Theory of Computation '23 Quiz 4
Time : 45 minutes

Name : **Roll :** **Marks :**

Problem.

$$B = \{(n,m) \mid \text{ Each } n - \text{state Turing machine with a fixed tape alphabet } M \text{ either}$$
$$\text{halts in less than } m \text{ steps on an empty input, or doesn't halt on an empty input } \}$$

(a) Show that B is not decidable. **Solution.** We show that if B is decidable, then we can construct a routine for deciding $HALT_{TM}$ which will be a contradiction. (+2 for writing the correct problem to reduce from and +2 for mentioning that you assume a decider for B and will design a decider for the selected problem)

Given an input M, w, we want to decide if M halts on w or not. We first construct a machine N, which just ignores its input and simulates M on w. Hence, N will halt on the empty input if and only if M halts on w. (+6 for doing till this construction correctly)

Let n be the number of states in N. We can now test if N halts on the empty input as follows:

(a) k = 1
(b) while (true)
(c)   if $(n,k) \in B$ break ;
(d)   else $k = k + 1$;
(e) run N on the empty input for k steps
(f) *accept* if N halts in at most k steps else *reject*

Since the number of n-state machines is finite (assuming a fixed alphabet), there must be some maximum k such that all such machines either halt in k steps or run forever. The above algorithm first finds this k and then simply checks if N halts in k steps. (+10 for this part. No credit for writing any other random stuff.)

(b) Show that B is not recognizable. (You can assume part (a) to be true to solve this)

Hint: Part (a) : You can try a mapping reduction from either of our two favorite undecidable problems. Follow the usual route of trying to construct a new TM inside the reduction. While designing the TM, note that n is a finite number

Solution. We show that $\overline{B}$ is recognizable. Since B is not decidable, this implies that B cannot be recognizable.

$$\overline{B} = \{(n, m) \mid \text{ some n-state machine halts on the empty input after more than m steps}\}$$

Since there are only a finite number of machines with n states, we can simulate all of them in parallel on the empty input. If $(n, m) \in B$, then at least one of the machines will halt after more than m steps and we will stop and accept.

(+10 for this solution or similar. Some of you have just mentioned that this can be done by mapping reduction from $\overline{(}A_{TM})$ or E_{TM} or $\overline{E_{TM}}$. I do not think this works and most proofs are completely wrong. But you have been awarded +2 just for writing this. Those who have reduced in the opposite direction gets a zero.)