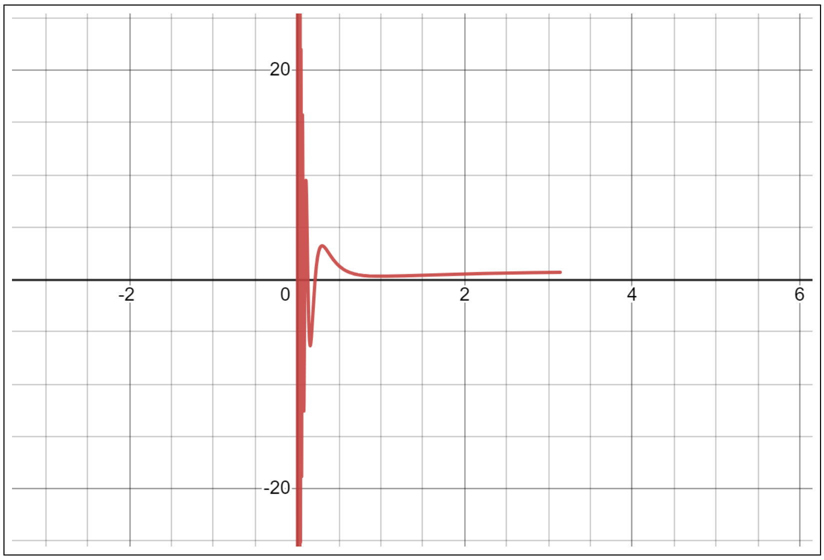
If any answer contains any of the following, they will be given **0 marks** for that section of the answer:

1. Pictorial representation or graph of a function on $\mathbb{R}$
2. Arithmetic operations on ∞ or $-\infty$ or similar undefined numbers on $\mathbb{R}$
3. Converse of known theorem, that is not true
4. Incorrect definitions used
5. L'Hôpital's rule or any of its variants
6. Differentiation or properties of differentiable functions
7. Integration of properties of Riemannn integrable functions

Q.4) a) Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$ for $x \neq 0$. Is f continuous at $x = 0$? (1.5 marks)

Ans: For $f : [0, \pi] \to \mathbb{R}$, such that

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Desmos : Graph of $f(x) : [0, \pi] \to \mathbb{R}$

$f(x)$ is not continuous at $x = 0$. **+0.5 marks**

Proof 1: Let us consider the sequence $\alpha_n = \dfrac{1}{2n\pi}\ \forall\ n \in \mathbb{N}$, **+0.25 marks**

$f(\alpha_n) = \dfrac{1}{2n\pi}\sin(2n\pi) - 2n\pi\cos(2n\pi) = -2n\pi\ \forall\ n \in \mathbb{N}$

Since $\alpha_n = \frac{1}{2\pi}\left(\frac{1}{n}\right)$ and $f(\alpha_n) = 2\pi(-n)$,
$\lim_{n\to\infty} \alpha_n = 0$ and $f(\alpha_n)$ diverges to $-\infty$ **+0.25 marks**

Since there exists a sequence α_n such that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} f(\alpha_n)$ does not exist, i.e. cannot be $f(0) = 0$. **+0.5 marks**

Hence $f(x)$ is not continuous at $x = 0$.

Note: This proof completely changes based on the choice of α_n; any valid sequence α_n in $[0, \pi]$ such that $\alpha_n \to 0$ but $f(\alpha_n) \not\to 0$ will be given **1.5 marks**

Proof 2: Let us assume for the sake of contradiction that $f(x)$ is continuous at $x = 0$,
$\implies \lim_{x\to 0} f(x) = 0$

Let $g : [0,\, \pi] \to \mathbb{R}$, such that

$$g(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Claim: $\lim_{x\to 0} g(x) = 0$ **+0.25 marks**

If $x = 0$, $g(x) = x = 0$
If $x \neq 0$, $g(x) = x\sin\left(\frac{1}{x}\right) \implies -x \leq g(x) \leq x$
$\implies \forall\ x \in [0, \pi]$, $-x \leq g(x) \leq x$
Therefore by Sandwich Theorem, $\lim_{x\to 0} g(x) = 0$ **+0.25 marks**

Let $h : [0,\, \pi] \to \mathbb{R}$, such that $h(x) = x(g(x) - f(x))$

$$h(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By Algebra of Limits, $\lim_{x\to 0} h(x) = 0$ **+0.25 marks**

