End Semester Exam

RA-I

Q4(a) $f(x) = \begin{cases} x^2-1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$

Let $x_0 \in \mathbb{R}$ be any arbitrary real no.

By sequential criteria for continuity,

f is continuous at x_0 iff for any sequence of real numbers $\{x_n\}$ converging to x_0, $(f(x_n))$ converges to $f(x_0)$ — (*)

(0.25)

By density property of rationals & irrationals in $\mathbb{R}$, we can construct a sequence of rationals $\{x_n\}$ converging to x_0, and a sequence of irrationals $\{y_n\}$ converging to x_0.

(choose $x_n \in \mathbb{Q}$ such that $x_0 < x_n < x_0 + 1/n \; \forall \, n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} x_n = x_0$)

Similarly, choose $y_n \in \mathbb{Q}^c$

(0.25)

Then, for f to be continuous at x_0, by (*),

$\lim_{n\to\infty} f(x_n) = f(x_0)$ and $\lim_{n\to\infty} f(y_n) = f(x_0)$

$\Rightarrow \lim_{n\to\infty} (x_n^2 - 1) = f(x_0)$ & $\lim_{n\to\infty} (0) = f(x_0)$

$(\because x_n \in \mathbb{Q} \; \forall \, n)$ $\quad (y_n \in \mathbb{Q}^c \; \forall \, n)$

$\Rightarrow \left(\lim_{n\to\infty} x_n\right)^2 - 1 = f(x_0)$ & $0 = f(x_0)$ — (2)

$\Rightarrow x_0^2 - 1 = f(x_0)$ — (1)

(By algebra of limits)

(0.5)

Case I If $x_0 \in \mathbb{Q} \Rightarrow f(x_0) = x_0^2 - 1$

Then ① holds trivially and

② $\Rightarrow x_0^2 - 1 = 0$

$\Rightarrow x_0 = \pm 1$

$\therefore$ The only possible rational points at which f is continuous are $x_0 = \pm 1$ (By ⊛)

(0.25)

Case II If $x_0 \notin \mathbb{Q} \Rightarrow f(x_0) = 0$

Then ① $\Rightarrow x_0^2 - 1 = 0$

$\Rightarrow x_0 = \pm 1 \in \mathbb{Q}$

Not possible as $x_0 \notin \mathbb{Q}$

② holds trivially.

$\therefore \nexists\ x_0 \in \mathbb{Q}^c$ such that f is continuous at x_0.

$\therefore$ The only points where f is continuous are $x = \pm 1$

(0.25)

Q4.(b) $f : [0, \pi] \longrightarrow \mathbb{R}$

$f(x) = \cos x$

Let $\varepsilon > 0$ be ~~an~~ arbitrary. Let $c \in [0, \pi]$ be an arbitrary no.

(0.5)

Consider

$$|f(x) - f(c)|$$

$$= |\cos x - \cos c|$$

$$= \left|-2\sin\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right|$$

$$\leq 2 \cdot 1 \cdot \left|\sin\left(\frac{x-c}{2}\right)\right| \quad \left(\because \left|\sin\left(\frac{x+c}{2}\right)\right| \leq 1\ \forall x\right)$$

$$\leq 2 \cdot 1 \cdot \left|\frac{x-c}{2}\right| \quad (\because |\sin x| \leq |x|\ \forall x \in \mathbb{R})$$

$$= |x - c|$$

(1)

Let $\delta = \epsilon > 0$

Then $|x-c| < \delta = \epsilon \Rightarrow |f(x) - f(c)| < \epsilon$ (by above argument)

(0.5)

$\because \epsilon > 0$ was arbitrary.

$\therefore$ for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$ (0.5)

Q4. (c) We say that $f(x)$ has a removable discontinuity at $x = a$ if

(1) $f(x)$ is defined everywhere in a domain D containing a except at $x = a$ and limit exists at $x = a$ i.e. $\lim_{x \to a} f(x)$ exists. (0.5)

(2) $f(x)$ is defined at $x = a$ and limit is not equal to function value at $x = a$

i.e. $\lim_{x \to a} f(x) \neq f(a)$. (0.25)

These functions can be extended as continuous by defining the value of f as the limit value at $x = a$

(0.25)

Q5. (a) Let f be continuous on the interval $I = [a, b]$ & let c be an interior point of I. Assume that f is differentiable on (a, c) & (c, b). Then: (0.25)

(a) If there is a neighborhood $(c-\delta, c+\delta)$ such that $f'(x) \geq 0$ $\forall$ x such that $f'(x) \geq 0$ $\forall$ $x \in (c-\delta, c)$ and $f'(x) \leq 0$ $\forall$ $x \in (c, c+\delta)$, then f has a local maximum at c. (0.5)

(b) If $\exists$ a neighbourhood $(c-\delta, c+\delta)$ such that $f'(x) \geq 0$ $\forall$ $x \in (c, c+\delta)$ and $f'(x) \leq 0$ $\forall$ $x \in (c-\delta, c)$, then f has a local minimum at c. (0.25)

