PES UNIVERSITY, BANGALORE-85

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Scheme and Solution

Q. No.		Marks
1a	Propane is a common gas used for cooking and home heating. Each molecule of propane is comprised of 3 atoms of carbon, and 8 atoms of hydrogen written as C_3H_8. When propane burns, it combines with oxygen gas O_2 to form carbon dioxide CO_2 and water H_2O. Balance the chemical equation $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$ that describes this process. Solution: Balance the chemical equation is $aC_3H_8 + b\,O_2 \rightarrow cCO_2 + d\,H_2O$ for atoms of carbon we have 3a=c for atoms of hydrogen we have a8=2d for atoms of oxygen we have 2b=2c+d thus expressing all the variables in terms of a we have c=3a, b=5a, d=4a $aC_3H_8 + 5a\,O_2 \rightarrow 3aCO_2 + 4a\,H_2O$ after cancelling the common factor a we have $C_3H_8 + 5\,O_2 \rightarrow 3CO_2 + 4\,H_2O$	2 2 2
b	Use the Gauss – Jordan method to invert the following matrices $$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$ Solution: $[A\colon I] = \begin{bmatrix} 1 & 2 & -1\colon 1 & 0 & 0 \\ -1 & 1 & 2\colon 0 & 1 & 0 \\ 2 & -1 & 1\colon 0 & 0 & 1 \end{bmatrix}$ $R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 - 2R_1$ $R_3 \rightarrow R_3 + 5R_2/3$, $R_1 \rightarrow R_1 + R_3/14$, $R_2 \rightarrow R_3 - R_2/14$ $R_1 \rightarrow R_1 - 28R_3/42$, After reducing to $[\mathrm{I}\colon \mathrm{A}^{-1}] = \begin{bmatrix} 1 & 0 & 0\colon 3/14 & -1/14 & 5/14 \\ 0 & 1 & 0\colon 5/14 & 3/14 & -1/14 \\ 0 & 0 & 1\colon -1/14 & 5/14 & 3/14 \end{bmatrix}$	1 4 2

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<td>c</td>
<td>Write down the elementary matrices E , F , G associated with the system of equations 2u + v + 3w = -1 , 4u + v + 7w = 5 , -6u – 2v – 12w = -2 . Also find the LU decomposition of A.
Solution: the given equations in the matrix form is Ax=b
Where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}, x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, b = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$
First: $R_2 \longrightarrow R_2 - 2R_1$
Second: $R_3 \longrightarrow R_3 + 3R_1$
Third: $R_3 \longrightarrow R_3 + R_2$
Then $U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$
$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
$L = G^{-1}F^{-1}E^{-1}$
$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$</td>
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<td>2a</td>
<td>Reduce these matrices to their echelon form to find their rank. Also find a special solution to each of the free variables.
$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$
Solution: step 1: $R_2 \longrightarrow R_2 - R_1$ step 2: $R_3 \longrightarrow R_3 - R_2$
The echelon form is $\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$R_1 \longrightarrow R_1 - 2R_2$
Row reduce form is $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Now the free variables are x_2, x_4, x_5
The special solutions are (-2, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)</td>
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1

3</td>
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b	For every c , find R and the special solutions to Ax = 0 : $$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$$ Solution: step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ We get $= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix}$ Case 1: If c=1, then the free variables are x_2, x_3, x_4 special solutions to Ax = 0 are (-1, 1, 0, 0), (-2, 0, 1, 0), (-2, 0, 0, 1) case 2: If c≠0, then the free variables are x_3, x_4 special solutions to Ax = 0 are (-2, 0, 1, 0), (-2, 0, 0, 1)	2 1 2 2
c	If the column space of A is spanned by the vectors (1, 4, 2) , (2, 5, 1) and (3, 6, 0) find all those vectors that span the left null space of A. Determine whether or not the vector b = (4, -2, 2) is in that subspace. What are the dimensions of C (A^T) and N (A^T) ? Solution: the left null space of A is N (A^T)={x: $A^Tx=0$} Now we solve the equation $A^Tx=0$ The aug[A^T:0]= $\begin{bmatrix} 1 & 4 & 2 & :0 \\ 2 & 5 & 1 & :0 \\ 3 & 6 & 0 & :0 \end{bmatrix}$ step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ step 2: $R_3 \to R_3 - R_2$ step 3: $R_1 \to R_1 + 4/3R_2$ step 4: $R_2 \to R_2/-3$ the row reduced form is $R = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ the free variables are x_3 the basis for N(A^T) is (2, -1, 1) the vector (4, -2, 2) is in N(A^T) since (4, -2, 2)=2(2, -1, 1).	1 1 3 2

