Real Variables

Notes for MATH 447, Aniket Deshpande

"It is *always* the triangle inequality." – Lee Deville

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Preface

These notes were compiled during the Spring 2025 semester for MATH 447: Real Variables at the University of Illinois at Urbana-Champaign. The course provides a rigorous undergraduate introduction to real analysis, with a focus on the theory of differentiation and integration of functions of one real variable.

The material presented here explores the foundations of mathematical analysis, beginning with the construction of the real number system and progressing through sequences, limits, continuity, and the fundamental theorems of calculus. Special emphasis is placed on developing mathematical rigor and proof techniques that form the backbone of modern analysis.

These notes, which closely follow *Elementary Analysis: The Theory of Calculus* by Kenneth Ross, are intended as a supplement to, not a replacement for, the assigned textbook and lecture materials. The insights and memorable quotes from Professor Lee Deville's lectures have been incorporated throughout. Any errors or omissions are my own.

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1 Introduction and Foundations

1.1 On $\mathbb{N}$

We begin with a discussion of the natural numbers, $\mathbb{N}$. Intuitively, these have a clear defintion.

Definition 1 ($\mathbb{N}$). The natural numbers are the set of all positive integers.

$$\mathbb{N} = \{1, 2, 3, \ldots\}. \tag{1}$$

The set of axioms that define the natural numbers are known as the *Peano* axioms.

Theorem 1 (The Peano Axioms of $\mathbb{N}$.). The natural numbers are defined by the following axioms:

1. $1 \in \mathbb{N}$.
2. For every $n \in \mathbb{N}$, $n \neq 1$, there exists a unique $m \in \mathbb{N}$ such that $n = m + 1$, called the *successor* of m.
3. 1 is not the successor of any element of $\mathbb{N}$.
4. If $m, n \in \mathbb{N}$ have the same successor, then $m = n$.
5. If $X \subseteq \mathbb{N}$ such that $1 \in X$ and for any $n \in X$, $n + 1 \in X$, then $X = \mathbb{N}$.

These axioms are *minimal*. Removing any one of them allows for a description of multiple sets and removes the uniqueness of the natural numbers. We now expand on the fifth axiom, known as the *principle of induction*.

Definition 2 (Induction). Given an argument that depends on a natural number n, $P(n)$, we can establish that $P(n)$ is true for all $n \in \mathbb{N}$ by proving the following:

1. $P(1)$ is true.
2. $P(k) \implies P(k+1)$ for all $k \in \mathbb{N}$.

Then, $\forall\, n \in \mathbb{N}$, $P(n)$ is true.

Example 1. The triangle number formula, stated below, can be proven using induction.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{2}$$

Proof. We first show that the formula holds for $n = 1$:

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}.$$

Now, assume that the formula holds for $n = k$. Then,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

We now show that the formula holds for $n = k + 1$:

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

Thus, by induction, the formula holds for all $n \in \mathbb{N}$. □

Example 2. We now use induction to prove the following, slightly unintuitive, statement.

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2. \tag{3}$$

Proof. We first show that the formula holds for $n = 1$:

$$P(1) \implies 1^3 = 1 = (1)^2.$$

Now, assume that the formula holds for $n = k$. Then induct on $n = k + 1$:

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} = \left(\sum_{i=1}^{k+1} i\right)^2.\end{aligned}$$

□

1.2 On $\mathbb{Z}$ and $\mathbb{Q}$.

We now define the integers, $\mathbb{Z}$.

Definition 3 ($\mathbb{Z}$). The integers are the set of all positive and negative whole numbers.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}. \tag{4}$$

Note that both the naturals and integers are *ordered*.

Definition 4 (Ordering). For any $n, m \in \mathbb{N}$, one of the following is true:

1. $n = m$.
2. $n < m$.
3. $n > m$.

The same is true for $\mathbb{Z}$.

The proof of this can be deduced using the Peano axioms, but it will be omitted for brevity (it is *quite* tedious).

The Algebra of $\mathbb{Z}$.

We now define the operations of addition and multiplication on $\mathbb{Z}$.

Definition 5 (Addition and Multiplication on $\mathbb{Z}$.). These are posulates related to the operations of addition and multiplication on $\mathbb{Z}$.

(A1) For all $x, y, z \in \mathbb{Z}$, $(x + y) + z = x + (y + z)$.

(A2) For all $x, y \in \mathbb{Z}$, $x + y = y + x$.

(A3) There exists an element $0 \in \mathbb{Z}$ such that $x + 0 = x$ for all $x \in \mathbb{Z}$.

(A4) For all $x \in \mathbb{Z}$, there exists an element $-x \in \mathbb{Z}$ such that $x + (-x) = 0$.

Note that this implies that $\mathbb{Z}$ is an *abelian group* under addition.

(M1) For all $x, y, z \in \mathbb{Z}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(M2) For all $x, y \in \mathbb{Z}$, $x \cdot y = y \cdot x$.

(M3) There exists an element $1 \in \mathbb{Z}$ such that $x \cdot 1 = x$ for all $x \in \mathbb{Z}$.

(D) For all $x, y, z \in \mathbb{Z}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Note that this implies that $\mathbb{Z}$ is a *commutative ring* under addition and multiplication.

The fundamental issue with the integers is that we cannot define multiplicative inverses. This leads us a construction of the rational numbers, $\mathbb{Q}$.

Define a set S such that

$$S = \{(x, y) \mid x, y, \in \mathbb{Z} \text{ and } y \neq 0\}. \tag{5}$$

Now define an *equivalence relation* in S.

$$(a, b) \sim (c, d) \iff ad = bc. \tag{6}$$

Equivalence. To prove that $\sim$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

1. *Reflexivity*: $(a, b) \sim (a, b)$ since $ab = ba$.

2. *Symmetry*: If $(a, b) \sim (c, d)$, then $ad = bc$. This implies that $cb = da$, so $(c, d) \sim (a, b)$.

3. *Transitivity*: If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ad = bc$ and $cf = de$. This implies that $adf = bcf = bde$, so $(a,b) \sim (e,f)$.

$\square$

We give this set a name, $\mathbb{Q}$.

Definition 6 ($\mathbb{Q}$). The rational numbers are the set of all numbers that can be expressed as a ratio of two integers.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}. \tag{7}$$

All the properties of $\mathbb{Z}$ hold for $\mathbb{Q}$, with the exception of the existence of multiplicative inverses. This is because $\mathbb{Q}$ is a *field*, which is a commutative ring with multiplicative inverses.

Proposition 1. Multiplicative inverses exist in $\mathbb{Q}$.

Proof. Let $x = (p,q) \in \mathbb{Q}$. Then, $x^{-1} = (q,p)$.

$$(p,q) \cdot (q,p) = (pq, pq) \sim (1,1). \tag{8}$$

The equivalence relation is true because $pq = pq$. Thus, $x^{-1} = (q,p)$. $\square$

Note that we know $p \neq 0$ because multiplicative inverses are only defined for $x \in \mathbb{Q}$, $x \neq 0$.

Theorem 2. $\mathbb{Q}$ is a field.

We encounter a new problem after constructing $\mathbb{Q}$. There are numbers that cannot be expressed as a ratio of two integers. These are the irrational numbers.

$\mathbb{Q}$ Has Holes.

We begin with some geometric motivation. We draw a right triangle with legs of length 1 and hypotenuse c. By the Pythagorean theorem, we have $c^2 = 2$.

Theorem 3. There is no rational number c such that $c^2 = 2$.

Proof. We prove this by contradiction. Assume $c = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Then, $c^2 = 2$ implies that $\frac{a^2}{b^2} = 2$. This implies that $a^2 = 2b^2$. Since a^2 is even, a is even. Let $a = 2k$ for some $k \in \mathbb{Z}$. Then, $4k^2 = 2b^2$, so $b^2 = 2k^2$. This implies that b is even. However, this contradicts the fact that a and b are coprime. Thus, there is no rational number c such that $c^2 = 2$. $\square$

Theorem 4. Let $x, y \in \mathbb{Q}$ such that $x < y$. Then there are infinitely many irrational numbers between x and y.

So, we have discovered an infinite amount of holes. In fact, this infinity is *larger* than the infinity of the natural numbers.

Lemma 1. Let $x, y \in \mathbb{Q}$ such that $x < y$. Then,

$$x < \frac{x+y}{2} < y. \tag{9}$$

Note that this implies that there are infinitely many *rational* numbers between x and y.

Lemma 2. Given $x, y \in \mathbb{Q}$ and $x < y$. Then

$$x + \frac{x_2 - x_1}{\sqrt{2}} \notin \mathbb{Q}. \tag{10}$$

1.3 On $\mathbb{Q}$ and towards $\mathbb{R}$.

Theorem 5 (Fraction Inequalities). We say for some $p, q, r, s \in \mathbb{Z}$ and $q, s \neq 0$ that

$$\frac{p}{q} \leq \frac{r}{s} \tag{11}$$

if and only if $ps \leq rq$.

Notice that this definition implies a relationship between the ordering of $\mathbb{Q}$ and $\mathbb{Z}$.

Theorem 6 (Ordering of $\mathbb{Q}$). The following are ordering properties of $\mathbb{Q}$. For all $x, y, z \in \mathbb{Q}$:

1. $x \leq y$ or $y \leq x$.
2. If $x \leq y$ and $y \leq x$, then $x = y$.
3. If $x \leq y$ and $y \leq z$, then $x \leq z$.
4. If $x \leq y$, then $x + z \leq y + z$.
5. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.

$\mathbb{Q}$ is an *ordered field*.

Proposition 2. For all $a \in \mathbb{Q}$, $a \cdot 0 = 0$.

Proof. We rewrite 0 as $0 + 0$.

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$$

Use the property stating that for all $a, b, c \in \mathbb{Q}$ with $c \geq 0$ and $a \leq b$, $ac \leq bc$. Then, $a \cdot 0 \leq a \cdot 0 + a \cdot 0$. Thus, $a \cdot 0 = 0$. □

Definition 7 (Absolute Value on $\mathbb{Q}$). The *absolute value* of a number $x \in \mathbb{Q}$ is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{12}$$

This will help us prove the following theorem.

Theorem 7. Given $a, b \in \mathbb{Q}$.

1. $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$.
2. $|ab| = |a| \cdot |b|$.
3. $|a+b| \leq |a| + |b|$ (*Triangle Inequality*).

Proof. We will prove the triangle inequality. Consider the following trivial statements.

$$\begin{aligned} -|a| \leq a \leq |a|, \\ -|b| \leq b \leq |b|. \end{aligned} \tag{13}$$

Now, we add these inequalities.

$$-|a| - |b| \leq a + b \leq |a| + |b|. \tag{14}$$

Now, look at the following lemma, which is a restatement of the absolute value definition.

Lemma 3. If $M > 0$ and $-M \leq x \leq M$, then $|x| \leq M$.