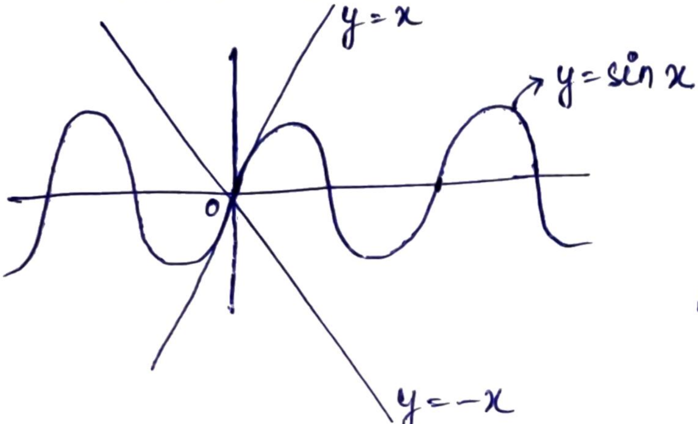
Q5(b). $K(x) = \cos x - 1 + \frac{1}{2}x^2$

$\because$ $\cos x$, 1 and x^2 are differentiable functions on $\mathbb{R}$.

$\therefore$ $K(x)$ is differentiable on $\mathbb{R}$.

$K'(x) = -\sin x + x \;\; \forall \; x \in \mathbb{R}$

$K'(0) = 0$ (0.5)



Observe that $\forall \; x \geq 0$,

$\sin x \leq x$

$\Rightarrow K'(x) \geq 0 \;\; \forall \; x \geq 0$ (0.5)

And for $x \leq 0$, $(-x) \geq 0$

~~$\sin x$~~ $\Rightarrow \sin(-x) \leq -x$

$\Rightarrow -\sin x \leq -x$

$\Rightarrow \sin x \geq x$

$\Rightarrow K'(x) \leq 0 \;\; \forall \; x \leq 0$. ~~(0.5)~~

Hence, $K'(x) \leq 0 \;\; \forall \; x \in (-\delta, 0)$

and $K'(x) \geq 0 \;\; \forall \; x \in (0, \delta)$ (0.5)

where δ is any arbitrary positive real.

$\therefore$ By first derivative test,

$x = 0$ is a point local minimum. (0.5)

Q5(c). Given $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous

$$\lim_{|x| \to \infty} f(x) = \infty$$

TP f is bounded on $\mathbb{R}$ and attains either an absolute max^m or an absolute min^m on $\mathbb{R}$.

Let $g(x) = |f(x)| \;\; \forall \; x \in \mathbb{R}$ ~~(0.25)~~

Case 1 $g(x) \equiv 0$

$\Rightarrow |f(x)| \equiv 0$

$\Rightarrow |f(x)| = 0 \; \forall \; x \in \mathbb{R}$

$\Rightarrow f(x) = 0 \; \forall \; x \in \mathbb{R}$

$\therefore$ f is clearly bounded by 0 on $\mathbb{R}$ and attains both absolute max^m & absolute min^m on R (0.5)

Case 2 Suppose $g(x) \not\equiv 0$

$\Rightarrow \exists \; c \in \mathbb{R}$ such that $g(c) \neq 0$

Also, $g(c) = |f(c)| > 0$ ~~(0.25)~~

As $\lim\limits_{|x| \to \infty} f(x) = 0$, by definition, for every $\varepsilon > 0$, $\exists$ a real no. $G_\varepsilon > 0$ such that $|f(x)| < \varepsilon \; \forall \; |x| > G_\varepsilon$

$\Rightarrow g(x) < \varepsilon \quad \forall \; |x| > G_\varepsilon$

Let $\varepsilon = g(c) > 0 \Rightarrow \exists \; G_{g(c)} > 0$ such that

$g(x) < g(c) \; \forall \; |x| > G_{g(c)}$

$\Rightarrow g(x) < g(c) \; \forall \; x > G_{g(c)}$ & $x < -G_{g(c)}$ (0.5)

Note that $c \in [-G_{g(c)}, G_{g(c)}]$

($\because$ If $c < -G_{g(c)}$ or $c > G_{g(c)}$

$\Rightarrow g(c) < g(c)$ $\times$)

$\because$ f is continuous on $\mathbb{R} \Rightarrow |f(x)|$ is continuous on R

$\Rightarrow g(x)$ is continuous on $\mathbb{R}$

$\Rightarrow g : [-G_{g(c)}, G_{g(c)}] \to \mathbb{R}$ is continuous.

$\therefore$ g being continuous on a closed & bounded interval of $\mathbb{R}$, attains its maximum & minimum on $[-G_{g(c)}, G_{g(c)}]$

$\Rightarrow \exists\ a, b \in [-G_{g(c)}, G_{g(c)}]$ such that

$$g(a) \le g(x) \le g(b) \quad \forall\ x \in [-G_{g(c)}, G_{g(c)}]$$ (0.5)

Note that $g(c) \le g(b)$

~~$\therefore g(x) \le g(b)$~~ $\therefore g(x) < g(c) \le g(b) \quad \forall\ |x| > G_{g(c)}$

and $g(x) \le g(b) \quad \forall\ x \in [-G_{g(c)}, G_{g(c)}]$

$\Rightarrow g(x) \le g(b) \quad \forall\ x \in R$

$\Rightarrow |f(x)| \le |f(b)| \quad \forall\ x \in R$

$\therefore f$ is bounded on $\mathbb{R}$ by $M = |f(b)|$

Also, $|f|$ attains maximum at $x = b$.

$\therefore f$ attains either absolute max^m or absolute minimum at $x = b$. (0.5)