PES UNIVERSITY ONLINE

DATA ANALYTICS

Unit 5: Markov chains contd

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Markov Chains with Absorbing States

A state *i* of a Markov chain is called an absorbing state when P_{ii} = 1, that is if the system enters this state, it will remain in the same state.

Many real-life problems such as bad debt (non-performing assets), bankruptcy, customer churn, employee attrition and so on can be modelled using absorbing state Markov chain.

Absorbing state Markov chain is a Markov chain in which there is at least one state k such that $P_{kk} = 1$.

Non-absorbing states in an absorbing state Markov chain are transient states.

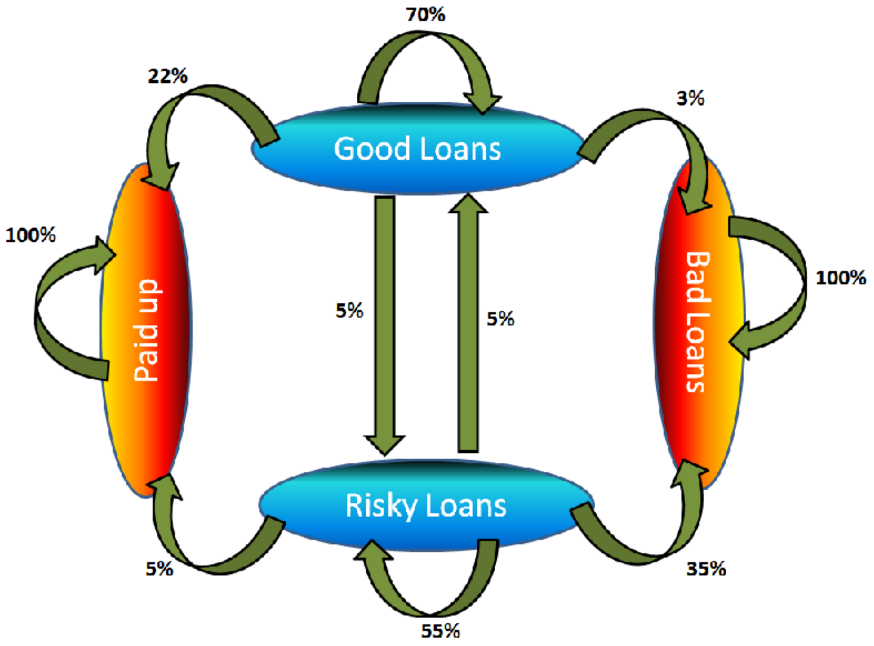
Markov Chains with Absorbing States

While using absorbing state Markov chains in analytics problem solving, we would like to learn the following from an absorbing state Markov chain:

1. The probability of eventual absorption to a specific absorbin state (when there are more than one absorbing states) from various transient states of the Markov chain.

2. The expected time to absorption from a transient state to absorbing states.

The above questions are answered using canonical form of the transition matrix.



https://www.analyticsvidhya.com/blog/2014/07/solve-business-case-simple-markov-chain/

Canonical Form of the Transition Matrix of an Absorbing State Markov Chain

The rows of the transition probability matrix of an absorbing state Markov chain can be rearranged such that the top rows are assigned for absorbing states followed by transient states (the idea here is to group the absorbing state and non-absorbing states).

Let A and T be the set of absorbing and transient states, respectively, in the Markov chain.

Then the transition probability matrix can be arranged such that

P =		A	T
	A	I	0
	T	R	Q

The Matrix P

The matrix P is divided into 4 matrices I, 0, R, and Q, where

- Matrix **I** is the identity matrix. It corresponds to transition within absorbing states.

- Matrix **0** is a matrix in which all elements are zero. Here the elements correspond to transition between an absorbing state and transient states.

- Matrix **R** represents the probability of absorption from a transient state to an absorbing state.

- Matrix **Q** represents the transition between transient states.

Expected time to absorption

To calculate the eventual probability of absorption, we would like to calculate the long-run (limiting probability) value of R in the above matrix. When we multiply the canonical form of the matrix, we get

$$\mathbf{P^n} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \left(\sum_{k=0}^{n-1} \mathbf{Q}^k\right)\mathbf{R} & \mathbf{Q^n} \end{pmatrix}$$

For large n, the matrix $\left(\sum_{k=0}^{n-1} \mathbf{Q}^k\right)\mathbf{R}$ will give the probability of eventual absorption to an absorbing state.