We apply this lemma to the inequality above.

$$|a+b| \leq |a| + |b|. \tag{15}$$

which is the triangle inequality. □

Theorem 8 (The Triangle Equality for n Numbers). Given $a_1, a_2, \dots, a_n \in \mathbb{Q}$, we have

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|. \tag{16}$$

Proof. We prove this by induction. The base case is trivial. Assume that the inequality holds for $n = k$. Then,

$$\left|\sum_{i=1}^{k} a_i + a_{k+1}\right| \leq \sum_{i=1}^{k} |a_i| + |a_{k+1}|. \tag{17}$$

This implies that

$$\left|\sum_{i=1}^{k+1} a_i\right| \leq \sum_{i=1}^{k+1} |a_i|. \tag{18}$$

Thus, by induction, the inequality holds for all $n \in \mathbb{N}$. □

1.4 Bounds of $\mathbb{R}$.

Let S be a non-empty set of numbers ($\mathbb{R}$ or $\mathbb{Q}$). If there exists $M \in S$ such that for any $x \in S$, $x \leq M$, then M is the maximum of S. The minimum is defined similarly.

Definition 8 (Maxes and Mins). We define the maximum and minimum of a set S as follows:

$$\max(S) = M \iff \forall x \in S, x \leq M \quad \min(S) = m \iff \forall x \in S, x \geq m. \tag{19}$$

However, consider the open interval $S = (a, b) = \{x \in \mathbb{Q} \mid a < x < b\}$. This set has no maximum or minimum.

Proposition 3. There is no maximum or minimum of the set $S = (a, b)$.

Proof. We prove this by contradiction. Assume $y = \max S$. Then, $y < b$. Let $z = \frac{y+b}{2}$. Then, $y < z < b$. This implies that $z \in S$, which contradicts the assumption that $y = \max S$. Thus, there is no maximum of S. □

Definition 9 (Upper and Lower Bounds). Let S be a non-empty set of numbers. If there exists $M \in \mathbb{R}$ such that for any $x \in S$, $x \leq M$, then M is an *upper bound* of S. The *lower bound* is defined similarly.

If a set S *has* an upper bound (there are sets that do not), we say S is *bounded above*. If S has a lower bound, we say S is *bounded below*. If S is bounded above and below, we say S is *bounded*.

Proposition 4. For a set $S = (a, b)$, b is the minimum of the set of upper bounds of S.

This gives us a way to define the *least upper bound* and *greatest lower bound* of a set.

Definition 10 (Supremum). . Let $S = (a, b)$ be a set that is bounded above. If there is a least upper bound, then this is called the *supremum* of S.

$$\sup S = \min\{M \in \mathbb{R} \mid \forall x \in S, x \leq M\}. \tag{20}$$

Definition 11 (Infimum). . Let $S = (a, b)$ be a set that is bounded below. If there is a greatest lower bound, then this is called the *infimum* of S.

$$\inf S = \max\{m \in \mathbb{R} \mid \forall x \in S, x \geq m\}. \tag{21}$$

This leads to the axiom that allows us to define the real numbers.

Theorem 9 (The Completeness Axiom for $\mathbb{R}$). Given a set in $\mathbb{R}$ that is bounded above, there exists a least upper bound. Given a set in $\mathbb{R}$ that is bounded below, there exists a greatest lower bound.

Intuitively, this axiom states that $\mathbb{R}$ has no holes. This is the defining property of the real numbers. No matter what set we take, there is always a number that is the least upper bound or greatest lower bound. This is what separates $\mathbb{R}$ from $\mathbb{Q}$.

Example 3. Let $S \subset \mathbb{R}$ be defined as follows.

$$S = \left\{x \in \mathbb{R} \mid x^2 < 2\right\} \tag{22}$$

Then, the least upper bound of S is $\sqrt{2}$. Note that in $\mathbb{Q}$, there is no such number.

Let us say we have a set $S \subset \mathbb{R}$ and S is *not* bounded above. Then,

$$\nexists \sup S \in \mathbb{R}.$$

We say that the supremum of S is ∞. This means there is no upper bound. Similarly, if S is not bounded below, then

$$\nexists\ \inf S \in \mathbb{R}.$$

We say that the infimum of S is $-\infty$. This means there is no lower bound.

2 Sequences

2.1 Sequences and Countability.

We define an important object in analysis, the sequence.

Definition 12 (Sequences). A sequence is a function $f : \mathbb{N} \to \mathbb{R}$. We denote the sequence as $\{a_n\}_{n=1}^{\infty}$, where $a_n = f(n)$.

Note that we *do not* need to start at $n = 1$, $n = 0$ is a very common starting point. We can also start at any $n \in \mathbb{N}$.

Example 4. The sequence $\{a_n\}_{n=0}^{\infty}$ defined as $a_n = \frac{1}{n+1}$ is the *harmonic sequence*.

$$a_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}. \tag{23}$$

Example 5. The sequence $\{a_n\}_{n=1}^{\infty}$ defined as $a_n = (-1)^n$ is an *alternating sequence*

$$a_n = \{-1, 1, -1, 1, \ldots\}. \tag{24}$$

Definition 13 (Limits of Sequences). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We say that $L \in \mathbb{R}$ is the limit of the sequence if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. We denote this as

$$\lim_{n\to\infty} a_n = L. \tag{25}$$

Here, ϵ is the "error" in the limit. We can approach L as close as needed by going far enough in the sequence.

Proposition 5. Given the sequence $\{x_n\}_{n=1}^{\infty}$ with each $x_n = \frac{1}{n^2}$,

$$\lim_{n\to\infty} x_n = 0.$$

Proof. We need to show for all $\epsilon > 0$, that

$$\left|\frac{1}{n^2} - 0\right| < \epsilon.$$

for sufficiently large n. This implies that

$$\frac{1}{n^2} < \epsilon \implies n > \frac{1}{\sqrt{\epsilon}}.$$

Therefore, choose $N > \frac{1}{\sqrt{\epsilon}}$ (like $N = \lceil \frac{1}{\sqrt{\epsilon}} \rceil$). Then, for all $n \geq N$, $|x_n - 0| < \epsilon$. $\square$

Theorem 10. Limits of sequences are unique.

Proof. Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Then, for all $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - L| < \epsilon$ and for all $n \geq N_2$, $|a_n - M| < \epsilon$. Choose $N = \max(N_1, N_2)$. Then, for all $n \geq N$, $|a_n - L| < \epsilon$ and $|a_n - M| < \epsilon$. This implies that $|L - M| < 2\epsilon$. Since ϵ is arbitrary, $L = M$. $\square$

Now, we define the countability and uncountability of sets.

Definition 14 (Countability). We say a set A is *countable* if

1. A is finite, or
2. There exists a bijection between A and $\mathbb{N}$.

Note that the second condition is equivalent to saying that elements of A can be written as a sequence. Any set that is not countable is *uncountable*.

Example 6. $\mathbb{Z}$ and $\mathbb{Q}$ are countable. We can write $\mathbb{Z}$ as the sequence

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}. \tag{26}$$

We can write $\mathbb{Q}$ as the sequence

$$\mathbb{Q} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \ldots\right\}. \tag{27}$$

Theorem 11. The set of real numbers $\mathbb{R}$ is uncountable.

We have not formally defined $\mathbb{R}$ yet, so we will not prove this theorem yet. However, a brief sketch of this proof is as follows. Assume that $\mathbb{R}$ is countable. Then, we can write $\mathbb{R}$ as a sequence. We can then construct a number that is not in this sequence, which is a contradiction. Thus, $\mathbb{R}$ is uncountable.

We remind ourselves of the definition of the limit of a sequence. Given a sequence $\{a_n\}_{n=1}^{\infty}$, we say that $L \in \mathbb{R}$ is the limit of the sequence if,

$$\forall\, \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N, |a_n - L| < \epsilon.$$

Proposition 6. The limit of

$$\frac{1}{n^p}, \quad p > 0$$

is 0.

Proof. Take the following inequality.

$$\left|\frac{1}{n^p} - 0\right| < \epsilon \implies \frac{1}{n^p} < \epsilon \implies n > \frac{1}{\epsilon^{1/p}}.$$

Now, choose $N > \frac{1}{\epsilon^{1/p}}$. Then, for all $n \geq N$, $|a_n - 0| < \epsilon$. □

Example 7. Let $a_n = \frac{3n+5}{7n-1}$. Then, the limit of a_n is $\frac{3}{7}$. To prove that, we can use the definition of the limit.

$$\left|\frac{3n+5}{7n-1} - \frac{3}{7}\right| < \epsilon \implies \left|\frac{7(3n+5) - 3(7n-1)}{7(7n-1)}\right| < \epsilon \implies \left|\frac{38}{49n-7}\right| < \epsilon.$$

Note that since n is always positive, we can ignore the absolute value.

$$\frac{38}{49n-7} < \epsilon \implies n > \frac{38}{49\epsilon} + \frac{1}{7}.$$

Thus, choose $N > \frac{38}{49\epsilon} + \frac{1}{7}$. Then, for all $n \geq N$, $|a_n - \frac{3}{7}| < \epsilon$.

Example 8. Let $a_n = \frac{\sin(3n)}{n}$. We claim that this sequence approaches 0. To prove this, we use the definition of the limit.

$$\left|\frac{\sin(3n)}{n} - 0\right| < \epsilon \implies \frac{|\sin(3n)|}{n} < \epsilon.$$

Note that we cannot explicitly solve for n here, but we can use the fact that the trigonometric function is bounded.

$$|\sin(3n)| \leq 1 \implies \frac{|\sin(3n)|}{n} \leq \frac{1}{n}$$

We can use this to find the N that satisfies the inequality.

$$\frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}.$$

If we choose $N > \frac{1}{\epsilon}$, then for all $n \geq N$, $|a_n - 0| < \epsilon$.

Example 9. Let $a_n = \frac{2n+4}{3n^4-2n+7}$. We claim that the limit of this sequence is 0. To prove this, we use the definition of the limit.

Proof. We need to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. This implies that

$$\left|\frac{2n+4}{3n^4-2n+7} - 0\right| < \epsilon \implies \left|\frac{2n+4}{3n^4-2n+7}\right| < \epsilon.$$

We wish to replace this sequence with a sequence that is bigger than it (proving a larger sequence is less than epsilon directly implies the original sequence is less than epsilon).

$$\frac{a_n}{b_n} \leq \frac{c_n}{d_n}$$

The conditions necessary for this are $c_n \geq a_n$ and $d_n \leq b_n$. Let us consider $3n$ and examine the numerator.

$$2n + 4 \leq 3n \implies n \geq 4.$$

Now consider $2n^4$ and examine the denominator.

$$3n^4 - 2n + 7 \geq 2n^4 \implies n^4 \geq 2n - 7 \implies n^4 \geq 2n \implies n \geq 2.$$

Thus, we have constructed a fraction that is larger than the original fraction.

$$\frac{2n+4}{3n^4-2n+7} \geq \frac{3n}{2n^4} = \frac{3}{2}n^{-3}.$$

Now we can continue with the limit definition.

$$\frac{3}{2n^3} < \epsilon \implies n > \left(\frac{3}{2\epsilon}\right)^{1/3}.$$

Thus, choose $N > \left(\frac{3}{2\epsilon}\right)^{1/3}$. Then, for all $n \geq N$, $|a_n - 0| < \epsilon$. ☐

Example 10. Let $a_n = n^2$. We say that there does not exist an L such that $\lim_{n\to\infty} a_n = L$. This is because the sequence is unbounded.

