2013 American Control Conference (ACC)
Washington, DC, USA, June 17-19, 2013

Termination Time of Multidimensional Hegselmann-Krause Opinion Dynamics*

Seyed Rasoul Etesami, Tamer Başar, Angelia Nedić, Behrouz Touri
Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801
Email: (etesami1, basar1, angelia, touri1)@illinois.edu

***Abstract*— We consider the Hegselmann-Krause model for opinion dynamics in higher dimensions. Our goal is to investigate the termination time of these dynamics, which has been investigated for a scalar case, but remained an open question for dimensions higher than one. We provide a polynomial upper bound for the termination time of the dynamics when the connectivity among the agents maintains a certain structure. Our approach is based on the use of an adjoint dynamics for the Hegselmann-Krause model and a Lyapunov comparison function that is defined in terms of the adjoint dynamics.**

***Index Terms*— Multidimensional Hegselmann-Krause model, opinion dynamics, non-linear time-varying dynamics, discrete time dynamics.**

I. INTRODUCTION

With the appearance of myriads of online social networks and availability of huge data sets, modeling of the opinion dynamics in a social network has gained a lot of attention in the recent years. Among many problems that arise in such an application, is the problem of modeling information diffusion over social networks. One of the models that addresses such dynamics is the Hegselmann-Krause model which is introduced in [1]. The same model has also been used for distributed rendezvous in a robotic network [2], [3].

It is known that, for any initial profile and any (uniform) confidence bound, the Hegselmann-Krause dynamics will terminate after finitely many steps [1], [4], [5]. However, depending on the initial profile and the confidence bound, the final state may or may not be a consensus. The existing studies on the behavior of the Hegselmann-Krause model in one dimension when the opinions of agents are scalars can be found in [6], [7]. It was shown in [2] that the termination time of the Hegselmann-Krause dynamics in one dimension is at least $O(n)$ and at most $O(n^5)$ and later in [7], the upper bound is shown to be $O(n^4)$.

In this paper, we consider the Hegselmann-Krause model in $\mathbb{R}^d$, where $d \geq 2$. Other than applications in modeling of opinion dynamics, the model has applications in the robotics rendezvous problem [2] in the plane and space. However, unlike the scalar the case, due to intrinsic complexity of the dynamics, a bound for the termination time of these dynamics is not known. This motivates us to consider the multidimensional Hegselmann-Krause dynamics and develop an upper bound for its termination time.

This paper is organized as follows: in Section II, we review the Hegselmann-Krause dynamics and some of its basic properties. In Section III, develop some preliminary results for later use. In Section IV, we present our main results providing some upper bounds for the termination time in higher dimensions. We conclude the paper in Section V.

Notations: For a vector $v \in \mathbb{R}^n$, we let v_i to be the ith entry of v. We say that v is stochastic if $v_i \geq 0$ for all $i \in [n]$ and $\sum_{i=1}^n v_i = 1$. Similarly, for a matrix A, we let A_{ij} to be the ijth entry of A. We say that A is stochastic (or row-stochastic) if each of its rows are stochastic. We denote the ith row of A by A_i. We denote the transpose of a matrix A by A', and the Euclidean norm of a vector v by $\|v\|$.

II. HEGSELMANN-KRAUSE DYNAMICS

In this section we describe the discrete-time Hegselmann-Krause opinion dynamics model as discussed in [1].

Let us assume that we have a set of n agents $[n] = \{1, \ldots, n\}$ and we want to model the interactions among their opinions. It is assumed that at each time $t = 0, 1, 2, \ldots$, the opinion of agent $i \in [n]$ can be represented by a vector $x_i(t) \in \mathbb{R}^d$ for some $d \geq 1$. According to this model, the evolution of opinion vectors can be modeled by the following discrete-time dynamics:

$$x(t+1) = A(t, x(t), \epsilon(t))x(t), \tag{1}$$

where $A(t, x(t), \epsilon(t))$ is an $n \times n$ row-stochastic matrix soon to be determined and $x(t)$ is the $n \times d$ matrix such that its ith row contains the opinion of the ith agent at time $t = 0, 1, 2, \ldots$, i.e., it is equal to $x_i(t)$. We refer to $x(t)$ as the *opinion profile* at time t. The entries of $A(t, x(t), \epsilon(t))$ are functions of time step t, current profile $x(t)$ and a parameter $\epsilon(t) > 0$.

In Hegselmann-Krause model, each agent updates its value at time $t = 0, 1, 2, \ldots$ by averaging its own value at time t and all the other agents which are in its $\epsilon(t)$-neighborhood. To be more specific, given a profile $x(t)$ at time t, define the matrix $A(t, x(t), \epsilon(t))$ in (1) by:

$$A(t, x(t), \epsilon(t)) = \begin{cases} \frac{1}{|\mathcal{N}_i(x(t), \epsilon(t))|} & \text{if } j \in \mathcal{N}_i(x(t)), \\ 0 & \text{else}, \end{cases} \tag{2}$$

*Research supported in part by the "Cognitive & Algorithmic Decision Making" project grant through the College of Engineering of the University of Illinois, and in parts by AFOSR MURI Grant FA 9550-10-1-0573 and NSF grant CCF 11-11342.

