TANISHQ Vyas H - Section

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3) Let $T : R^3 \to R^3$ defined by
$T(x, y, z) = (x+2y-z, \; y+z, \; x+y-2z)$

i) Find matrix T relative to standard basis of R^3
ii) Find basis for 4 fundamental subspaces of T.
iii) Find the eigen values & eigen vectors of T.
iv) Decompose $T = QR$.

Now for $(1,0,0) \to T(1,0,0) = 1, 0, 1$
$(0,1,0) \to T(0,1,0) = 2, 1, 1$
$(0,0,1) \to T(0,0,1) = -1, 1, -2$

Writing them in columns

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \text{Ans (i)}$$

Now,

$$T' = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus Basis are as follows $\to$

Column Space $(C(T)) = \{(1,0,1), (2,1,1)\}$
Row Space $(C(T)^T) = \{(\cancel{1}, 2, -1), (0,1,1)\}$
Null space $= (N(T)) = \{(3,-1,1)\}$
Left Null space $(N(T)^T) = \{(-1, 1, 1)\}$

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Now, $|T - \lambda I|x = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} x = 0$$

$\Rightarrow |T - \lambda I| = 0$

$$\Rightarrow (1-\lambda)\big((1-\lambda)(-2-\lambda) - 1\big) - 2(-1) - \big(-(1-\lambda)\big) = 0$$

$$\Rightarrow (1-\lambda)[-2-\lambda+2\lambda+\lambda^2-1] + 2 + 1 - \lambda = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2+\lambda-3] + (3-\lambda)$$

$$= \cancel{\lambda^2} + \cancel{\lambda} - \cancel{3} - \lambda^3 - \cancel{\lambda^2} + 3\lambda + \cancel{3} - \cancel{\lambda} = 0$$

$$= -\lambda^3 + 3\lambda = 0$$

$$\Rightarrow \lambda(3-\lambda^2) = 0$$

$\Rightarrow \lambda = 0, \pm\sqrt{3}$ $\rightarrow$ 3 values

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = \sqrt{3}, \quad \lambda_3 = -\sqrt{3}$

Check $\lambda_1 \lambda_2 \lambda_3 = \det(T) = 0$

for $\lambda = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow (3, -1, 1)$ Eigen Vector

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for $\lambda = \sqrt{3}$

$$\begin{vmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -0.732 & 2 & -1 \\ 0 & -0.732 & 1 \\ 1 & 1 & -3.732 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (0,0,0)$ Eigen Vector

for $\lambda = -\sqrt{3}$

$$\begin{vmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{2} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow (0,0,0)$ Eigen Vector

iv] $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$a = (1,0,1)$
$b = (2,1,1)$
$c = (-1,1,-2)$

Now using Gram-Schmidt $\rightarrow$

$q_1 = \dfrac{a}{\|a\|} = \dfrac{(1,0,1)}{\sqrt{2}}$, $q_2 = \dfrac{B}{\|B\|}$

where $B = b - (q_1^T b) q_1$

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$q_2 = \frac{B}{||B||} = \frac{1}{\sqrt{6}}(1,2,-1)$

$q_3 = \frac{C}{||C||}$ where $C = c - (q_2^T c) q_2 - (q_1^T c) q_1$

$\Rightarrow q_3 = (0,0,0)$

QR factorisation $\rightarrow$

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$\Rightarrow R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, Q is $[q_1 \; q_2 \; q_3]$

So $T = QR$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix}$$

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a)

x	-4	1	2	3
y	4	6	10	8

$y = c + dx$

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \cancel{772/116} \\ \cancel{80/116} \end{bmatrix} \quad (A^T A)^{-1} A^T b = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 772/116 \\ 80/116 \end{bmatrix}$$

$$\Rightarrow y = \frac{772}{116} + x \frac{80}{116}$$

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5] We have $\mathcal{E}_{\ell}^{4}$ →

$$x_1 + x_2 + 3x_3 + 4x_4 = 0$$
$$x_1 = -x_2 - 3x_3 - 4x_4$$
$$x_2 = c_2 \quad x_3 = c_3 \quad x_4 = c_4$$

$$\forall \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right.$$

$$A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times3}$$

$$\text{Proj} = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 10 & 12 \\ 4 & 12 & 17 \end{bmatrix}$$

