LINEAR ALGEBRA
ESA MODEL PAPER.
B.TECH IV SEMESTER (MAY 2020)

SCHEME & SOLUTION

1.a)

$$A:b = \left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 1 & 3 & 2+2a & 7b \\ 3 & 1 & 3+3a & 11b \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & -2 & 3 & 5b \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & 0 & 5+a & 10b \end{array}\right)$$

—(1M) —(1M) —(1M)

(i) $a \neq -5$ and any b gives unique non-trivial solution
(ii) $a \neq -5$, $b = 0$ gives trivial solution
(iii) $a = -5$, $b \neq a$ gives no solution
(iv) $a = -5$, $b = 0$ gives infinite number of solutions

—(4M)

1 b)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 4 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{pmatrix} = U \quad \text{—(3M)}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -3 & \\ & & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ — (3M)}$$

$L = U^T$ & $U = L^T$ — (1M)

1c)

$$A : I = \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 + 2R_1]{R_2 - 4R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & 4 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_2} \text{—(2M)}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 6 & 10 & -2 & 1 \end{array}\right) \xrightarrow[R_2 + \frac{1}{6}R_3]{R_1 - \frac{1}{6}R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -2/3 & 1/3 & -1/6 \\ 0 & 1 & 0 & -7/3 & 2/3 & 1/6 \\ 0 & 0 & 6 & 10 & -2 & 1 \end{array}\right) \xrightarrow{R_1 + 2R_2} \text{—(2M)}$$

—(1M)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/3 & 5/3 & 1/6 \\ 0 & 1 & 0 & -7/3 & 2/3 & 1/6 \\ 0 & 0 & 6 & 10 & -2 & 1 \end{array}\right) \xrightarrow{R_3/6} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/3 & 5/3 & 1/6 \\ 0 & 1 & 0 & -7/3 & 2/3 & 1/6 \\ 0 & 0 & 1 & 5/3 & -1/3 & 1/6 \end{array}\right) = (I : A^{-1}) \text{—(1M)}$$

—(1M)

$$A^{-1} = \begin{pmatrix} -16/3 & 5/3 & 1/6 \\ -7/3 & 2/3 & 1/6 \\ 5/3 & -1/3 & 1/6 \end{pmatrix}$$

2a) Consider $c_1(w_2-w_3)+c_2(w_1-w_3)+c_3(w_1-w_2)=0$ (1M)

$\Rightarrow (c_2+c_3)w_1+(c_1-c_3)w_2+(-c_1-c_2)w_3=0$ —(1M)

Since w_1, w_2, w_3 are linearly independent

$k_1w_1+k_2w_2+k_3w_3=0 \Rightarrow k_1=k_2=k_3=0$ —(1M)

Hence
$$\left.\begin{aligned} k_1&=c_2+c_3=0 \Rightarrow c_2=-c_3 \\ k_2&=c_1-c_3=0 \Rightarrow c_1=c_3=-c_2 \\ k_3&=-c_1-c_2=0 \Rightarrow c_1=-c_2 \end{aligned}\right\} c_1=c_2=c_3 \text{ may or may not be zero.}$$
(1M) (1M)

∴ given vectors are linearly dependent —(1M)

2b)
$$(A:b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 3 & 3 & b_4-3b_1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 3 & 3 & b_4-3b_1 \end{array}\right)$$
(3M) (1M)

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_4-3b_3+3b_1 \end{array}\right) \text{—(2M)}$$

System is solvable for $b_4-3b_3+3b_1=0$ —(1M)

2c)
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow[R_3'=R_3-2R_1]{R_2'=R_2-4R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{pmatrix} \xrightarrow{R_3''=R_3'-R_2'} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \text{—(2+1)M}$$

$R_3''=R_3'-R_2'$
$=(R_3-2R_1)-(R_2-4R_1)$
$=R_3-R_2+2R_1$ (1M)

$(2,-1,1)$ is the vector that spans the left nullspace of A (1M)

$b=(4,-2,2)=2(2,-1,1)\in N(A^T)$ — (1M)

$\dim(C(A^T))=2 \quad \dim(N(A^T))=1.$

3a)
$$A=\begin{pmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{aligned} &Ax=0 \\ &x+4y+4z+t=0 \\ &y=0 \end{aligned}\right\} \text{(1M)}$$
(1M)

$z=1, t=0 \Rightarrow x=-4$ $\begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ & $z=0, t=1 \Rightarrow x=-1$ $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ —(2M)

∴ $\begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are vectors orthogonal to given vectors (1M)

3b) $T(1,1,1) = (3,-3,3)$

$$\Rightarrow a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \left.\begin{matrix} a=3 \\ b=-6 \\ c=6 \end{matrix}\right\} -(2M)$$

Illy. $T(1,1,0) = (2,-3,3) \Rightarrow a=3, b=-6, c=5$ — (2M)

& $T(1,0,0) = (0,1,3) \Rightarrow a=3, b=-2, c=-1$ — (2M)

Matrix of transformation $T = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix}$ — (1M)

