Theory of Computation '23
Problem Set 1

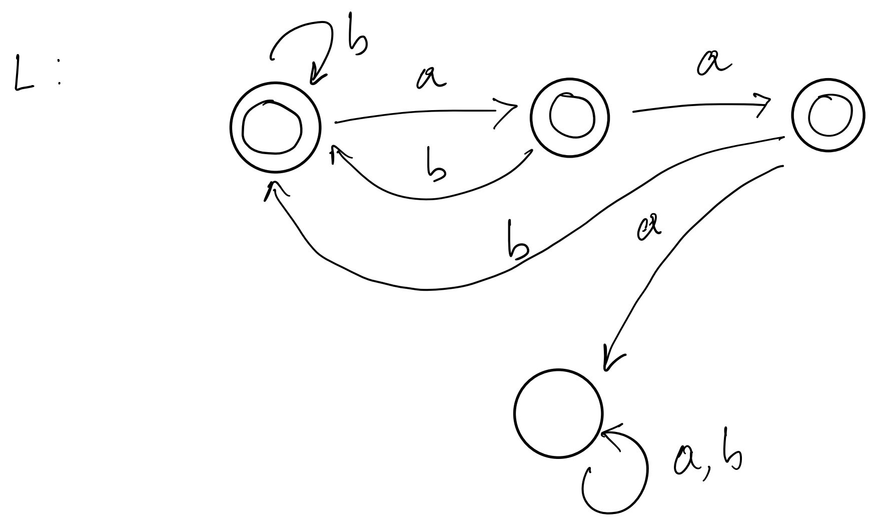
Problem 1. Recall the theorem we proved in the lecture - If a language A is recognized by an NFA, then A is regular. The way we proved this is as follows. Given an NFA M that recorgnizes A, we constructed a DFA which recognizes exactly A. We had left out the part where you need to take care of the ε-transitions. Modify the construction to accomodate this. What kind of states do you think you will add to your DFA for this? What will be the transitions etc...

Solution. Read from Sipser (3rd Edition) Theorem 1.39

Problem 2. Give a DFA over the alphabet a, b for each of the following:

a. Accepts strings which do not contain aaa as a substring.

Solution, The best way to solve this one is by eyeballing. What you need to realize is that there should be states to 'memorize' how many a's have been encountered so far.



An alternative way to solve this one would be to use the closure property under complements. This means, if a language L which is a subset of $\Sigma^\star$ is described by a DFA M, then the language $\tilde{L} = \Sigma^\star \setminus L$ is accepted by a DFA M' which has the same states and transitions as M, but the accept and non-accept states are reversed. You can use this to solve the above by first constructing a DFA for the language

$$\tilde{L} = \{w \in \Sigma^\star | w \text{ contains the substring } aaa\}$$

and then construct the complement DFA.

Additional Remark: The above fact does not hold for NFAs. Construct an example (that needs at least two states) to show that it does not. The solution here is trivial. Just think of an NFA where the alphabet is $\Sigma = \{a, b\}$ and it accepts only the single-length string a. An NFA for this will have two states : q_0, q_1. $q1$ is an accepting state. The only transition we allow is $\delta(q_0, a) = q_1$. Clearly reversing the accepting and non-accepting state does not make the NFA accept the language $\Sigma^\star \setminus a$

b. Accepts all strings which contain aab as a substring but do not contain aaa as a substring.

Solution. Usually, to solve these kind of exercises, you would be expected to break down the language in to simpler components. Here, the language L is an *intersection* of two different languages as follows

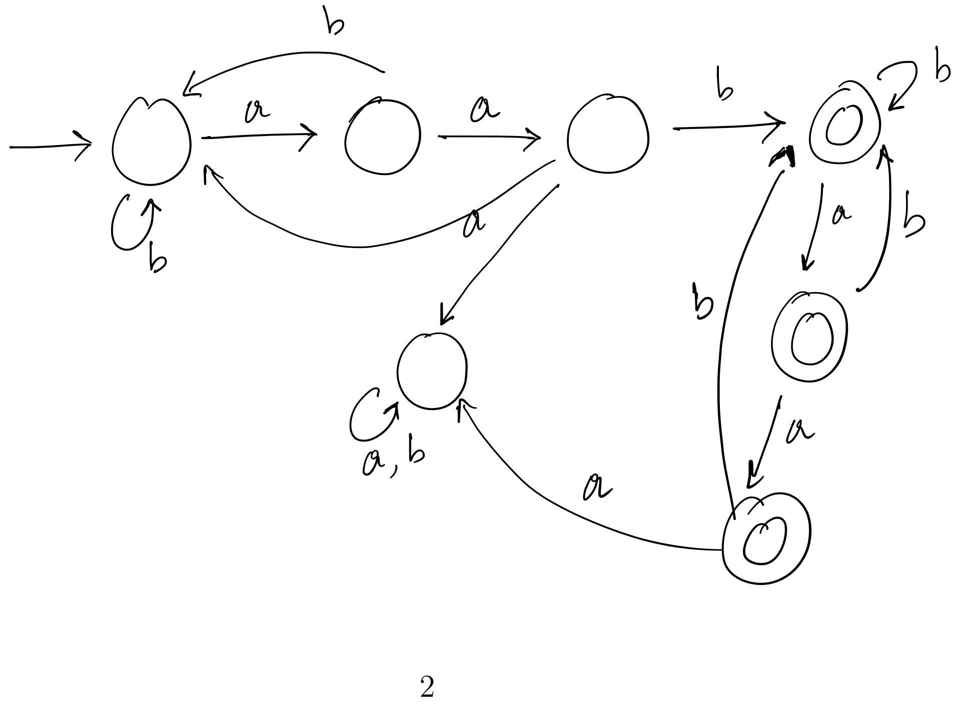
$$L_1 = \{w \in \Sigma^\star | w \text{ does not contain } aaa\}$$
$$L_2 = \{w \in \Sigma^\star | w \text{ contains } aab\}$$

