because$ This can be done $\forall\ n \in \mathbb{N}$

$\Rightarrow$ ~~$x_n \in (c-\delta$~~ $x_n \in (c - 1/n, c + 1/n) \cap [a,b]$ and $f(x_n) \leq \beta$
$\forall\ \cancel{x}\ n \in \mathbb{N}$

(0.5 marks)

$\therefore \{x_n\}$ is a sequence in $[a,b]$ such that

$$c - 1/n < x_n < c + 1/n \quad \forall\ n \in \mathbb{N} \quad \text{——— (1)}$$

$$\text{and } f(x_n) \leq \beta \quad \forall\ n \in \mathbb{N} \quad \text{——— (2)}$$

Applying squeeze theorem in eqⁿ (1), we get

$$\lim_{n\to\infty} x_n = c$$

$\Rightarrow \{x_n\}$ is a sequence in $[a,b]$ converging to c. (0.25 marks)

As f is continuous ~~at~~ at $c \in [a,b]$, by sequential criteria of continuity,

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(c)$$

Taking limit $n \to \infty$ on both sides of eqⁿ (2), we get $\lim_{n\to\infty} f(x_n) \cancel{\geq\beta} \leq \beta$

$\Rightarrow f(c) \cancel{\geq\beta} \leq \beta$

This contradicts the given hypothesis that $f(c) > \beta$.

(0.5 marks)

$\therefore$ Our assumption is wrong.

$\therefore \exists\ \delta' > 0$ such that $f(x) > \beta\ \forall\ x \in (c-\delta', c+\delta') \cancel{\subseteq [a,b]}$

Let $\delta = \min\{\delta', b-c\}$

$\Rightarrow f(x) > \beta\ \forall\ x \in (c-\delta, c+\delta) \subseteq [a,b]$.

(0.25 marks)

Q3(b). Given $1 - \frac{x^2}{4} = \cos x$

To find: A root other than 0.

Pf Let $g(x) = 1 - \frac{x^2}{4} - \cos x\ \forall\ x \in \mathbb{R}$

Clearly, ~~g(x)~~ $g(0) = 0$.

We need to ~~find~~ locate a non-zero root of g

$$g(\pi/2) = 1 - \frac{\pi^2}{16} - \cos(\pi/2)$$

$$= \frac{16 - 9.87}{16}\ (> 0)$$

(0.5 marks)

$$g(\pi) = 1 - \frac{\pi^2}{4} - \cos\pi$$

$$= 2 - \frac{9.87}{\cancel{16}\,4} = \frac{8 - 9.87}{4}\ (< 0)$$

(0.5 marks)

Observe that $g(x)$ is continuous on $\mathbb{R}$ ~~by algebra~~ being sum of polynomial & trigonometric functions.

In particular, f is continuous on $[\pi/2, \pi]$.

By Intermediate value theorem, as $g(\pi/2)\, g(\pi) < 0$

$\Rightarrow \exists\ c \in (\pi/2, \pi)$ such that $g(c) = 0$. (0.25 marks)

$\therefore$ There lies a root for the given eq^n in the interval $(\pi/2, \pi)$. (0.25 marks)

Q5 (b). Define $f: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

(0.5 marks for mentioning the func^n only)

Claim f is continuous only at 0.

$\because \lim_{x \to 0} f(x) = 0 = f(0)$

$\Rightarrow f$ is continuous at 0.

Let $y \neq 0$ ($\in \mathbb{Q}$) be arbitrary

By density theorem, $\exists\ c_n \in \mathbb{Q}^c$ such that

$$y < c_n < y + 1/n \quad \forall n.$$

(0.5 marks for explaination)

$$\Rightarrow \lim_{n \to \infty} c_n = y \quad \text{(By squeeze theorem)}$$

But $\lim_{n \to \infty} f(c_n) = 0$. $(\because f(c_n) = 0\ \forall n)$

But $f(y) = y \neq 0$.

$\Rightarrow f$ can't be continuous at $y \in \mathbb{Q}$ $(y \neq 0)$

Similarly, let $y (\neq 0) \in \mathbb{Q}^c$ and construct a sequence $\{c_n\}$ of rationals converging to y

Then $0 = f(y) \neq \lim_{n \to \infty} f(c_n) = \lim_{n \to \infty} c_n = y$ $\therefore f$ can't be cts. at $y \in \mathbb{Q}^c$ $(y \neq 0)$.