Live Class : 17/3/2020. at 8.10 - 9.40.

Topic : Properties of Eigen values & The Largest Eigen vectors & Values.

Properties :-

1] eigen values are unique.

2] For a λ, eigen vectors are not unique.

3] A & A^T have the same eigenvalues & eigenvectors.

4] Eigen values of a diagonal matrix are diagonal entries.

5] Products of the diagonal elements = determinant of the matrix.

6] Sum of the eigenvalues = Trace of the matrix = Sum of the elements of the principal diagonal.

7] If λ is an eigen value of A, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

8] If $\lambda_1, \lambda_2, \dots, \lambda_m$ are eigen values of A, then $\lambda_1^n, \lambda_2^n, \dots, \lambda_m^n$ are eigenvalues of A^n. where $n \in I^+$.

9] If A is Singular matrix, then zero's are the eigen values.

Problems:

Find the eigen values and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that trace equals to the sum of the eigenvalues and the determinant equals to their product. If we shift A to $A - 7I$, then what are the eigenvalues ~~are~~ of $A - 7I$, and how are they related to those of A.

Ans:-

The eigen values are given by

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\left[\lambda^2 - (\text{sum of the diagonal elements})\lambda + |A| = 0\right]$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \lambda = 3.$$

The eigenvalues are $\lambda = 2, \lambda = 3$.

The eigen vectors are given by

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$,

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1$

$$\begin{array}{c} \begin{array}{cc} PC & FV \end{array} \\ \begin{bmatrix} \boxed{-1} & -1 \\ 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x - y = 0 \Rightarrow x = -y$$

y is free variable. Let $y = 1$,

Then $x = -1$.

$\therefore$ The eigen vector is $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

When $\lambda = 3$,

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-2x - y = 0$$

y is free variable. Let $y = 1$.

$$X_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \text{ or } K \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

$$\left.\begin{array}{l} \text{Trace} = 1+4 = 5 \\ \text{Sum of the eigenvalues} = 2+3 = 5 \end{array}\right\} \text{equal.}$$

$$\left.\begin{array}{l} \text{Product of the eigenvalues} = 2\times 3 = 6. \\ \text{Det. } |A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4+2 = 6. \end{array}\right\} \text{equal.}$$

Construct $A - \lambda I$.

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$$

$\underline{\lambda = 7}$

$$A - 7I = \begin{pmatrix} 1-7 & -1 \\ 2 & 4-7 \end{pmatrix}$$

$$A - 7I = \begin{pmatrix} -6 & -1 \\ 2 & -3 \end{pmatrix} = B \text{ (say)}.$$

The eigen values are given by

$$|B - \lambda I| = 0.$$

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (-9)\lambda + 20 = 0$$

$$\lambda^2 + 9\lambda + 20 = 0$$

$$(\lambda+4)(\lambda+5) = 0$$

$$\lambda = -4, -5$$

The eigenvectors are given by

$(B-\lambda I)X = 0.$

$$\begin{pmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

When $\lambda = -4$, $\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow R_2 + R_1 \to R_2$

$$\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2x - y = 0$ Let $y = 1$

$-2x = 1 \Rightarrow x = -1/2$

$X_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or $K\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$

When $\lambda = -5$, $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-x - y = 0 \Rightarrow x = -y.$

Let $y = 1$, $x = -1.$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $K\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Eigen values of A are : $2, 3$

Eigenvalues of $B = A - 7I$ are : $-4, -5$.

$-4 = 2 -$

Problems :-

Find the eigenvalues of the matrixes A, A^2, A^{-1} and $A+4I$, given $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Ans :-

The eigenvalues of A are

$\lambda = 1, \lambda = 3.$

The eigenvalues of A^2 are : $1^2, 3^2 = 1, 9.$

" " " A^{-1} : $\frac{1}{1}, \frac{1}{3}$

" " " $A+4I$: $1+4, 3+4 = 5, 7.$

Problem :

Write the 3 different 2×2 matrices whose eigenvalues are 4 and 5.

Ans :- Given, eigenvalues are $4, 5.$

$\therefore |A| = 20.$

1) $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ (diagonal matrix)

2) $A = \begin{bmatrix} 4 & 10 \\ 0 & 5 \end{bmatrix}$ (upper Δ matrix)

3) $A = \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$ (lower Δ matrix).

$\times$

Rayleigh's Power Method

1] What 2] Why 3] Procedure. 4] Adv. & Disadv.
5] Applications.

Finding largest eigen value and corresponding eigen vector.

→ It is an iterative process. Writing an algorithms is easy!

Procedure :-

1] Given :- A, initial approximation of eigen vector.

2] $Ax = \lambda x$

3] Apply initial eigen vector as x_0.

$Ax_0 = \lambda_1 x_1$, $Ax_1 = \lambda_2 x_2$, ... $Ax_{n-1} = \lambda_n x_n$

4] Stop the process when two successive iterations are same.

Problem :-
Find the largest eigenvalue and eigenvector corresponding to the matrix $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$.

Ans :-
Since initial eigen vector is not given, assume it one of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Let $x^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

First Iteration:-

$$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda_1 x^{(1)}$$

Second Iteration:-

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda_2 x^{(2)}$$

Third Iteration:-

$$AX^{(2)} = \begin{bmatrix} 4 & +1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6 \begin{bmatrix} 1 \\ 0.9285 \\ -0.9285 \end{bmatrix}$$

$$= 5.6 \begin{bmatrix} 1 \\ 0.929 \\ -0.929 \end{bmatrix}$$

$$= \lambda_3 x^{(3)}$$

Fourth Iteration:-

$$AX^{(3)} = \begin{bmatrix} 5.857 \\ 5.714 \\ -5.714 \end{bmatrix} = 5.857 \begin{bmatrix} 1 \\ 0.976 \\ -0.976 \end{bmatrix} = \lambda_4 X^{(4)}$$

Fifth Iteration:-

$$AX^{(4)} = \begin{bmatrix} 5.9512 \\ 5.9024 \\ +5.9024 \end{bmatrix} = 5.9512 \begin{bmatrix} 1 \\ 0.9908 \\ -0.9908 \end{bmatrix} = \lambda_5 X^{(5)}$$

Sixth iteration:-

$$AX^{(5)} = \begin{bmatrix} 5.9816 \\ 5.9632 \\ -5.9632 \end{bmatrix} = 5.9816 \begin{bmatrix} 1 \\ 0.9969 \\ -0.9969 \end{bmatrix} = \lambda_6 X^{(6)}$$

Seventh Iteration:-

$$AX_6 = \begin{pmatrix} 5.9938 \\ 5.9876 \\ -5.9876 \end{pmatrix} = 5.9938 \begin{pmatrix} 1 \\ 0.999 \\ -0.999 \end{pmatrix} = \lambda_7 X^{(7)}$$

Stop the process

The Largest eigen value $= 5.994 \cong 6$.

" " " vector $\approx \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

* Google, twitter uses this method.

* Smallest eigen value of A = Largest eigen value of A^{-1}.