ME681 Assignment 2 solution

January 28, 2017

Question 1. Which of the following subsets of R^3 are actually subspaces?
(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.
(b) The plane of vectors b with $b_1 = 1$.
(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) All combinations of two given vectors (1,1,0) and (2,0,1).
(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

Ans. (a) The plane of vectors $(0, b_2, b_3)$ is the non empty subset of vector space R^3 and linear combination stay in the subspace. Therefore this is the subspace of R^3.
(b) This plane $(1, b_2, b_3)$ does not passes through the origin. Therefore this is not the subspace.
(c) This subset is not closed under addition so not a subspace.
(d) Two given vectors (1,1,0) and (2,0,1) are linearly independent so they span the subspace of R^3.
(e) These vectors span the 2-d plane which is closed under addition and multiplication. Therefore this is a subspace.

Question 2. (a) Under what conditions on scalars ξ and η are the vectors $(1, \xi)$ and $(1, \eta)$ in R^2 linearly dependent.
(b) Under what conditions on scalars ξ, η, and ζ are the vectors $(1, \xi, \xi^2)$, $(1, \eta, \eta^2)$, and $(1, \zeta, \zeta^2)$ in R^3 linearly dependent.
(c) Guess a generalization of the above to R^n.

Ans. (a) $\xi = \eta$
(b) $\xi = \eta = \zeta$
(c) for R^n, all the corresponding components of the vector must be equal.

Question 3. The four types of subspaces of R^3 are planes, lines, R^3 itself, or Z containing only (0,0,0).
(a) Describe the three types of subspaces of R^2.
(b) Describe the five types of subspaces of R^4.

Ans.(a) The three types of subspaces of R^2 are lines through (0,0), R^2 itself, or Z containing only (0,0,0).
(b) The five types of subspaces of R^4 are three dimensional planes $(n.v = 0)$,two dimensional

subspaces ($n_1.v = 0$ and $n_2.v = 0$) ,lines through (0,0) , R^4 itself, or Z containing only (0,0,0).

Question 4. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space M of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector -A. What matrices are in the smallest subspace containing A?

Ans. zero vector in this space is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the vector $\frac{1}{2}A$= $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and the vector -A= $\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The matrices which are in the smallest subspace containing A are $\begin{bmatrix} -2c & 2c \\ -2c & 2c \end{bmatrix}$. where c is a real number.

Question 5. Decide the linear dependence or independence of
(a) the vectors (1,3,2), (2,1,3), and (3,2,1).
(b) the vectors (1,-3,2), (2,1,-3), and (-3,2,1).

Ans. (a) independent because $\alpha(1,3,2)+\beta(2,1,3)+\gamma(3,2,1) = 0$ implies $\alpha = 0, \beta = 0, \gamma = 0$.
(b) dependent because -(1,-3,2) - (2,1,-3) = (-3,2,1)

Question 6. Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:
$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.
Ans. $\alpha v_1 + \beta v_2 + \gamma v_3 = 0 \Rightarrow \alpha = 0, \beta = 0, \gamma = 0$.
$v_4 = 4v_3 - v_1 - v_2$.

Question 7. Find the dimensions of (a) the column space of A, (b) the column space of U, (c) the row space of A, (d) the row space of U. Which two of the spaces are the same?
$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
Ans. (a) 2 (b) 2 (c) 2 (d) 2. Both the matrices have rank = 2 so dimension of row space= dimension of column space= rank. Row space of both the matrices are same.

Question 8. Find a basis for each of these subspaces of R^4:
(a) All vectors whose components are equal.
(b) All vectors whose components add to zero.
(c) All vectors that are perpendicular to (1,1,0,0) and (1,0,1,1).
(d) The column space (in R^2) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Ans. (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. (b) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. (c) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
These basis are not unique! there can be other basis too.

Question 9. Find a basis for each of these subspaces of 3 by 3 matrices:
(a) All diagonal matrices.
(b) All symmetric matrices ($A^T = A$).
(c) All skew-symmetric matrices ($A^T = -A$).

Ans. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

Question 10. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.
Ans. $p(x) = a + bx + cx^2 + dx^3$ has basis $1, x, x^2, x^3$.

$$\begin{gathered} p(1) = 0 \\ a + b + c + d = 0 \Rightarrow a = -(b + c + d) \\ p(x) = -(b + c + d) + bx + cx^2 + dx^3 \\ p(x) = (1 - x)b + (1 - x^2)c + (1 - x^3)d \\ \text{so basis of } p(1) = 0 \text{ is } (1 - x), (1 - x^2), (1 - x^3) \end{gathered}$$