Fundamentals of Hypothesis Testing: One Sample Tests

Learning Objectives

In this chapter, you will learn:

- The basic principles of hypothesis testing
- How to use hypothesis testing to test a mean or proportion
- The assumptions of each hypothesis-testing procedure, how to evaluate them, and the consequences if they are seriously violated
- How to avoid the pitfalls involved in hypothesis testing
- Ethical issues involved in hypothesis testing

The Hypothesis

- A hypothesis is a claim (assumption) about a population parameter:
  - population mean

    Example: The mean monthly cell phone bill of this city is $\mu = \$52$

  - population proportion

    Example: The proportion of adults in this city with cell phones is P^ = .68

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

  Example: The mean number of TV sets in U.S. Homes is equal to three.

$$H_0: \mu = 3$$

- Is always about a population parameter, not about a sample statistic.

The Null Hypothesis, H_0

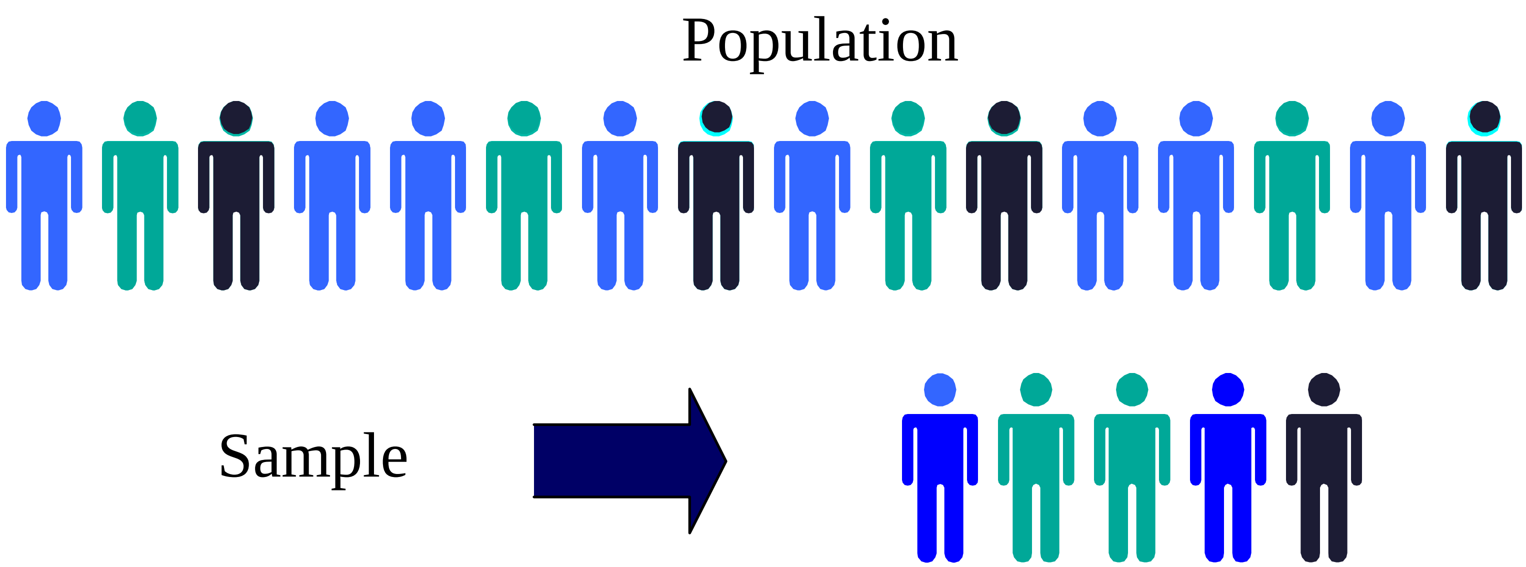
- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- It refers to the status quo
- Always contains "$=$" , "$\leq$" or "$\geq$" sign
- May or may not be rejected

The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The mean number of TV sets in U.S. homes is not equal to 3 (H_1: $\mu \neq 3$)
- Challenges the status quo
- Never contains the "=" , "≤" or "≥" sign
- May or may not be proven
- Is generally the hypothesis that the researcher is trying to prove

