Moment-Based Regression in FitzHugh–Nagumo Neuronal Dynamics

Derivations

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1 Introduction

The FitzHugh–Nagumo (FHN) equations model the spiking behavior of excitable neurons using a two-dimensional reduction of the Hodgkin-Huxley system. In these notes, we consider a stochastic formulation of the FHN model, where neuron dynamics are subject to both intrinsic noise and static heterogeneity in input parameters. Our goal is to derive and analyze moment-based approximations to this stochastic population, and to build surrogate regression models that efficiently predict population-level summaries conditioned on static input variation.

Let $x(t) = (v(t), w(t))^\top \in \mathbb{R}^2$ represent the neuronal state, governed by the canonical FHN dynamics

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I(t), \qquad \frac{dw}{dt} = \epsilon(v + a - bw), \tag{1}$$

with $I(t)$ representing a time-varying input, modulated by a heterogeneous parameter λ. We model the system as an Ito stochastic differential equation (SDE):

$$dx = f(x, \lambda)dt + G(x, \lambda)dW_t, \tag{2}$$

where W_t is a multivariate Wiener process and G captures the state-dependent noise.

2 Moment Evolution Equations

We are interested in the evolution of conditional and marginal statistics of the population:

$$\mu(t) := \mathbb{E}[x(t) \mid \lambda], \qquad \Sigma(t) := \mathrm{Cov}[x(t) \mid \lambda], \qquad C(t) := \mathrm{Cov}[x(t), \lambda]. \tag{3}$$

We proceed by deriving the evolution equations for these moments under the assumption that f is smooth and $x(t)$ evolves according to an SDE with drift $f(x, \lambda)$ and diffusion $G(x, \lambda)$.

2.1 Conditional Mean

Define the conditional mean:

$$\mu_k(t \mid \lambda) := \mathbb{E}[x_k(t) \mid \lambda]. \tag{4}$$

Applying conditional expectation to each component of the SDE,

$$\mathbb{E}[dx_k(t) \mid \lambda] = \mathbb{E}[f_k(x(t), \lambda) \mid \lambda]dt + \mathbb{E}\left[\sum_{j=1}^{r} G_{kj}(x(t), \lambda)dW_j \mid \lambda\right].$$

Since $\mathbb{E}[dW_j \mid \lambda] = 0$, the stochastic term vanishes:

$$\frac{d}{dt}\mu_k = \mathbb{E}[f_k(x,\lambda) \mid \lambda] \quad \Longleftrightarrow \quad \dot{\mu} = \mathbb{E}[f(x,\lambda)]. \tag{5}$$

This equation is exact but intractable. We approximate f via a second-order Taylor expansion:

$$f(x,\lambda) \approx f(\mu,\lambda) + J_f(x-\mu) + \tfrac{1}{2}(x-\mu)^\top H_f(x-\mu).$$

Taking expectation, and using $\mathbb{E}[(x-\mu)^\top H(x-\mu)] = \mathrm{Tr}(H\Sigma)$, we obtain:

$$\frac{d}{dt}\mu \approx f(\mu,\lambda) + \tfrac{1}{2}\,\mathrm{Tr}(H_{xx}\Sigma). \tag{6}$$

2.2 Conditional Covariance

Using the definition $\Sigma := \mathrm{Cov}[x \mid \lambda]$, we differentiate:

$$\frac{d}{dt}\Sigma = \frac{d}{dt}\mathbb{E}[(x-\mu)(x-\mu)^T] = \mathbb{E}[\dot{x}x^T + x\dot{x}^T + GG^T] - \dot{\mu}\mu^T - \mu\dot{\mu}^T.$$

Substituting the expansion $f(x,\lambda) \approx f(\mu,\lambda) + J_f(x-\mu)$ and simplifying, we recover:

$$\frac{d}{dt}\Sigma \approx J_f\Sigma + \Sigma J_f^T + GG^T, \tag{7}$$

which is the standard Lyapunov equation.

2.3 Cross-Covariance with Static Input

Define the cross-covariance $C(t) = \mathrm{Cov}[x(t),\lambda]$. Since λ is time-invariant, we differentiate:

$$\frac{d}{dt}C = \frac{d}{dt}\mathbb{E}[(x-\mu)(\lambda-\bar{\lambda})^T] = \mathbb{E}[f(x,\lambda)(\lambda-\bar{\lambda})^T].$$

Using a first-order Taylor expansion about $(\mu,\bar{\lambda})$:

$$f(x,\lambda) \approx f(\mu,\bar{\lambda}) + J_x(x-\mu) + J_\lambda(\lambda-\bar{\lambda}),$$

we obtain:

$$\frac{d}{dt}C \approx J_x C + J_\lambda\Lambda, \tag{8}$$

where $\Lambda := \mathrm{Cov}(\lambda)$.

3 Closed-Form Approximations

In practice, we may not have access to individual states x, but only their means and covariances. To approximate $\mathbb{E}[f(x,\lambda)]$, we expand f about $(\mu,\bar{\lambda})$ to second-order:

$$\begin{aligned} f(x,\lambda) \approx\ & f(\mu,\bar{\lambda}) + J_x(x-\mu) + J_\lambda(\lambda-\bar{\lambda}) \\ & + \tfrac{1}{2}(x-\mu)^\top H_{xx}(x-\mu) + \tfrac{1}{2}(\lambda-\bar{\lambda})^\top H_{\lambda\lambda}(\lambda-\bar{\lambda}) + (\lambda-\bar{\lambda})^\top H_{x\lambda}(x-\mu). \end{aligned}$$

Taking expectations and using properties of Gaussian moments:

$$\mathbb{E}[f(x,\lambda)] \approx f(\mu,\bar{\lambda}) + \frac{1}{2}\operatorname{Tr}(H_{xx}\Sigma) + \frac{1}{2}\operatorname{Tr}(H_{\lambda\lambda}\Lambda) + \operatorname{Tr}(H_{x\lambda}C). \tag{9}$$

This yields a closed system of ODEs for $\mu(t), \Sigma(t), C(t)$, based entirely on known summary statistics.