Proof. We prove this by contradiction. Assume such L exists.

$$|n^2 - L| < \epsilon, \text{ pick } \epsilon = 1.$$

Then, $|n^2 - L| < 1$. This implies that $n^2 < L + 1$. However, we can choose $n > \sqrt{L+1}$. This is a contradiction, so there does not exist an L such that $\lim_{n\to\infty} a_n = L$. ☐

2.2 Limit Theorems

Definition 15 (Convergence). We say that a sequence (x_n) is *convergent* if it converges to a limit $L \in \mathbb{R}$.

$$\lim_{n\to\infty} x_n = L. \tag{28}$$

Definition 16 (Bounded). We say that a sequence (x_n) is *bounded* if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

These definitions are connected by the following theorem.

Theorem 12. A sequence that converges is bounded.

Proof. Let our sequence be (x_n) and assume $\lim_{n\to\infty} x_n = L$. Choose $\epsilon = 1$. Then, there exists N such that $|x_n - L| < 1$ for all natural n.

$$|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L|.$$

by the triangle inequality. So, $\forall\ n > N$, $|x_n| < |L| + 1$. Let M be

$$M = \max(|x_1|, |x_2|, \ldots, |x_N|, |L| + 1).$$

So, for all $n \in \mathbb{N}$, $|x_n| \leq M$. ☐

Note that we can use the contrapositive to the theorem to show that unbounded sequences do not converge.

Theorem 13 (Limit Theorems). These are theorems regarding limits that will prove to be very useful.

1. If $\lim_{n\to\infty} x_n = L$, $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \alpha x_n = \alpha L$.
2. If $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$, then $\lim_{n\to\infty}(x_n + y_n) = L + M$.
3. If $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$, then $\lim_{n\to\infty}(x_n y_n) = LM$.
4. If $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$, $M \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L}{M}$.
5. If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L$, then if some sequence z_n exists such that

$$x_n \leq z_n \leq y_n$$

then $\lim_{n\to\infty} z_n = L$.

The last theorem is known as the *Squeeze Theorem*.

Proposition 7. Consider the sequence below.

$$x_n = \begin{cases} 2 \text{ if } n \text{ is odd.} \\ 5 \text{ if } n \text{ is even.} \end{cases}$$

We claim the limit of this sequence does not exist.

Proof. For all $\epsilon > 0$, there exists N such that $n > N$, $|x_n - L < \epsilon$.

$$\text{Even case: } |5 - L| < \epsilon$$
$$\text{Odd case: } |2 - L| < \epsilon$$

FINISH. □

Proof. We now prove the 1st limit theorem. Use the limit definition with α.

$$|\alpha x_n - \alpha L| < \epsilon$$
$$|\alpha(x_n - L)| < \epsilon$$
$$|\alpha||x_n - L| < \epsilon \implies |x_n - L| < \frac{\epsilon}{\alpha}.$$

We can assume $\alpha \neq 0$, as that is a trivial case. Now apply the limit defintion. $\forall \epsilon > 0, \exists N$ such that $n > N$. □

Proof. We now prove the 2nd limit theorem. Use the limit definition with x_n and y_n.

$$\exists N_1 \text{ such that } n > N_1, |x_n - L| < \frac{\epsilon}{2}$$
$$\exists N_2 \text{ such that } n > N_2, |y_n - M| < \frac{\epsilon}{2}.$$

Choose $N = \max(N_1, N_2)$. Then, for all $n > N$,

$$|x_n + y_n - L - M| \le |x_n - L| + |y_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Applying the limit definition, we have proven the 2nd limit theorem. $\square$

LIMIT THEOREMS ON EXAM .

2.3 Divergence.

There are two types of *non-converging* sequences. For now, we will focus on the simpler of the two.

Definition 17 (Divergence). Given a sequence (x_n), we say that the sequence *diverges to infinity* if for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $x_n > M$. We denote this as

$$\lim_{n\to\infty} x_n = \infty. \tag{29}$$

Note that here, ∞ is not a number, but notation to show that the sequence grows without bound.

Equivalently, we can define divergence with a set. Given our sequence (x_n), for any $M \in \mathbb{R}$, define

$$S_M = \{n \in \mathbb{N} \mid x_n \le M\} \implies \lim_{n\to\infty} x_n = \infty \iff \forall M \in \mathbb{R}, S_M \text{ is finite.}$$

Example 11. Let $x_n = n^2$. We claim that $\lim_{n\to\infty} x_n = \infty$. To prove this, we use the definition of divergence.

$$n^2 > M \implies n > \sqrt{M}.$$

So, choose any $M > 0$. Then, let $N = \lceil \sqrt{M} \rceil$. Then, for all $n \ge N$, $x_n > M$.

Example 12. Let $x_n = n + \sqrt{n} - 7$. We claim that $\lim_{n\to\infty} x_n = \infty$. To prove this, we use the definition of divergence.

$$\sqrt{n} - 7 > 0 \implies n > 49.$$

If $n > 49$, then add n to both sides to get $n + \sqrt{n} - 7 > n$. Thus, choose $N = \max(M, 49)$ for any $M > 0$. Then, for all $n \ge N$, $x_n > M$.

Theorem 14. If $\lim_{n\to\infty} x_n = \infty$ and $y_n \ge x_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} y_n = \infty$.

Proof. We use the definition of divergence. Given $M > 0$, there exists N such that for all $n \ge N$, $x_n > M$. Since $y_n \ge x_n$, $y_n > M$ for all $n \ge N$. Thus, y_n also satisfies the definition of divergence. $\square$

Definition 18 (Divergence to $-\infty$). Given a sequence (x_n), we say that the sequence *diverges to negative infinity* if for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$. We denote this as

$$\lim_{n\to\infty} x_n = -\infty. \tag{30}$$

Equivalently, we can write this in terms of $-M$ to develop a better intuition for $-\infty$. If for all $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < -M$, then $\lim_{n\to\infty} x_n = -\infty$.

Theorem 15. Given a sequence (x_n), the following is true.

$$\lim_{n\to\infty} x_n = \infty \iff \lim_{n\to\infty} -x_n = -\infty.$$

This transitions into our limit theorems for divergent sequences.

Theorem 16 (Limit Theoreoms). Consider two sequences (x_n) and (y_n).

1. If $x_n \to \infty$ and $y_n \to L$, with $L \in \mathbb{R}$, then $x_n + y_n \to \infty$.
2. If $x_n \to \infty$ and $y_n \to \infty$, then $x_n + y_n \to \infty$.
3. If $x_n \to \infty$ and $y_n \to L$, with $L > 0$, then $x_n y_n \to \infty$.
4. If $x_n \to \infty$ and $y_n \to \infty$, then $x_n y_n \to \infty$.
5. For $x_n > 0$, $x_n \to \infty$ if and only if $\frac{1}{x_n} \to 0$.

Proof. We will prove the first limit theorem. Given that $x_n \to \infty$ and $y_n \to L$, we wish to show that $x_n + y_n \to \infty$. Using the definition of divergence, we want to show that

$$\forall M \in \mathbb{R},\ \exists N \text{ such that } n \geq N \implies x_n + y_n > M.$$

First, pick $\epsilon = 1$, implying $|y_n - L| < 1$.

$$-1 < y_n - L < 1 \implies y_n > L - 1.$$

We know that $x_n > M$, so

$$x_n + y_n > M + L - 1.$$

Thus, choose N such that $n \geq N \implies x_n + y_n > M + L - 1$. This proves the first limit theorem. $\square$

The proof of the second limit theorem is a bit easier, conceptually, as we do not need to consider a finite limit.

Proof. We will prove the 2nd limit theorem. Given that $x_n \to \infty$ and $y_n \to \infty$, we wish to show that $x_n + y_n \to \infty$. Using the definition of divergence, we want to show that

$$\forall M \in \mathbb{R},\ \exists N \text{ such that } n \geq N \implies x_n + y_n > M.$$

We know that, separately, $x_n > M_1$ and $y_n > M_2$. Thus, $x_n + y_n > M_1 + M_2$. Choose N such that $n \geq N \implies x_n + y_n > M_1 + M_2$. This proves the 2nd limit theorem. $\square$

Proof. We will now prove the 3rd limit theorem. Given that $x_n \to \infty$ and $y_n \to L$, we wish to show that $x_n y_n \to \infty$. Using the definition of divergence, we want to show that

$$\forall M \in \mathbb{R},\ \exists N \text{ such that } n \geq N \implies x_n y_n > M.$$

Since $y_2 \to L$, for all $\epsilon > 0$, there exists a N_2 such that for all $n > N_2$, $|y_n - L| < \epsilon$. Let $\epsilon = L/2$.

$$|y_n - L| < \frac{L}{2}$$
$$-\frac{L}{2} < y_n - L < \frac{L}{2} \implies y_n > \frac{L}{2}.$$

We know that $x_n > M$, so $x_n y_n > \frac{ML}{2}$, which is greater than M. Choose N such that $n \geq N \implies x_n y_n > M$. This proves the 3rd limit theorem. □

2.4 Monotonic and Cauchy Sequences.

Definition 19 (Increasing and Decreasing Sequences). A sequence (x_n) is *increasing* if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. A sequence is *decreasing* if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. A sequence is *monotonic* if it is either increasing or decreasing.

Note here, that we are using $\leq$ and $\geq$. This means that the sequence can be constant and still be considered increasing or decreasing.

Example 13. The sequence $x_n = 1$ is both increasing and decreasing.

A function is *strictly increasing* if $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. Similarly, a function is *strictly decreasing* if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

Example 14. The sequence $x_n = 1 - \frac{1}{n}$ is strictly increasing.

Theorem 17. Let (x_n) be a sequence that is bounded and monotonic. Then, (x_n) converges to a limit $L \in \mathbb{R}$.

Proof. We will show that if a sequence is bounded above and monotonic, it converges. Let (x_n) be increasing and bounded above.

$$S = \{x_n \mid n \in \mathbb{N}\}.$$

Since S is bounded above, it has a supremum (by the Completeness axiom). Let $L = \sup S$. We wish to show that $\lim_{n\to\infty} x_n = L$.

Let $\epsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $x_N > L - \epsilon$ (otherwise, $L - \epsilon$ would be the supremum). So, we have the following inequality.

$$L - \epsilon < x_N \leq L.$$

Since x_n is increasing, for all $n > N$

$$\begin{aligned} L - \epsilon < x_n &\leq L \\ -\epsilon < x_n - L &\leq 0 \\ |x_n - L| &< \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} x_n = L$. The proof for decreasing sequences is similar. □

Example 15. Let $x_n = 1 - \frac{1}{n}$. We claim that this sequence converges to 1. To prove this, we show that the sequence is increasing and bounded above.

It is clear that the sequence is increasing, and we can show that the supremum is 1. Let $\epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N \implies 1 - \frac{1}{n} > 1 - \epsilon$. Thus, the sequence is bounded above by 1. Thus, the sequence converges to 1.

Proposition 8. The sequence $x_{n+1} = \left(\sqrt{2}\right)^{x_n}$ is increasing and bounded above by 2. Thus, the sequence converges.

