18/03/2020

Diagonaliyation.

It is the process of finding an Invertible matrix S and diagonal matrix Λ, such that the given matrix A satisfies

$A = S\Lambda S^{-1}$

Examples :- (i) All diagonal matrix are diagonalizable

Eg:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(ii) All Symmetric Matrix are diagonalizable

Eg:- $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Non Examples :- (i) Shear matrix $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $(a \neq 0)$

$\left(K\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right.$ are Eigen vectors$)$

(ii) A Jordan block $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$\left(K\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right.$ are Eigen vectors$)$

(iii) A rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta \neq 0, \pi$

(No Eigen Vectors)

$A = S\Lambda S^{-1}$

$\Lambda \rightarrow$ Eigen value matrix.

$S \rightarrow$ Eigen matrix → Vector

Diagonalizability of A $\Leftrightarrow$ Existence of a basis of Eigen vectors

$A = S\Lambda S^{-1}$

S — Column form a basis of Eigen spaces

NOTE : If A is $n \times n$ matrix has n distinct Eigen values, then A must be diagonalizable.

The condition is not necessary for e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda = 1, 1, 2$

Problems:-

(1) Show that $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalizable.

Soln:- For this matrix $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 15$ (Eigen values)

$V_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $V_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $V_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ (Eigen vectors)

Set $S = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

Scaling the column of S, we get the orthogonal matrix P,

$A = P \Lambda P^T$.

$\Rightarrow$ A is diagonalizable.

(2) Show that $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

Since it is an upper $\triangle$lar matrix $\lambda = 1, 1, 3$ are the Eigen values

$\lambda = 1$, $(A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Thus $(x_1, 0, 0)$ are the Eigen vectors

Thus there are not Enough ~~vectors~~ Eigen vectors (we need two independent Eigen vectors)

The number of independent Eigen vectors is no more than the multiplicity of the respective Eigen value.

$\Rightarrow$ It is not diagonilizable.

(3) Show that $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.

Since it is upper Δ^{le} matrix $\lambda = 1, 1, 3$ are the Eigen values.

$\lambda = 1$ $(A - \lambda I)x = 0$, $\lambda = 1$ $\Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} x = 0$ Eigen vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3$ $(A - 3I)x = 0$ $\Rightarrow \begin{pmatrix} -2 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$ Eigen vector $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore$ A is diagonalizable.

We can compute powers of a diagonalizable matrix A as follows,

$A = S\Lambda S^{-1}$ then $A^n = (S\Lambda S^{-1})^n$

$= S\Lambda \cancel{S^{-1}} \cancel{S}\Lambda \cancel{S^{-1}} \cancel{S}\Lambda \cancel{S^{-1}} \underbrace{\ldots\ldots S\Lambda S^{-1}}_{(n\text{times})}$

$= S\Lambda^n S^{-1}$

Examples :-

① Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$, Find A^4.

Solution :- Here $\lambda = 2, 2, 4$ are the Eigen values and

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are the Eigen Vectors.

$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Matrix A Satisfy $A = S \Lambda S^{-1}$

$$\therefore A^4 = S \Lambda^4 S^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 256 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 136 & -120 & 120 \\ 0 & 16 & 0 \\ 120 & -120 & 136 \end{pmatrix}$$

If A and B are diagonalizable matrices Can we compute the product the same way?

Not unless A and B have the same Eigen vectors.

Cayley Hamilton theorem :-

Every Square matrix A, satisfies its own Characteristic Equation.

Eg:- If A has characteristic polynomial

$$\lambda^2 - 3\lambda + 2 = 0$$

then $A^2 - 3A + 2I = [0] \longrightarrow$ zero matrix

Thus $A - 3A + 2A^{-1} = [0]$ (Multiply by A^{-1})

Then $A^{-1} = \dfrac{3I - A}{2}$

If $A = S\Lambda S^{-1}$, $B = S\Gamma S^{-1}$ then $AB = BA = S\Lambda\Gamma S^{-1}$

Theorem :- $AB = BA$ for diagonalizable A and B iff A & B have the same Eigen vectors.