ME681 Assignment 1

Due Date: 17/01/17 (Next Tuesday)

January 25, 2017

Question 1. Which number q makes this system singular and which right hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

Solution: Subtracting the first equation from the second gives $3y - 4z = 5$, we have a singular system for $q = -4$. $t = 5$: infinitely many solutions. $z = 1$ gives $y = 3$ and equation 1 then gives $x = -9$.

Question 2. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3. \end{aligned}$$

Solution: The given system can be converted to:

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ (a-2)y + z &= b_2 - b_1 \\ (a-4)z &= b_3 - b_2. \end{aligned}$$

So, elimination will fail for $a = 0$, $a = 2$, $a = 4$.

Question 3. Which three matrices E_{21}, E_{31}, E_{32} put A into a triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \textbf{ and } E_{32}E_{31}E_{21}A = U$$

Multiply those $E's$ to get one matrix M that does elimination: $MA = U$.

Solution:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}.$$

Question 4. Decide whether the following systems are singular or non singular and whether they have no solution, one solution or infinitely many solutions:

$$\begin{aligned} v - w &= 2 \\ u - v &= 2 \\ u - w &= 2 \end{aligned}$$

$$\begin{aligned} v - w &= 0 \\ u - v &= 0 \\ u - w &= 0 \end{aligned}$$

$$\begin{aligned} v + w &= 1 \\ u + v &= 1 \\ u + w &= 1 \end{aligned}$$

Solution: System 1: The first two equations give $u - w = 4$. No Solution. System 2: The first two equations give $u - w = 0$. Infinitely many solutions. System 3: The first two equations give $u - w = 0$. One solution: $u = \frac{1}{2}$, $v = \frac{1}{2}$, $w = \frac{1}{2}$.

Question 5. Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Solution:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$$

Conditions for four pivots: $a \neq 0$, $b \neq a$, $c \neq b$, $d \neq c$.

Question 6. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

Solution: $L = I$,

$$D = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}.$$

$$U = D^{-1}A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 7. Use Gauss-Jordan elimination on $[A \quad I]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Question 8. The following matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \quad I]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Question 9. Solve $Ax = b$ by solving the triangular systems $Lc = b$ and $Ux = c$:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

What part of A^{-1} have you found with this particular b?

Solution:

$$c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

This solution gives the third column of A^{-1}.

Question 10. Solve by elimination or show that there is no solution:

$$\begin{aligned} u + v + w &= 0 \\ u + 2v + 3w &= 0 \\ 3u + 5v + 7w &= 1 \end{aligned} \qquad\qquad \begin{aligned} u + v + w &= 0 \\ u + v + 3w &= 0 \\ 3u + 5v + 7w &= 1. \end{aligned}$$

Solution: System 1: The first two equations give $3u + 5v + 7w = 0$ which means the system has no solution. System 2: $u = \frac{-1}{2}$, $v = \frac{1}{2}$, $w = 0$.