Proof. We will use some techniques from calculus to prove this. Let $f(x) = \left(\sqrt{2}\right)^x$. We wish to show that $f(x) > x$ for all $x \in \mathbb{R}$. We can show this by taking the derivative of $f(x)$.

$$f'(x) = \ln(\sqrt{2})\left(\sqrt{2}\right)^x > 0.$$

Thus, the function is increasing. To show that it is both increasing *and* convex, we can take the second derivative.

$$f''(x) = \ln^2(\sqrt{2})\left(\sqrt{2}\right)^x > 0.$$

Now, define the function $g(x) = f(x) - x$ and consider its roots. These are the intersection points of $f(x)$ and x, one of them being $x = 2$. Thus, $f(x) > x$ when $x < 2$. Thus, the sequence is increasing and bounded above by 2. □

Decimal Expansions

Let $D = \{0, 1, \ldots, 9\}$. Now, consider the set D^*, defined below.

$$D^* = \{(a_0, a_1, a_2, \ldots) \mid a_n \in D\}.$$

This is the set of infinite sequences of elements in D. We wish to construct a map that maps these sequences to a real, $\varphi : D^* \to [0,1]$.

$$(d_1, d_2, \ldots) \mapsto \sum_{i=1}^{\infty} \frac{d_i}{10^i}.$$

Proposition 9. Let $x_n = \sum_{i=1}^{n} \frac{d_i}{10^i}$. Then, the sequence (x_n) is increasing and bounded above by 1.

Proof. We will show that the sequence is increasing. Let $x_n = \sum_{i=1}^{n} \frac{d_i}{10^i}$ and $x_{n+1} = \sum_{i=1}^{n+1} \frac{d_i}{10^i}$. Then, $x_{n+1} - x_n = \frac{d_{n+1}}{10^{n+1}} > 0$. Thus, the sequence is increasing.

We will now show that the sequence is bounded above by 1. Let $x_n = \sum_{i=1}^{n} \frac{d_i}{10^i}$. Then, $x_n \le \sum_{i=1}^{n} \frac{9}{10^i} = \frac{1-\frac{1}{10^n}}{9} < 1$. Thus, the sequence is bounded above by 1. □

Theorem 18. If a sequence (x_n) is increasing and not bounded above, then the sequence diverges to ∞.

$$\lim_{n\to\infty} x_n = \infty. \tag{31}$$

If a sequence (x_n) is decreasing and not bounded below, then the sequence diverges to $-\infty$.

$$\lim_{n\to\infty} x_n = -\infty. \tag{32}$$

Proof. First, we will prove this for increasing sequences.

If (x_n) is not bounded, then for all M, there exists an index N such that $x_N > M$. Since the sequence is increasing, for all $n \ge N$, $x_n > x_N > M$. Thus, the sequence diverges to ∞. The proof for decreasing sequences is similar.

We will now prove the second part of the theorem. If (x_n) is decreasing and not bounded below, then for all M, there exists an index N such that $x_N < M$. Since the sequence is decreasing, for all $n \ge N$, $x_n < x_N < M$. Thus, the sequence diverges to $-\infty$. □

$\limsup$ and $\liminf$

Definition 20 (Limit Superior). Given a sequence (x_n), we define the *limit superior* as

$$\limsup x_n = \lim_{N\to\infty} \sup\{x_n \mid n \ge N\}. \tag{33}$$

In other words, the limit superior is the limit of the supremum of the tail of the sequence.

Intuitively, the limit superior is the largest limit point of the sequence. It describes the largest value that the tail of the sequence approaches.

Definition 21 (Limit Inferior). Given a sequence (x_n), we define the *limit inferior* as

$$\liminf x_n = \lim_{N\to\infty} \inf\{x_n \mid n \ge N\}. \tag{34}$$

In other words, the limit inferior is the limit of the infimum of the tail of the sequence.

If (x_n) is unbounded above, then $\limsup x_n = \infty$. If (x_n) is unbounded below, then $\liminf x_n = -\infty$.

Intuitively, the limit inferior is the smallest limit point of the sequence. It describes the smallest value that the tail of the sequence approaches.

Example 16. Consider the sequence $x_n = (-1)^n + 1$. This sequence oscillates between 0 and 2. Thus, $\limsup x_n = 2$ and $\liminf x_n = 0$.

Let us formally unpack the definition of the limit superior. Let (x_n) a sequence, e.g. bounded. For any N, the set $\{x_n \mid n > N\}$ is the tail of the sequence and is bounded above. Therefore, the supremum of this set exists.

$$b_N = \sup\{x_n \mid n > N\}.$$

Note that b_N is a decreasing sequence: $b_{N+1} \leq b_N$. So, b_N is bounded and decreasing, implying that it converges to some real number L.

Theorem 19. For any bounded sequence (x_n), the limit superior and limit inferior exist.

Note that these limits are not necessarily equal! The proof of this is conceptually covered by the above discussion.

Example 17. Consider the simple alternating sequence, $x_n = (-1)^n$. The limit superior is 1 and the limit inferior is -1.

Theorem 20. For a bounded sequence (x_n), the following is true.

$$\lim_{n\to\infty} x_n = L \iff \limsup x_n = \liminf x_n = L.$$

Proof. ($\implies$) Assume $\lim_{n\to\infty} = L$. This means that for all $\epsilon > 0$, there exists an N such that if $n > N$, $|x_n - L| < \epsilon$.

$$n > N \implies L - \epsilon < x_n < L + \epsilon.$$

We can use our unpacking of the limit superior to show that $\limsup x_n = L$.

$$b_N = \sup\{x_n \mid n > N\} \leq L + \epsilon.$$

Since b_N is decreasing, $b_n \leq L + \epsilon$ for all $n > N$. This implies that

$$\lim_{N\to\infty} b_n \leq L + \epsilon \implies \limsup x_n \leq L + \epsilon.$$

Since this is true for any $\epsilon > 0$, $\limsup x_n \leq L$. The proof for $\liminf x_n = L$ is similar.

($\impliedby$) Assume $\limsup x_n = \liminf x_n = L$.

$$\limsup x_n = L \implies \lim_{N\to\infty} b_N = L$$

Since $b_N \to L$, for all $\epsilon > 0$, there exists an N_1 such that

$$\begin{aligned} n > N_1 &\implies |b_n - L| < \epsilon \\ &\implies b_n < L + \epsilon \\ &\implies \sup\{x_k \mid k > n\} < L + \epsilon \end{aligned}$$

Now consider the limit inferior. If $\lim_{N\to\infty} a_N = L$, then for all $\epsilon > 0$, there exists an N_2 such that

$$\begin{aligned} n > N_2 &\implies |a_n - L| < \epsilon \\ &\implies a_n > L - \epsilon \\ &\implies \inf\{x_k \mid k > n\} > L - \epsilon. \end{aligned}$$

Now, for all $n > \max(N_1, N_2)$, we have

$$L - \epsilon < x_n < L + \epsilon \implies |x_n - L| < \epsilon.$$

Thus, $\lim_{n\to\infty} x_n = L$. □

Definition 22 (Cauchy Sequence). A sequence (x_n) is a *Cauchy sequence* if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|x_n - x_m| < \epsilon. \tag{35}$$

Consider this a limit "in pairs". General convergent sequences get close to a limit, but Cauchy sequences get close to each other.

Theorem 21. A sequence that converges is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence with limit $L \in \mathbb{R}$. Then, for all $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|x_n - L| < \epsilon$.

$$\begin{aligned} |x_n - x_m| = -|x_n - L + L - x_m| &= |(x_n - L) - (x_m - L)| \\ &\leq |x_n - L| + |x_m - L| < 2\epsilon. \end{aligned}$$

Thus, (x_n) is a Cauchy sequence. □

Theorem 22. A Cauchy sequence is bounded.

Proof. Let $\epsilon = 1$. Then, there exists an N such that $|x_n - x_m| < 1$ for all $n, m \geq N$.

$$\begin{aligned} &|x_n - xN + 1| < 1 \\ &|x_n| < |x_{N+1}| + 1.|x_n| \qquad \leq \max\{|x_1|, |x_2|, \ldots, |x_N|, |x_{N+1}| + 1\}. \end{aligned}$$

Thus, (x_n) is bounded. □

Theorem 23. Let (x_n) be a sequence in $\mathbb{R}$ and is Cauchy. Then, (x_n) converges.

Proof. For all $\epsilon > 0$, there exists $\hat{N}$ such that for all $n, m > \hat{N}$, $|x_n - x_m| < \epsilon$. This implies the following statements.

$$\begin{aligned} x_n &\leq x_{\hat{N}+1} + \epsilon, \\ b_{\hat{N}} &\leq x_{\hat{N}+1} + \epsilon, \\ \limsup x_n &\leq b_{\hat{N}}. \end{aligned}$$

Additionally, we have

$$\begin{aligned} x_n &\geq x_{\hat{N}+1} - \epsilon, \\ a_{\hat{N}} &\geq x_{\hat{N}+1} - \epsilon, \\ \liminf x_n &\geq a_{\hat{N}}. \end{aligned}$$

So we have that $\limsup x_n \leq b_{\hat{N}} \leq a_{\hat{N}} \leq \liminf x_n + 2\epsilon$. Since this is true for all $\epsilon > 0$, $\limsup x_n = \liminf x_n = L$. Thus, (x_n) converges. □

2.5 Subsequences

Definition 23 (Subsequences). Let (x_n) be a sequence. Consider an increasing list of integers $n_1 < n_2 < \ldots n_k < \ldots$. Then, the sequence (y_k) defined by $y_k = x_{n_k}$ is a *subsequence* of (x_n).

A subsequence of (x_n) is a sequence formed by choosing an integer list n_k and defining $y_k = x_{n_k}$. This is a new sequence that is a subset of the original sequence.

Example 18. Let $(x_n) = (-1)^n$. Then, the following are subsequences.

$$\begin{aligned} y_k &= x_{2k} = 1, \\ z_k &= x_{2k+1} = -1. \end{aligned} \tag{36}$$

We will also define a subsequence with a restrictive condition.

Example 19. Let $x_n = \frac{(-1)^n}{n^2}$. Choose the subsequence of positive terms.

$$y_k = \{1/4, 1/16, 1/36, \ldots\}.$$

Definition 24 (Satisfying Subsequence). Let (x_n) be asequence and let $P(x)$ be a boolean statement depending on x, such that $P(x_n)$ is true for infinitely many n. We say that when a subsequence of (x_n) satisfies $P(x)$, it is a *satisfying subsequence*. Let n_1 be the smallest n such that $P(x_n)$ is true. Then,

$$n_{k+1} = \text{smallest } n > n_k \text{ such that } P(x_n) \text{ is true} = \text{argmin}_{n > n_k} P(x_n). \tag{37}$$

Note that, by definition, $n_{k+1} > n_k$ and n_{k+1} always exists, as $P(x)$ is true for infinitely many n.

Theorem 24. Let (x_n) be a sequence in $\mathbb{R}$ and let $L \in \mathbb{R}$. Then, there exists a subsequence $(y_n) = (x_{n_k})$ such that $(y_n) \to L$ if and only if for all $\epsilon > 0$, the set

$$\{n \in N \mid |x_n - L| < \epsilon\} \tag{38}$$

is infinite.

