Homework problems for smooth closed curves on 2-dimensions

Useful definitions:

1. *Smooth curve in 2-dimensions*: can be defined parametrically as a vector
$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}, \tag{1}$$
where t is the parameter, and $x(t), y(t)$ are differentiable functions of t.

2. *Tangent vector*:
$$\vec{T} \equiv \frac{d\vec{r}}{dt} \tag{2}$$

3. *Arc length*:
$$\mathrm{d}s = |\vec{T}|\,dt. \tag{3}$$

4. *Geodesic curvature, κ*: κ, at any point of the curve is a measure of how quickly the tangent turns with respect to the arc length. It is given by
$$\kappa = \frac{d\hat{T}}{ds}. \tag{4}$$

5. *Unit normal vector, $\hat{n}$*: given by
$$\hat{n} = \frac{1}{|\frac{d\hat{T}}{dt}|}\frac{d\hat{T}}{dt}. \tag{5}$$

Exercises:

1. Draw the curve defined by $x(t) = a\cos t,\ y(t) = b\sin t$ on the $x-y$ plane for $0 \le t \le 6$. Draw for a few values of a, b that you choose.

2. Show that the equation of an ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
can be recast in parametric form of a closed curve as
$$r(t) = a\cos t\,\hat{\imath} + b\sin t\,\hat{\jmath}.$$

3. Express the parametrization of the curve given by Eq. (1) in terms of s.

4. Show that the unit tangent vector can be expressed in terms of the arc length as
$$\hat{T} = \frac{d\vec{r}}{ds}.$$

5. Show that in terms of derivatives with respect to t of $\dot{x},\ \ddot{x},\ \dot{y},\ \ddot{y},\ \kappa$ becomes
$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{6}$$

6. Show that for a circle of radius R, κ is constant and is equal to $1/R$.

7. Calculate $\vec{T},\ \hat{T},\ \hat{n},\ \kappa$ for the ellipse.

8. Calculate the perimeter of the ellipse.

9. Show that $W_2 == \frac{1}{2\pi}\int_C \kappa \mathrm{d}s$ for any closed curve C on the plane.

10. For an ellipse whose semi-major axis is aligned with x axis and semi-minor axis is aligned with y-axis, show that $W_1^{0,2}$ can be put in the form
$$W_2^{1,1} = \begin{pmatrix} f(a,b) & 0 \\ 0 & f(b,a) \end{pmatrix} \tag{7}$$
where
$$f(a,b) = \frac{1}{2}a^2b^2\int_0^{2\pi} dt \frac{cos^2 t}{[a^2 - (a^2 - b^2)cos^2 t]^{3/2}}. \tag{8}$$

11. Now choose coordinates x, y such that the semi-minor axis lies along the $x-$axis and the semi-major axis lies along the $y-$axis. Show that in this case $W_1^{0,2}$ is again of diagonal form, but with the positions of the two eigenvalues interchanged in comparision to the above case.

12. Prove that for two identical ellipses placed such that their semi-major axes are parallel to each other, we get $\alpha = \beta$. If they are placed perpendicular to each other, $\alpha = 1$. If the relative angle between the axes is between zero and $\pi/2$ then $\beta < \alpha < 1$.

13. Consider 3 identical ellipses. Which configurations would correspond to $\alpha = 1$?

14. What about for any N number of identical ellipses?