Linear Algebra

FIITJEE

Consider we have m equations with n variables each

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 \; --- \quad a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 \; --- \quad a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{2m}x_2 \; --- \quad a_{mn}x_n &= b_n \end{aligned}\right\} m \text{ Eqn's}$$

Then these equations can be shown/represented using matrices →

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix}}_{\text{Co-efficient Matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{Variables}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\text{Constant}}$$

* Multiplicative Representation.

$$Ax = b$$

- Assume we have a matrix with 'm' rows & 'n' columns then we say that its order is $m \times n$.
- Say we have A & B two matrices then the resulting matrix

$$aA + bB = C$$ where a & b are scalar co-eff. will alway lie in the space spanned by A & B.

- Assume we have a system

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where c_1, c_2, c_3 are const.

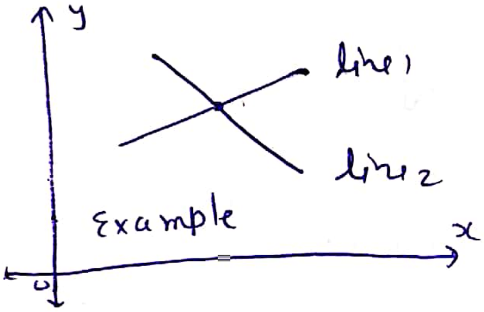
then we can make same changes both the sides (LHS & RHS) in order to perform simplification operations.

- **Row & Col Pictures** ⇒

Say we have

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

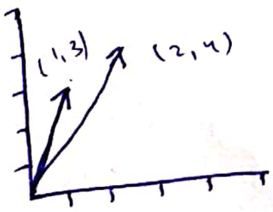
- Row picture simply means that we draw the lines
$x + 2y = c_1$ & $3x + 4y = c_2$
on a geometric plane and look for their intersection to find solⁿ.



$$a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

constants

- Column Picture means that we take the column vectors in the matrix and try to find out that for which linear combination of these vectors we can get the constant matrix.



★ Basic vector Addⁿ

Matrix

- **In Consistent** (No Solution) → $|A| = 0$ → Singular
- **Consistent** (Sol^n exists)
  - 1 Sol^n → $|A| \neq 0$ → Non-Singular
  - ∞ Sol^n → $|A| = 0$ → Singular

- Singular matrix has determinant zero. $|A| = 0$.
  A square matrix with $|A| = 0$ is non-invertible. (A^{-1} does not exist)
- When we say trivial sol^n $\Rightarrow$ obvious sol^n $\Rightarrow$ $(0, 0)$ //
- Rank of a matrix $\Rightarrow$ It is nothing but $\rightarrow$
  1) Max. no. of linearly independent rows in a matrix

  or

  2) Max. no. of linearly independent colums in a matrix

Denotion : $\rho(A)$

Say we have $Ax = b$ then

if $\rho(A) = \rho([A\ b]) \Rightarrow$ Unique sol^n

$\rho(A) = \rho([A\ b]) = n \Rightarrow 1\ Sol^n$

$\rho(A) = \rho([A\ b]) \neq n \Rightarrow \infty\ Sol^n$

$\rho(A) \neq \rho([A\ b]) \Rightarrow 0\ Sol^n$

n : num of variable/unknown

Fundamental Way to check Linear Dependency

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Then if

$$c_1V_1 + c_2V_2 + c_3V_3 \cdots c_kV_k = 0 \quad \text{such that}$$

$$c_1 = c_2 = c_3 \cdots = c_k = 0 \quad * \text{ all are '0'}$$

Then we say that $V_1, V_2 \ldots V_k$ are linearly Independent.

Transpose of a Matrix →

The transpose of a matrix A is a matrix whose columns are the rows of matrix A in the same order.

Denotion : A^T or A^{tr}

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & - & a_{2n} \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n} \Longrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n\times m}$$

* Basically a flip along diagonal

Co-factor Matrix → It is a matrix of the signed minors of the elements of a given matrix.