3c) Least squares method for $Ax=b$ is

$A^TA\hat{x} = A^Tb \Rightarrow \hat{x} = (A^TA)^{-1}A^Tb$ — ~~(1M)~~

$A^TA = \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}$ — (2M) $\quad {A^TA}^{-1} = \frac{1}{11}\begin{pmatrix} 22 & 11 \\ 11 & +6 \end{pmatrix}$ — (1M) $\quad A^Tb = \begin{pmatrix} -4 \\ 11 \end{pmatrix}$ — (2M)

$\hat{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ — (2M)

4a) Characteristic equation for A is $\lambda^3 + 3\lambda^2 - 4 = 0$ —

$\therefore \lambda = 1, -2, -2.$ are its eigen values — (1M)

$\lambda_1 = 1 \Rightarrow x_1 = (1,-1,1) \quad \lambda_2 = \lambda_3 = -2 \Rightarrow (-1,0,1) = x_2$ & $(-1,1,0) = x_3$ — (3M)

$u = (1,a,1) = x_1 \Rightarrow a = -1$ — (1M) $\rightarrow \lambda_1 = 1$

$v = (-1,b,1) = x_2 \Rightarrow b = 0$ — (1M) $\rightarrow \lambda_2 = -2$ } — (1M)

4b) $q_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})$ (1M) $\quad q_1^T a_2 = 1/\sqrt{2}$ (1M) $\quad p = (q_1^Ta_2)q_1 = (1/2, 0, 1/2)$ — (1M)

$e = (1/2, 0, -1/2) \quad \therefore q_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ — (1M)

$q_1^Ta_1 = \sqrt{2}$; $q_1^Ta_2 = 1/\sqrt{2}$; $q_2^Ta_2 = 1$ (1M) $\quad R = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{pmatrix}$ — (1M)

$A = QR = \begin{pmatrix} 1/\sqrt{2} & 1 \\ 0 & 0 \\ 1/\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{pmatrix}$ — (1M)

4c) Characteristic polynomial is $\lambda^2-4\lambda+3=0$.

$\Rightarrow \lambda = 3, 1$. —(1M)

$\lambda_1 = 3 \Rightarrow x_1 = \begin{pmatrix}2\\1\end{pmatrix}$ —(1M) $\lambda_2 = 1 \Rightarrow x_2 = \begin{pmatrix}3\\1\end{pmatrix}$ —(1M)

$S = \begin{pmatrix}2&3\\1&1\end{pmatrix}$ —(1M) $S^{-1} = \begin{pmatrix}-1&3\\1&-2\end{pmatrix}$ —(1M) $\Lambda = \begin{pmatrix}3&0\\0&1\end{pmatrix}$

$\therefore A = S\Lambda S^{-1} = \begin{pmatrix}2&3\\1&1\end{pmatrix}\begin{pmatrix}3&0\\0&1\end{pmatrix}\begin{pmatrix}-1&3\\1&-2\end{pmatrix}$ —(1M)

$A^{47} = S\Lambda^{47}S^{-1} = \begin{pmatrix}3-2(3^{47}) & 6(3^{47})-6\\ 1-3^{47} & 3(3^{47})-2\end{pmatrix}$ —(1M)

5a) $AA^T = \begin{pmatrix}3&1\\1&3\end{pmatrix}$ —(1M). char polynomial of $AA^T = \lambda^2-6\lambda+8=0$

$\Rightarrow \lambda = 2, 4$ —(1M) are eigen values of AA^T

$\lambda_1 = 4$ $x_1 = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow u_1 = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ —(1M) $\lambda_2 = 2 \Rightarrow x_2 = \begin{pmatrix}-1\\1\end{pmatrix}$ $u_2 = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ —(1M)

Eigen values of A^TA are 4, 2, 0 $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$ —(1M)

$v_1 = \frac{u_1^T A}{\sigma_1} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$ —(2M) $v_2 = \frac{u_2^T A}{\sigma_2} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ —(2M)

~~$A = U\Sigma V^T$~~ $v_3 = \frac{1}{\sqrt{2}}(-1, 1, 0) = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$ is orthogonal to v_1 & v_2 —(1M)

$A = U\Sigma V^T = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}2&0&0\\0&\sqrt{2}&0\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0&0&1\\ -1/\sqrt{2} & 1/\sqrt{2} & 0\end{pmatrix}$ —(3M)

5b) Eigen values of A are 16, 4. —(1M)

$\lambda_1 = 16 \Rightarrow x_1 = \begin{pmatrix}1\\1\end{pmatrix}$ —(1M) $\lambda_2 = 4 \Rightarrow x_2 \begin{pmatrix}-1\\1\end{pmatrix}$ —(1M)

$Q = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$ —(1M) $\Lambda = \begin{pmatrix}16&0\\0&4\end{pmatrix}$ —(1M) $Q^T = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$,

$R = Q\sqrt{\Lambda}Q^T = \begin{pmatrix}3&1\\1&3\end{pmatrix}$ —(2M) Eigen values of R are 4, 2 —(1M) which are positive

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