Proof. ($\implies$) Assume that there exists a subsequence (y_n) such that $(y_n) \to L$. Then, for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $|y_n - L| < \epsilon$. Since $y_n = x_{n_k}$, this implies that for all $n \geq N$, $|x_{n_k} - L| < \epsilon$. Thus, the set of n such that $|x_n - L| < \epsilon$ is infinite.

($\impliedby$) Assume that the set of n such that $|x_n - L| < \epsilon$ is infinite for all $\epsilon > 0$. Then, we can construct a subsequence (y_n) such that $|y_n - L| < 1$ with induction. Let n_1 be the smallest n such that $|x_n - L| < 1$. Then, let n_2 be the smallest $n > n_1$ such that $|x_n - L| < 1/2$. Continue this process to get a subsequence (y_n) such that $|y_n - L| < 1/k$ for all $k \in \mathbb{N}$. Thus, $(y_n) \to L$. $\square$

Note that this construction is completed *because* the subsequence is monotonic. This is a key part of the proof.

Proposition 10. $\mathbb{Q}$ is countable.

Proof. Pick any listing in $\mathbb{Q}$, $(q_1, q_2, \ldots)$. $\square$

Theorem 25. If a sequence (x_n) converges to L, then every subsequence of (x_n) also converges to L.

Proof. Let $(y_n) = (x_{n_k})$ be a subsequence of (x_n). Since $(x_n) \to L$, for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $|x_n - L| < \epsilon$. Since $n_k \geq k$, for all $k \geq N$, $|x_{n_k} - L| < \epsilon$. Thus, $(y_n) \to L$. $\square$

Theorem 26. Every sequence has a monotonic subsequence.

Proof. Let (x_n) be a sequence. Define $n_1 = 1$. Then, define n_{k+1} to be the smallest $n > n_k$ such that $x_n \geq x_{n_k}$ or $x_n \leq x_{n_k}$. This defines a monotonic subsequence. $\square$

Definition 25 (Dominant Sequence). Given a sequence (x_n), we say that a term x_n is *dominant* if for all $m > n$, $x_n > x_m$. We say that x_n is the "greatest" for the "rest" of the sequence.

We now prove Theorem 26.

Proof. Let $D = \{n \in \mathbb{N} \mid x_n \text{ is dominant}\}$ for a sequence (x_n). If D is infinite, then we can construct a decreasing subsequence. If D is finite, then there exists a dominant term x_N such that for all $n > N$, $x_N > x_n$. Thus, we can construct a decreasing subsequence. $\square$

This theorem and definition follows into the *Bolzano-Weierstrass Theorem*.

Theorem 27 (Bolzano-Weierstrass Theorem). Every bounded sequence has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence. By Theorem 26, there exists a monotonic subsequence (y_n). Since (y_n) is bounded and monotonic, it converges. Thus, every bounded sequence has a convergent subsequence. □

Definition 26 (Subsequential Limit). Let (x_n) be a sequence. A number L is a *subsequential limit* of (x_n) if there exists a subsequence (y_n) such that $(y_n) \to L$.

Example 20. If $(x_n) \to L$, then the set of all subsequential limits is $\{L\}$.

Example 21. Let (r_k) be in $\mathbb{Q}$. Then, the set of all subsequential limits is $\mathbb{R} \cup \{\pm\infty\}$.

We now provide some theorems on subsequential limits.

Theorem 28. With (x_n) being a sequence and Λ being the set of all subsequential limits, the following is true.

1. $\Lambda \neq \varnothing$.
2. $\limsup x_n \in \Lambda$.
3. $\sup \Lambda = \limsup x_n$.
4. $\liminf x_n \in \Lambda$.
5. $\inf \Lambda = \liminf x_n$.
6. Suppose $z_n \in \Lambda \cap \mathbb{R}$, then for all n and $z = \lim z_n$ exists, then Λ is closed (i.e. $z \in \Lambda$).
7. If the limit of x_n exists, then $\Lambda = \{L\}$.

Proof. We prove each part of the theorem.

1. A direct consequence of the Bolzano-Weierstrass Theorem.

2. Recall the definition of the $\limsup$.

$$\limsup x_n = \lim_{k\to\infty} \left(\sup_{m\geq k} x_m \right).$$

   We must show that there is a subsequence that converges to $\limsup x_n$. Let $y_k = \sup\{x_m \mid m \geq k\}$. Then, $\lim y_k = \limsup x_n$. For each k, because y_k is the supremum of $\{x_m \mid m \geq k\}$, there exists some index $n_k \geq k$ such that x_{n_k} is within $1/k$ of y_k. In other words,

$$\left| x_{n_k} - y_k \right| < \frac{1}{k} \implies \lim_{k\to\infty} x_{n_k} = \limsup x_n \implies \limsup x_n \in \Lambda.$$

3. From the previous part, we know that $\limsup x_n \leq \sup \Lambda$. Let $L \in \Lambda$. By definition, there is a subsequence (x_n) such that $(x_{n_k}) \to L$. By the definition of $\limsup$,

$$\limsup x_n = \lim_{k\to\infty} \left(\sup_{m\geq k} x_m \right) \geq \lim_{k\to\infty} x_{n_k} = L.$$

   Because this is true for every $L \in \Lambda$, we can conclude that $\sup \Lambda \leq \limsup x_n$. Thus, $\sup \Lambda = \limsup x_n$.

4. This is completely analogous to the $\limsup$ case.

5. This is completely analogous to the $\limsup$ case.
6. This prove is omitted for brevity, but the basic idea is that if $z_n \to z$, then for all subsequential limits z_k, $z_k \to z$.
7. If a sequence (x_n) converges to L, every subsequence also converges to L. So, there can be *no other* subsequential limits. Thus, $\Lambda = \{L\}$.

□

Theorem 29. Let (x_n) and (y_n) such that $(x_n) \to L$ and the convergence of (y_n) is arbitrary.

1. $\limsup x_n y_n = L \limsup y_n$.
2. $\liminf x_n y_n = L \liminf y_n$.

3 Functions

3.1 Continuity

FINISH LATER.

3.2 Limits of Functions

We have not yet formally defined limits on functions. However, we use them all the time in examples such as the following.

Example 22. $\lim_{x \to 2} x + 3 = 5$

Now, let us formally define limits on functions. We will use the following definition.

Definition 27 (Limit of a Function). Let $f : A \to B$ be a function, and let $L \in \mathbb{R} \cup \{\pm\infty\}$. We say that

$$\lim_{x \to a} f(x) = L \tag{39}$$

for $a \in A$ if we have a sequence $x_n \in A$ such that $x_n \to a$ and $f(x_n) \to L$.

Note that we require that $x_n \in A$. In a standard calculus class, the following limit does not exist on one side:

Example 23. $\lim_{x \to 0} \sqrt{x} = 0$

However, we require the x_n to be in the domain of the function, so all x_n are positive.

Theorem 30. This theorem is twofold.

1. If $f : A \to B$ is continuous at x, then

$$\lim_{x \to a} f(x) = f(a)$$

2. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

Additionally, limits of functions are linear in their arguments.

Theorem 31. Let $f : A \to B$ and $g : A \to B$ be functions. Then if f and g are continuous at a, we have

1. $\lim_{x \to a}(\alpha f(x) + \beta g(x)) = \alpha \lim_{x \to a} f(x) + \beta \lim_{x \to a} g(x)$
2. $\lim_{x \to a}(f(x) g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
3. $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

provided that $\lim_{x \to a} g(x) \neq 0$.

Example 24. Let $f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ be

$$f(x) = \frac{1}{(x-3)^5}$$

Then,

$$\lim_{x\to\infty} f(x) = 0$$

Additionally,

$$\lim_{x\to-\infty} f(x) = 0$$

This is because as x approaches ∞, the denominator approaches ∞, and thus the whole function approaches 0. The same is true for $-\infty$. Formally, we look at the sequence.

$$f(x_n) = \frac{1}{(x_n - 3)^5}$$

For all M, there exists an N such that for all $n > N$, we have $x_n > 3 + M$. Thus, we have

$$0 \leq f(x_n) < \frac{1}{M^5}$$

Thus, we have

$$\lim_{x_n\to\infty} f(x_n) = 0$$

Proposition 11. Let $f : \mathbb{R} to \mathbb{R}$ be a function, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$$

This is because as x approaches ∞, $\frac{1}{x}$ approaches 0.

Theorem 32. Let $f : A \to B$ and $g : B \to C$ with both being continuous. Let

$$\lim_{x\to a} f(x) = L$$

If g is continuous at L, then

$$\lim_{x\to a}(g \circ f)(x) = g(L) = \lim_{y\to L} g(y)$$

Proof. We know that g is continuous at L, so we have, meaning $y_n \to L$. Then, $g(y_n) \to g(L)$. If $x_n \to a$, then $f(x_n) \to L$. Thus, we have

$$g(f(x_n)) \to g(L)$$

Note that we do not need f to be continuous at a. This is because we are not looking at the limit of $f(x)$, but rather the limit of $g(f(x))$. However, we do need f to be continuous at L. This is because we are looking at the limit of $g(f(x))$ as $x \to a$, and we need $f(x)$ to be in the domain of g. $\square$

Proposition 12. Let $f : A \to B$. Then we claim that

$$\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right|$$

Proof. The proof of this is a direct result of the previous theorem. Let g be the absolute value function $g(x) = |x|$. Then, we have

$$\lim_{x\to a} |f(x)| = g\left(\lim_{x\to a} f(x)\right) = g(L) = |L| = \left|\lim_{x\to a} f(x)\right|$$

Thus, we have

$$\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right|$$

□

3.3 Power Series

Definition 28 (Power Series). Let (a_n) be a sequence. The sum

$$\sum_{n=0}^{\infty} a_n x^n \tag{40}$$

is called the *formal* power series with coefficients (a_n).

Note that here, *formal* does not mean what it means in conversational English. This is not a rigorous definition, we are simply giving the definition a "form".

We will be using the convention that $0^0 = 1$. In this section, we want to answer some big questions about the power series.

1. When does the formal power series make sense?
2. If we define a function
$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{41}$$
what are the properties of f?
3. What does calculus on a power series look like?

Theorem 33. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Then, define $\beta := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ and $R := \frac{1}{\beta}$.
1. For all $|x| < R$, the series converges absolutely.
2. For all $|x| > R$, the series diverges.
3. For all $|x| = R$, this theorem cannot make a claim.

Here, R is called the *radius of convergence* of the power series.

Proof. We will prove this theorem using the ratio test. We have

$$\limsup_{n\to\infty} |a_n x_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} |x| = |x| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \beta|x|$$

If $|x| > \frac{1}{\beta}$, then we have that $\beta|x| > 1$: the series diverges. If $|x| < \frac{1}{\beta}$, then we have that $\beta|x| < 1$: the series converges. If $|x| = \frac{1}{\beta}$, then we have that $\beta|x| = 1$: then we cannot conclude anything. Thus, we have shown that the series converges absolutely for all $|x| < R$ and diverges for all $|x| > R$. □

Example 25. Let $a_n = 1$ for all n. Then we have the following series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

So, we apply the theorem.

$$|a_n|^{\frac{1}{n}} = 1, \quad \beta = 1, \quad R = 1$$

Thus, the series converges on the interval $(-1, 1)$ and diverges on the intervals $(-\infty, -1)$ and $(1, \infty)$.

