DATA ANALYTICS

Unit 5: Markov chains in Predictive Analytics

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Markov Chains in Predictive Analytics

One of the primary applications of Markov chain is predicting the values of Xn in the future.
For example, assume that the initial distribution of customers in 4 states is

$$P_I = (450, 225, 175, 150).$$

Assume that the one-step transition matrix P is as

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.8189 & 0.0882 & 0.0472 & 0.0457 \\ 2 & 0.1128 & 0.7180 & 0.0902 & 0.0789 \\ 3 & 0.2077 & 0.0984 & 0.6011 & 0.0929 \\ 4 & 0.0663 & 0.0964 & 0.0964 & 0.7410 \end{array}$$

Markov Chains in Predictive Analytics

Using Chapman–Kolmogorov relationship [Eq. (16.12)], we can show that the distribution of customers after n periods is given by $P_I \times P^n$, where P_I is the initial distribution of customers across various states and P is the one-step transition matrix. For example, the distribution of customers after 4 weeks is $P_I \times P^4$. That is

$$\begin{pmatrix}450 & 225 & 175 & 150\end{pmatrix} \times \begin{pmatrix}0.8189 & 0.0882 & 0.0472 & 0.0457 \\ 0.1128 & 0.7180 & 0.0902 & 0.0789 \\ 0.2077 & 0.0849 & 0.6011 & 0.0929 \\ 0.0663 & 0.0964 & 0.0964 & 0.7410\end{pmatrix}^4 = \begin{pmatrix}417.84 & 243.06 & 150.69 & 188.41\end{pmatrix}$$

So after 4 periods, the distribution of customers will be

State 1: 417.84; State 2: 243.06; State 3: 150.69; and State 4: 188.41

Stationary Distribution in a Markov Chain

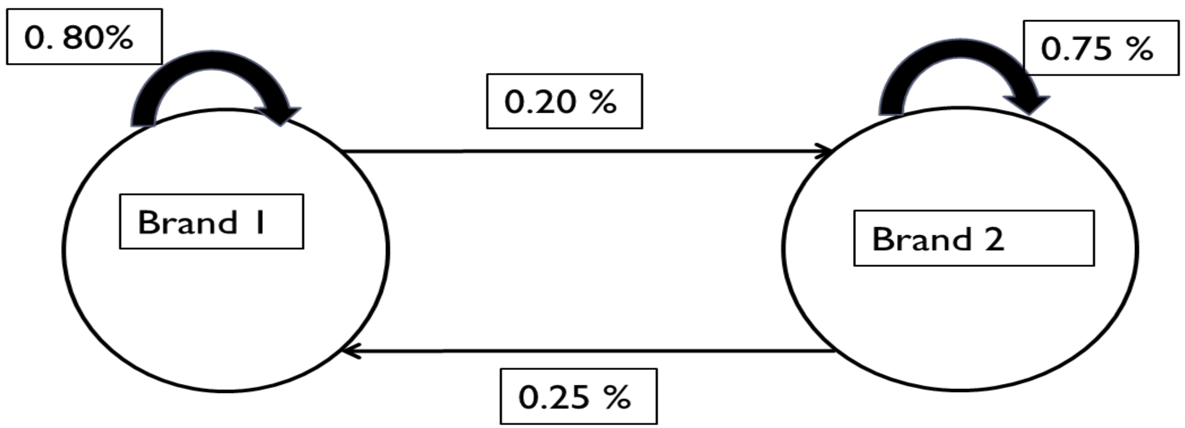
Consider brand switching between two brands (B1 and B2) and let the initial market share be as shown in the following vector:

$P_I = (0.2\ 0.8)$

Transition probability

	Brand 1	Brand 2
Brand 1	0.80	0.20
Brand 2	0.25	0.75

State transition diagram between brands.



Stationary Distribution in a Markov Chain

In Table given, both rows of the matrix **P^n** converge to 0.555556 and 0.444444 as the value of n increases.

The market share of brands 1 and 2 converges to 0.555556 and 0.444444, respectively.

The values (0.555556, 0.444444) are the stationary probability distribution of the Markov chain or equilibrium probabilities.

The values can be interpreted as long-run market shares of the brands.

TABLE 16.7 Shows the values of P^n and the market share of brands after *n* periods ($P_I P^n$)

		Brand 1	Brand 2	Market Share *n* Periods		
P^I		0.2	0.8			
		Brand 1	Brand 2		Brand 1	Brand 2
P	Brand 1	0.8	0.2	1 ($P_I P^1$)	0.36	0.64
	Brand 2	0.25	0.75			
P^2	Brand 1	0.69	0.31	2 ($P_I P^2$)	0.448	0.552
	Brand 2	0.3875	0.6125			
P^4	Brand 1	0.596225	0.403775	4 ($P_I P^4$)	0.52302	0.47698
	Brand 2	0.504719	0.495281			
P^8	Brand 1	0.559277	0.440723	8 ($P_I P^8$)	0.552578	0.447422
	Brand 2	0.550904	0.449096			
P^{16}	Brand 1	0.555587	0.444413	16 ($P_I P^{18}$)	0.555531	0.444469
	Brand 2	0.555517	0.444483			
P^{32}	Brand 1	0.555556	0.444444	32 ($P_I P^{32}$)	0.555556	0.444444
	Brand 2	0.555556	0.444444			
P^{64}	Brand 1	0.555556	0.444444	64 ($P_I P^{64}$)	0.555556	0.444444
	Brand 2	0.555556	0.444444			