where $|S|$ represents the cardinality of the set S, and $\mathcal{N}_i(x(t), \epsilon(t))$ is the set of agents in the $\epsilon(t)$-neighborhood of agent i, i.e.

$$\mathcal{N}_i(x(t), \epsilon(t)) = \{j \in [n] \mid \|x_i(t) - x_j(t)\| \leq \epsilon(t)\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

The parameter $\epsilon(t)$ is referred to as the *confidence bound* or *observation radius* at time t. Throughout this paper, our focus is on the homogeneous case of the dynamics, i.e., the case where $\epsilon(t) = \epsilon$ for some $\epsilon > 0$ and all $t = 0, 1, 2, \ldots$. Also, for the sake of simplicity of notation we drop the dependency of $\mathcal{N}_i(x, \epsilon)$ on ϵ. Moreover, for a fixed $x(0) \in \mathbb{R}^{n\times d}$, we simply write $\mathcal{N}_i(t)$ instead of $\mathcal{N}_i(x(t))$. Similarly, we drop the dependency of $A(t)$ on $x(t)$ and ϵ.

According to the above setting, the Hegselmann-Krause dynamics can be simply written as

$$x(t+1) = A(t)x(t), \qquad \text{for all } t \geq 0,$$
$$A(t) = \begin{cases} \frac{1}{|\mathcal{N}_i(t)|} & \text{if } j \in \mathcal{N}_i(t), \\ 0 & \text{else.} \end{cases}$$

By the definition of $A(t)$, one can easily see that the positive entries in each row of $A(t)$ are the same.

III. Preliminary Results

In this section, we briefly discuss some preliminary results which will be used to prove our main results. We start our discussion by defining some notations that will be used throughout this work.

Definition 1: For a fixed bound of confidence $\epsilon > 0$ and for any time instance $t = 0, 1, 2, \ldots$, we define:

$$d_i(t) = \max\{\|x_p(t) - x_q(t)\| : p, q \in \mathcal{N}_i(t)\},$$
$$S_0(t) = \{i : d_i(t) = 0, \}$$
$$S_1(t) = \{i : 0 < d_i(t) < \tfrac{\epsilon}{n}\},$$
$$S_2(t) = \{i : d_i(t) \geq \tfrac{\epsilon}{n}\}.$$

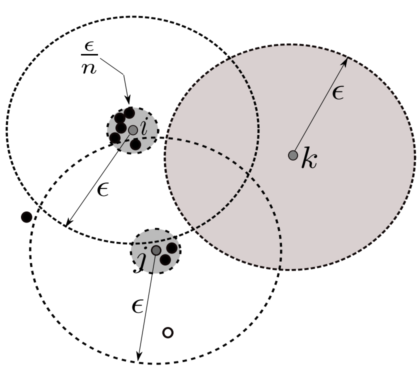

Fig. 1. Three different types of agents: $k \in S_0(t),\ i \in S_1(t),\ j \in S_2(t)$.

From the above definition, it follows that $S_0(t)$, $S_1(t)$ and $S_2(t)$ partition the set $[n]$ for all $t = 0, 1, 2, \ldots$.

One of the fundamental concepts and properties of the Hegselmann-Krause dynamics that will be used extensively throughout this work is that such dynamics admits a quadratic time-varying Lyapunov function. This Lyapunov function was introduced and studied in [5], [7], [8], [9], where it has been shown that, associated with the trajectories $\{x(t)\}$ of the dynamics, there exists a sequence $\{\pi(t)\} \subset \mathbb{R}^n$ of stochastic vectors such that the function $V(t)$, given by

$$V(t) = \sum_{i=1}^{n} \pi_i(t)\|x_i(t) - \pi'(t)x(t)\|^2, \tag{3}$$

is non-increasing along the trajectories of the dynamics. Moreover, the decrease in the Lyapunov function can be estimated as stated in the following lemma.

Lemma 1: For the quadratic Lyapunov function defined in (3), we have for all $t \geq 0$,

$$V(t) - V(t+1) \geq \frac{1}{2n}\sum_{i=1}^{n} \pi_i(t+1)d_i^2(t).$$

Proof: The result follows by using the same line of analysis as in the proof of Lemma 2 in [7]. ■

In analyzing the Hegselmann-Krause dynamics, the agents in the set $S_2(t)$ are favorite agents as they have a relatively wide local spread of opinion. To see this, using the quadratic Lyapunov function and Lemma 1, we have for all $t \geq 0$,

$$V(t) - V(t+1) \geq \frac{1}{2n}\sum_{i=1}^{n} \pi_i(t+1)d_i^2(t)$$
$$\geq \frac{1}{2n}\sum_{i\in S_2(t)} \pi_i(t+1)d_i^2(t) \geq \frac{\epsilon^2}{2n^3}\sum_{i\in S_2(t)} \pi_i(t+1),$$

and hence, the following corollary holds.

