Assignment - 6

Solutions

① let $f:[a,b] \to R$ be a Continouse function. Then f is uniformly continouse.

Soln

Assume the Contrary that f is not uniformly Continouse,

therefore
there exist an $\varepsilon_0 > 0$ and two Sequence $\{x_n\}, \{y_n\}$ in $[a,b]$ Such that

$$|x_n - y_n| < \frac{1}{n}$$

but $|f(x_n) - f(y_n)| > \varepsilon_0$ for $n \in N$. Since $\{x_n\}$ is in $[a,b]$, by Bolzano Weierstrass Theorem there

exist a Subsequence $\{x_{n_i}\}$ of $\{x_n\}$
and there exist $x_0 \in [a,b]$

Such that $\{x_{n_i}\} \to x_0$ as $i \to \infty$

Hence $\{y_{n_i}\} \to x_0$ as $i \to \infty$

By continuity of f it follows that $f(x_{n_i}) \to f(x_0)$ an

$f(y_{n_i}) \to f(x_0)$ as $i \to \infty$

therefore $|f(x_{n_i}) \to f(y_{n_i})| \to 0$ as $i \to \infty$

This Contradicts the fact that $|f(x_{n_i}) - f(y_{n_i})| >$
Therefore f is uniformly Continouse

Then

Q-2 let $f: A \to R$ be a uniformly Continouse function and $\{x_n\}$ is a Cauchy Sequence in A, then $\{f(x_n)\}$ is also a Cauchy sequence.

Ans Soln

let $\varepsilon > 0$ as f is uniformly Continouse there exist $\delta > 0$ Such that

$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad -(i)$$

$y_n\}$

Since $\{x_n\}$ is a Cauchy Sequence, there exist $N \in \mathbb{N}$ Such that

$$|x_n - x_m| < \delta \quad \forall\ m, n \geq N$$

in

therefore by (i)

$$|f(x_n) - f(x_m)| < \varepsilon \quad \forall\ m, n \geq N.$$

e

Q-3 Uniformly Continouse maps bounded Sets to bounded Sets.

Soln

Suppose S is not bounded.

$\therefore$ For every $n \in \mathbb{N}$ there exist $x_n \in A$ st $|f(x_n)| > n$

So $\{x_n\}$ is a Sequence in A

and A is bounded and $\{x_n\} \subseteq A$. So

So $\{x_n\}$ is bounded.

By Bolzano - W - theorem $\{x_n\}$ has a Convergent Sub Sequence $\{x_{n_k}\}$ Converging to x_0 (say)

As $\{x_{n_k}\}$ is convergent.
So it is a Cauchy Sequence and as f is uniformly Continouse so by Q 3.

$\{f(x_{n_k})\}$ is Cauchy

So $\{f(x_{n_k})\}$ is bounded (as Cauchy Sequence are bounded)

but by Construction ①

$|f(x_{n_k})| > n_k$ ~~So~~
So $\{f(x_{n_k})\}$ cannot be bounded. as a Contradiction.

Q-4 Given f be continouse on $[a,b]$. $c \in (a,b)$
f is diff on (a,c) & (c,b)

a) given there exists a ~~nb~~ boundance of c i.e an interval $(c-\delta, c+\delta)$ st $f'(x) > 0$ for all x
$c-\delta < x < c$
& $f'(x) \leq 0$ for all x st $c < x < c+\delta$.

R.T.P f is local maximum at c.

Soln

let $x \in (c-\delta, c)$ then by LMVT

$\exists$ $d_x \in (x, c)$

applying LMVT

$$f(x) - f(c) = (x-c) f'(d_x)$$

but $f'(t) \geq 0$ $\forall t$ st $c-\delta < t < c$
in particular $t = d_x$ we have $f'(d_x) \geq 0$

so $f(x) - f(c) = (x-c) f'(d_x) \leq 0$
as $x \in (c-\delta, c)$ implies $c-\delta < x < c$ so $x-c < 0$

so $f(x) - f(c) \leq 0$ so $f(x) \leq f(c)$

as x is arbitary in $(c-\delta, c)$ so $x-c < 0$
$f(x) \leq f(c)$ so $f(x) \leq f(c)$

as x is arbitary in $(c-\delta, c)$ so,
$f(x) \leq f(c)$ for all $x \in (c-\delta, c)$ — (1)

Now take $x \in (c, c+\delta)$

by LMVT.