We need to calculate the behavior of the series at the endpoints. We have

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$$

$$\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \dots = \infty$$

Both of these series diverge. Thus, we have determined that the *domain of convergence* of the series is $(-1, 1)$.

Example 26. Let $a_n = \frac{1}{n!}$. Then we have the following series.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots := e^x$$

We will cover this series in more detail later. For now, we will apply the ratio test, as the root test requires some nuance using Sterling's approximation.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \left|\frac{1}{n+1}\right| \to 0$$

So $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$. And since $\limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$, we have that the series converges for all x. Thus, we have that the series converges for all $x \in \mathbb{R}$.

3.4 Uniform Convergence

Definition 29. Let (f_n) be a sequence of functions $f : A \to \mathbb{R}$. We say that $f_n \to f$ pointwise if

$$f(x) = \lim_{n\to\infty} f_n(x), \text{ for all } x \in A \tag{42}$$

We also say that f is the *pointwise limit* of (f_n).

Note that this is *not* a condition for continuity for $f(x)$.

Example 27. Let $f_n : [0,1] \to \mathbb{R}$ be defined as $f_n(x) = x^n$. If $x < 1$, then $x^n \to 0$.

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Let us unpack this. Our definition of the pointwise limit means that for all x, $f_n(x) \to f(x)$. Formally,

$$\forall\, x \in A, \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies |f_n(x) - f(x)| < \epsilon$$

Note that N might be dependant on x!

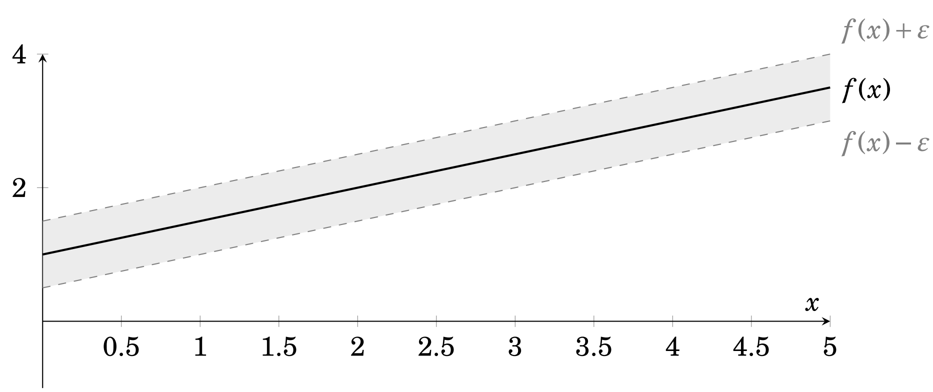
Definition 30. We say that f_n converges uniformly to f on A if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies |f_n(x) - f(x)| < \epsilon \tag{43}$$

Another formal definition is as follows.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies \sup_{x\in A} |f_n(x) - f(x)| < \epsilon \tag{44}$$

The *entire* function $f_n(x)$ lives in the band $|f_n(x) - f(x)| < \epsilon$.



Example 28. Let $f_n : [0,2\pi] \to \mathbb{R}$ be defined as $f_n(x) = \frac{1}{n}\sin(nx)$. This function oscillates rapidly and then the amplitude shrinks to 0. Note that since $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$, $|\sin(nx)| \le 1$ for all $x \in [0,2\pi]$. This implies

the following.

$$0 \le \sup_{x\in[0,2\pi]} \left|\frac{\sin(nx)}{n}\right| \le \frac{1}{n}$$

So, by the squeeze theorem, $f_n(x) \to 0$ uniformly on $[0, 2\pi]$.

Recall $f_n(x) = x^n$. We will show that it does not converge to

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Proposition 13.

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \text{ for all } n \in \mathbb{N} \tag{45}$$

Pick an n and pick an $\epsilon > 0$. If there exists an x such that $x^n > 1-\epsilon$, then we are done. If not, choose x such that $(1-\epsilon)^{\frac{1}{n}} < x < 1$. Then, we are done.

Theorem 34. If $f_n(x)$ is continuous on its domain for all n and $f_n \to f$ uniformly, then f is continuous on its domain.

Proof. Let $x_0 \in A$ be a point in the domain of f. We want to show that f is continuous at x_0. So, we need to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Since f_n converges uniformly to f, we have that for all $\epsilon > 0$, there exists an N such that for all $n > N$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Since f_n is continuous at x_0, we have that for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

Thus, we have

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

□

Why did we need uniform convergence? Note that we picked N first, and we knew however we changed x by choosing δ small, this would not mess up the outer two terms.

Definition 31 (Uniformly Cauchy). Let (f_n) be a sequence of functions $f : A \to \mathbb{R}$. We say that (f_n) is uniformly Cauchy if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sup_{x\in A} |f_n(x) - f_m(x)| < \epsilon \tag{46}$$

for all $n, m > N$.

Theorem 35. Let (f_n) be a sequence of functions, and let $f : A \to \mathbb{R}$ be a function uniformly Cauchy on A. Then, there exists a function $f : A \to \mathbb{R}$ such that f_n converges to f uniformly.

Proof. The basic idea of this proof is to first find the candidate function f and then prove that f_n converges to f uniformly. Choose $x_0 \in A$. Then consider the sequence $(f_n(x_0))$. We claim that this is Cauchy. For all $\epsilon > 0$, there exists an N such that $\sup_{x\in A} |f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$. But,

$$|f_n(x_0) - f_m(x_0)| \le \sup_{x\in A} |f_n(x) - f_m(x)|$$

This implies that for all $\epsilon > 0$, there exists an N such that for all $n, m > N$, we have that $|f_n(x_0) - f_m(x_0)| < \epsilon$, implying that $(f_n(x_0))$ is Cauchy. Since this sequence is Cauchy, it has a limit. Define $f(x_0)$ to be this limit. In other words, f is the pointwise limit of $(f_n(x_0))$.

Now, we need to show that f_n converges to f uniformly. Pick an $epsilon > 0$. Then, there exists an N such that for all $n, m > N$, we have $\sup_{x\in A} |f_n(x) - f_m(x)| < \epsilon$. If we choose $m > N$, then we have that

$$f_m(x) - \epsilon < f_n(x) < f_m(x) + \epsilon$$

Take this limit to infinity.

$$f_m(x) - \epsilon \le f(x) \le f_m(x) + \epsilon \implies \sup_{x\in A} |f_m(x) - f(x)| \le \epsilon$$

This is the definition of uniform convergence. Thus, we have shown that f_n converges to f uniformly. $\square$

4 Calculus

Now, we begin with our rigorous review of calculus.

4.1 Series of Functions

Definition 32. Let $g_n : A \to \mathbb{R}$ be a sequence of functions. Define the following series.

$$f_n(x) = \sum_{k=1}^{n} g_k(x) = g_1(x) + g_2(x) + \ldots + g_n(x) \tag{47}$$

Say f_n converges uniformly to f on A. Then we say $f(x) = \sum_{k=1}^{\infty} g_k(x)$ is the *uniformly convergent series* of functions $g_k(x)$ on A.

A basic example of this is the general power series.

Theorem 36. Assume $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on A, and $g_n(x)$ is continuous for all $x \in A$. Then, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ is continuous on A.

Example 29. For a power series, each $g_n = a_n x^n$ is continuous on any interval $[a, b]$.

This implies something powerful: if we can show a power series converges unformly, then it is continuous! Now, recall the Cauchy criterion for a series.

$$\forall \epsilon > 0, \exists N \in N \text{ such that } n > m > N \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon$$

Similarly, we can define the Cauchy criterion for a series of functions.

Definition 33. For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > m > N$, we have

$$\sup_{x \in A} \left| \sum_{k=m}^{n} g_k(x) \right| < \epsilon \tag{48}$$

Again, we are just saying that the partial sums are uniformly Cauchy.

Now, we define the *Weierstrass M-test*.

Theorem 37 (Weierstrass M-test). Let (M_k) be a sequence in $\mathbb{R}$, with $\sum_{k=1}^{\infty} M_k$ converging. If we have a sequence of functions $g_k : A \to \mathbb{R}$ such that $|g_k(x)| \leq M_k$ for all $x \in A$, then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on A.

Proof.

$$\left| \sum_{k=m+1}^{n} g_k(x) \right| \leq \sum_{k=m+1}^{n} |g_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

So, (g_k) satisifes the Cauchy criterion for uniform convergence, implying that M_k satisfies the Cauchy criterion, implying that $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on A. $\square$

We will now *formally* define differentiation and integration on power series, which leads us into the final section of this course, calculus.

Definition 34 (Derivatives and Integrals of Power Series). Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series. We define the *formal* derivatives and integrals of f as follows.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n \tag{49}$$

Note that this not a rigorous definition of the integral, we still have to show these correspond to the correct functions. They will though, mostly. Let us informally define a function as *integrable* if the integral of it "makes sense".

1. Let two functions g and h be integrable on $[a,b]$ such that $g(x) \leq h(x)$ for all $x \in [a,b]$. Then,

$$\int_a^b g(x)dx \leq \int_a^b h(x)dx.$$

2. Integrals of power series are linear in their arguments.

$$\int_a^b \alpha g + \beta h \; dx = \alpha \int_a^b g(x)dx + \beta \int_a^b h(x)dx$$

3. If g is integrable on $[a,b]$,

$$\left| \int_a^b g(x)dx \right| \leq \int_a^b |g(x)|dx \qquad \text{(Triangle Inequality)}$$

We will soon prove that all continuous functions are integrable.

Theorem 38. If a sequence $f_n \to f$ uniformly on $[a,b]$, then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx$$

Proof. Since $f_n \to f$ uniformly, for all $\epsilon > 0$, there exists an N such that for all $n > N$,

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

Then, we have

$$\begin{aligned} \left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \int_a^b |f_n(x) - f(x)|dx &\leq \int_a^b \sup_{x \in [a,b]} |f_n(x) - f(x)|dx \\ &< (b-a) \cdot \frac{\epsilon}{b-a} = \epsilon \end{aligned}$$

as needed. ☐

Theorem 39. Let $\sum_n a_n x^n$ be a power series with radius of convergence R. Then, for any $\widetilde{R} < R$, the power series converges uniformly on the interval $[-\widetilde{R}, \widetilde{R}]$.