Corollary 1: Suppose that $V(t)$ is the Lyapunov function defined in (3); then, for all $t \geq 0$,

$$V(t) - V(t+1) \geq \frac{\epsilon^2}{2n^3}\sum_{i\in S_2(t)} \pi_i(t+1).$$

Next, we show that if i is in $S_1(t)$ for some time t, then its neighborhood increases at time $t+1$. This follows from the uniform averaging property of the Hegselmann-Krause dynamics.

Lemma 2: If $d_i(t) \leq \frac{\epsilon}{n}$ for some time instant t, then $\mathcal{N}_i(t) \subseteq \mathcal{N}_i(t+1)$.

Proof: Let $d_i(t) \leq \frac{\epsilon}{n}$ for some $i \in [n]$ and $t = 0, 1, 2, \ldots$. It suffices to show that for an arbitrary element $j \in \mathcal{N}_i(t)$, we have $j \in \mathcal{N}_i(t+1)$. To show this, let us assume that $|\mathcal{N}_j(t) \setminus \mathcal{N}_i(t)| = r$ and $|\mathcal{N}_i(t)| = m$ and without loss of generality, assume that the labeling of agents are such that $i, j > r + m - 2$, $\mathcal{N}_j(t) \setminus \mathcal{N}_i(t) = \{1, \ldots, r\}$, and $\mathcal{N}_i(t) = \{r+1, \ldots, r+m-2, i, j\}$. Therefore, by applying

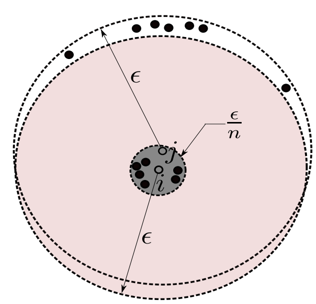

Fig. 2. A configuration of agents for illustration of Lemma 2.

the dynamic rule we obtain

$$x_i(t+1) = \frac{1}{m}\left(x_i(t) + x_j(t) + \sum_{k=r+1}^{r+m-2} x_k(t)\right),$$

$$x_j(t+1) = \frac{1}{m+r}\left(x_i(t) + x_j(t) + \sum_{k=1}^{r+m-2} x_k(t)\right).$$

Thus, we have:

$$\begin{aligned}&\|x_j(t+1) - x_i(t+1)\| \\ &= \frac{1}{m(m+r)}\Big\| \sum_{\ell=1}^{r}(x_i(t) - x_\ell(t)) + \sum_{\ell=1}^{r}(x_j(t) - x_\ell(t)) + \\ &\quad + \sum_{k=r+1}^{r+m-2}\sum_{\ell=1}^{r}(x_k(t) - x_\ell(t)) \Big\|. \end{aligned} \tag{4}$$

For the terms involved in (4) we can see that

$$\begin{aligned}&\sum_{\ell=1}^{r}\|x_i(t) - x_\ell(t)\| \le r(\epsilon + \frac{\epsilon}{n}),\\ &\sum_{\ell=1}^{r}\|x_j(t) - x_\ell(t)\| < r\epsilon,\\ &\sum_{k=r+1}^{r+m-2}\sum_{\ell=1}^{r}\|x_\ell(t) - x_k(t)\| \le (m-2)r\left(\epsilon + \frac{\epsilon}{n}\right). \end{aligned} \tag{5}$$

Therefore, by applying the triangle inequality on the right hand side of (4) and summing up both sides of the in inequalities (5), we obtain:

$$\|x_j(t+1) - x_i(t+1)\| \le \left(\frac{rm + \frac{r}{n}(m-1)}{rm + m^2}\right)\epsilon < \epsilon,$$

which follows from the fact that $\frac{r}{n} \le 1$. ■

In the next few intermediate results, we establish some relations for the sets $S_0(t), S_1(t)$ and $S_2(t)$. This allows us to analyze the variation in the entries of the sequence $\{\pi(t)\}$ among these sets and, hence, come up with a bound on the decrease rate for the Lyapunov function $V(t)$ in (3).

Lemma 3: Suppose that $\ell_0 \in S_0(t)$ and $\ell_0 \notin S_0(t+1)$; then $\ell_0 \in S_2(t+1)$.

Proof: Note that if $\ell_0 \in S_0(t)$, then $x_{\ell_0}(t+1) = x_{\ell_0}(t)$. Thus, for every $\ell \in [n]$ we have

$$\|x_\ell(t+1) - x_{\ell_0}(t+1)\| = \|x_\ell(t+1) - x_{\ell_0}(t)\|.$$

Note that $|\mathcal{N}_{\ell_0}(t+1)| > 1$ since $\ell_0 \notin S_0(t+1)$, so it follows that $\mathcal{N}_{\ell_0}(t+1) \setminus \{\ell_0\} \neq \emptyset$. Consider an arbitrary $\ell_1 \in \mathcal{N}_{\ell_0}(t+1)$. Then, we further have

$$\|x_{\ell_1}(t+1) - x_{\ell_0}(t+1)\| = \|x_{\ell_1}(t+1) - x_{\ell_0}(t)\|. \tag{6}$$