Stationary Distribution in a Markov Chain

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ be the stationary distribution. Then it satisfies the following system of equations:

$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj} \tag{16.20}$$

$$\sum_{k=1}^{m} \pi_k = 1 \tag{16.21}$$

The system of equations in Eq. (16.20) can be written as

$$\pi = \pi \mathrm{P} \tag{16.22}$$

Stationary Distribution in a Markov Chain

The stationary distribution equation for the matrix in Transition Table is given by

$$(\pi_1 \quad \pi_2) = (\pi_1 \quad \pi_2)\begin{pmatrix} 0.80 & 0.20 \\ 0.25 & 0.75 \end{pmatrix}$$

That is

$$\pi_1 = 0.80\pi_1 + 0.25\pi_2$$
$$\pi_2 = 0.20\pi_1 + 0.75\pi_2$$

Since π_1 and π_2 are probabilities, we have

$$\pi_1 + \pi_2 = 1$$
$$0.20\pi_1 - 0.25\pi_2 = 0$$
$$\pi_1 + \pi_2 = 1$$

Solving the above system of equations, we get $\pi_1 = 0.555556$ and $\pi_2 = 0.444444$. That is, in the long run, the markets shares of brand 1 and brand 2 will converge to 0.555556 and 0.444444, respectively. The stationary distribution will be independent of the initial probability distribution P_I.

Regular Matrix

A matrix P is called a regular matrix, when for some *n*, all entries of P^n will be greater than zero, that is for some *n*

$$P_{ij}^{n} > 0$$

Regular Matrix

Consider the matrix:

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0 & 0.8 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

Then

$$\mathbf{P}^2 = \begin{pmatrix} 0.28 & 0.56 & 0.16 \\ 0.25 & 0.35 & 0.4 \\ 0.41 & 0 & 0.59 \end{pmatrix} \text{ and } \mathbf{P}^3 = \begin{pmatrix} 0.384 & 0.112 & 0.504 \\ 0.345 & 0.280 & 0.375 \\ 0.259 & 0.413 & 0.328 \end{pmatrix}$$

Note that although the matrix P has zero entries ($P_{12} = P_{22} = P_{33} = 0$), in $\mathbf{P}^3$ all entries are greater than zero, thus matrix **P** is a **regular matrix**. A regular matrix will have stationary distribution and satisfy the system of equations as shown below,

$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj}$$

$$\sum_{k=1}^{m} \pi_k = 1$$

Example

The number of flights cancelled by an airline daily is modelled using a Markov chain. The states of the chain and the description of states are given in Table 16.8. The revenue loss (in millions of rupees) due to cancellation of flights in various states is given in Table 16.9. The transition probability matrix between states is shown in Table 16.10.

(a) If there are no cancellations initially, what is the probability that there will be at least one cancellation after 2 days?

(b) Calculate the steady-state expected loss due to cancellation of flights.

TABLE 16.8 States representing cancellation of flights

State	Description
0	No cancellations
1	One cancellation
2	Two cancellations
3	More than 2 cancellations

TABLE 16.9 Revenue loss due to cancellations

State	0	1	2	3
Loss	0	4.5	10.0	16.0

TABLE 16.10 State transition matrix between flight cancellations

	0	1	2	3
0	0.45	0.30	0.20	0.05
1	0.15	0.60	0.15	0.10
2	0.10	0.30	0.40	0.20
3	0	0.10	0.70	0.20

Example

Solution:

(a) If there are no cancellations initially, then the initial state vector is $\mathbf{P}_I = [1\ 0\ 0\ 0]$. The probability distribution after two days is

$$\mathbf{P}_I\mathbf{P}^2 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0.45 & 0.30 & 0.20 & 0.05\\ 0.15 & 0.60 & 0.15 & 0.10\\ 0.10 & 0.30 & 0.40 & 0.20\\ 0 & 0.10 & 0.70 & 0.20\end{pmatrix}^2 = \begin{pmatrix}0.2675 & 0.38 & 0.25 & 0.1025\end{pmatrix}$$

Probability that there will be at least one cancellation after 2 days $= 0.38 + 0.25 + 0.1025$
$= 0.7325$.

