1 Adjoining Dynamics

It has been shown that for the Hegselmann-Krause dynamics there exists a sequence of stochastic vectors $\{\pi(t)\}$ such that:

$$\pi^{'}(t) = \pi^{'}(t+1)A(t) \quad \forall t \geq 0$$

We can view the above dynamics as the adjoint for the original HK dynamics. Note that this dynamics evolves backward in time. We can derive the relation between HK trajectory $\{x(t)\}$ and adjoining stochastic vectors $\{\pi(t)\}$ as:

$$\pi^{'}(t+1)x(t+1) = \pi^{'}(t)x(t) \quad \forall t \geq 0$$

It can be seen that the adjoint process for the Hegselmann- Krause dynamics is not unique in general. We will construct one such process for later use.

$$\pi^{'}(t) = \pi^{'}(t+1)A(t) \quad \forall \quad 0 \leq t < T$$
$$\pi^{'}(t) = \pi^{'}(t+1) \quad \forall t \geq T$$

One of the fundamental concepts and properties of the Hegselmann-Krause dynamics that will be used extensively is that such dynamics admits a quadratic time-varying Lyapunov function which is non decreasing along the trajectory of the dynamics.

$$V(t) = \sum_{i=1}^{n} \pi_i(t) \|x_i(t) - \pi^{'}(t)x(t)\|^2$$

What can be shown about this Lyapunov function is:

$$V(t) - V(t+1) \geq \frac{1}{2n} \sum_{i=1}^{n} \pi_i(t+1) d_i^2(t)$$

where:
$d_i(t) := \max\{\|x_p(t) - x_q(t)\| : p, q \in N_i(t)\}$ Suppose we partition the set of vertices into:
$S_0(t) := \{i : d_i(t) = 0\}$
$S_1(t) := \{i : 0 < d_i(t) < \frac{1}{n}\}$
$S_2(t) := \{i : d_i(t) \geq \frac{1}{n}\}$
Then the above the decrement of the Lyapunov function can be written as:

$$V(t) - V(t+1) \geq \frac{1}{2n^3} \sum_{i \in S_2(t)} \pi_i(t+1)$$

To use this analysis to prove an upper bound on the convergence of HK dynamics, we have to upper and lower bound the following quantities:

$$\sum_{i \in S_0(t)} \pi_i(t+1) \leq value_1$$

$$\sum_{i \in S_1(t)} \pi_i(t+1) \leq value_2$$

$$\sum_{i \in S_2(t)} \pi_i(t+1) \geq 1 - (value_1 + value_2) = value_3$$

Then we get:

$$V(t) - V(t+1) \geq \frac{1}{2n^3} value_3$$

The current bound for $value_3 + value_2 > \frac{1}{(1+\frac{|S_0(t)|}{h})^h}$ Finally after doing some messy calculations we get

$$\frac{1}{8n^4} \sum_{t=0}^{T_n} (\frac{1}{2})^{|S_0(t)|} \leq V(0) - V(T_n) \leq V(0)$$

We can hope to do better for one dimension, why ???? because:
1) Can assume $|S_0(t)| = 0$, why ??.
Can actually get $O(n^4)$ result by this method for 1-dimension (think!, Hint: need to change the definition of $S_1(t)$ and $S_2(t)$).

Just the technique on how you prove bound on these

$$\sum_{i \in S_1(t)} \pi_i(t+1) \leq value_2$$

Consider any $j \in S_1(t)$, what is $\pi_j(t+1)$, it is exactly:

$$\pi_j(t+1) = \sum_{k \in N_j(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|}$$

Express $N_j(t+1)$ in terms of $S_2(t)$(is possible).