Also,

$$\begin{aligned}&\|x_{\ell_1}(t+1) - x_{\ell_1}(t)\| = \|\frac{\sum_{k\in N_{\ell_1}(t)\setminus\{\ell_1\}}(x_k(t) - x_{\ell_1}(t))}{|N_{\ell_1}(t)|}\| \\ &\le \frac{\sum_{k\in N_{\ell_1}(t)\setminus\{\ell_1\}}\|x_k(t) - x_{\ell_1}(t)\|}{|N_{\ell_1}(t)|} \le \frac{|N_{\ell_1}(t)|-1}{|N_{\ell_1}(t)|}\epsilon \le \frac{n-1}{n}\epsilon. \end{aligned} \tag{7}$$

Therefore, by combining (6) and (7) and using triangle inequality we get:

$$\begin{aligned}&\|x_{\ell_1}(t+1) - x_{\ell_0}(t)\| \ge \|x_{\ell_1}(t) - x_{\ell_0}(t)\| \\ &\quad - \|x_{\ell_1}(t+1) - x_{\ell_1}(t)\| \ge \epsilon - \frac{n-1}{n}\epsilon = \frac{\epsilon}{n}. \end{aligned} \tag{8}$$

Using (8) in (6) shows that $\|x_{\ell_1}(t+1) - x_{\ell_0}(t+1)\| \ge \frac{\epsilon}{n}$, hence $\ell_0 \in S_2(t+1)$, which follows from the fact that $\ell_0 \in S_0(t)$. This shows that $d_{\ell_0}(t+1) \ge \frac{\epsilon}{n}$, i.e. $\ell_0 \in S_2(t+1)$. ■

The next step in our development is to analyze the dynamics at some non-critical time instances. For this we define merging times of the dynamics.

Definition 2: We say that a time instance t is a merging time for the dynamics if there are two different agents $i \neq j$ such that $\mathcal{N}_i(t-1) \neq \mathcal{N}_j(t-1)$ and $\mathcal{N}_i(t) = \mathcal{N}_j(t)$. We also say that at such a time the agents i and j merge.

Based on this definition, we can see that if two agents i and j merge at time instant t, then they will have the same opinion at time $t+1$ and onward, while their common opinion may be varying with time. Moreover, prior to the termination time of the dynamics, we cannot have more than n merging times since there are n agents in the model.

Lemma 4: Suppose that t is not a merging time. Then, for every $j \in S_1(t)$, we have:

$$\mathcal{N}_j(t+1) \subset \Big(S_0(t) \cap S_2(t+1)\Big) \cup \Big(\cup_{i\in S_2(t)} \mathcal{N}_i(t+1)\Big). \tag{9}$$

Proof: Assume that $r \in \mathcal{N}_j(t+1)$. If $r \in S_0(t)$ and at time $t+1$ we have $r \in \mathcal{N}_j(t+1)$, then it follows that $r \notin S_0(t+1)$. Thus by Lemma 3, we have $r \in S_2(t+1)$ and $r \in S_0(t) \cap S_2(t+1)$.

Now, suppose that $r \notin S_0(t)$. Therefore, $r \in S_1(t) \cup S_2(t)$. If $r \in S_2(t)$, it follows that $r \in \cup_{i\in S_2(t)}\mathcal{N}_i(t+1)$. If $r \in S_1(t)$, we claim that for every $q \in N_r(t)$, $q \neq r$ we have $q \in S_2(t)$. In fact, this is true because q has at least one neighbor which is r, hence, $q \notin S_0(t)$. Also, $q \notin S_1(t)$, otherwise, q and r will see the same neighbours at time t and it means that they will merge in the next step. Since we assumed that t is not a merging time, the only option is that $q \in S_2(t)$. Furthermore, using Lemma 2 we see that q will remain as a neighbor of r at time $t+1$ and hence, $r \in \cup_{i\in S_2(t)}\mathcal{N}_i(t+1)$. ■

Now, we turn our attention to the adjoint dynamics. Let $\{\pi(t)\}$ be an adjoint dynamics for $\{x(k)\}$, i.e.,

$$\pi'(t+1)A(t) = \pi'(t) \qquad \text{for } t = 0, 1, 2, \ldots,$$

which exists as proven in [10]. The following lemma assists us to bound the decrease in the quadratic Lyapunov function.

Theorem 1: Suppose that t is not a merging time, then

$$\sum_{j\in S_1(t)}\pi_j(t+1) \le |S_1(t)|\Big(\sum_{i\in S_2(t)}\pi_i(t+1) + \sum_{i\in S_2(t+1)}\pi_i(t+2)\Big).$$

Proof: Using Lemma 4, for $j \in S_1(t)$, we have

$$\begin{aligned}
\pi_j(t+1) &= \sum_{k\in\mathcal{N}_j(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} \\
&\leq \sum_{k\in(S_0(t)\cap S_2(t+1))\cup_{i\in S_2(t)}\mathcal{N}_i(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} \\
&\leq \sum_{k\in S_0(t)\cap S_2(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} + \sum_{k\in\cup_{i\in S_2(t)}\mathcal{N}_i(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} \\
&\leq \sum_{k\in S_0(t)\cap S_2(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} + \sum_{i\in S_2(t)}\sum_{k\in\mathcal{N}_i(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} \\
&= \sum_{k\in S_0(t)\cap S_2(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} + \sum_{i\in S_2(t)} \pi_i(t+1).
\end{aligned} \tag{10}$$

