Assignment → 4

Q→1. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $\boxed{f(x+y) = f(x) + f(y)}$; $x, y \in \mathbb{R}$ —— (+)

Assume $\lim_{x \to 0} f(x) = L$ exist —— (*)

then ① $L = 0$

② f has a limit at every point $c \in \mathbb{R}$.

Given: $f(x+y) = f(x) + f(y)$

Put $y = x$, we get

$f(2x) = 2f(x)$ —— ①

Observe: if $x \to 0$ then $2x \to 0$ So $\lim_{x \to 0} f(2x) = L$ (by *)

Taking $\lim x \to 0$ in eqⁿ-①, we get

$L = 2L$

$\Rightarrow \boxed{L = 0}$

Let $c \in \mathbb{R}$ be any point.

One can write $f(x) = f((x-c)+c)$

$= f(x-c) + f(c)$ [by (+) Taking $x = x-c$, $y = c$.]

Taking $\lim x \to c$, we get

$$\lim_{x \to c} f(x) = \lim_{x \to c} f(x-c) + f(c)$$

$$\lim_{x \to c} f(x) = 0 + f(c)$$ (by Part-1)

H.P.

Q→2. (a) If $\lim_{x\to c} f(x)$ & $\lim_{x\to c} [f(x)+g(x)]$ exist

then $\lim_{x\to c} g(x)$ exist

Hint : $g(x) = [f(x)+g(x)] - f(x)$

Given: $\lim_{x\to c} f(x)$ & $\lim_{x\to c} [f(x)+g(x)]$ exist ~~say L & l~~

Using Algebra of limits, $\lim_{x\to c} g(x)$ exist

(b) $\lim_{x\to c} f(x)$ & $\lim_{x\to c} f(x)g(x)$ exist

Does it follow that $\lim_{x\to c} g(x)$ exists?

Ans No

Take $f(x) = x^2$ $\qquad \lim_{x\to 0} f(x) = 0$

$g(x) = \frac{1}{x}$; $x \neq 0$ $\qquad \lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x = 0$.

But $\lim_{x\to 0} g(x)$ does Not exist.

Q→3. Give eg of f & g : 1) f & g donot have a lt at c

2) $f+g$ & fg have a lt at c.

$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$ $\qquad g(x) = 1 - f(x)$

$= \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{else} \end{cases}$

Observe:-

$f(x)g(x) = 0 \quad \forall x \in \mathbb{R}$ $\qquad$ | $\qquad f(x)+g(x) = 1 \quad \forall x$

$\lim_{x\to c} f(x)g(x) = 0 \quad ; \forall c \in \mathbb{R}$ $\qquad$ | $\qquad \lim_{x\to c} [f(x)+g(x)] = 1 \quad \forall c \in \mathbb{R}$

But $f(x)$ & $g(x)$ donot have a limit at $c \in \mathbb{R}$ (any).

Q→4. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

One can prove that $\sin^2\theta + \cos^2\theta = 1$. ; $\forall \theta$

$\Rightarrow -1 \le \sin\theta \le 1$

$\& \; -1 \le \cos\theta \le 1$ ——— (1)

Taking integration w.r.t. θ, we get

$\cancel{-\int_0^x 1\, d\theta \le \int_0^x \sin\theta\, d\theta \le \int_0^x 1\, d\theta}$

$\cancel{-x \le \cos x}$.

$-\int_0^x 1\, d\theta \le \int_0^x \cos\theta\, d\theta \le \int_0^x 1\, d\theta$

$-x \le +\sin x \le x$

Again integrate, we get

$-\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$

i.e $1 - \frac{x^2}{2} \le \cos x \le 1 + \frac{x^2}{2}$ ——— (2)

By (1), (2) $1 - \frac{x^2}{2} \le \cos x \le 1$

Again integrate we get $\boxed{x - \frac{x^3}{6} \le \sin x \le x}$; $\forall x \ge 0$

||ly, one can obtain $\boxed{x \le \sin x \le x - \frac{x^3}{6}}$; $x < 0$

Hence $1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1$

$\downarrow \qquad\qquad \downarrow$

$1 \qquad\qquad 1$ as $x \to 0$

Hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (by Squeeze Th^m)

Q→5. If $f : (a, \infty) \to \mathbb{R}$ is s.t. $\lim_{x \to \infty} x f(x) = L$; $L \in \mathbb{R}$

then $\lim_{x \to \infty} f(x) = 0$

Use fact :- If $x_n \to 0^+$ iff $\frac{1}{x_n} \to \infty$ — ★

Given :- $\langle x_n \rangle \in (a, \infty)$ diverging to $+\infty$.

then $\lim_{n \to \infty} x_n f(x_n) = L$

Show: $\lim_{n \to \infty} f(x_n) = 0$

Given $\varepsilon > 0$ be any arb. $\mathbb{R}$

Since $\lim_{n \to \infty} x_n f(x_n) = L$

$\exists$ $N_\varepsilon \in \mathbb{N}$: $|x_n f(x_n) - L| < \varepsilon$; $\forall n \geq N_\varepsilon$

$$\frac{L - \varepsilon}{x_n} < f(x_n) < \frac{L + \varepsilon}{x_n}$$

Using Squeeze Thm & ★ , we get $\lim_{n \to \infty} f(x_n) = 0$

H.P.

Q→6. $f(x) = \dfrac{\sqrt{1 + 3x^2} - 1}{x^2}$; $x \neq 0$

$\lim_{x \to 0} f(x)$?

$$f(x) = \frac{1 + 3x^2 - 1}{x^2 \left(\sqrt{1 + 3x^2} + 1\right)}$$

$$f(x) = \frac{3}{\sqrt{1 + 3x^2} + 1}$$

$$\lim_{x \to 0} f(x) = \frac{3}{2} \text{ Ans}$$