

Data Structures and Algorithms

Lecture 17

Aniket Basu Roy

BITS Pilani Goa Campus

2026-02-27 Fri

Agenda

Data Structures

Heaps

Heapsort

Priority Queues

Recap: Heaps

- ▶ A heap is a nearly complete binary tree stored as an array.
- ▶ Max-heap property: $A[\text{PARENT}(i)] \geq A[i]$ for all i .

MAX-HEAPIFY

The Problem

- ▶ $A[i]$ may be smaller than its children.
- ▶ But subtrees rooted at $\text{LEFT}(i)$ and $\text{RIGHT}(i)$ are valid max-heaps.

MAX-HEAPIFY "floats" $A[i]$ down to restore the max-heap property.

MAX-HEAPIFY

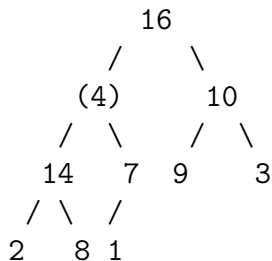
Pseudocode

MAX-HEAPIFY(A, i)

1. $l = \text{LEFT}(i)$
2. $r = \text{RIGHT}(i)$
3. if $l \leq A.\text{heap-size}$ and $A[l] > A[i]$
4. $\text{largest} = l$
5. else $\text{largest} = i$
6. if $r \leq A.\text{heap-size}$ and $A[r] > A[\text{largest}]$
7. $\text{largest} = r$
8. if $\text{largest} \neq i$
9. exchange $A[i]$ with $A[\text{largest}]$
10. MAX-HEAPIFY(A, largest)

MAX-HEAPIFY: Example

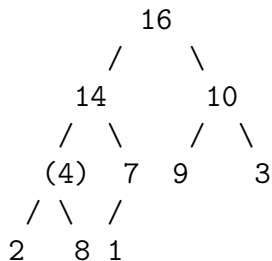
MAX-HEAPIFY(A, 2):



largest = LEFT(2) = 14, since $A[4] = 14 > A[2] = 4 \Rightarrow$ swap

MAX-HEAPIFY: Example (contd.)

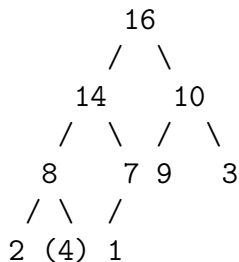
After swap, recurse MAX-HEAPIFY(A, 4):



$A[9] = 8 > A[4] = 4 \Rightarrow \text{swap } A[4] \text{ and } A[9]$

MAX-HEAPIFY: Example (contd.)

Final:



MAX-HEAPIFY: Time Complexity

- ▶ $O(1)$ work per level, recurse on a subtree.
- ▶ Subtree has at most $2n/3$ elements (worst case: bottom level half full).

Recurrence:

$$T(n) \leq T(2n/3) + \Theta(1)$$

By Master Theorem (Case 2):

$$T(n) = O(\lg n) = O(h)$$

BUILD-MAX-HEAP

Key Observation

Elements $A[\lfloor n/2 \rfloor + 1], \dots, A[n]$ are **leaves** — trivially valid max-heaps.

Call MAX-HEAPIFY only on internal nodes, bottom to top.

Pseudocode

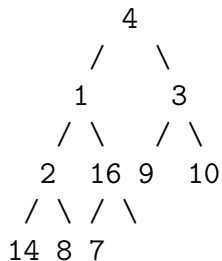
BUILD-MAX-HEAP(A, n)

1. $A.\text{heap-size} = n$
2. for $i = \text{floor}(n/2)$ downto 1
3. MAX-HEAPIFY(A, i)

BUILD-MAX-HEAP: Example

Input: [4, 1, 3, 2, 16, 9, 10, 14, 8, 7]

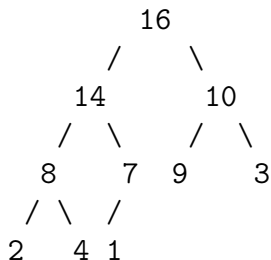
Initial tree:



$\text{floor}(10/2) = 5$, call MAX-HEAPIFY on $i=5,4,3,2,1$

BUILD-MAX-HEAP: Example

Result:



BUILD-MAX-HEAP: Time Complexity

► Naive: n calls $\times O(\lg n)$ each = $O(n \lg n)$ — **not tight**.

Tighter: nodes at height h cost $O(h)$; at most $\lceil n/2^{h+1} \rceil$ such nodes.

$$\sum_{h=0}^{\lceil \lg n \rceil} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) = O(n \cdot 2) = O(n)$$

(Using $\sum_{h=0}^{\infty} hx^h = \frac{x}{(1-x)^2}$ with $x = 1/2$.)

BUILD-MAX-HEAP runs in $\Theta(n)$ time.

Summary

Operation	Time Complexity
MAX-HEAPIFY	$O(\lg n)$
BUILD-MAX-HEAP	$\Theta(n)$

Heapsort: The Idea

- ▶ Build a max-heap from the input array. $(\Theta(n))$

Heapsort: The Idea

- ▶ Build a max-heap from the input array. $(\Theta(n))$
- ▶ The maximum element is at $A[1]$.

Heapsort: The Idea

- ▶ Build a max-heap from the input array. $(\Theta(n))$
- ▶ The maximum element is at $A[1]$.
- ▶ Swap $A[1]$ with $A[n]$, shrink the heap by one, restore with MAX-HEAPIFY.