On the other hand, we observe that

$$\sum_{k\in S_0(t)\cap S_2(t+1)} \frac{\pi_k(t+2)}{|N_k(t+1)|} \leq \sum_{k\in S_0(t)\cap S_2(t+1)} \pi_k(t+2). \tag{11}$$

Combining (10) and (11), it follows that

$$\begin{aligned}
\pi_j(t+1) &\leq \sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{k\in S_0(t)\cap S_2(t+1)} \pi_i(t+2) \\
&\leq \sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{k\in S_2(t+1)} \pi_i(t+2).
\end{aligned}$$

Therefore, by summing up both sides of the above inequality for all $j \in S_1(t)$, the assertion follows. ∎

A. Estimation on the Lyapunov Function Decrease

As stated in Corollary 1, the decrease in the quadratic Lyapunov function is proportional to $\sum_{i\in S_2(t)} \pi_i(t+1)$ and our aim is to provide bounds on this quantity.

Theorem 2: Let $T \geq 2$ and t_0 be such that there is no merging time in the time interval $[t_0, t_0+T]$. Then, we have:

$$V(t_0) - V(t_0+T) \geq \frac{\epsilon^2}{4n^4}\Big(\sum_{t=t_0}^{t_0+T-2}(1 - \sum_{i\in S_0(t)} \pi_i(t))\Big).$$

Proof: We prove the theorem by induction. Let $T \geq 2$ and $t \in [t_0, T-2]$ be arbitrary. Using Corollary 1, for time t and $t+1$, we obtain

$$\begin{aligned}
V(t)-V(t+2) &\geq \frac{\epsilon^2}{2n^3}\Big(\sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{i\in S_2(t+1)} \pi_i(t+2)\Big) \\
&= \frac{\epsilon^2}{2n^3(1+|S_1(t)|)}\left(\sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{i\in S_2(t+1)} \pi_i(t+2)\right. \\
&\qquad \left. + |S_1(t)|\Big(\sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{i\in S_2(t+1)} \pi_i(t+2)\Big)\right).
\end{aligned}$$

By Theorem 1, we have

$$|S_1(t)|\Big(\sum_{i\in S_2(t)} \pi_i(t+1) + \sum_{i\in S_2(t+1)} \pi_i(t+2)\Big) \geq \sum_{i\in S_1(t)} \pi_i(t+1),$$

thus implying

$$\begin{aligned}
V(t) - V(t+2) &\geq \frac{\epsilon^2}{2n^3(1+|S_1(t)|)}\Big(\sum_{i\in S_1(t)\cup S_2(t)} \pi_i(t+1) \\
&\qquad + \sum_{i\in S_2(t+1)} \pi_i(t+2)\Big) \\
&\geq \frac{\epsilon^2}{2n^4}\Big((1 - \sum_{i\in S_0(t)} \pi_i(t+1)) + \sum_{i\in S_2(t+1)} \pi_i(t+2)\Big).
\end{aligned}$$

The last inequality holds since $\pi(t+1)$ is a stochastic vector and $1 + |S_1(t)| \leq n$. Therefore, we have shown so far that

$$V(t) - V(t+2) \geq \frac{\epsilon^2}{2n^4}\left(1 - \sum_{i\in S_0(t)} \pi_i(t+1)\right).$$

By summing up the above inequalities for odd and even time instances t, we obtain

$$2(V(t_0) - V(t_0+T)) \geq \frac{\epsilon^2}{2n^4}\Big(\sum_{t=t_0}^{t_0+T-2}(1 - \sum_{i\in S_0(t)} \pi_i(t+1))\Big), \tag{12}$$

and hence, the result follows. ∎

Now, we define the notion of termination time over all the initial profiles.

Definition 3: For every $n \geq 1$ we define the uniform termination time, or simply termination time T_n to be the maximum number of iterations before reaching to steady state over all the initial profiles when the number of agents is n.

Lemma 5: Suppose that we start the Hegselmann-Krause model over a set of n with the initial profile $x(0)$, then, either $V(0) \leq n^2\epsilon^2$ or there exist at least two groups of agents which will stay separated from each other until the termination of the dynamics.

Proof: By the definition of $V(\cdot)$ and the fact that $\pi(0)$ is a stochastic vector we have:

$$\begin{aligned}
V(0) &= \sum_{i=1}^{n} \pi_i(0)\|x_i(0) - \pi'(0)x(0)\|^2 \\
&\leq \max_{i,j} \|x_i(0) - x_j(0)\|^2.
\end{aligned}$$

Without loss of generality assume that $x_1(0)$ and $x_n(0)$ have the maximum distance. For each $i \in [n]$ let $y_i(0)$ be the projection of $x_i(0)$ on the line segment connecting $x_1(0)$ to $x_n(0)$. By moving from $x_1(0)$ toward $x_n(0)$ on the line segment $[x_1(0), x_n(0)]$ we start to relabel $y_i(0)$ in an increasing order. In this relabling if multiple of them collide on the same point we pick only one of them arbitrarily. Let us assume this relabling to be $x_1(0) = y_{\ell_1}(0), y_{\ell_2}(0), \ldots, y_{\ell_s}(0) = x_n(0)$, for some $1 \leq s \leq n$. The reason that we use s instead of n is that there might be some agents whose projections are out of the line segment $[x_1(0), x_n(0)]$ or their projections are essentially one point and hence, they did not appear in relabling. If the distance between every two consecutive