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$$(A^TA)^{-1} = \begin{bmatrix} 26/27 & -1/4 & -4/27 \\ -1/9 & 2/3 & -4/9 \\ -4/27 & -4/9 & 11/27 \end{bmatrix}$$

$$A(A^TA)^{-1} = \begin{bmatrix} -1/27 & -1/9 & -4/27 \\ 26/27 & -1/9 & -4/27 \\ -1/9 & 2/3 & -4/9 \\ -4/27 & -4/9 & 11/27 \end{bmatrix}$$

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 26/27 & -1/27 & -1/9 & -4/27 \\ -1/27 & 26/27 & -1/9 & -4/27 \\ -1/9 & -3/27 & 6/9 & -12/27 \\ -4/27 & -4/27 & 4/9 & 1/27 \end{bmatrix}$$

Now, wkt

$$I = \text{Proj}_L + \text{Proj}_{v_1}.$$

$$\therefore \text{Projection} = \begin{bmatrix} 1/27 & 1/27 & 1/9 & 4/27 \\ 1/27 & 1/27 & 1/9 & 4/27 \\ 1/9 & 3/27 & 3/9 & 12/27 \\ 4/27 & 4/27 & 4/9 & 16/27 \end{bmatrix}$$

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6)

i) $\begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \approx \begin{bmatrix} a & 2 & 2 \\ 0 & \frac{a^2-4}{a} & 2a-4/a \\ 0 & \frac{2a-4}{a} & \frac{a^2-4}{a} \end{bmatrix}$

$a > 0 \quad , \quad \frac{a^2-4}{a} > 0$

Say $a > 2$

$a(a^2-4) - 2(2a-4) + 2(4-2a)$

$\Rightarrow a^3 - 12a + 16 > 0$

$a > -4, 2, 2$

thus

$\boxed{2 < a < \infty}$

ii) $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\begin{bmatrix} a_{11} y_1 + a_{21} y_2 + a_{31} y_3 \\ a_{21} y_1 + a_{22} y_2 + a_{23} y_3 \\ a_{31} y_1 + a_{32} y_2 + a_{33} y_3 \end{bmatrix}$

thus

$a_{12} + a_{21} = -2$

$a_{31} + a_{13} = 0$

$a_{33} + a_{32} = -2$

Since the matrix is symmetric using that we can say that

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$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

7)

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Now finding the Eigen Values →

$|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{bmatrix} = 0$$

$= (81-\lambda)(9-\lambda) - (729) = 0$

$\Rightarrow \lambda^2 - 90\lambda - 729 + 729 = 0$

$\Rightarrow \lambda = 0, 90$

for $\lambda = 0$

$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 3x \rightarrow (1, 3)$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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for $\lambda = 90$

$$\begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad (-3, 1)$

Now,

$$V = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

thus we get $\sigma = \sigma_1 \sqrt{90}$

$$u_1 = \frac{A v_1}{\sigma_1} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} / 0$$

$$u_2 = \frac{A v_2}{\sigma_2} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} / \sqrt{90}$$

$u_3 = 10x - 20y - 20z = 0$

$\Rightarrow$ we get $x = 1,\ y = 1/4,\ z = 1/4$

$$\Rightarrow u_3 = \begin{bmatrix} 4/\sqrt{18} \\ 1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

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Thus

$$A = \begin{bmatrix} 10/\sqrt{900} & 4/\sqrt{18} \\ -20/\sqrt{900} & 4/\sqrt{18} \\ -20/\sqrt{900} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{90} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$V = \begin{bmatrix} \cdot 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Singular values of A are
$\sigma = \sqrt{90}, 0$

Eigen values of $AA^T \Rightarrow 90, 0, 0$

$$u_1 = \frac{Av_1}{\sigma_1} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

Now other sol^n are found by finding vector sol^n to $u_1^T x = 0$ are they are orthogonal to u_1.

$$u_1^T x = 0 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow X_n = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

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Making use of Gram Schmidt to get u_2, u_3

$$u_2 = \frac{a_2}{\|a_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

Now,

$$e = a_2 - (u_2^T a_2) u_2$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{e}{\|e\|} = \left(\frac{2}{3\sqrt{5}}, \frac{-4}{3\sqrt{5}}, \frac{\sqrt{5}}{3} \right)$$

$$= \left(\frac{2}{\sqrt{45}}, \frac{-4}{\sqrt{45}}, \frac{5}{\sqrt{45}} \right)$$

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ \frac{1}{\sqrt{10}} & 3/\sqrt{10} \end{bmatrix}$$