The Hypothesis Testing Process

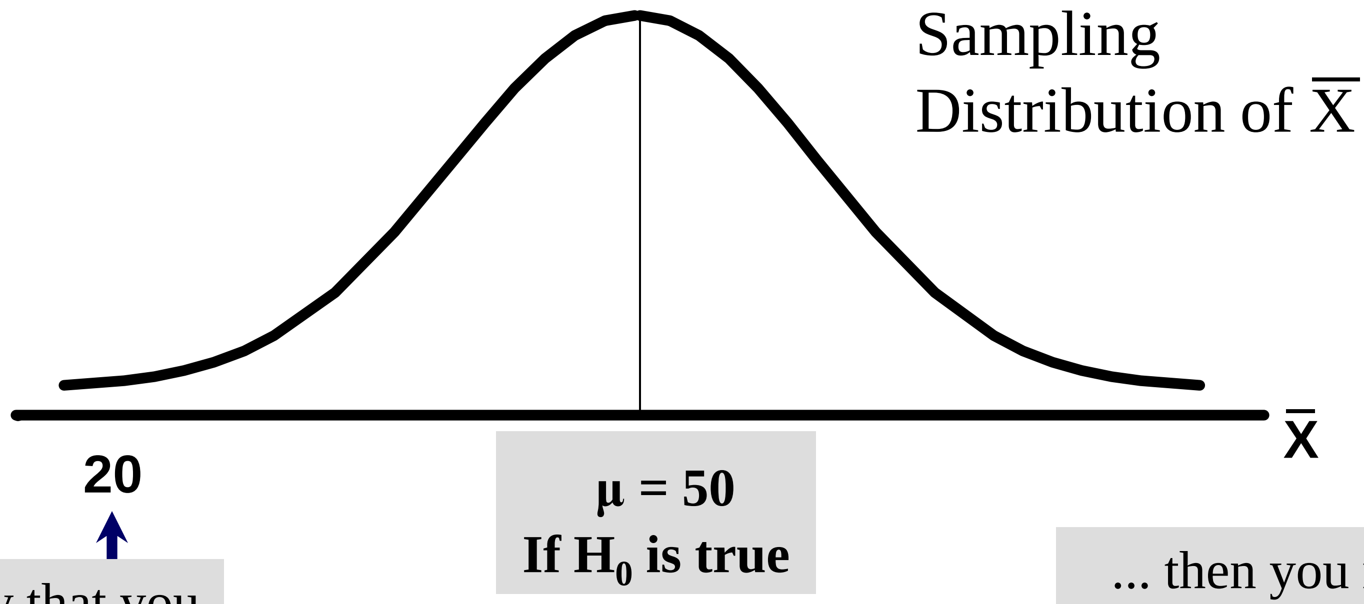
- Claim: The population mean age is 50.
  - H_0: $\mu = 50$, H_1: $\mu \neq 50$
- Sample the population and find sample mean.



The Hypothesis Testing Process

- Suppose the sample mean age was $\overline{X}$ = 20.
- This is significantly lower than the claimed mean population age of 50.
- If the null hypothesis were true, the probability of getting such a different sample mean would be very small, so you reject the null hypothesis .
- In other words, getting a sample mean of 20 is so unlikely if the population mean was 50, you conclude that the population mean must not be 50.

The Hypothesis Testing Process



If it is unlikely that you would get a sample mean of this value ...

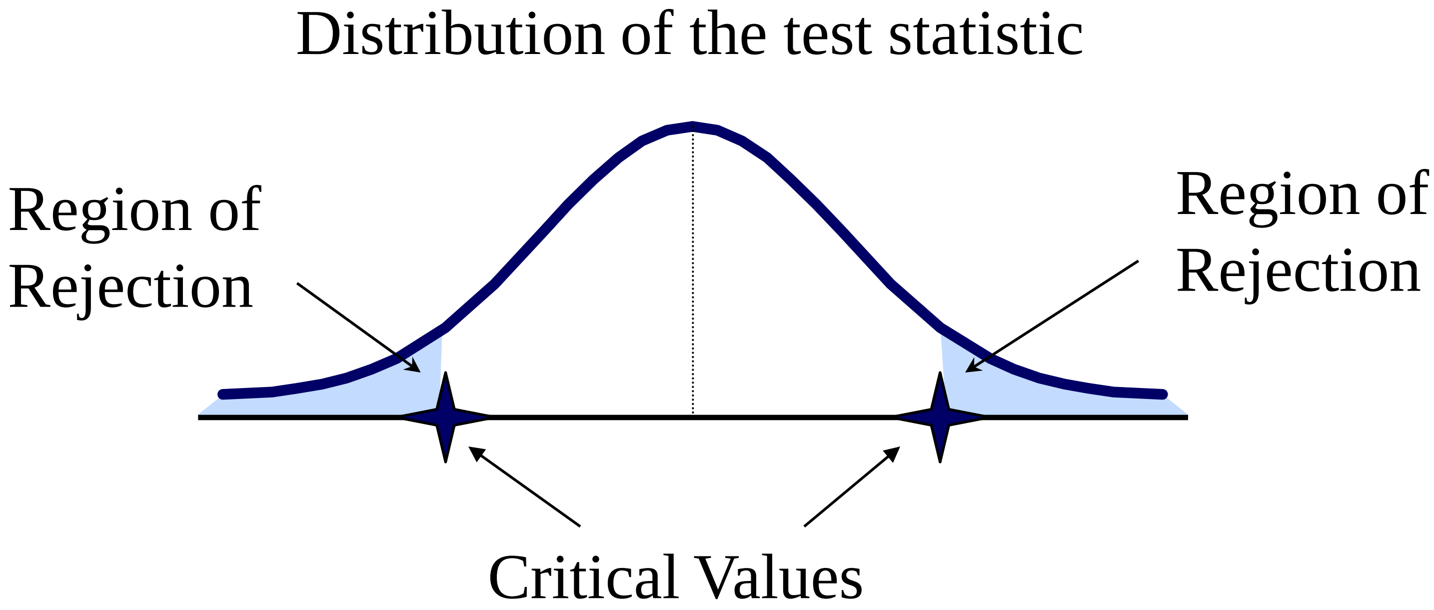
... if in fact this were the population mean…