So first we will construct the individual DFAs for L_1, say M_1 and L_2, say M_2 as follows.

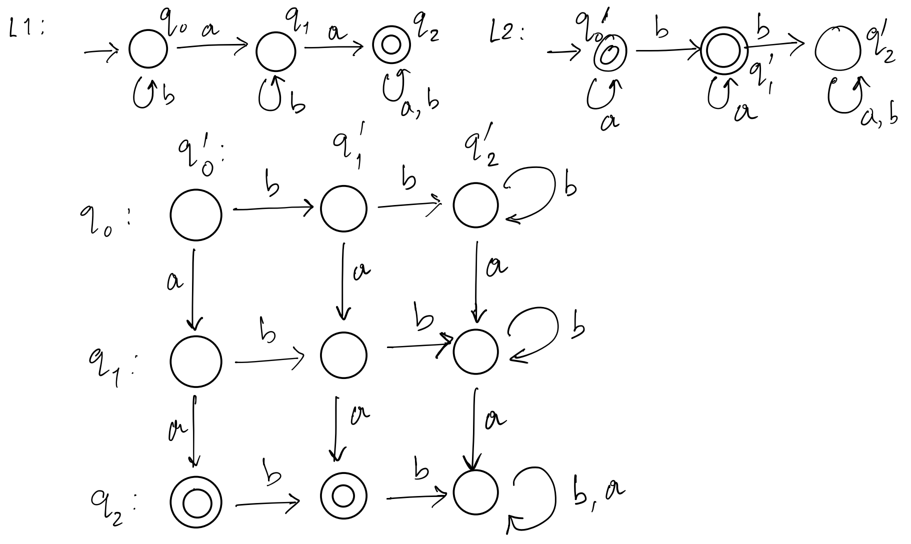
Now one option is to apply the same construction as the proof of union closure done in class. The only difference is, here, the accepting states would be those that contain accept states from both M_1 and M_2. The final construction will have 16 states. However, as you can imagine, this will be quite complicated, especially the transitions.

So for this case, its is better to not take the intersection blindly but try to imagine what is going to happen. One potential thought process is as follows. Let us first take the DFA for the L_2 since it is somewhat of an 'must include' condition. Now once this reaches the accepted state for L_2, all you need to make sure that you satisfy L_1 - in other words, no matter what symbols you see, you will remain in accepted state, unless - you see aaa. The overall construction is shown below.

$L = \{ w \in \{a,b\}^* \mid w$ contains aab but not aaa $\}$

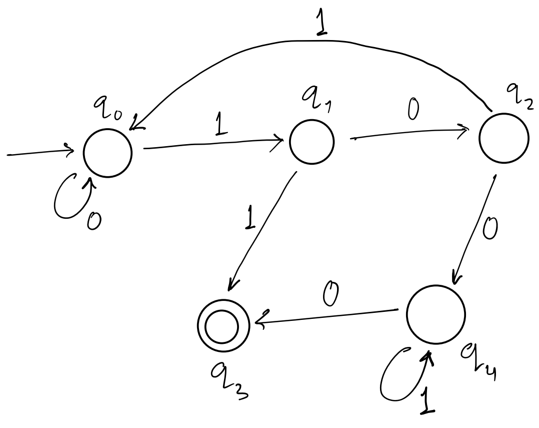


c. Accepts $L = \{w \in \{a, b\} | w \text{ contains at least two } a's \text{ and at most one } b's\}$ **Solution.** Apply the 'intersection' as above. For the second language, it might be easier to use the complementing technique.



d. Accepts all strings that are binary representations of numbers that are 3(mod 5), that is leaves a remainder 3 when divided by 5.