Example

(b) To calculate the steady-state expected loss, we have to calculate the steady-state distribution. The steady-state distribution will satisfy the following system of equations:

$$\pi_0 = 0.45\pi_0 + 0.15\pi_1 + 0.10\pi_2$$
$$\pi_1 = 0.30\pi_0 + 0.60\pi_1 + 0.30\pi_2 + 0.10\pi_3$$
$$\pi_2 = 0.20\pi_0 + 0.15\pi_1 + 0.40\pi_2 + 0.70\pi_3$$
$$\pi_3 = 0.05\pi_0 + 0.10\pi_1 + 0.20\pi_2 + 0.20\pi_3$$
$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

Solving the above system of equations we get

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3\end{pmatrix} = \begin{pmatrix}0.163 & 0.390 & 0.311 & 0.137\end{pmatrix}$$

The steady-state expected loss is $\sum_{i=0}^{3} \pi_i \times L_i$, where L_i is the expected revenue loss in state i (Table 16.9). Hence

$$\sum_{i=0}^{3} \pi_i \times L_i = 0.163\times 0 + 0.390\times 4.5 + 0.311\times 10 + 0.137\times 16 = 7.05$$

Classification of States in a Markov Chain

Not all Markov chains will have stationary probability distribution. To derive the necessary and sufficient conditions for existence of stationary distribution of a Markov chain, we have to understand different classes of states that exist in a Markov chain.

The classes of states are

- Accessible state
- Communicating state
- Recurrent and Transient state
- Positive recurrent and Null-Recurent state
- Periodic and Aperiodic state

Accessible State

A state *j* is accessible (or reachable) from state *i* if there exists a *n* such that

$$P_{ij}^{n} > 0.$$

That is, there exists a path from state *i* to state *j*.

Communicating States

Two states i and j are communicating states when there exists n and m such that

$$P_{ij}^{n} > 0 \text{ and } P_{ji}^{m} > 0.$$

That is, state j can be reached (accessible) from state i and similarly state i can be reached from state j.

A Markov chain is called irreducible if all states of the chain communicate with each other.

Recurrent and Transient States

A state i of a Markov chain is called a recurrent state when

$$\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$$

That is, if the state i is recurrent then the Markov chain will visit state i infinite number of times in the long run.

If state i is recurrent and states i and j are communicating states, then state j is also a recurrent state.

A state k of a Markov chain is called a transient state when

$$\sum_{n=1}^{\infty} P_{kk}^{n} < \infty$$

That is, state k is called a transient state w $\sum_{n=1}^{\infty} P_{kk}^{n} < \infty$ is finite.

This means it is possible that the Markov chain may not return to state k in the long run.

First Passage Time and Mean Recurrence Time

First passage time is the probability that the Markov chain will enter state *i* exactly after *n* steps for the first time after leaving state *i*, that is

$$f_{ii}^{n} = P\left[X_n = i, X_k \neq i, k = 1, 2, \ldots, n-1 \mid X_0 = i\right]$$

Mean recurrence time is the average time taken to return to state *i* after leaving state *i*. Mean recurrence time μ_{ii} is given by

$$\mu_{ii} = \sum_{n=1}^{\infty} n \times f_{ii}^{n}$$

If the mean recurrence time is finite (μ_{ii} is finite), then the recurrent state is called a **positive recurrent state** and if it is infinite then it is called **null-recurrent state**.

Periodic and Aperiodic State

Periodic state is a special case of recurrent state in which *d*(*i*) is the greatest common divisor of *n* such that

$$P_{ii}^{n} > 0$$

If *d*(*i*) = 1, it is called aperiodic state and if *d*(*i*) ≥ 2, then it is called a periodic state.

In the matrix shown in Table, for state 1

$P_{11}^{3} = 1, P_{11}^{6} = 1, P_{11}^{9} = 1$, and $P_{11}^{n} = 0$

when n is not a multiple of 3. That is, the greatest common divisor is 3 which means that the periodicity is 3.

TABLE 16.11 Transition matrix

		1	2	3
	1	0	1	0
P=	2	0	0	1
	3	1	0	0

Ergodic Markov Chain

A state *i* of a Markov chain is ergodic when it is positive recurrent and aperiodic. Markov chain in which all states are positive recurrent and aperiodic is called an ergodic Markov chain.

For an ergodic Markov chain, a stationary distribution exists that satisfies the system of equations

$$\pi_j = \sum_{k=1}^{m} \pi_k P_{kj}$$

$$\sum_{k=1}^{m} \pi_k = 1$$

Note : A Markov chain is called an *ergodic chain* if it is possible to go from every state to every state (not necessarily in one move).

Limiting Probability

In a Markov chain, the limiting probability is given by:

$$\underset{n \to \infty}{Lt} \; p_{ij}^{n}$$

The main difference between limiting probability and stationary distribution is that, stationary distribution when exists is unique. Whereas limiting probability may not be unique.

References

Text Book:

"Business Analytics, The Science of Data-Driven Decision Making", U. Dinesh Kumar, Wiley 2017

Markov chains in Predictive Analytics [ch 16.4.5- ch 16.5.7]

THANK YOU

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