3a

On the space P_3 of cubic polynomials, what matrix represents $\frac{d^2}{dt^2}$? Find its null space and column space. What do they mean in terms of polynomials?

Solution: the cubic polynomials is $P_3=1+x+x^2+x^3$. 3

$$\frac{d^2}{dt^2}(P_3) = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3

$$N\left(\frac{d^2}{dt^2}\mathrm{P3}\right) = (1,0,0,0),(0,1,0,0)$$

$$C\left(\frac{d^2}{dt^2}\mathrm{P3}\right) = (2,0,0,0),(0,6,0,0)$$

It is a linear combination of standard basis.

b

What multiple of a = (1, 1, 1) is closest to b = (2, 4, 4) ? Find also the point on the line through b that is closest to a.

Solution: the projection matrix is $\mathrm{p1} = \hat{x}\mathrm{a}$, where $\hat{x} = \frac{a^T b}{a^T a}$ 1

We get $\hat{x} = \frac{10}{3}$ 2

b must be a multiple of 10/3 such that, a is closest to the point b=(2, 4, 4). 2

the point on the line through b that is closest to a 2

the projection matrix is $\mathrm{p2} = \hat{x}\mathrm{b}$, where $\hat{x} = \frac{b^T a}{b^T b}$

We get $\hat{x} = \frac{10}{36}$

a must be a multiple of 10/36 such that, b is closest to the point a=(1, 1, 1).

c

An ice- cream vendor records the number of hours of sun shine (x) versus the number of ice- creams sold in an hour (y) at his shop from Monday to Friday and found the following data :

x	2	3	5	7	9
y	4	5	7	10	15

Find the best values of m and c that suit the equation y = mx + c. If there is a weather forecast that says there would be 8 hours of sun shine the next day, estimate the number of ice- creams that he expects to sell on that day.

Solution: y = mx+c 2

	The equations are 4=2m+c, 5=3m+c, 7=5m+c, 10=7m+c, 15=9m+c The matrix form is Ax=b, where $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{bmatrix}, x = \begin{bmatrix} m \\ c \end{bmatrix},\ b = \begin{bmatrix} 4 \\ 5 \\ 7 \\ 10 \\ 15 \end{bmatrix}$ The normal equation is $A^TAx=A^Tb$ After solving we get $\begin{bmatrix} 168 & 26 \\ 26 & 5 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 263 \\ 41 \end{bmatrix}$ Solution of m=1.613, c=0.305 y = 1.613x+0.305 at x=8, y=13.209	3 2
4a	Find the largest Eigen value and the corresponding Eigen vector of a matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by using the initial vector x_0=(1, 1, 1). Solution: the power method is Step1; $Ax_0=\lambda\, x_1$ $Ax_0= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ This process is continued We get λ=3.4146, x=(0.7071, -1, 0.7071)	2 4
b	Find orthogonal vectors q_1, q_2 , q_3 by Gram- Schmidt method from a = (1, 1 , 0) , b = (1, 0, 1) and c = (0, 1, 1) solution: the G-S method $q_1 = \frac{a}{\|a\|}, \quad q_2 = \frac{e2}{\|e2\|}, where\ e2 = b - ({q_1}^T b) q_1,$ $q_3 = \frac{e3}{\|e3\|}, \quad where\ e3 = c - ({q_1}^T c) q_1 - ({q_2}^T c) q_2$ $q_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),\ q_2 = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right),\ q_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	3 2 2
c	Factor the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in to $S\Lambda S^{-1}$. Also find SAS^{-1}. Solution: the Eigen values are λ=0, 2.	2 1

