End Semester Examination

Course Title : *Real Analysis 1* Time Duration: 2 h 30 min
Date: December 8, 2023 Total Mark: 40
Course Code: MTH-240 Time: 2.30-5 pm

**Give proper justifications for your answer. Mention the results or theorems which you are using.**

**Write True or False**

**Q.1)a)** If $h(x) = 0$ for $x < 0$ and $h(x) = 1$ for $x \geq 0$. Then there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = h(x)$ for all $x \in \mathbb{R}$.
**Q.1)b)** We can apply L'Hospital's rule to evaluate $\lim_{x\to+\infty} \frac{x-\sin x}{2x+\sin x}$.
**Q.1)c)** If $w = x^2 + y^2$ and $x = r - s$ and $y = r + s$, then $w_r = 4r$.
**Q.1)d)**Consider the function $f(x) = |x^3|$ for $x \in \mathbb{R}$. Then $f''(0)$ does not exists.
**Q.1)e)** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |\sin x|$ is differentiable at all points in its domain.
**Q.1)f)** The limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^6+y^2}$ exists and equal to 0.
**Q.1)g)** There does not exist a continuous function from $[a, b]$ onto $\mathbb{R}$.
**Q.1)h)** $\lim_{x\to c} g(x)f(x)$ exists always implies $\lim_{x\to c} g(x)$ and $\lim_{x\to c} f(x)$ exist.
**Q.1)i)** $g \circ f$ is differentiable at all points in its domain always implies $g$ and $f$ are differentiable at all points in their respective domains.
**Q.1)j)** If $f$ and $g$ are Lipschitz functions on the domain $A$ and both of them are bounded on $A$, then the product $fg$ is a Lipschitz function on $A$. $10 \times 1 = 10$-marks

**Q.2)a)** Let $m$ be the slope of the tangent line to the graph of $y = \frac{x^2}{x+2}$ at the point $(-3, -9)$. Express $m$ as a limit (Do not compute $m$).
**Q.2)b)** Suppose that $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then show that $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.
**Q.2)c)** Find all tangent lines to the graph of $y = x^3$ that pass through the point $(2, 4)$. $1.5 + 1.5 + 2 = 5$-marks

**Q.3)a)** What is equivalent criterion for differentiability for $z = f(x, y)$ at any point in its domain?
**Q.3)b)** Determine all values of the constant $\alpha > 0$ for which the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{|x|^3+|y|^\alpha}$ exists?
**Q.3)c)** Show by an example that the existence of partial derivatives at a given point does not imply continuity at that point. 1.25+2.5+1.25 =

5-marks

**Q.4)a)** Find the set of points of continuity of $f$ given by

$$f(x) = \begin{cases} x^2 - 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$$

**Q.4)b)** Prove using $(\varepsilon - \delta)$ definition that $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = \cos x$ is continuous function.
**Q.4)c)** What is a removable discontinuity? $1.5 + 2.5 + 1 =$ 5-marks

**Q.5)a)** What is the first derivative test?
**Q.5)b)** Determine whether $x = 0$ is a point of local extremum of $k(x) = \cos x - 1 + \frac{x^2}{2}$.
**Q.5)c)** Suppose $f$ is a function from $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{|x|\to\infty} f(x) = 0$. Prove that $f$ is bounded on $\mathbb{R}$ and attains either an absolute maximum or an absolute minimum!
$1 + 2 + 2 =$ 5-marks

**Q.6)a)** Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function having at least three distinct zeros in $[a, b]$ then the equation $f(x) + f''(x) = 2f'(x)$ has at least one root in $[a, b]$.
**Q.6)b)** What is the definition of the directional derivative of $z = f(x, y)$ at any point $(x_0, y_0)$ (in its domain) in the direction of the vector $\bar{v} = v_1\bar{i} + v_2\bar{j}$.
**Q.6)c)** Give the geometric interpretation of the directional derivative (no need to draw). $2 + 1.5 + 1.5 =$ 5-marks

**Q.7)a)** Let $f(x) = x^3y - xy^2 + cx^2$ where $c$ is a constant. Find $c$ if $f$ increases fastest at the point $P(3, 2)$ in the direction of the vector $\bar{v} = 2\bar{i} + 5\bar{j}$.
**Q.7)b)** Can you prove $\nabla\frac{f}{g}|_{(x_0,y_0)} = \frac{g\nabla f - f\nabla}{g^2}|_{(x_0,y_0)}$ with necessary conditions.
$2.5 + 2.5 =$ 5-marks