Heapsort: The Idea

- ▶ Build a max-heap from the input array. $(\Theta(n))$
- ▶ The maximum element is at $A[1]$.
- ▶ Swap $A[1]$ with $A[n]$, shrink the heap by one, restore with MAX-HEAPIFY.
- ▶ Repeat until the heap has size 1.

Heapsort: Pseudocode

HEAPSORT(A, n)

1. BUILD-MAX-HEAP(A, n)
2. for i = n downto 2
3. exchange A[1] with A[i]
4. A.heap-size = A.heap-size - 1
5. MAX-HEAPIFY(A, 1)

Heapsort: Example Walkthrough

Input: [4, 1, 3, 2, 16, 9, 10, 14, 8, 7]

After BUILD-MAX-HEAP:

Heap: [16, 14, 10, 8, 7, 9, 3, 2, 4, 1]

i=10: swap A[1]=16 \leftrightarrow A[10]=1, heap-size=9

Heap: [14, 8, 10, 4, 7, 9, 3, 2, 1]

Sorted: [16]

i=9: swap A[1]=14 \leftrightarrow A[9]=1, heap-size=8

Heap: [10, 8, 9, 4, 7, 1, 3, 2]

Sorted: [14, 16]

...

Final: [1, 2, 3, 4, 7, 8, 9, 10, 14, 16]

Correctness: Loop Invariant

At the start of each iteration with index i :

- ▶ $A[1 : i]$ is a **max-heap** containing the i smallest elements of A .
- ▶ $A[i + 1 : n]$ contains the $n - i$ largest elements in **sorted order**.

Initialization

After BUILD-MAX-HEAP, $A[1 : n]$ is a max-heap; $A[n + 1 : n]$ is empty. ✓

Maintenance

$A[1]$ is maximum of $A[1 : i]$. Swap places it at $A[i]$.
MAX-HEAPIFY restores the heap on $A[1 : i - 1]$. ✓

Termination

$i = 1$: $A[2 : n]$ sorted, $A[1]$ is the minimum. Array fully sorted.

Time Complexity of Heapsort

Step	Cost	Invocations
BUILD-MAX-HEAP	$\Theta(n)$	1
MAX-HEAPIFY	$O(\lg n)$	$n - 1$
Total	$O(n \lg n)$	

Time Complexity of Heapsort

Step	Cost	Invocations
BUILD-MAX-HEAP	$\Theta(n)$	1
MAX-HEAPIFY	$O(\lg n)$	$n - 1$
Total	$O(n \lg n)$	

Lower Bound

Heapsort is comparison-based $\Rightarrow \Omega(n \lg n)$ worst case.

\Rightarrow Heapsort runs in $\Theta(n \lg n)$ time.

Properties of Heapsort

- ▶ **In-place**: $O(1)$ extra space beyond the input array.
- ▶ **Not stable**: equal elements may not preserve original order.

Properties of Heapsort

- ▶ **In-place**: $O(1)$ extra space beyond the input array.
- ▶ **Not stable**: equal elements may not preserve original order.

Comparison

Algorithm	Time	Space	Stable?
Merge Sort	$\Theta(n \lg n)$	$O(n)$	Yes
Quicksort	$O(n \lg n)$ exp.	$O(\lg n)$	No
Heapsort	$\Theta(n \lg n)$	$O(1)$	No

Priority Queues

Definition

A **max-priority queue** supports:

Operation	Description
INSERT(S, x, k)	Insert x with key k into S
MAXIMUM(S)	Return element with max key
EXTRACT-MAX(S)	Remove & return element with max key
INCREASE-KEY(S, x, k)	Increase the key of element x to k

HEAP-MAXIMUM

HEAP-MAXIMUM(A)

1. if A.heap-size < 1
2. error "heap underflow"
3. return A[1]

Time: $\Theta(1)$

HEAP-EXTRACT-MAX

HEAP-EXTRACT-MAX(A)

1. `max = HEAP-MAXIMUM(A)`
2. `A[1] = A[A.heap-size]`
3. `A.heap-size = A.heap-size - 1`
4. `MAX-HEAPIFY(A, 1)`
5. `return max`

Time: $O(\lg n)$

HEAP-INCREASE-KEY

Increase key of x to k (require $k \geq x.key$). Bubble x upward.

HEAP-INCREASE-KEY(A, x, k)

1. if $k < x.key$
2. error "new key is smaller than current key"
3. $x.key = k$
4. while $x \neq A[1]$ and $x.key > x.parent.key$
5. exchange x with $x.parent$
6. $x = x.parent$

Time: $O(\lg n)$ — at most height-many swaps.

MAX-HEAP-INSERT

```
MAX-HEAP-INSERT(A, key, n)
1.  if A.heap-size == n
2.      error "heap overflow"
3.  A.heap-size = A.heap-size + 1
4.  x = A[A.heap-size]
5.  x.key = -infinity
6.  HEAP-INCREASE-KEY(A, x, key)
```

Time: $O(\lg n)$

Summary: Priority Queue Operations

Operation	Time Complexity
HEAP-MAXIMUM	$\Theta(1)$
HEAP-EXTRACT-MAX	$O(\lg n)$
HEAP-INCREASE-KEY	$O(\lg n)$
MAX-HEAP-INSERT	$O(\lg n)$