Proof. First, let us show that the two series $\sum_n a_n x^n$ and $\sum_n |a_n| x^n$ have the same radius of convergence. The radius of convergence is determined by the growth of the coefficients. Taking the absolute value of the coefficients does not change the growth rate. Now, choose an $\widetilde{R} < R$. We know that

$$\sum_{n=0}^{\infty} |a_n| \widetilde{R}^n < \infty$$

meaning the series converges *absolutely* at $x = \widetilde{R}$. For all $x \in [-\widetilde{R}, \widetilde{R}]$, we have

$$\left| a_n x^n \right| \leq |a_n| \widetilde{R}^n$$

because $|x| \leq \widetilde{R}$. Thus, we define $M_n := |a_n| \widetilde{R}^n$. This is a bounding sequence such that

$$\sum_n M_n < \infty$$

This condition satisfies the Weierstrass M-test, so we have that the series converges uniformly on $[-\widetilde{R}, \widetilde{R}]$. Thus, we have shown that the series converges uniformly on $[-\widetilde{R}, \widetilde{R}]$. □

Corollary 1. If a power series $\sum_n a_n x^n$ has a radius of convergence R, the series converges to a continuous function on the interval $(-R, R)$.

Proof. If $-R < x_0 < R$, then we can choose an $\widetilde{R} < R$ such that

$$-R < -\widetilde{R} < x_0 < \widetilde{R} < R$$

implying uniform convergence on $[-\widetilde{R}, \widetilde{R}]$. By the previous theorem, we have that the series converges to a continuous function on $[-\widetilde{R}, \widetilde{R}]$. Since x_0 is in the interval, we have that the series converges to a continuous function on $(-R, R)$. □

Lemma 4. If a power series $\sum_n a_n x^n$ has a radius of convergence R, then so does

$$\sum_n n a_n x^{n-1} \quad \text{and} \quad \sum_n \frac{a_n}{n+1} x^{n+1}$$

Theorem 40. Define a function f as follows.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{50}$$

Suppose f has a radius of convergence R. Then, we define the integral of f as follows.

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \tag{51}$$

Definition 35. Let $f:[a,b]\to\mathbb{R}$. If there exists a function $g:[a,b]\to\mathbb{R}$ such that

$$\int_a^x g(t)dt = f(x) - f(a) \tag{52}$$

then we say g is the *derivative* of f at x.

Theorem 41. Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the derivative of f is continuous on the interval $(-R,R)$.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}. \tag{53}$$

Proof. Consider the power series $g(x)$.

$$g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

This function also has a radius of convergence R. Therefore, this converges to a continuous function on $(-R,R)$. Then, by the integral theorem, we have that

$$\int_0^x g(t)dt = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

with $|x|<R$. □

Let us get into some examples.

Example 30. Let $f(x)=\sum_{n=0}^{\infty} x^n$ with $|x|<1$.

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \to f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

Theorem 42 (Abel's Theorem). Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. If the series converges at $x=R$, then, the following limit exists.

$$\lim_{x\to R^-} f(x) = \sum_{n=0}^{\infty} a_n R^n \tag{54}$$

The proof for this theorem is tedious, so we will not cover it in detail. The basic idea is to show that the series converges uniformly on $[0,R]$. Then, we can use the uniform convergence theorem to show that the series converges to a continuous function on $[0,R]$. Finally, we can use the continuity of the function to show that the limit exists.

Definition 36 (The Exponential Function).

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{55}$$

We define the formal derivative of the exponential function as follows.

$$\frac{d}{dx}e^x = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \tag{56}$$

So, the exponential function is its *own* derivative. Let us prove some interesting properties of the exponential function using its power series representation. (Later, we will show that this is the Taylor series for e and is, in fact, *the* definition of e.)

Proposition 14.

$$e^{x+y} = e^x e^y$$

Proof. Let us write the power series for e^{x+y} as follows.

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^k y^m}{k!m!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{m=0}^{\infty} \frac{y^m}{m!}$$

This is exactly the product of e^x and e^y, as needed. □

The next example is slightly out of the scope of this course, but it is a direct (and important) use of the exponential function.

Proposition 15.

$$e^{ix} = \cos(x) + i\sin(x) \tag{57}$$

Proof. Let us write the power series for e^{ix} as follows.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \cos(x) + i\sin(x)$$

□

This is the Taylor series for $\cos(x)$ and $\sin(x)$, respectively. This is *Euler's formula*. Note that we use somewhat cyclic reasoning here, as we are using the Taylor series for $\cos(x)$ and $\sin(x)$ to prove the Taylor series for e^{ix}.

This result is, in my opinion, proof that studying the real numbers opens more questions than answers. After seeing this result in lecture, I knew that the "answers" I hoped real analysis would have were actually found in complex analysis.

4.2 Derivatives

Now, we will rigorous define the derivative of a function.

Definition 37 (Derivatives). Let $f : I \to \mathbb{R}$ and $x_0 \in I$. The *derivative* of f at x_0 is defined as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ or } \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{58}$$

If this derivative exists, we denote it by $f'(x_0)$ or $\frac{df}{dx}\Big|_{x=x_0}$. We say f is *differentiable* at x_0 if the derivative exists.

For polynomials, we have the following theorem.

Theorem 43 (Power Rule). Let $f(x) = \sum_{n=0}^{N} a_n x^n$ be a polynomial. Then, the derivative of f is given by

$$f'(x) = \sum_{n=1}^{N} n a_n x^{n-1} \tag{59}$$

This is known as the *power rule*.

Proof. Take a certain term x^n and apply the definition of the derivative.

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2 + \ldots$$

As $h \to 0$, all of the terms with h in them go to 0. Thus, we have the derivative is nx^{n-1} □

Theorem 44. If f is differentiable at x_0, then f is continuous at x_0.

Proof. We write f in the following form.

$$f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$$

We take the limit as $x \to x_0$ on both sides.

$$0 + \lim_{x \to x_0} f(x_0) = f(x)$$

This is the definition of continuity. Thus, we have shown that if f is differentiable at x_0, then f is continuous at x_0. □

Proposition 16. The derivative is linear.

$$\left(\alpha f + \beta g\right)'(x) = \alpha f'(x) + \beta g'(x) \tag{60}$$

Proof. Let f and g be differentiable at x_0. Then, we have

$$\begin{aligned}
\left(\alpha f + \beta g\right)'(x_0) &= \lim_{h\to 0} \frac{\alpha f(x_0+h) + \beta g(x_0+h) - (\alpha f(x_0) + \beta g(x_0))}{h} \\
&= \lim_{h\to 0} \left(\alpha \frac{f(x_0+h) - f(x_0)}{h} + \beta \frac{g(x_0+h) - g(x_0)}{h} \right) \\
&= \alpha f'(x_0) + \beta g'(x_0)
\end{aligned}$$

as needed. □

Proposition 17. The derivative of a product is given by the following.

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{61}$$

This is known as the *product rule*.

Proof. Let f and g be differentiable at x_0. Then, we have

$$\begin{aligned}
(fg)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} (f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0))
\end{aligned}$$

We can split this into two limits.

$$\begin{aligned}
(fg)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0) \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

as needed. □

Proposition 18. The derivative of a quotient is given by the following.

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \tag{62}$$

This is known as the *quotient rule*.

Proof. Let f and g be differentiable at x_0. Then, we have

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{hg^2(x_0+h)} \\
&= \lim_{h\to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} \cdot \frac{1}{g(x_0+h)} - f(x_0) \cdot \frac{g(x_0+h) - g(x_0)}{hg^2(x_0+h)} \right)
\end{aligned}$$

We can split this into two limits.

$$\begin{aligned}\left(\frac{f}{g}\right)'(x_0) &= f'(x_0)\cdot\frac{1}{g(x_0)} - f(x_0)\cdot\lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{hg^2(x_0+h)} \\ &= f'(x_0)\cdot\frac{1}{g(x_0)} - f(x_0)\cdot\frac{g'(x_0)}{g^2(x_0)}\end{aligned}$$

as needed. □

Proposition 19. The derivative of a composition is given by the following.

$$(f\circ g)'(x_0) = f'(g(x_0))g'(x_0) \tag{63}$$

This is known as the *chain rule*.

We will omit this proof for brevity.

Theorem 45 (Mean Value Theorem). Let $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$. If x_0 is an extremum of f and $f'(x_0)$ exists, then $f'(x_0)=0$.

Proof. Let us assume that x_0 is a maximum of f. Then there are three cases.

$$f'(x_0)>0,\quad f'(x_0)<0,\quad f'(x_0)=0$$

Let us assume that $f'(x_0)>0$. Then, we have

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

So, there exists a $\delta>0$ such that for all $x\in(x_0-\delta,x_0+\delta)$, we have

$$\frac{f(x)-f(x_0)}{x-x_0}>0$$

Let $x_0<x<x_0+\delta$. Then, $x-x_0>0$ implies that $f(x)-f(x_0)>0$. This implies that $f(x)>f(x_0)$, which is a contradiction. Thus, we have shown that if $f'(x_0)>0$, then x_0 cannot be a maximum. The same argument holds for the case where $f'(x_0)<0$. Thus, we have shown that if x_0 is an extremum of f, then $f'(x_0)=0$. □

Theorem 46 (Rolle's Theorem). Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$.

Note that this theorem does *not* imply that c is the only point such that $f'(c)=0$.

Proof. If f achieves an extremum in $[a,b]$ i.e. there exists an x_{min} or x_{max} such that

$$f(x_{min})\le f(x)\le f(x_{max})$$

for all $x \in [a,b]$. If x_{min} and x_{max} are both at end points of $[a,b]$, then $f(x)$ is constant on $[a,b]$. If x_{min} and x_{max} are both in (a,b), then we can apply the mean value theorem to show that there exists a point $c \in (a,b)$ such that $f'(c) = 0$. ☐

We now restate the mean value theorem in a more useful form.

Theorem 47. Assume f is continuous on $[a,b]$ and differentiable on (a,b). Then, there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{64}$$

This statement of the mean value theorem says that the *instantaneous rate of change* of f at c is equal to the *average rate of change* of f on the interval $[a,b]$.

Proof. Let $L(x)$ be a linear function such that $L(a) = f(a)$ and $L(b) = f(b)$.

$$\begin{aligned}
L(x) &= \alpha(x-a) + \beta \\
L(a) &= \alpha(a-a) + \beta = f(a) \\
L(b) &= \alpha(b-a) + \beta = f(b) \implies \alpha = \frac{f(b)-f(a)}{b-a} \\
L(x) &= \frac{f(b)-f(a)}{b-a}(x-a) + f(a)
\end{aligned}$$

Let $g(x) = f(x) - L(x)$ and $g'(x) = f'(x) - L'(x)$. Then, we have

$$\begin{aligned}
g(a) &= f(a) - L(a) = 0 \\
g(b) &= f(b) - L(b) = 0
\end{aligned}$$

By Rolle's theorem, we have that there exists a point $c \in (a,b)$ such that $g'(c) = 0$. This implies that $f'(c) - L'(c) = 0$. Thus, we have

$$f'(c) = L'(c) = \frac{f(b)-f(a)}{b-a}$$

as needed. ☐

Corollary 2. If f is differentiable on (a,b) and continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$.

In physics terms, we can understand this as "if my speed is never larger than v, than my average speed is also never larger than v". This is a very useful theorem, as it allows us to show that a function is uniformly continuous on an interval.

Corollary 3. If f is differentiable on (a,b) and there is an $x \in (a,b)$ such that $f'(x) = 0$, then f is constant on (a,b).

Corollary 4. If $f'(x) = g'(x)$ on (a,b), then $f(x) = g(x) + C$ for some constant C for all $x \in (a,b)$.