	The vectors are $x_1 = k\begin{bmatrix}-1\\1\end{bmatrix}, x_2 = k\begin{bmatrix}1\\1\end{bmatrix}$ $$S = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}, \Lambda = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}, S^{-1} = \frac{1}{2}\begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$$ $S\Lambda S^{-1}=A$, $SAS^{-1}=\Lambda$.	2 2
5a	Let A be an n × d matrix with right singular vectors $v_1, v_2, \ldots, v_r$, left singular vectors $u_1, u_2, \ldots, u_r$, and corresponding singular values $\sigma_1, \sigma_2, \ldots, \sigma_r$. Then $A = \sum_{i=1}^{r} \sigma_i u_i v^T{}_i$. Solution: For each singular vector v_j , $A = \sum_{i=1}^{r} \sigma_i u_i v^T{}_i\, v_j$. Since any vector v can be expressed as a linear combination of the singular vectors plus a vector perpendicular to the vi , $A_v = \sum_{i=1}^{r} \sigma_i u_i v^T{}_i\, v_j$ and by Lemma 1.4, $A = \sum_{i=1}^{r} \sigma_i u_i v^T{}_i$. The decomposition is called the singular value decomposition, SVD, of A. In matrix notation $A = UDV^T$ where the columns of U and V consist of the left and right singular vectors, respectively, and D is a diagonal matrix whose diagonal entries are the singular values of A. For any matrix A, the sequence of singular values is unique and if the singular values are all distinct, then the sequence of singular vectors is unique also. However, when some set of singular values are equal, the corresponding singular vectors span some subspace. Any set of orthonormal vectors spanning this subspace can be used as the singular vectors.	
b	Test the following matrices for positive or semi definite $$A = \begin{bmatrix}5 & 2 & 1\\2 & 2 & 2\\1 & 2 & 5\end{bmatrix}, \quad B = \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}$$ Solution: Case A: $A = \begin{bmatrix}5 & 2 & 1\\2 & 2 & 2\\1 & 2 & 5\end{bmatrix}$, we get the $U = \begin{bmatrix}5 & 2 & 1\\0 & 6/5 & 8/5\\0 & 0 & 8/3\end{bmatrix}$ $$L^{-1}x = \begin{bmatrix}1 & 2/5 & 1/5\\0 & 1 & 4/3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$ $x^TAx = 5(u+2v/5+w/5)^2+6/5(v+4w/3)^2+8w^2/3>0$ $\lambda_1\lambda_2\lambda_3=16$ $\lambda_1 + \lambda_2 + \lambda3=12$	2 2 3

	pivots are 5, 6/5, 8/3 > 0 Therefore A is Positive definite. Case B: $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ we get the $U = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$ pivots are 2, 3/2, 0 ≥ 0 Therefore B is semi definite.	
c	Find SVD for Matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$. Solutions: $A^T A = \begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}$, Eigen values of A^TA are 10, 8. Eigen vectors are $x_1 = (1, -1)$, $x_2 = (1, 1)$ Matrix $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Singular values of A are $\sigma_1 = \sqrt{10}$, $\sigma_2 = \sqrt{8}$ Eigen values of AA^T are 10, 8, 0. Therefore $$u_1 = \frac{AV_1}{\sigma_1} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}$$ Similarly $$u_2 = \frac{AV_2}{\sigma_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ So u_3 has to be orthogonal to u_2 *and* u_1, we get. $$u_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$ Therefore $A = \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $$A = U\Sigma V^T$$	2 1 2 2