PES University, Bangalore

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MAY 2020: END SEMESTER ASSESSMENT (ESA) B.TECH. IV SEMESTER

UE18MA251- LINEAR ALGEBRA

Solution and Scheme of Evaluation for Model Question Paper

1a	[A b] = $\begin{bmatrix} 6 & -3 & 3 & -2 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 6 & -3 & 3 & -2 \\ 0 & 0 & 0 & 5/3 \\ 0 & 7/2 & -11/2 & 5 \end{bmatrix} \sim \begin{bmatrix} 6 & -3 & 3 & -2 \\ 0 & 7/2 & -11/2 & 5 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$ From the last row, we see that the system is inconsistent and the given planes do not have a common point of intersection. If the number 1 is changed to -2/3 then the reduced system is $\begin{bmatrix} 6 & -3 & 3 & -2 \\ 0 & 7/2 & -11/2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The set of all solutions is (2k/3 , 3k, k) , $k \neq 0$ is real.	3 1 3
1b	A $\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 4 & 1 \\ 0 & -4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & -4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ = U L = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. The permutation matrices used for elimination are P_{23} and P_{34}	3 2 1+1
1c	**A**: [A I] = $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 1 & a & 2 & 0 & 1 & 0 \\ 1 & 0 & b & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 0 & 2-b & -1 & 1 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & b/2-b & -b/2-b & 1 \\ 0 & 1 & 0 & 1/a & 0 & -1/a \\ 0 & 0 & 1 & -1/2-b & 1/2-b & 0 \end{bmatrix}$ Comparing with given inverse of A we get a= 1 and b = 1.	2 1+1 1+1
2a	[A b] = $\begin{bmatrix} 1 & 1 & 2 & u \\ 1 & 2 & 4 & v \\ 2 & 4 & 8 & w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & u \\ 0 & 1 & 2 & v-u \\ 0 & 2 & 4 & w-2u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & u \\ 0 & 1 & 2 & v-u \\ 0 & 0 & 0 & w-2v \end{bmatrix}$ From the last row we see that C(A) contains vectors of the form (u, v, w) for which w = 2v. The given vector b = (2 , 3, 5) does not satisfy this condition and hence is not in C(A). If the number 5 is replaced by 6 then the system Ax = b has infinitely many solutions of the form (k, -k, k). Hence (2 , 3, 6) = 1 (1 , 1, 2) – 1 (1 , 2, 4) + 1 (2 , 4, 8)	3 1 1 1 1
2b	A $\sim \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & c-4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & c-2 \end{bmatrix}$ (i) C(A) is a plane for c = 2 (ii) C(A) is the whole of R^3 for $c \neq 2$	1 1+1

	When c = 2 we have $\begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables are z and t and special solutions are (-1, -1, 1, 0) , (2, -2, 0, 1)	1 1+2
2c	A = $\begin{bmatrix} 1 & 3 & 3 & 2 & a \\ 2 & 6 & 9 & 7 & b \\ -1 & -3 & 3 & 4 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 2 & a \\ 0 & 0 & 3 & 3 & b-2a \\ 0 & 0 & 6 & 6 & c+a \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 2 & a \\ 0 & 0 & 3 & 3 & b-2a \\ 0 & 0 & 0 & 0 & c+5a-2b \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ = R Dim N(A) = 2 and a basis is { (-3, 1, 0, 0) , (1, 0, -1 , 1) } Dim N(A^T) = 1 and a basis is { (5, -2 , 1 }	2 1 1+1 1+1
3a	Ax = [1 1 1 -1] $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ = 0. The special solutions are (-1, 1, 0, 0) , (-1, 0, 1 ,0) , (1, 0, 0, 1) Which form a basis for S. A basis for $S^{\perp}$ is (1, 1, 1, -1). The projection of (1, 1, 1, 1) on $S^{\perp}$ is (½ , ½, ½, - ½) = b Hence, a = (1, 1, 1, 1) – (½, ½, ½, - ½) = (½, ½, ½ , 3/2).	3 1 2 1
3b	The normal eqns are $A^TA\hat{x} = A^Tb$ which gives $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Solution is $\hat{x}$ = (1/3, 1/3) p = $A\hat{x}$ = (1/3, 1/3, 1/3).	4 2+1
3c	T ($\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$) = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$ T ($\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$) = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$ T ($\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$) = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$ T ($\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$) = $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ The required matrix is A = $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}$	1 1 1 1 2
4a	$q_1 = \frac{a}{\sqrt{2}}$, $B = 1/2(1,0,-1)$, $q_2 = (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})$, C = (0, 1, 0) = q_3 1 mark each (5 marks)	5

	A = $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$ = QR	2
4b	The successive approximations are 10 (-0.1, 0.4, 1) , 11.7 (0. 0085, 0.3846 , 1), 11.5299 (0.0141, 0.4159, 1), 11.6494 (0.0225, 0.4184, 1), 11.6511 (0.0238, 0.4209, 1), 11.6599(0.0243, 0.4214, 1) 1 mark each The largest eigenvalue is 11.660	6 1
4c	The characteristic equation of A is $\lambda^2 - 10\lambda + 9 = 0$. Hence λ = 1, 9. The eigenvectors are (1, -1) and (1, 1) respectively. A = $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$ = $S \Lambda S^{-1}$. $A^{1/2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.	2 2 2
5a	The matrix of the quadratic form is A = $\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ The eigenvalues are 3 and 7 Since both eigenvalues are positive the matrix is positive definite. The eigenvectors are respectively v1 = (1 , 1) and v2 = (-1 , 1) Normalizing them we get u1 = $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, u2 = $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ Now Q = $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and D = $\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ Therefore A = Q D Q^T	1 1 1 1 1 1+1 1
5b	For the given matrix A we have $A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$ The eigenvalues of A^TA are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Corresponding unit eigenvectors are, respectively, $\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$ The square roots of the eigenvalues are the singular values: $\sigma_1 = 6\sqrt{10}, \quad \sigma_2 = 3\sqrt{10}, \quad \sigma_3 = 0$ The nonzero singular values are the diagonal entries of D. The matrix Σ is the same size as A, with D in its upper left corner and with 0's elsewhere. $D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}, \quad \Sigma = [\, D \quad 0 \,] = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$ $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$ $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$ Note that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is already a basis for $\mathbb{R}^2$. Thus no additional vectors are needed for U, and $U = [\, \mathbf{u}_1 \quad \mathbf{u}_2 \,]$. The singular value decomposition of A is $A = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}_{V^T}$	1 3 3 1 1 1 1 1