Minor : In order to find minor of an element. Hide the respective row & col. The find the determinant of the remaining matrix.

Signs →

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times 3}$$

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Eg → $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ⇒ Co factor matrix = ?

$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$\begin{bmatrix} & \\ & 4 \end{bmatrix}$

Hide rows for 1 & find | | ⇒ 4

Look for sign → $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ ⇒ Co fac = $[4$

• Now hide row & col for 2

$\begin{bmatrix} & \\ 3 & \end{bmatrix}$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ ⇒ Co fac = $\begin{bmatrix} 4 & -3 \end{bmatrix}$

for all elements

Cofactor matrix = $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

<u>Eg-2</u>

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ ⇒ Co factor = $\begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$

◉ Adjoint of a matrix ⇒

It is simply the transpose of the co-factor matrix.

Denotion ⇒ Adj (A).

★ also called <u>adjugate</u>

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Finding Inverse of a sqr matrix using Adjoint.

$$\Rightarrow \boxed{A^{-1} = \frac{adj(A)}{|A|}}$$

adj(A) → adjoint

|A| → Determinant

Operations that can be performed →

1) Row Swaps

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

2) Row Changes $\Rightarrow R_j = R_j + \lambda R_i$; $\lambda \in$ Const.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

when we say $R_2 = R_2 - R_1$

3) Scalar multiplication in a Row

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \xrightarrow[\text{scalar} = 2]{} \begin{bmatrix} 2 & 4 & 6 \\ 7 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

- Augmented form

$Ax = b$ can be written as $\begin{bmatrix} A & b \end{bmatrix}$

#JaisiJiskiConvenience

Gaussian Elimination ⇒

It is a way of solving the linear system of eqⁿ by bringing the matrix in an upper triangular form.

→ All entries ~~above~~ below the diagonal are zero.

we use basic operations to apply gaussian elimination.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 2 \\ 3 & 4 & 5 & 8 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & -4 \\ 3 & 4 & 5 & 8 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$$\downarrow \text{swap}(R_2, R_3)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & -4 \end{bmatrix}$$

↳ upper Δar form

* A temporary breakdown is where a pivot element becomes zero, which can be fixed by row swaps. But permanent breakdown cannot be fixed. In such cases system becomes singular ($|A| = 0$).

- Some Stuff regarding ~~Elimination~~ Elementary Matrices

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Say we have a matrix A.

Then an elementary matrix is obtained by performing one single row operation.

$$E_{32}E_{31}E_{21}A = V$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}V$$

Say if $V = U$ then $A = LU = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

Say $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -h & 1 \end{bmatrix}$$

Bhauna Samjho

- **Permutation Matrix** ⇒

If I_n is an identity matrix of order n then permutation matrix can be obtained by swapping any 2 rows. to swap m^{th} & n^{th} row of ur matrix multiply with P_{mn}.

Eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P_{12}}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$P^{-1} = P^T$ Always ⚠

Triangular Factorization $\Rightarrow$ (LU Decomposⁿ)

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$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}_{U}$$

{uses row swaps}

- Bhavna $\rightarrow$ • make U outta A $\rightarrow$ use the const. from row change opr $R_i = R_i - \underbrace{\lambda R_j}$ $\rightarrow$ as ele for L. where diag $=1$;

LDU decomposⁿ $\rightarrow$

$$A = L \cdot D \quad U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} \text{pivot}1 & 0 & 0 \\ 0 & \text{pivot}2 & 0 \\ 0 & 0 & \text{pivot}3 \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Basically Divide all rows of U in LU is divided by pivots in U and made and pivots are made into

Checking if matrix is Invertible

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Say we have a matrix A (order n)

then $\boxed{AA^{-1} = A^{-1}A = I}$ $\quad A^{-1}$ is unique

If A^{-1} exists then we can find x when $Ax = b$

$\Rightarrow \boxed{x = A^{-1}b}$ {Premultiply A^{-1} both sides}

* A sqr matrix of order n is invertible if & only if gaussian Eliminatⁿ produces 'n' pivots.