$y_{\ell_j}(0), j = 1, \ldots, s$ is less than ϵ, then it follows that:

$$\|x_n(0) - x_1(0)\| = \|y_{\ell_s}(0) - y_{\ell_1}(0)\|$$
$$= \|\sum_{j=1}^{s-1}(y_{\ell_{j+1}}(0) - y_{\ell_j}(0))\| \le s\epsilon \le n\epsilon,$$

where the last inequality follows by the triangle inequality. Therefore, in this case, $V(0) \le n^2\epsilon^2$.

Otherwise, there exists $i^* \in [n-1]$ such that $y_{\ell_{i^*+1}}(0), y_{\ell i^*}(0)$ are two consecutive projection points on this line but $\|y_{\ell_{i^*+1}}(0) - y_{\ell i^*}(0)\| > \epsilon$. Let $\mathcal{P}_1$ be a hyperplane which includes $x_{\ell i^*}(0)$ and $y_{\ell i^*}(0)$ and is perpendicular to the line segment $[x_1(0), x_n(0)]$. Similarly let $\mathcal{P}_2$ be a hyperplane which includes $x_{\ell_{i^*+1}}(0)$ and $y_{\ell_{i^*+1}}(0)$ and is perpendicular to the line segment $[x_1(0), x_n(0)]$. Note that $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel and hence they divide the whole space into two disjoint sets. Furthermore, since $\|y_{\ell_{i^*+1}}(0) - y_{\ell i^*}(0)\| > \epsilon$, hence, the gap between $\mathcal{P}_1$ and $\mathcal{P}_2$ is at least ϵ. This means that the convex hull of the agents in either sides of $\mathcal{P}_1$ and $\mathcal{P}_2$ are separated from each other by a distance of at least ϵ. Since running the dynamics will shrink each of these two convex hulls, the gap between them will always remain more than ϵ. This proves that these two components will stay separated for the rest of the time. ■

Lemma 5 assures that if $V(0)$ is greater than $n^2\epsilon^2$, we can analyze the Hegselmann-Krause dynamics over two smaller sets of agents.

In what follows, we will use this result to derive an upper bound for the termination time.

IV. Main Results

We start this section by the following lemma.

Lemma 6: For any $t \ge 0$, we have:

$$\sum_{j\in S_0^c(t)} \pi_j(t) = \sum_{j\in S_0^c(t)} \pi_j(t+1),$$

where S^c is the complement of the set S.

Proof: The proof follows by summing the sides of $\pi'(t+1)A(t) = \pi'(t)$ over $j \in S_0^c(t)$ and stochasticity of $A(t)$. ■

Now let us assume that all the elements of $S_0(t)$ will leave this set as t passes. In other words, there exists an integer T such that $\cap_{t'=t}^{t+T} S_0(t') = \emptyset$ for all $t = 0, 1, 2, \ldots$. Moreover, assume that at time t, $S_0(t)$ can be partitioned into h clusters of agents with the same opinion. The reason that we consider clusters of singletons is that if at time t, some agents are already merged, then they will have the same sequence of π's from time t onward. To be more specific, we can write

$$S_0(t) = \bigcup_{\ell=1}^{h} P_\ell(t),$$

where

- $P_\ell(t) \cap P_{\ell'}(t) = \emptyset$ for all $\ell \ne \ell' \in [h]$,
- for any $i, j \in P_\ell(t)$, $x_i(t) = x_j(t)$,
- for any $i \in P_\ell(t)$ and $j \in P_{\ell'}(t)$ with $\ell \ne \ell'$, $x_i(t) \ne x_j(t)$.

Note that all the agents in the same cluster $P_\ell(t)$ will follow the same dynamics from time t onward and hence, all of them will leave the set $S_0(t)$ at the same time. For a fixed time t, let k_ℓ be the first time instance after t that $P_\ell(t)$ leaves the set $S_0(t)$, i.e. $P_\ell(t) \subseteq \cap_{t'=t}^{t+k_\ell} S_0(t')$ and $P_\ell(t) \cap S_0(t+k_\ell+1) = \emptyset$. Without loss of generality assume that $0 \le k_1 \le k_2 \le \ldots \le k_h$.

For a fixed ℓ, let $i \in P_\ell(t)$. Then,

$$\mathcal{N}_i(t+k_\ell+1) \subseteq \cup_{j\in S_0^c(t+k)} \mathcal{N}_j(t+k_\ell+1).$$

The reason is that if we know that agent i is not a singleton at time $t+k+1$ then, all of its neighbors must be non-singleton at time $t+k$. This implies that

$$\pi_i(t+k+1) \le \sum_{j\in S_0^c(t+k)} \pi_j(t+k+1).$$

Also, since $i \in \cap_{t'=t}^{t+k} S_0(t)$, we have $\pi_i(t) = \pi_i(t+1) = \ldots = \pi_i(t+k) = \pi_i(t+k+1)$. Therefore, we have that

$$\pi_i(t) \le \sum_{j\in S_0^c(t+k)} \pi_j(t+k). \tag{13}$$

Based on the above discussion, we have the following result.

