Theory of Computation '24
Problem Set 2

Notations. Let $\Sigma = \{a, b\}$. For $w \in \Sigma^\star$, let $|w|$ denote the length of w. Let $\#_a(w)$ denote the number of as in w and let $\#_b(w)$ denote the number of bs in w.

1. **Definition 1** *Let Σ and Γ be two finite alphabets. A function $f : \Sigma^* \longrightarrow \Gamma^*$ is called a string homomorphism if for all $x, y \in \Sigma^*$, $f(x \cdot y) = f(x) \cdot f(y)$.*

   Prove that the class of regular languages is closed under homomorphisms. That is, prove that if $L \subseteq \Sigma^*$ is a regular language, then so is $f(L) = \{f(x) \mid x \in L\}$. Here, it is advisable to informally describe how you will turn a DFA for L into an NFA for $f(L)$.

2. Let $L_k = \{x \in \{0,1\}^* \mid |x| \geq k$ and the k'th character of x from the end is a $1\}$. Prove that every DFA that recognizes L_k has at least 2^k states. Also show that, on the other hand, there is an NFA with $O(k)$ states that recognizes L_k.

3. A *devilish NFA* is same as an NFA, except that we define the acceptance criterion of a devilish NFA as follows. We say that a devilish NFA N accepts x if and only if every run of N on x ends in an accepting state. Prove that a language is recognized by a devilish NFA if and only if the language is regular.

4. If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that

   $$A_{\frac{1}{2}-} = \{x \mid \text{for some } y,\ |x| = |y| \text{ and } xy \in A\}$$

   Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

5. Write the regular expressions corresponding to the following languages.

   a. $L = \{\#_a(w) = 1 (\mod 2)\}$.

   b. $L = \{\text{every other letter in } w \text{ is } a\}$

   c. $L = \{w \text{ contains an odd number of } a\text{'s and an even number of } b\text{'s}\}$