Assignment → 3

Q→1 Let $x_n \geq 0$ th

$\lim_{n\to\infty} (-1)^n x_n$ exist

SHOW: $\lim_{n\to\infty} x_n$ exist

Consider $|x_n - x_m| = \big| |x_n| - |x_m| \big|$ $\quad (\because x_n \geq 0)$

$= \big| |(-1)^n x_n| - |(-1)^m x_m| \big|$

$= |y_n - y_m|$

$< \varepsilon$ $\quad \forall n, m \geq N_\varepsilon$.

Hence $\langle x_n \rangle$ is chy seq.

Q→2: Let $\langle x_n \rangle$ be a bdd seq.

$s = \sup\{x_n ; n \in \mathbb{N}\}$

(a) If $s \notin \{x_n ; n \in \mathbb{N}\}$

SHOW: $\exists$ a subseq. of $\langle x_n \rangle$: which cgs to s.

$\because s = \sup\{x_n ; n \in \mathbb{N}\}$

for any $\varepsilon > 0$, $s - \varepsilon \neq$ u.b. for set

$\exists\ x_{n_1} \in$ Set : $s - 1 < x_{n_1} < s$ $\quad [\because s \notin \text{Set} \Rightarrow x_{n_1} \neq s]$

→ As $s - \frac{1}{2} < s$ Again Apply defn of sup.

$\exists\ n_2 \in \mathbb{N}$: $x_{n_2} \in$ Set : $s - \frac{1}{2} < x_{n_2} < s$

$\boxed{\text{Claim: } n_2 > n_1}$ If ~~there~~ were n_2 : $n_2 > n_1$

s.t. $s - \frac{1}{2} < x_{n_2} < s$

$\Rightarrow$ $x_n < s - \frac{1}{2}$ $\quad \forall n > n_1$

$\forall n \in \mathbb{N}$ we get $x_n \leq \max\{s - \frac{1}{2}, x_1, x_2, \ldots, x_{n_1}\} < s$

So $\sup(\text{set}) < s$

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$\exists\, n_2 > n_1 : \quad s - \frac{1}{2} < x_{n_2} < s$

Suppose we found $n_1 < n_2 < n_3 \dots < n_k : \quad s - \frac{1}{j} < x_{n_j} < s \quad \forall j = 2 \text{ to } k$

Just as we proceed from n_1 to n_2
we can find a ~~seq.~~ $n_{k+1} : n_{k+1} > n_k$
& $s - \frac{1}{k+1} < x_{n_{k+1}} < s.$ —— $\star$

In this way we obtain subseq. $\langle n_k \rangle_{k \in \mathbb{N}}$ $\uparrow$sing
s.t. $\langle x_{n_k} \rangle$ satisfy $\star$ $\forall k \in \mathbb{N}$

Hence apply Sandwich Th^m we get $x_{n_k} \to s.$

$\longrightarrow \exists\, x_{n_1} \in$ Set : $s - 1 < x_{n_1} < s.$

One can choose x_{n_j} : $\max\{s - \frac{1}{j}, x_{n_{j-1}}\} < x_{n_j} < s.$ $\quad \forall j \geq 2.$

In this way we get a subseq. $\langle x_{n_j} \rangle$ s.t.

1) $\langle x_{n_j} \rangle$ is Mon $\uparrow$sing

2) $\underset{\downarrow s}{s - \frac{1}{j}} < x_{n_j} < \underset{\downarrow s}{s}$; $\forall j \in \mathbb{N}$

(b) Suppose if $s \in \{x_n ; n \in \mathbb{N}\}$ then again $\exists$ subseq. $\xrightarrow{cgs} s.$
$\langle x_{n_k} \rangle = s$ i.e. const. seq.

Q→3 Let s_n, t_n be 2 seq. s.t. $s_n \le t_n \quad \forall n > N_0$

(then) 1) $\lim\inf s_n \le \lim\inf t_n$.

2) $\lim\sup s_n \le \lim\sup t_n$.

Results used :-

1 ① $\lim\inf(a_n) + \lim\inf b_n \le \lim\inf\{a_n + b_n\}$

② $\lim\sup\{a_n+b_n\} \le \lim\sup a_n + \lim\sup b_n$.