Lemma 7: Assume that $S_0(t) = \bigcup_{\ell=1}^{h} P_\ell$ and $0 \le k_1 \le \cdots \le k_h < \infty$. Then,

$$\sum_{i\in S_0(t)} \pi_i(t) \le 1 - \Big(\frac{1}{|P_r|}\sum_{i\in P_r}\pi_i(t) - \sum_{i\in\cup_{j=1}^{r-1}P_j}\pi_i(t)\Big),$$

for all $r = 1, 2, \ldots, h$.

Proof: Let $r \in \{1, 2, \ldots, h\}$ and $i_r \in P_r$. According to the definition of P_r, we have $\pi_i(t) = \pi_{i_r}(t)$ for all $i \in P_r$. Using relation (13) we have:

$$\begin{aligned}
\pi_{i_r}(t) &\le \sum_{j\in S_0^c(t+k_r)} \pi_j(t+k_r) = \Big(1 - \sum_{j\in S_0(t+k_r)} \pi_j(t+k_r)\Big) \\
&= \Big(1 - |P_r|\pi_{i_r}(t+k_r) - \sum_{j\in S_0(t+k_r)\setminus P_r} \pi_j(t+k_r)\Big) \\
&= \Big(1 - |P_r|\pi_{i_r}(t) - \sum_{j\in S_0(t+k_r)\setminus P_r} \pi_j(t+k_r)\Big) \\
&\le \Big(1 - |P_r|\pi_{i_r}(t) - \sum_{j\in S_0(t)\setminus\cup_{i=1}^{r}P_i} \pi_j(t)\Big),
\end{aligned} \tag{14}$$

where the last inequality follows from the fact that $S_0(t) \setminus \cup_{i=1}^{r} P_i \subseteq S_0(t+k_r) \setminus P_r$. Therefore, using (14), we get

$$\pi_{i_r}(t) \le \frac{1}{1+|P_r|}\Big(1 - \sum_{j\in S_0(t)\setminus\cup_{i=1}^{r}P_i} \pi_j(t)\Big).$$

Furthermore, since all the agents $i \in P_r$ have the same $\pi_i(t)$, it follows that

$$\sum_{i\in P_r} \pi_i(t) = |P_r|\pi_{i_r}(t) \le \frac{|P_r|}{1+|P_r|}\Big(1 - \sum_{j\in S_0(t)\setminus\cup_{i=1}^{r}P_i} \pi_j(t)\Big).$$

Therefore, using the above inequality, we have

$$\sum_{i\in S_0(t)} \pi_i(t) \le \frac{|P_r|}{1+|P_r|}\Big(1 - \sum_{j\in S_0(t)\setminus\cup_{i=1}^{r}P_i} \pi_j(t)\Big) + \sum_{i\in S_0(t)\setminus P_r} \pi_i(t)$$

$$= \frac{|P_r|}{1+|P_r|}\Big(1 - \sum_{j\in S_0(t)\setminus\cup_{i=1}^{r}P_i} \pi_j(t) + \frac{1+|P_r|}{|P_r|}\sum_{i\in S_0(t)\setminus P_r}\pi_i(t)\Big)$$
$$= \frac{|P_r|}{1+|P_r|}\Big(1 + \frac{1}{|P_r|}\Big[|P_r|\sum_{i\in\cup_{j=1}^{r-1}P_j}\pi_i(t) + \sum_{i\in S_0(t)\setminus P_r}\pi_i(t)\Big]\Big)$$
$$= \frac{|P_r|}{1+|P_r|}\Big(1 + \sum_{i\in\cup_{j=1}^{r-1}P_j}\pi_i(t) - \frac{1}{|P_r|}\sum_{i\in P_r}\pi_i(t)\Big)$$
$$+ \frac{1}{1+|P_r|}\sum_{i\in S_0(t)}\pi_i(t).$$

By moving the term $\frac{1}{1+|P_r|}\sum_{i\in S_0(t)}\pi_i(t)$ to the left hand side of the above inequality, the result follows. ■

Based on the above estimation, we have the following result.

Theorem 3: Suppose that $k_h < \infty$. Then,

$$\sum_{i\in S_0^c(t)}\pi_i(t) \geq \frac{1}{\prod_{k=1}^{h}(1+|P_k|)} \geq \frac{1}{(1+\frac{|S_0(t)|}{h})^h}.$$

Proof: Suppose that the inequality does not hold. Then, we have

$$\sum_{i\in S_0^c(t)}\pi_i(t) < \frac{1}{\prod_{k=1}^{h}(1+|P_k|)}, \tag{15}$$

or equivalently,

$$1 - \frac{1}{\prod_{k=1}^{h}(1+|P_k|)} < \sum_{i\in S_0(t)}\pi_i(t). \tag{16}$$