Solution. This one is slightly hard unless you know the trick. The idea is the following. For any non-negative integer x, $x \mod 5$ can take 5 possible values {0, 1, 2, 3, 4 }. We will now have 5 states in our DFA, $q_0, \cdots q_4$, each representing one of these 5 possibilities. The state $q_i, i = 0, 1, 2, 3, 4$ is supposed to 'memorize' the fact that the substring seen by the DFA so far is a binary string whose decimal is $i \mod 5$. Now the transitions should not be hard to figure out. Suppose you are at state q_i, Then $\delta q_i, 0$ 'means' that you multiply the number by 2. Hence, this should take you to the state that represents $2i \mod 5$. Similarly, $\delta q_i, 1$ takes you to the state that represents $(2i + 1) \mod 5$. Finally, it is obvious that q_3 is the accept state here.



Problem 3. Give a NFA for the following

$$L = \{wc | w \in \{0,1,2\}^{\star}, c \in 0,1,2 \text{ and c occurs in w } \}$$

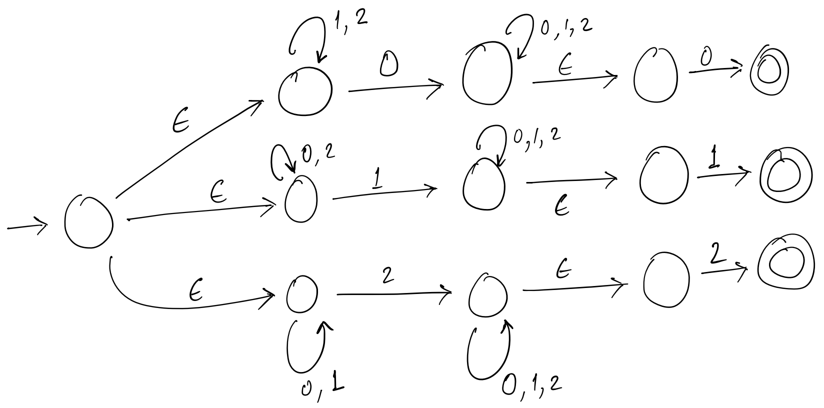
Solution. This one is very easy ones you know how to use the closure constructions properly. Again, it is easy to observe that $L$is a union of languages L_c, c=0,1,2 such that

$$L_c = \{wc | w \in \{0,1,2\}^{\star} \text{ and contains } c\}$$

Ok, now how to construct L_c ? Again, think of breaking it down in to concat of two languages - $\{wc | w \in \{0,1,2\}^{\star}$ and contains $c\}$ and $\{c\}$. Now construct an NFA for the first one. This is almost trivial to construct. You just need two states : q_0 and q_1, where q_1 is accepting. Now q_0 will have transitions labelled with everything other than c back to itself. $\delta(q_0, c) = q_1$. q_1 has all transitions pointing back to itself.

Now the NFA for $\{c\}$ is even easier. Again two states q_0, q_1 and just one transition out of q_0 to q_1 which is labelled with c. q_1 is the only accepting state, Note that you do not need all transitions for an NFA.

One you have the above pieces, concatenation is easy. Finally, take the union for $c = 0,1,2$.



Problem 4. For language L in (c) above, what is the language $L \circ L$?

Additional Problems : **Problem 5.** Suppose $\Sigma = \{a_1, a_2, \cdots a_k\}$. Given an NFA with $k+1$ states that accept the following language

$$L = \{w | \exists i, 1 \leq i \leq k, w \text{ does not contain } a_i\}$$

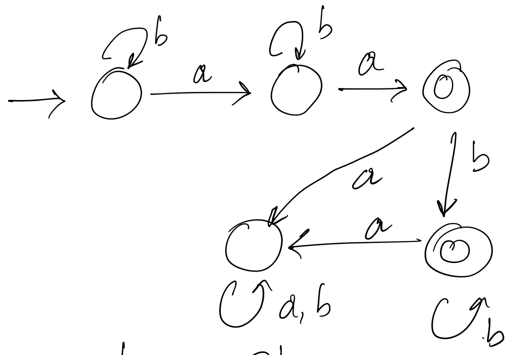
Solution. The trick is to again break this down in to smaller components. First design NFAs for the individual $i = 1, 2, \cdots k$ such that it does not accept string which contains a_i. This can be easily done with only one state. That state itself is the starting and final one. The only trick is that there would be *no transtions* labelled with a_i. This means, whenever it sees an a_i, the computation will be dead. Then just take the union of these - you just need one additional state.

Problem 6. Construct a DFA for

$$L = \{w \in \{a,b\}^{\star} | w \text{ does not contain exactly two } a\}$$

Solution Again, one can first construct a DFA for the complement language and then just swap the states.

$\tilde{L}$:



L :