Theory of Computation '23 Quiz 2
Time : 60 minutes

Name : **Roll :** **Marks :**

Problem 1. (5 points) Let A be the set of strings over $\{0,1\}$ that can be written in the form $1^k y$ where y contains at least k 1s, for some $k \geq 1$. Show that A is a regular language by constructing a regular expression that generates precisely A. Give an explanation of your construction in less than or equal to two sentences.

Solution. It is easy to see that any string in A must start with a 1, and contain at least one other 1 (in the matching y segment). Conversely, any string that starts with a 1 and contains at least one other 1 matches the description for k = 1. Hence, A is described by the regular expression

$$1 \cdot 0^{\star} \cdot 1 \cdot (0 \cup 1)^{\star}$$

, and is therefore regular.

(+3 for the correct regular expression and +2 for the short reasoning)

Problem 2. (10 points) Consider the language $A_m = \{1^{m-2}\}$ over the alphabet $\Sigma = \{0,1\}$.

1. Describe a DFA with m states that accepts A_m.
2. Show that there cannot exist a DFA with less than m states that accepts the above language.

Solution.

1. A DFA with m states which has states $0, \cdots, m-2$ for counting the number of 1s seen so far, and an additional "crash state" into which it enters on ever seeing a 0 or more than $m-2$ 1s, is a DFA with m states which recognizes this language.
2. On the other hand, any two strings of the form $1^i, 1^j$ for $0 \leq i < j \leq m-1$ are distinguished by 1^{m-2-i} and hence, any DFA must have at least m states

(+5 for each part)

Problem 3.

$$F = \{a^i b^j c^k \mid i,j,k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$$

1. (5 points) Show that F acts like a regular language in the pumping lemma i.e. for a pumping length $p = 2$, F satisfies the conditions of the lemma.

   Solution. The pumping lemma says that for any string s in the language, with length greater than the pumping length p, we can write $s = xyz$ with $\mid xy \mid \leq p$, such that $xy^i z$ is also in the language for every $i \geq 0$. We take p = 2. Consider any string $a^i b^j c^k$ in the language. If $i = 1 or i > 2$, we take $x = \varepsilon, y = a$. If $i = 1$, we must have $j = k$ and adding any number of a's still preserves the membership in the language. For $i > 2$, all strings obtained by pumping y as defined above, have two or more a's and hence are always in the language. For $i = 2$, we can take $x = \varepsilon$ and $y = aa$. Since the strings obtained by pumping in this case always have an even number of a's, they are all in the language. Finally, for the case $i = 0$, we take $x = \varepsilon, y = b$ if $j > 0$ and $y = c$ otherwise. Since strings of the form $b^j c^k$ are always in the language, we satisfy the conditions of the pumping lemma in this case as well. (+1 points for writing/understanding pumpimg lemma , +1 for each of the 4 cases)

2. (5 points) Show that F is not regular. We claim all strings of the form $ab^i, i = 0, 1, 2 \cdots$ must be pairwise distinguishable by F . This is because any two strings $ab^m, ab^n, m \neq n$ can be distinguished by c^m , since $ab^m c \in F$, while $ab^n c \notin F$. Since there are infinitely many pairwise distinguishable strings, we conclude by the Myhill-Nerode theorem that no DFA can recognize F. (+3 for the set, +2 for the correct distinguishing extension)