Theory of Computation '23
Problem Set 3

Notations. Let $\Sigma = \{a, b\}$. For $w \in \Sigma^\star$, let $|w|$ denote the length of w. Let $\#_a(w)$ denote the number of a's in w and let $\#_b(w)$ denote the number of b's in w.

Problem 1. Show that regular languages are closed under the **repeat** operation, where **repeat** operation on a language L is given by

$$\textbf{repeat}(L) = \{\ell_1\ell_1\ell_2\ell_2 \dots \ell_k\ell_k \mid \ell_1\ell_2 \dots \ell_k \in L\}$$

Problem 2. If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that

$$A_{\frac{1}{2}-} = \{x \mid \text{for some } y,\ |x| = |y| \text{ and } xy \in A\}$$

Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

Problem 3. For every string $x \in \{0, 1\}^+$ consider the number

$$0.x = x[1] \cdot \frac{1}{2} + x[2] \cdot \frac{1}{2^2} + \cdots + x[|x|] \cdot \frac{1}{2^{|x|}}$$

where $|x|$ is the length of x. For a real number $\theta \in [0, 1]$ let

$$L_\theta = \{x : 0.x \leq \theta\}$$

Prove that L_θ is regular if and only if θ is rational.

Problem 4 Prove or disprove that the following languages are regular.

a) $\textsc{Twice} = \{w \in \{a, b\}^\star | \#_a(w) = 2\#_b(w)\}$

b) $\text{NEQ} = \{0^i1^j | i \neq j\}$ [This is a slightly hard problem.]

Problem 5 Consider the language $L = \{w \in \Sigma^\star |$ 2nd letter from the end is $a\}$.

a) Draw an NFA for L.

b) Using the ideas of subset construction draw a DFA for L.

c) Using the DFA minimization idea discussed in class, check whether the DFA thus constructed is minimal or not. If it not a minimal DFA then draw the correspoding minimal DFA for it.

d) Let $L_k = \{w \in \Sigma^\star | k\text{th letter from the end is } a\}$. Prove using pigeon hole principle (or by any other method) that any DFA accepting L_k must have $\Omega(k)$ states.

Problem 6 Let L be a regular language. One of the following languages is regular and the other is not. Give a proof and provide a counterexample, respectively.

a) $\{w \in \{a, b\}^\star | \exists n \geq 0, \exists x \in L, x = w^n\}$

b) $\{w \in \{a, b\}^\star | \exists n \geq 0, \exists x \in L, w = x^n\}$