But this leads to a a contradiction by taking $\beta_n = \frac{1}{2n\pi}\ \forall\ n \in \mathbb{N}$,
$h(\beta_n) = \cos(2n\pi) = 1\ \forall\ n \in \mathbb{N}$
Since $\beta_n = \frac{1}{2\pi}\left(\frac{1}{n}\right)$ and $h(\beta_n) = 1$, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} h(\beta_n) = 1$ **+0.25 marks**
Since there exists a sequence β_n such that $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} h(\beta_n) = 1$, i.e. not be $h(0) = 0$, we contradict the fact that $\lim_{x\to 0} h(x) = 0$

Hence, $f(x)$ is not continuous at $x = 0$.

Note: This proof completely changes based on the choice of β_n;if all the steps before substitution and any valid sequence β_n in $[0, \pi]$ such that $\beta_n \to 0$ but $f(\beta_n) \not\to 0$ will be given **1.5 marks**

Proof 3: Let us assume for the sake of contradiction that $f(x)$ is continuous at $x = 0$,
$\implies \lim_{x \to 0} f(x) = 0$
$\implies \forall \varepsilon > 0, \exists\ \delta_\varepsilon \in \mathbb{R}_{>0}$ such that $|f(x) - 0| < \varepsilon\ \forall\ x \in (-\delta_\varepsilon, \delta_\varepsilon)$

Taking $\varepsilon = \dfrac{1}{2}$,
For any possible $\delta_\varepsilon > 0$, $\exists\ j \in \mathbb{N}$ such that $0 < \dfrac{1}{2j\pi} < \delta_\varepsilon$ (By Archimedean Property)
But $\left| f\left(\dfrac{1}{2j\pi}\right) \right| = |-2j\pi| = 2j\pi \geq 2\pi > \dfrac{1}{2}$ **(Contradiction)**
Hence, $f(x)$ is not continuous at $x = 0$ **+1 mark**

Note: This proof completely changes based on the choice of substitution; all the previous steps and any valid substitution will be given **1.5 marks**

Q.4) b) If $\lim_{x \to 0^+} f(x) = A$ and $\lim_{x \to 0^+} f(x) = B$ and A may not be equal to B, then what is $\lim_{x \to 0^+} f(x^3 - x)$? (1.5 marks)

Ans: The final limit is B. **+0.5 marks**
Let x_n be any sequence converging to 0 such that $\exists\ N \in \mathbb{N}$, $x_n \geq 0\ \forall\ n \geq N$ $(x_n \to 0^+)$,
Claim: $x_n^3 - x_n \to 0^-$ **+0.25 marks**

Proof 1:
$\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that $x_n = |x_n - 0| < \varepsilon\ \forall\ n \geq N_\varepsilon$
Taking $\varepsilon = 1$, $\exists\ N_1$ such that $x_n < 1\ \forall\ n \geq N_1$
And for all $x \in [0, 1)$, $x^3 \leq x^2 \leq x \leq 1 \implies x^3 - x \leq 0$
$\implies x_n^3 - x_n \leq 0\ \forall\ n \geq N_1$
Furthermore, $y_n = x_n^3 - x_n$ converges to 0 (By Algebra of Convergent Sequences)
$\implies y_n$ converges to 0 from the left hand side or $x_n^3 - x_n \to 0^-$ **+0.5 marks**

Proof 2:
$\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that $x_n = |x_n - 0| < \varepsilon\ \forall\ n \geq N_\varepsilon$
Taking $\varepsilon = 1$, $\exists\ N_1$ such that $x_n < 1\ \forall\ n \geq N_1$
And for all $x \in [0, 1)$, $x^3 \leq x^2 \leq x \leq 1 \implies x^3 - x \leq 0$
$\implies x_n^3 - x_n \leq 0\ \forall\ n \geq N_1$
$\implies 0 \leq |x_n^3 - x_n| = x_n - x_n^3 \leq x_n = |x_n - 0|\ \forall\ n \geq N_1$
$\implies \forall\ \varepsilon > 0$, $\exists\ N_\varepsilon^* = \max\{N_\varepsilon, N_1\} \in \mathbb{N}$ such that $|x_n^3 - x_n - 0| \leq |x_n - 0| < \varepsilon\ \forall\ n \geq N_\varepsilon^*$
$\implies x_n^3 - x_n$ converges to 0 and $\exists\ N_1 \in \mathbb{N}$ such that $x_n^3 - x_n \leq 0\ \forall\ n \geq N_1$
$\implies x_n^3 - x_n \to 0^-$ **+0.5 marks**