As $n \to \infty$, we can show that $\sum_{k=0}^{n-1} \mathbf{Q}^k = \mathbf{F} = (\mathbf{I}-\mathbf{Q})^{-1}$. The matrix F is called the fundamental matrix and the matrix FR is the probability of eventual absorption into an absorbing state from a transient state. The expected time to absorption is given by

Expected time to absorption = Fc

where c is a unit vector. That is, the row sum of the fundamental matrix gives the expected duration for absorption (that is, expected time it takes to reach an absorbing state from a transient state).

Example 16.5

Airwaves India (AI) is a mobile phone service provider based in Allahabad, India that provides several value-added services such as mobile data, video conferencing, etc. The market is highly competitive and AI faces high churn rate among its customers. The customers of AI are categorized into different states as listed below:

STATE 1 : Customer churn that generated no revenue/profit
STATE 2 : Customer churn that generated INR 200 profit per month on average (customer uses the service only for incoming calls and data)
STATE 3 :Customer state that generated INR 300 profit per month on average
STATE 4 :Customer state that generated INR 400 profit per month on average
STATE 5 :Customer state that generated INR 600 profit per month on average
STATE 6 :Customer state that generated INR 800 profit per month on average

Example 16.5

The transition probability values between different states are shown in Table 16.12.

TABLE 16.12 Transition probability matrix (based on monthly data)

	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0.05	0.05	0.90	0	0	0
4	0.10	0.05	0	0.80	0.05	0
5	0.20	0.10	0	0.05	0.60	0.05
6	0.10	0.20	0	0	0	0.70

(a) If a customer is in state 6, calculate the probability of eventual absorption in state 2?
(b) Calculate the expected value of time taken to absorption if the current state is 4.

Solution: 16.5

(a) To calculate the probability of absorption of a customer in state 6 to state 2, we have to calculate **FR**.

The matrix **Q** is given by

$$\mathbf{Q} = \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0.05 & 0 \\ 0 & 0.05 & 0.6 & 0.05 \\ 0 & 0 & 0 & 0.7 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.8 & 0.05 & 0 \\ 0 & 0.05 & 0.6 & 0.05 \\ 0 & 0 & 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.2 & -0.05 & 0 \\ 0 & -0.05 & 0.4 & -0.05 \\ 0 & 0 & 0 & 0.3 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{1} - \mathbf{Q})^{-1} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5.1613 & 0.6452 & 0.1075 \\ 0 & 0.6452 & 2.5806 & 0.4301 \\ 0 & 0 & 0 & 3.3333 \end{bmatrix}$$

Solution: 16.5

Probability of absorption FR is given by

$$FR = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5.1613 & 0.6452 & 0.1075 \\ 0 & 0.6452 & 2.5806 & 0.4301 \\ 0 & 0 & 0 & 3.3333 \end{bmatrix} \times \begin{bmatrix} 0.05 & 0.05 \\ 0.1 & 0.05 \\ 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.6559 & 0.3441 \\ 0.6237 & 0.3763 \\ 0.3333 & 0.6667 \end{bmatrix}$$

That is, if the current customer state is 6, the probability of absorption into churn state 2 is 0.6667.

(b) Expected value of time to absorption is given by Fc:

$$Fc = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5.1613 & 0.6452 & 0.1075 \\ 0 & 0.6452 & 2.5806 & 0.4301 \\ 0 & 0 & 0 & 3.3333 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5.91 \\ 3.65 \\ 3.33 \end{bmatrix}$$

Expected value of time to absorption when the current state is 4 is 5.91 months.

Expected Duration to Reach a State from Other States

The expected duration to reach a state *j* from state *i* can be calculated by solving the following system of equations:

Then, $E_{i,j}$ satisfies the following system of equations:

$$E_{i,j} = 1 + \sum_{k} P_{i,k} E_{k,j} \quad \forall\, i,\ i \neq j$$

$$\mathbf{E_{j,j} = 0}$$

Example 16.6

The transition probability matrix calculated based on monthly data is shown in Table 16.13. Calculate the expected duration (in months) for the process to reach state 7 from state 4.

The percentage of non-performing assets at a bank is classified into the following seven states:

State	State Description
1	NPA is less than 1%
2	NPA is between 1% and 2%
3	NPA is between 2% and 3%
4	NPA is between 3% and 4%
5	NPA is between 4% and 5%
6	NPA is between 5% and 6%
7	NPA greater than 6%

TABLE 16.13 Transition probability matrix between NPA states

	1	2	3	4	5	6	7
1	0.95	0.05	0	0	0	0	0
2	0.10	0.85	0.05	0	0	0	0
3	0	0.10	0.80	0.10	0	0	0
4	0	0	0.15	0.70	0.15	0	0
5	0	0	0	0.15	0.65	0.20	0
6	0	0	0	0	0.20	0.60	0.20
7	0	0	0	0	0	0.10	0.90