2 If $a_n \le b_n$ (then) $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

Pf ①

$\because a_n \le b_n ; \forall n > N_0$

$0 \le b_n - a_n \quad ; \forall n > N_0$

Set $= \{b_n - a_n ; n > N_0\}$

we get 0 is l.b. for set S

Hence $0 \le \inf\{b_n - a_n ; n > N_0\}$

Use Result →2

$0 \le \lim\inf\{b_n - a_n\}$ —— ① eq^n

Use Result → 1 ①

Consider

$b_n = a_n + (b_n - a_n)$

$\lim\inf b_n = \lim\inf[(a_n) + (b_n - a_n)]$

$\ge \lim\inf[a_n] + \lim\inf(b_n - a_n)$

Use eq^n ① $\ge \lim\inf(a_n)$

Hence

$\lim\inf(a_n) \le \lim\inf(b_n)$

H.P.

$\because a_n \le b_n \quad \forall n > N_0$

$a_n - b_n \le 0 \quad ; \forall n > N_0$

Set $= \{a_n - b_n ; n > N_0\}$

we get 0 is u.b. for set

Hence $\sup\{a_n - b_n ; n > N_0\} \le 0$

Use Result ②

$\lim\inf\{a_n - b_n\} \le 0$ —— ②

Use Result → 1 ②

$a_n = b_n + (a_n - b_n)$

$\lim\sup a_n = \lim\sup[b_n + a_n - b_n]$

$\le \lim\sup b_n + \lim\sup(a_n - b_n)$

$\le \lim\sup b_n$ (Use eq^n ②)

Hence

$\lim\sup(a_n) \le \lim\sup(b_n)$

Q→4 (a) $a_n = \left(1+\frac{1}{n}\right)^n$

AM > GM

$a_1 = 1,\ a_2 = \dots = a_{n+1} = 1+\frac{1}{n}$

$$\frac{1+n\left(1+\frac{1}{n}\right)}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$\frac{1+n+1}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

ie $a_{n+1} > a_n$

$a_{n+1} > a_n ;\ \forall n \geq 1$

(b) $b_n = \left(1+\frac{1}{n}\right)^{n+1}$

$HM = \frac{n}{\frac{1}{a_1}+\dots+\frac{1}{a_n}}$

HM < GM

$a_1 = 1,\ a_2 = \dots = a_{n+1} = 1+\frac{1}{n-1}$

$$\frac{n+1}{1+n\left(\frac{1}{1+\frac{1}{n-1}}\right)} < \sqrt[n+1]{\left(1+\frac{1}{n-1}\right)^n}$$

$$\frac{n+1}{1+n\left(\frac{n-1}{n}\right)}$$

$$\frac{n+1}{1+n-1} = \frac{n+1}{n} < \sqrt[n+1]{\left(1+\frac{1}{n-1}\right)^n}$$

$$\left(1+\frac{1}{n}\right)^{n+1} < \left(1+\frac{1}{n-1}\right)^n.$$

$b_n < b_{n-1} ;\ \forall n > 2$

One can observe $a_n < b_n$

$$a_1 < a_2 < \dots < a_n < b_n < \dots < b_1$$

$\langle a_n \rangle$ is bdd by b_1

Hence it cgs to sup.

H.P

$\langle b_n \rangle$ is bdd by a_1.

Hence it cgs to infimum.

H.P.

We have show $\langle a_n \rangle$ is Mon. st ↑ing & bdd above

& $\langle b_n \rangle$ is Mon st ↓ing & bdd below

Hence Both cgs

Q→5 let $x_n = \left(1+\frac{x}{n}\right)^n$

(a) If $x>0$, $\langle x_n \rangle$ is bdd & Mon. ↑ seq (st)

AM > GM

$a_1 = 1, a_2 = \dots = a_{n+1} = 1+\frac{x}{n}$

Hence $\langle x_n \rangle$ is Mon. ↑ seq — (A)

~~(b) If $n > -x$, $x \in \mathbb{R}$~~
~~then seq is bdd, st ↑ seq~~

(a) Case-1 If $0 < x < 1$

$x_n = \left(1+\frac{x}{n}\right)^n < \left(1+\frac{1}{n}\right)^n < e$ [In previous Q→4 $\lim_{n\to\infty} a_n = \left(1+\frac{1}{n}\right)^n = e$]

Hence $\langle x_n \rangle$ is cgt

Case-2 If $x > 1$

Archimed prop. $\exists n_0$: $ny > x$; $x \geq 0$ ($y=1$)

$\exists n_0 \in \mathbb{N}$: $n_0 > x$

ie $x < n_0$.