Corollary 5. If $f'(x) > 0$ on (a, b), then for any x, y such that $a < x < y < b$, $f(x) < f(y)$. This implies that f is strictly increasing on (a, b).

Theorem 48. Let $f : I \to f(I)$ be bijective and continuous. If $f'(x_0)$ exists and is not 0, then f^{-1} is differentiable at $f(x_0)$ and

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)} \tag{65}$$

This is known as the *inverse function theorem*.

Proof. Using composition, note that $f^{-1} \circ f = \text{id}$.

$$f^{-1}(f(x)) = x$$

We take the derivative of both sides and apply the chain rule.

$$\left(f^{-1}\right)'(f(x))f'(x) = 1$$

□

4.3 Taylor Series

Recall that if a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence R, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ also has a radius of convergence R. By induction,

$$f^{(k)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \tag{66}$$

also has a radius of convergence R. Let us evaluate this derivative at $x = 0$.

$$f^{(k)}(0) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k! a_k$$

We have expanded $f(x)$ around $x = 0$, this is a *Taylor series* about $x = 0$.

Definition 38 (Taylor Series). Let $f : I \to \mathbb{R}$ be a function that is infinitely differentiable on I. Then, we define the *Taylor series* of f at x_0 as follows.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tag{67}$$

Definition 39 (Taylor Remainder). Let f be defined on an open interval containing a point c.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(x) + R_n(x) \tag{68}$$

Here, $R_n(x)$ is the *remainder* of the Taylor series.

$$R_n(x) = f(x) - \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Note that f being equivalent to its Taylor series $\iff R_n(x) \to 0$ as $n \to \infty$.

Theorem 49 (Taylor Theorem). Let $f : (a,b) \to \mathbb{R}$ with all derivatives existing on (a,b). Then, if there exists a point c such that

$$\max_n \sup_{x \in (a,b)} \left| f^{(n)}(x) \right| < c \tag{69}$$

then $R_n(x) \to 0$ as $n \to \infty$ for all $x \in (a,b)$.

Lemma 5. Let $f : (a,b) \to \mathbb{R}$. Let $c \in (a,b)$. Suppose $f^{(n)}$ exist on (a,b), then for all $x \in (a,b)$ with $x \neq c$, there exists a $y \in (c,x)$ or (x,c) such that

$$R_n(x) = \frac{1}{n!} f^{(n)}(y)(x-c)^n. \tag{70}$$

Proof. Let us fix $x \neq c$ and $n \geq 1$. Then, choose an M such that

$$M = \frac{n!}{(x-c)^n} \underbrace{\sum_{k=0}^{n-1} \frac{(x-c)^k}{k!} - f(x)}_{R_n(x)}. \tag{71}$$

Rearranging this gives us

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{M}{n!}(x-c)^n$$

If we can show that there exists a y between x and c such that $f^{(n)}(y) = M$, then we have shown that the remainder of the Taylor series converges to 0 as $n \to \infty$.

Define g to be

$$g(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-c)^k + \frac{M}{n!}(t-c)^n - f(t)$$

$$\begin{aligned} g(c) &= f(0) + 0 - f(0) = 0 \\ g'(c) &= f'(0) + 0 - f'(0) = 0 \\ g''(c) &= f''(0) + 0 - f''(0) = 0 \\ &\vdots \\ g^{(k)}(c) &= 0 \end{aligned}$$

for all $k < n$. By a repeated use of Rolle's theorem, there exists an $x_n \in (c, x_n)$ such that $g^{(n)}(x_n) = 0$. □

Now, we will prove the theorem

Proof. If $\max_n \sup_{x \in (a,b)} |f^{(n)}(x)| < c$, then we have that

$$|R_n(x)| \leq \frac{0}{n!}|x - c|^n \implies \lim_{n \to \infty} R_n(x) = 0$$

as needed. □

Example 31. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ be the exponential function. Then, on the interval $(-M, M)$, we have

$$\max_n \sup_{x \in (-M,M)} \left| f^{(n)}(x) \right| \leq e^M$$

So, the Taylor series converges to e^x on the interval $(-M, M)$. This works for any M. Therefore, we have that the Taylor series converges to e^x for all $x \in \mathbb{R}$.

Consider the following limit.

$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ when } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \tag{72}$$

Theorem 50. Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f'(x) \neq 0$, $\lim_{x \to a} g'(x) \neq 0$. Then, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} \tag{73}$$

We can generalize the derivative to higher order derivatives.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f^{(n)}(x)}{\lim_{x \to a} g^{(n)}(x)} \tag{74}$$

This is known as L'Hospital's rule.

Proof. Let us Taylor expand f at $x = a$.

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Given that $f(a) = 0$ and $f'(a) \neq 0$, we have

$$f(x) = f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$
$$g(x) = g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \dots$$

Now, let us take the ratio of the functions.

$$\frac{f(x)}{g(x)} = \frac{f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots}{g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \dots} = \frac{f'(a) + \frac{f''(a)}{2}(x-a) + \dots}{g'(a) + \frac{g''(a)}{2}(x-a) + \dots}$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a) + 0 + \dots}{g'(a) + 0 + \dots} = \frac{f'(a)}{g'(a)}$$

To prove the general case, we can use the same argument. We can Taylor expand f and g at $x = a$.

$$f(x) = \frac{f^{(n)}(x-a)^n}{n!} + \mathcal{O}(x-a)^{n+1}$$
$$g(x) = \frac{g^{(n)}(x-a)^n}{n!} + \mathcal{O}(x-a)^{n+1}$$

Now, we can take the ratio of the functions and take the limit.

$$\frac{f(x)}{g(x)} = \frac{\frac{f^{(n)}(x-a)^n}{n!} + \mathcal{O}(x-a)^{n+1}}{\frac{g^{(n)}(x-a)^n}{n!} + \mathcal{O}(x-a)^{n+1}} = \frac{f^{(n)}(x-a)^n + \mathcal{O}(x-a)^{n+1}}{g^{(n)}(x-a)^n + \mathcal{O}(x-a)^{n+1}}$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f^{(n)}(a)(x-a)^n + 0}{\lim_{x\to a} g^{(n)}(a)(x-a)^n + 0} = \frac{\lim_{x\to a} f^{(n)}(a)}{\lim_{x\to a} g^{(n)}(a)}$$

as needed. □

Example 32. To calculate the following limit, we can use L'Hospital's rule.

$$\lim_{x\to 0} \frac{(\sin x - x)^2}{(\cos x - 1)^3}$$

Let us expand the numerator and denominator using Taylor series.

$$\sin x - x = -\frac{x^3}{6} + \mathcal{O}(x^5) \to \frac{x^6}{36} + \mathcal{O}(x^8)$$
$$\cos x - 1 = -\frac{x^2}{2} + \mathcal{O}(x^4) \to -\frac{x^6}{8} + \mathcal{O}(x^8)$$

We take the ratio of the two functions.

$$\frac{(\sin x - x)^2}{(\cos x - 1)^3} = \frac{\frac{x^6}{36} + \mathcal{O}(x^8)}{-\frac{x^6}{8} + \mathcal{O}(x^8)} = \frac{\frac{1}{36}}{-\frac{1}{8}} = -\frac{2}{9}$$

Proposition 20.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{75}$$

Proof. Let the limit be L.

$$\log L = \lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = \frac{1}{n}\log(1+n) = 1$$

So, we have $\log L = 1$. Thus, we have $L = e$. □

We have ignored an important fact about functions and their Taylor series. We showed that the Taylor series of a function f converges on the domain of convergence of f. However, we did not show that the Taylor series converges to f. In fact, this is not true in general.

Definition 40 (Analytic Functions). A function f is said to be *analytic* at a point x_0 if there exists a radius of convergence $R > 0$ such that the Taylor series converges to $f(x)$ for all $x \in (x_0 - R, x_0 + R)$.

4.4 The Riemann Integral

The Riemann integral (or the Darboux integral) was the first rigorous definition of the integral on an interval. We first define some other useful terms.

Definition 41 (Partition). Let $f : [a,b] \to \mathbb{R}$. Assume f is bounded. We define the following.

$$M(f,S) = \sup_{x\in S} f(x), \quad m(f,S) = \inf_{x\in S} f(x) \tag{76}$$

A *partition* of $[a,b]$ is a choice of points

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b \tag{77}$$

The *upper Darboux sum* of f with respect to a partition P is given by

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \tag{78}$$

The *upper Darboux integral* of f is given by

$$U(f) = \inf_P U(f,P) \tag{79}$$

Intuitively, this is the area of rectangles "above" the graph of f.

The *lower Darboux sum* of f with respect to a partition P is given by

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \tag{80}$$

The *lower Darboux integral* of f is given by

$$L(f) = \sup_P L(f,P) \tag{81}$$

Example 33. Let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = 2$. Then, for any partition P, we have

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\ &= 2\sum_{k=1}^{n}(t_k - t_{k-1}) \\ &= 2\sum_{k=1}^{n}(t_k - t_{k-1}) = 2(t_n - t_0) = 2(b-a) \end{aligned}$$

Similarly, $L(f) = 2(b-a)$.

The upper and lower Darboux sums being equal is not a coincidence. In fact, this is the definition of the Riemann integral.

Theorem 51. For any partition P,

$$m(f,[a,b])(b-a) \le L(f,p) \le U(f,P) \le M(f,[a,b])(b-a) \tag{82}$$

Proof. By definition, we have

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \le \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1}) = M(f,[a,b])(b-a)$$

Since $m(f,[a,b]) \le M(f,[t_{k-1},t_k])$, we have

$$\sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) \le \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \le U(f,P)$$

So, we have

$$m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a)$$

□

Corollary 6. If f is bounded, then $U(f)$ and $L(f)$ are bounded as well.

Proof. By the previous theorem, we have that there exists A,B such that

$$A \leq L(f,P) \leq U(f,P) \leq B$$

This implies that

$$A \leq L(f) \leq B \qquad A \leq U(f) \leq B$$

as needed. □

Is it obvious that $L(f) \leq U(f)$, always?

Definition 42 (Integrability). A function f is said to be *integrable* on $[a,b]$ if $L(f) = U(f)$. In this case, we define the *Riemann integral* of f as

$$\int_a^b f(t)dt = L(f) = U(f) \tag{83}$$

If $L(f) \neq U(f)$, then we say that f is *non-integrable* on $[a,b]$.

Example 34. We have previously shown that the constant function $f(x) = c$ is integrable on $[a,b]$.

$$\int_a^b f(t)dt = \int_a^b c\,dt = c(b-a)$$

Now, let us define the *width* of a partition P as follows.

$$w(P) = \max_k (t_k - t_{k-1}) \tag{84}$$

Lemma 6. Let $f : [a,b] \to \mathbb{R}$ be bounded and define $B := \sup_x |f(x)|$. Then, if P and Q are partitions of $[a,b]$ with $P \subseteq Q$ and Q has exactly ℓ more points than P, then

1. $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$
2. $0 \leq L(f,Q) - L(f,P) \leq 2\ell B w(P)$
3. $0 \leq U(f,P) - U(f,Q) \leq 2\ell B w(P)$