$|A| \neq 0$ & gaussian Eliminatⁿ produces 'n' pivots.
(i.e. has <u>unique</u> Solⁿ)

$(AB)^{-1} = B^{-1}A^{-1}$

$[A\ I] \xrightarrow{\text{transform}} [I\ A^{-1}]$ # used to find A^{-1}

Let A & B be Invertible matrices of order 'n' then $(A \pm B)$ may or maynot be invertible.

AB is always invertible.

$(A^T)^T = A$

$(A \pm B)^T = A^T \pm B^T$

$(AB)^T = B^T A^T$

$(A^T)^{-1} = (A^{-1})^T$

if $(A^T) = A \Rightarrow A$ is symmetric

• if A is symmetric
$\Rightarrow L^T = U$, $U^T = L$

Proof →

$A = LDU$

$A^T = (LDU)^T$

$A^T = U^T D^T L^T$

$\downarrow$

$A = U^T D L^T$ # Proofed

Gauss - Jordon - Method

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Now we know that $A = LU$

$\Rightarrow A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$ — ①

also iff

$Ax = b \Rightarrow x = A^{-1}b$ — ②

Using these 2

$$[A : I] \Rightarrow [LU : I] \Rightarrow [I : (LU)^{-1} I] \Rightarrow [I : A^{-1}]$$

Boom its here!
(Poof)

Steps →

$$[A\ I] \longrightarrow [U\ L^{-1}] \longrightarrow [I\ A^{-1}]$$

to make $U \rightarrow I$ make all ele. 0
then divide each row by pivots.

Bhauna
↓

$$AI = A \xrightarrow{\text{transform}} AA^{-1} = I$$

isko I bana do toh I automatically A^{-1} banjayega

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Vector Spaces

- A vector space is non empty set V of objects called vectors, together with the following oprⁿ ⇒

1) If $x, y \in V$ then $x+y \in V$ ⇒ V is closed under addⁿ
2) If $c \in R$ & $x \in V$ then $cx \in V$ ⇒ V is closed under mulⁿ
3) $x+y = y+x$ {Commutative}
4) $(x+y)+z = x+(y+z)$ {associative}
5) There is a unique zero vector such that
   $0+x = x+0 = x$ {identity law}
6) For each vector $x \in V$ there is a unique vector $-x$
   $x+(-x) = 0$ {Inverse law}
7) $c_1(x+y) = c_1x + c_1y$ $\quad c_1 \in R$
8) $(c_1+c_2)x = c_1x + c_2x$ $\quad c_1, c_2 \in R$
9) $c_1(c_2x) = (c_1x)c_2$ $\quad c_1, c_2 \in R$
10) $1 \cdot x = x \cdot 1 = x$

A set of all $R^1, R^2, R^3 \dots R^n$ are going to form vector space

All polynomials in 'x' of degree 'n' can be vector space

Origin is the smallest possible subspace

Say S (non empty) is a subset of $(V, +, \cdot)$ such that $(S, +, \cdot)$ then S is a subspace of V.

Say we have a matrix of order $m \times n$ then the column space lies in R^m and null space in R^n.

To find Null space we always solve for $Ax = 0$

⊙ Subspaces ⇒ {ye badi shi cheez hai}

A non-empty subset 'w' of a vectorspace V is called a subspace of V if 'w' is itself a vector space under the same operations of addition and scalar mul^n as defined in V.

★ NOTE w is subspace of V if →

- if $0 \in w$ {origin}
- if $x, y \in w \Rightarrow x + y \in w$
- if $x \in w \Rightarrow cx \in w$; $c \in R$

Every vector space V has atleast 2 subspaces →
- V itself
- Zerospace {containing only zero vector}

V → largest , {0} → smallest.