Solution for Example 16.6

Let $E_{4,7}$ be the expected number of duration for the process to reach state 7 from state 4. Then it satisfies the following system of equations:

$$E_{4,7} = 1 + 0.15E_{3,7} + 0.70E_{4,7} + 0.15E_{5,7}$$
$$E_{3,7} = 1 + 0.10E_{2,7} + 0.80E_{3,7} + 0.10E_{4,7}$$
$$E_{5,7} = 1 + 0.15E_{4,7} + 0.65E_{5,7} + 0.20E_{6,7}$$
$$E_{6,7} = 1 + 0.20E_{5,7} + 0.60E_{6,7} + 0.20E_{7,7}$$
$$E_{2,7} = 1 + 0.10E_{1,7} + 0.85E_{2,7} + 0.05E_{3,7}$$
$$E_{1,7} = 1 + 0.95E_{1,7} + 0.05E_{2,7}$$
$$E_{7,7} = 0$$

Solving the system of equations we get $E_{4,7}$ = 206.6667.
That is, it takes approximately 207 months on average for the process to reach state 7 from state 4.

Calculation of Retention Probability and Customer Lifetime Value using Markov Chains

CLV is the net present value (NPV) of the future margin generated from its customers or customer segments. CLV is calculated usually at a customer segment level.

Ching *et al.* (2004) showed that the steady-state retention probability can be calculated using

$$R_t = \sum_{i=1}^{n} \frac{\pi_i}{\left(\sum_{j=1}^{n} \pi_j\right)} (1 - P_{i0}) = 1 - \frac{\pi_0 (1 - P_{00})}{1 - \pi_0}$$

where R_t is the steady-state retention probability.

Calculation of Retention Probability and Customer Lifetime Value using Markov Chains

The customer lifetime value for *N* periods is given by (Pfeifer and Carraway, 2000):

$$\mathrm{CLV} = \sum_{t=0}^{N} \frac{\mathbf{P_I} \times \mathbf{P^t R}}{(1+i)^t}$$

where P_I is the initial distribution of customers in different states, P is the transition probability matrix, R is the reward vector (margin generated in each customer segments).

Example :16.7

The customers of Dubai Data Services (DDS) are classified into five categories as shown in Table 16.14 along with transition probability matrix. State 0 represents non-customers and the remaining states are different customer segments created based on the revenue generated. The average margin generated in different states is shown in Table 16.15 along with initial distribution of customers in millions.

Calculate the steady-state retention probability and CLV for 6 periods (N = 5) using a discount factor of d = 0.95.

TABLE 16.14 Customer states of DDS and transition matrix

	0	1	2	3	4
0	0.80	0.10	0.10	0	0
1	0.10	0.60	0.20	0.10	0
2	0.15	0.05	0.75	0.05	0
3	0.20	0	0.10	0.60	0.10
4	0.30	0	0	0.05	0.65

TABLE 16.15 Margin generated in different states

State	0	1	2	3	4
Average Margin	0	120	300	450	620
Customers (in millions)	55.8	6.5	4.1	2.3	1.6

Solution

The stationary distribution equations are

$\pi_0 = 0.8\pi_0 + 0.10\pi_1 + 0.15\pi_2 + 0.20\pi_3 + 0.30\pi_4$

$\pi_1 = 0.1\pi_0 + 0.60\pi_1 + 0.05\pi_2$

$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.75\pi_2 + 0.10\pi_3$

$\pi_3 = 0.10\pi_1 + 0.05\pi_2 + 0.60\pi_3 + 0.05\pi_4$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

Solving the above system of equations, we get $\pi_0 = 0.4287$. The steady-state retention probability R_t is

$$R_t = 1 - \frac{\pi_0(1 - P_{00})}{1 - \pi_0} = 1 - \frac{0.4287 \times (1 - 0.80)}{1 - 0.4287} = 0.85$$

Customer lifetime value for N = 5 is

$$\text{CLV} = \sum_{t=0}^{5} \frac{\mathbf{P_I} \times \mathbf{P}^t \mathbf{R}}{(1+i)^t}$$

where

P_I =

$$\begin{pmatrix} 55.8 & 6.5 & 4.1 & 2.3 & 1.6 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 0 \\ 120 \\ 300 \\ 450 \\ 620 \end{pmatrix}$$

Substituting the values in CLV equation, we get CLV = 40181.59.

References

Text Book:

"Business Analytics, The Science of Data-Driven Decision Making", U. Dinesh Kumar, Wiley 2017

Markov chains contd (absorbing states, expected duration to reach a state) [ch 16.6 - ch 16.8]

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THANK YOU

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