$\exists\ c_x \in (c, x)$

st

$$f(x) - f(c) = (x-c)\, f'(c_x) \quad \text{— (i)}$$

but $f'(t) \le 0$ for all t in $(c, c+\delta)$ (given)

In particular $t = c_x$. implies $f'(c_x) \le 0$

now $x \in (c, c+\delta)$ implies $c < x < c+\delta$. So

$$x - c > 0$$

So by (i) $f(x) - f(c) \le (x-c)\, f'(c_x) \le 0$

$\therefore\ f(x) \le f(c)$

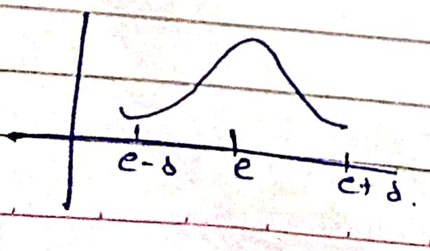
as x is arbitary so

$f(x) \le f(c)$ for all $x \in (c, c+\delta)$ — (2)

Combining (1) and (2)

$f(x) \le f(c)$ for all $x \in (c-\delta, c+\delta)$

So for f has local max at $x = c$



Part b is similar.

↳ but the converse of this statement is not true

let given $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R$

Consider $f(x) = \begin{cases} x^2 \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

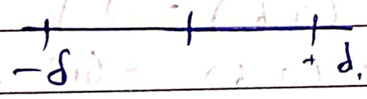
$f(x) \geq 0 = f(0)$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So $f(x) \geq 0 = f(0)$ $\forall$ $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So $x = 0$ is an absolute maximum.

Now $f'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$

take $x_n = \frac{1}{\pi n}$



$y_n = -\frac{1}{\pi n}$

$\{x_n\}$ lies in R.H.S $\{x_n\}$ is Converging Seq. to 0

but $f'(x_n) = \frac{2}{\pi n}(-1)^n$

$f'(x_n) > 0$ if n is even
$f'(x_n) < 0$ if n is odd

So f' takes both +ve and -ve. in R HS of 0

and $\{y_n\}$ is on the lhs of 0

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and $f'(y_n) = \frac{2}{\pi n} (-1)^n$

$f'(y_n) > 0$ if n is even, & $f'(y_n) < 0$ if n is odd so f' takes $-ve$ & $+ve$ values on the lhs of 0.

So f' takes $+ve$ and $-ve$ values in RHS and LHS of 0

Q-5

let $f: [a,b] \to R$ be a continuous function and differentiable on (a,b) if the derivative $f'(x) \neq 0$ for all $x \in (a,b)$, then either $f'(x) > 0$ for all $x \in (a,b)$ or $f'(x) < 0$ for all $x \in (a,b)$

Contrapositive of given statement is

if $f'(x) \nless 0$ for all $x \in (a,b)$ — (i)
and $f'(x) \ngtr 0$ for all $x \in (a,b)$ — (ii)
then $f'(x) = 0$

from (i) and (ii) we can implies that
$f'(x) = 0$

∴ Hence proved.

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Q-6 Suppose f is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} f'(x)=0$ prove that $\lim_{x\to\infty}[f(x+1)-f(x)]=0$

Soln ~~Applying LMVT $\exists$ $c_x \in (x,x+1)$~~

~~$\lim_{x\to\infty}(f(x+1))$~~

applying LMVT on $[x, x+1]$

$\therefore \quad f(x+1)-f(x) = x+1-x \; f'(c_x)$

by LMVT $\exists$ $c_x \in (x,x+1)$

$\lim_{x\to\infty}(f(x+1)-f(x)) = \lim_{x\to\infty} f'(c_x) \rightarrow$ ~~t~~ (1)

Now $\lim_{t\to\infty} f'(t)=0$ (given) $-$ (2)

as $x\to\infty$ implies $c_x\to\infty$

$\therefore \lim_{x\to\infty} f'(c_x) = \lim_{t\to\infty} f'(t) = 0$ $-$ (3) (by (2))

So by (3) an (1) $\lim_{x\to\infty}[f(x+1)-f(x)] = 0$

X ——— X