We claim that if this is true, then there exists at least one $r_0 \in [h]$ such that

$$\frac{1}{|P_r|}\sum_{i\in P_r}\pi_i(t) - \sum_{i\in\cup_{j=1}^{r-1}P_j}\pi_i(t) \geq \frac{1}{\prod_{k=1}^{h}(1+|P_k|)}. \tag{17}$$

If (17) does not hold for all $r \in [h]$, then by induction on r successively, we have:

$$\sum_{i\in P_r}\pi_i(t) \leq \frac{|P_r|\prod_{j=1}^{r-1}(1+|P_j|)}{\prod_{k=1}^{h}(1+|P_k|)}, \tag{18}$$

for all $r \in [h]$. By adding up the above inequality for all $r \in [h]$, we get:

$$\sum_{i\in S_0(t)}\pi_i(t) = \sum_{r=1}^{h}\sum_{i\in P_r}\pi_i(t) \leq \frac{\sum_{r=1}^{h}|P_r|\prod_{j=1}^{r-1}(1+|P_j|)}{\prod_{k=1}^{h}(1+|P_k|)}$$
$$= \frac{\prod_{k=1}^{h}(1+|P_k|)-1}{\prod_{k=1}^{h}(1+|P_k|)} \leq 1 - \frac{1}{\prod_{k=1}^{h}(1+|P_k|)},$$

which contradicts (16). Thus, at least for one $r_0 \in \{1,2,\dots h\}$, (17) holds. The existence of such an r_0 and relation (15) contradict Lemma 7; hence, (15) is not true and the stated result follows. ■

Using the above results, we obtain the following corollary.

Corollary 2: Suppose that T_n denotes the termination time of the Hegselmann-Krause dynamics in $\mathbb{R}^d$. Then, for the quadratic Lyapunov function (3), we have:

$$\frac{\epsilon^2}{8n^4}\sum_{t=0}^{T_n}\frac{1}{(1+\frac{|S_0(t)|}{h(t)})^{h(t)}} < V(0) - V(T_n),$$

where $|S_0(t)|$ and $h(t)$ denote, respectively, the number of singletons and the number of singleton clusters at time t.

Proof: The inequality follows by Theorem 3 and Theorem 2. ■

Using the same line of analysis as in the proof of Theorem 3 and considering each singleton as a cluster set, we have $|P_k| = 1$ and $h(t) = |S_0(t)|$. Thus, we arrive at the following corollary.

Corollary 3: For the termination time T_n of the dynamics we have:

$$\frac{\epsilon^2}{8n^4}\sum_{t=0}^{T_n}(\frac{1}{2})^{|S_0(t)|} < V(0) - V(T_n) \leq V(0).$$

As a particular result, if for a particular instance of the dynamics, the agents maintain the connectivity throughout the dynamics and they do not admit more than one cluster of singletons, we conclude that $T_n \leq \frac{16V(0)}{\epsilon^2}n^4$. Also, from the above corollary it can be seen that as long as the cardinality of singletons is not larger that $\log_2(n)$, then, the upper bound for the termination time would be $\frac{8V(0)}{\epsilon^2}n^5$.

V. Conclusion

In this paper, we studied the termination time of the Hegselmann-Krause dynamics in higher dimensions. We provided a polynomial bound for the termination time of the model when agents maintain their connectivity within a logarithmic bound. As future work, one can try to improve the bounds given in this paper independent of the logarithmic connectivity among agents. An other interesting problem is to consider the randomized version of this model when the initial profile is randomized over a unit ball.

References

[1] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Artificial Societies and Social Simulation*, vol. 5, pp. 1–33, 2002.

[2] F. Bullo, J. Cortes, and S. Martinez, *Distributed Control of Robotic Networks*. Princeton University Press, 2009.

[3] S. Martinez, F. Bullo, J. Cortes, and E. Frazzoli, "On synchronous robotic networks - part I: Models, tasks and complexity," *IEEE Transactions on Automatic Control*, vol. 52, pp. 2199–2213, 2007.

[4] J. Lorenz, "Repeated averaging and bounded-confidence, modeling, analysis and simulation of continuous opinion dynamics," Ph.D. dissertation, Univ. Bremen, 2007.

[5] B. Touri, *Product of random stochastic matrices and distributed averaging*, Springer, 2012.

[6] J. Lorenz, "Consensus strikes back in the Hegselmann-Krause model of continuous opinion dynamics under bounded confidence," *Journal of Artificial Societies and Social Simulation*, vol. 9, pp. 227–263, 2006.

[7] B. Touri and A. Nedić, "Discrete-time opinion dynamics," in *Signals, Systems and Computers (ASILOMAR), 2011 Conference Record of the Forty Fifth Asilomar Conference on. IEEE*, 2011, pp. 1172–1176.

[8] B. Touri and A. Nedić, "On existence of a quadratic comparison function for random weighted averaging dynamics and its implications," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on. IEEE*, 2011, pp. 3806–3811.

[9] B. Touri and A. Nedić, "Product of random stochastic matrices," *arXiv preprint arXiv:1110.1751*, 2011.

[10] A. Kolmogoroff, "Zur Theoric der Markoffschen Ketten," *Mathematische Annalen*, vol. 112, pp. 155–160, 1935.