Proof 3:
$g(x) : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3 - x$ is a continous function
$\implies \lim_{n \to \infty} x_n^3 - x_n = \lim_{n \to \infty} g(x_n) = g\left(\lim_{n \to \infty} x_n\right) = g(0) = 0$

But for $x \in [0, 1)$, $x^3 \leq x \implies g(x) \leq 0$

Since x_n is a convergent sequence,
$\implies \exists\ N_1 \in \mathbb{N}$ such that $0 \leq x_n = |x_n - 0| < 1\ \forall\ n \geq N_1$
$\implies \exists\ N_1 \in \mathbb{N}$ such that $x_n \in [0, 1)\ \forall\ n \geq N_1$
$\implies \exists\ N_1 \in \mathbb{N}$ such that $g(x_n) = x_n^3 - x_n \leq 0\ \forall\ n \geq N_1$
$\implies g(x_n)$ converges to 0 from the left hand side or $g(x_n) = x_n^3 - x_n \to 0^-$ **+0.5 marks**

Therefore, since $x_n^3 - x_n \to 0^-$ for any arbitrary $x_n \to 0^+$,
$\lim\limits_{x\to 0^+} f(x^3 - x) = \lim\limits_{n\to\infty} f(x_n^3 - x_n) = \lim\limits_{x\to 0^-} f(x) = B$ **+0.25 marks**

Q.5) a) Using the ε-δ definition, can you prove that $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{\sqrt{x}}$ is a continuous function (2 marks)

Ans: $f : (0, \infty) \to \mathbb{R}$, such that $f(x) = \dfrac{1}{\sqrt{x}}$

Let $c \in (0, \infty)$ and ε such that $\varepsilon > 0$,
We take $\delta_\varepsilon = \min\{\frac{c}{2}, \frac{c\sqrt{c}\varepsilon}{\sqrt{2}}\}$ **+0.5 marks**

For any $x \in (c - \delta_\varepsilon, c + \delta_\varepsilon)$,
$\frac{c}{2} \leq c - \delta_\varepsilon < x < c + \delta_\varepsilon \leq \frac{3c}{2}$ $(\delta_\varepsilon \leq \frac{c}{2})$
$\implies \frac{c}{2} < x < \frac{3c}{2}$ **+0.25 marks**

$$\begin{aligned}
|f(x) - f(c)| &= |\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}| \\
&\leq \frac{|\sqrt{x} - \sqrt{c}|(\sqrt{x} + \sqrt{c})}{\sqrt{x}\sqrt{c}(\sqrt{x} + \sqrt{c})} \\
&< \frac{\sqrt{2}|x - c|}{\sqrt{c}\sqrt{c}(\sqrt{x} + \sqrt{c})} \text{ since } x > \frac{c}{2} \implies \sqrt{x} > \frac{\sqrt{c}}{\sqrt{2}} \\
&\leq \frac{\sqrt{2}|x - c|}{c\sqrt{c}} \text{ since } \sqrt{x} + \sqrt{c} \geq \sqrt{c} \\
&< \frac{\sqrt{2}\delta_\varepsilon}{c\sqrt{c}} \text{ since } |x - c| < \delta_\varepsilon \\
&\leq \varepsilon \text{ since } \delta_\varepsilon \leq \frac{c\sqrt{c}\varepsilon}{\sqrt{2}}
\end{aligned}$$

+1 mark for all inequalities to hold
-0.25 marks for any one incorrect inequality

Thus, $\forall\ \varepsilon > 0,\ \exists\ \delta_\varepsilon = \min\left\{\dfrac{c}{2}, \dfrac{c\sqrt{c\varepsilon}}{\sqrt{2}}\right\}$ such that $|f(x) - f(c)| < \varepsilon\ \forall\ x \in (c - \delta_\varepsilon, c + \delta_\varepsilon)$
$\implies f(x)$ is at continuous at $x = c$
And since our choice of point $c \in (0, \infty)$, i.e. domain of f was arbitrary,
$f(x)$ is a continuous function **+0.25 marks**

Note: Since the question explicitly mentions that you must use the $\varepsilon - \delta$ definition, this is the only correct way to get **1.5 marks**. However if any other method is used, the solution would be graded out of **0.75 marks**, half of the original **1.5 marks**.