The space R^n consists of $(n+1)$ subspaces

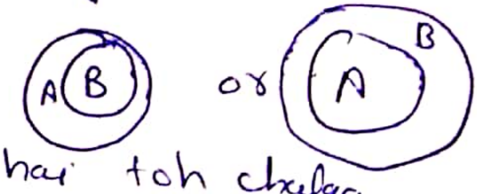
There cant be any subspace without origin

If A and B are two subspaces then

$A \cup B \Longrightarrow$ ✗ not a subspace

$A \cap B \Longrightarrow$ ✓ is a subspace

NOTE $A \cup B$ can be a subspace of V iff $A \subset B$ or $B \subset A$



Subset hai toh chalega

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diagonal matrices, upper & lower Δar matrices are subspaces of $n \times n$ matrix along with 0 matrix.

Column Space $\Rightarrow$

Let A be a matrix of order $m \times n$ then the column space of A is the set of all linear combinations of the columns of matrix A.

$$C(A) = c_1 \, Col_1(A) + c_2 \, Col_2(A) \cdots c_n \, Col_n(A) \; ; \; c_i \in R$$

working space $= R^m$

{Self note: pivots ka kuch khel hai/tha yad nhi arra}

Null Space $\Rightarrow$

Let A be a matrix of order $m \times n$ then the null space of A is the set of all solutions of $Ax = 0$. It is a sub space of R^n.

$$N(A) = \{ s \mid As = 0 \}$$

Echelon and Row Reduced Form ⇒

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A rectangular matrix is said to be in Echelon form if it has the following characterizations →

i) All zero rows are below non-zero rows

ii) Each pivot lies to the right of the pivot in above rows. Thus forming kinda staircase pattern.

iii) All elements below the pivot are zero.

- The matrix is said to be in row reduced form if along with the above features it also has →

iv) Pivots should be 1 and all elements above pivot must be zero.

The process of reducing a matrix to row Echelon form is called Gaussian Elimination and that of reducing it to reduced row-Echelon form is Gauss-Jordan Elimination.

{ Self note : Cols with pivot = colspace, other cols = nullspace }

$Rx = 0$ has same solutions as $Ax = 0$.
R reveals the solutions immediately
→ Cols with pivots → Col. space
→ Other cols. → null space

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Pivot and Free Variables ⇒

Say we have a matrix $Rx = 0$

$$\begin{bmatrix} 8 & 3 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In this example
$x, z \rightarrow$ Pivot Variables
$y, \lambda \rightarrow$ Free Variables

The variables are divided into →

- Pivot variables → which corresponds to column with pivot.
- Free Variables → " " " " " " without pivot.

In order to find the most general solⁿ to $Rx = 0$, we may assign random values to free variables. And the pivot variables are determined completely in terms of free variables.

$$8x + 3y - \lambda = 0 \quad -①$$ $$x = (-3y + \lambda)/8$$

$$z + \lambda = 0 \quad -②$$ $$\Rightarrow \boxed{z = -\lambda}$$

Then

$$X_n = \begin{bmatrix} x \\ y \\ z \\ \lambda \end{bmatrix} = \begin{bmatrix} (-3y+\lambda)/8 \\ y \\ -\lambda \\ \lambda \end{bmatrix} = \begin{bmatrix} -\frac{3y}{8} \\ y \\ \cancel{0}\,0 \\ \cancel{0}\,0 \end{bmatrix} + \begin{bmatrix} \lambda/8 \\ 0 \\ -\lambda \\ \lambda \end{bmatrix}$$

↓

Complete Solⁿ or general Solⁿ

$$= y \begin{bmatrix} -3/8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1/8 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

special Solⁿ

The complete Solⁿ is the linear combination of 2 special or more solⁿ

Linear Dependency →

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Say we have vectors $v_1, v_2, v_3 \cdots v_n$ and a vector space V. Then vectors of the form →

$c_1 v_1 + c_2 v_2 + c_3 v_3 \cdots c_n v_n$ where $c_i \in R$ —①

are linear combination of vectors.

- If $c_1 v_1 + c_2 v_2 + c_3 v_3 \cdots c_n v_n = 0$

↳ Implies $c_1 = c_2 = c_3 \cdots = c_n = 0$ {all are 0}

↳ Implies $v_1, v_2, v_3, \cdots v_n$ are Linearly INDEPENDENT

else they are dependent.

in other words any vector cant be written as a linear combination of other vectors.