$x_n = \left(1+\frac{x}{n}\right)^n \leq \left(1+\frac{n_0}{n}\right)^n$ — (1)

We know $\langle x_n \rangle$ is Mon. ↑ seq ; $\forall x > 0$ (by (A))

In particular for $x = n_0$, $\langle b_n \rangle = \left(1+\frac{n_0}{n}\right)^n$ is Mon ↑ seq

We know $n_0 n \geq n$; $\forall n \geq 1$

$\because \langle b_n \rangle$ is Mon ↑ seq $\Rightarrow b_{n_0 n} \geq b_n$

ie $\left(1+\frac{n_0}{n n_0}\right)^{n_0 n} \geq \left(1+\frac{n_0}{n}\right)^n$

$\left[\left(1+\frac{1}{n}\right)^n\right]^{n_0} \geq \left[1+\frac{n_0}{n}\right]^n$ — (2).

Combining (1), (2)

$x_n \leq \left[\left(1+\frac{1}{n}\right)^n\right]^{n_0} \leq e^{n_0}$; $\forall n \geq 1$

Hence $\langle x_n \rangle$ is bdd by e^{n_0}

Hence $\langle x_n \rangle$ is cgt.

(b) If $n > -x$; $x \in \mathbb{R}$, (then) seq. is bdd & st $\uparrow$ing.

If $x \le 0$

$1 + \frac{x}{n} > 0$ if $1 > -\frac{x}{n}$ if $n > -x$

For $n > -x$ we apply $AM > GM$

with $a_1 = 1$, $a_2 = \dots = a_{n+1} = 1 + \frac{x}{n}$

we get $\left(1 + \frac{x}{n+1}\right) > \sqrt[n+1]{1 + \frac{x}{n}}$ $\left(\because 1 + \frac{x}{n} > 0\right)$

$$\left(1 + \frac{x}{n+1}\right)^{n+1} > \left(1 + \frac{x}{n}\right)$$

Hence $\langle x_n \rangle = \left(1 + \frac{x}{n}\right)^n$ is Mon $\uparrow$ing ; $\forall n > -x$ $(\because n > -x)$

Observe $\langle x_n \rangle$ is bdd by 1

$$\left[\begin{array}{l} 0 < 1 + \frac{x}{n} < 1 \\ \Rightarrow \left(1 + \frac{x}{n}\right)^n < 1 \end{array}\right]$$

(Hence) lt x_n exist

H.P.

(c) **Show:** $\sum_{J=0}^{\infty} \frac{x^J}{J!}$ is cgt to e^x.

$a_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ i.e. Seq. of Partial Sum of Series $\sum_{J=0}^{\infty} \frac{x^J}{J!}$

$$\left| a_n - \left(1+\frac{x}{n}\right)^n \right| = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$- \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 - \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3 + \cdots - \frac{n!}{n!} x^n$$

$$= \left[1 - \frac{n(n-1)}{n^2}\right]\frac{x^2}{2!} + \left[1 - \frac{n(n-1)(n-2)}{n^3}\right]\frac{x^3}{3!} + \cdots + \left[1 - \frac{n(n-1)\cdots[n-(n-1)]}{n^n}\right]\frac{x^n}{n!}$$

$$\cancel{= \sum_{k=} 1 - 1}$$

$$= \left[1 - \left(1-\frac{1}{n}\right)\right]\frac{x^2}{2!} + \left[1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right]\frac{x^3}{3!} + \cdots + \left[1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\right]\frac{x^n}{n!}$$

$$\le \sum_{k=2}^{n} \left[1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]\frac{|x|^k}{k!}$$

Use $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \ge 1 - \sum_{J=1}^{k-1} \frac{J}{n}$ — (★)

where we have used: if $a_k > -1$ $\forall k = 1, 2, \ldots, n$ are all +ve (or) all are −ve

then $(1+a_1)(1+a_2)\cdots(1+a_n) \ge 1 + a_1 + a_2 + \cdots + a_n$.

Use (★), we get $\le \sum_{k=2}^{n} \left[1 - \left\{1 - \sum_{J=1}^{k-1} \frac{J}{n}\right\}\right]\frac{|x|^k}{k!}$

$$\le \sum_{k=2}^{n} \frac{k(k-1)}{2n} \frac{|x|^k}{k!}$$

$$= \frac{x^2}{2n} \sum_{k=2}^{n} \frac{|x|^{k-2}}{(k-2)!} = \frac{x^2}{2n} \sum_{m=0}^{n-2} \frac{x^m}{m!}$$

$$= x_n \, S_n \quad \text{say}$$

$$0 \le \left| a_n - \left(1+\frac{x}{n}\right)^n \right| \le x_n S_n$$

$\downarrow$ 0 $\qquad\qquad$ $\downarrow$ 0

Hence $a_n \to e^x$. $\qquad$ $\because x_n \to 0$ & S_n is cgt seq (by Part Test) say it $= M$.