A set of n vectors in R^m must be dependent if $n > m$.

Any set of vectors including zero vector are always dependent.

Columns with pivots are always Independent.

Span ⇒

Let V be a vector space,

The set of vectors $v_1, v_2, v_3, v_4 \cdots v_n$ are said to span V if for all v ; $v \in V$.

$v = c_1 v_1 + c_2 v_2 \cdots c_n v_n$

Eg → $[0]$ spans $[0]$ in R_1

$[3]$ spans R_1 in R_1

$\begin{bmatrix}1\\0\end{bmatrix}$ spans x-axis in R_2

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Basis ⇒ A set of vectors say $S = \{V_1, V_2, V_3 - - V_n\}$ of a vector space V is called a **Basis** of V if the vectors $V_1, V_2, - - V_n$ are linearly independent and $V = \text{span}(S)$.

- The dimension of a Vector space is the number of basis vectors
- Basis isn't unique. There can be ∞ Basis.

The 4 fundamental Subspaces ⇒

Let A be a matrix of order $m \times n$. Then →

1) Column Space : The column space of A is set of all the linear combinations of the colums of A.

$$C(A) = c_1 \,Col_1(A) + c_2 \,Col_2(A) - - - c_n \,Col_2(A) \; ; \; c_i \in R$$

working space $= R^m$
If $\rho(A) = k$, then $\dim(C(A)) = k$
Basis of C(A) corresponds to the set of vectors which are colums of A corresponding to the columns having pivots. in echelon form of A.

2) Row space : The row space of A is set of all the linear combinations of the rows of A.

In other words Row space $= C(A^T)$

Working space R^n
If $\rho(A) = k$, then $\dim C(A^T) = k$
The Basis of $C(A^T)$ corresponds to set of vectors which are rows of A corresponding to the rows having pivots. in echelon form of A.

3) Null Space : The null space of A is the set of all the vectors which are solution of $Ax=0$ **FIITJEE**

It is a subspace of R^n

$$N(A) = \{x \mid Ax = 0\}$$

If $\rho(A) = k$, then $\dim(N(A)) = n-k$

The Basis for $N(A)$ is obtained by solving the system $Vx = 0$, identifying the pivot variables and free variables. V: row reduced form of A.

4) Left Null space : The left null space of A is the set of all vectors which are solution of $A^T x = 0$.

It is a subspace of R^m

$$N(A^T) = \{x \mid A^T x = 0\}$$

If $\rho(A) = k$, $\Rightarrow \dim N(A^T) = m-k$

The Basis for $N(A^T)$ is obtained by solving the matrix to get row reduced form and looking at the zero rows of the matrix and then looking for corresponding rows in A.

* $\dim(C(A)) + \dim(N(A)) = n$

$\dim(C(A^T)) + \dim(N(A^T)) = m$

Existence of Inverses -

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If $\rho(A) = m$, ($m \times n$ matrix A), then A will have right inverse of order ~~mxn~~ $n \times m$

$$A_{m\times n} \, A^{R^{-1}}_{n \times m} = I_{m \times m}$$

★ Right Inverse $= A^T (A A^T)^{-1}$

If $\rho(A) = n$, ($m \times n$ matrix A), then A will have left inverse of order ~~mxn~~ $n \times m$

$$A^{L^{-1}}_{n \times m} \, A_{m \times n} = I_{n \times n}$$

★ Left Inversse $= (A^T A)^{-1} A^T$

Note

1) In the existence case (right Inverse) the no. of sol^n when the col. span R^n is 1 or ∞.

2) In the uniqueness case (left Inverse) the no. of sol^n is 0 or 1.

3) $m \leq n$, $\rho(A) = m$ Right Inverse [Full row rank]

4) $n \leq m$, $\rho(A) = n$ Left Inverse [Full Col. rank]