... then you reject the null hypothesis that $\mu = 50$.

The Test Statistic and Critical Values

- If the sample mean is close to the assumed population mean, the null hypothesis is not rejected.
- If the sample mean is far from the assumed population mean, the null hypothesis is rejected.
- How far is "far enough" to reject H_0?
- The critical value of a test statistic creates a "line in the sand" for decision making.

The Test Statistic and Critical Values



Errors in Decision Making

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error
  - The probability of a Type I Error is $\boldsymbol{\alpha}$
    - Called level of significance of the test
    - Set by researcher in advance
- **Type II Error**
  - Failure to reject false null hypothesis
  - The probability of a Type II Error is $\boldsymbol{\beta}$

Errors in Decision Making

Possible Hypothesis Test Outcomes		
	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	No Error Probability $1 - \alpha$	Type II Error Probability β
Reject H_0	Type I Error Probability α	No Error Probability $1 - \beta$

Level of Significance, α

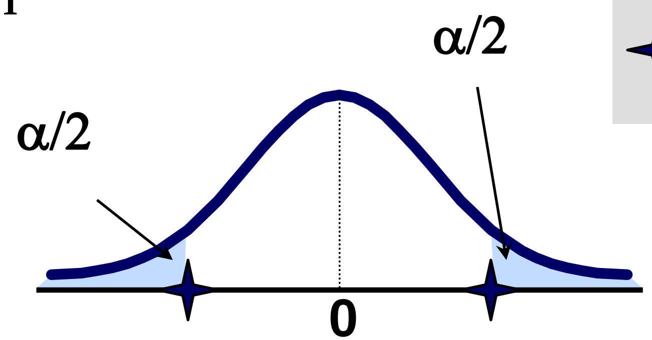
Claim: The population mean age is 50.

Represents critical value

Rejection region is shaded

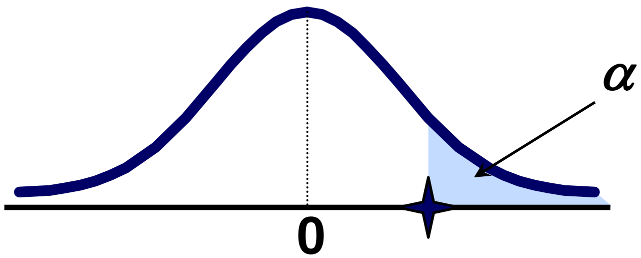
H_0: μ = 50
H_1: μ ≠ 50

Two-tail test



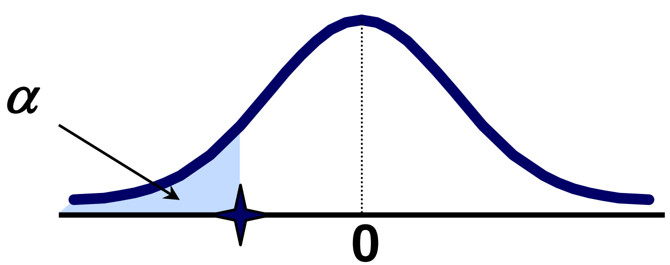
H_0: μ ≤ 50
H_1: μ > 50

Upper-tail test



H_0: μ ≥ 50
H_1: μ < 50

Lower-tail test



Hypothesis Testing: σ Known

For two tail test for the mean, σ known:

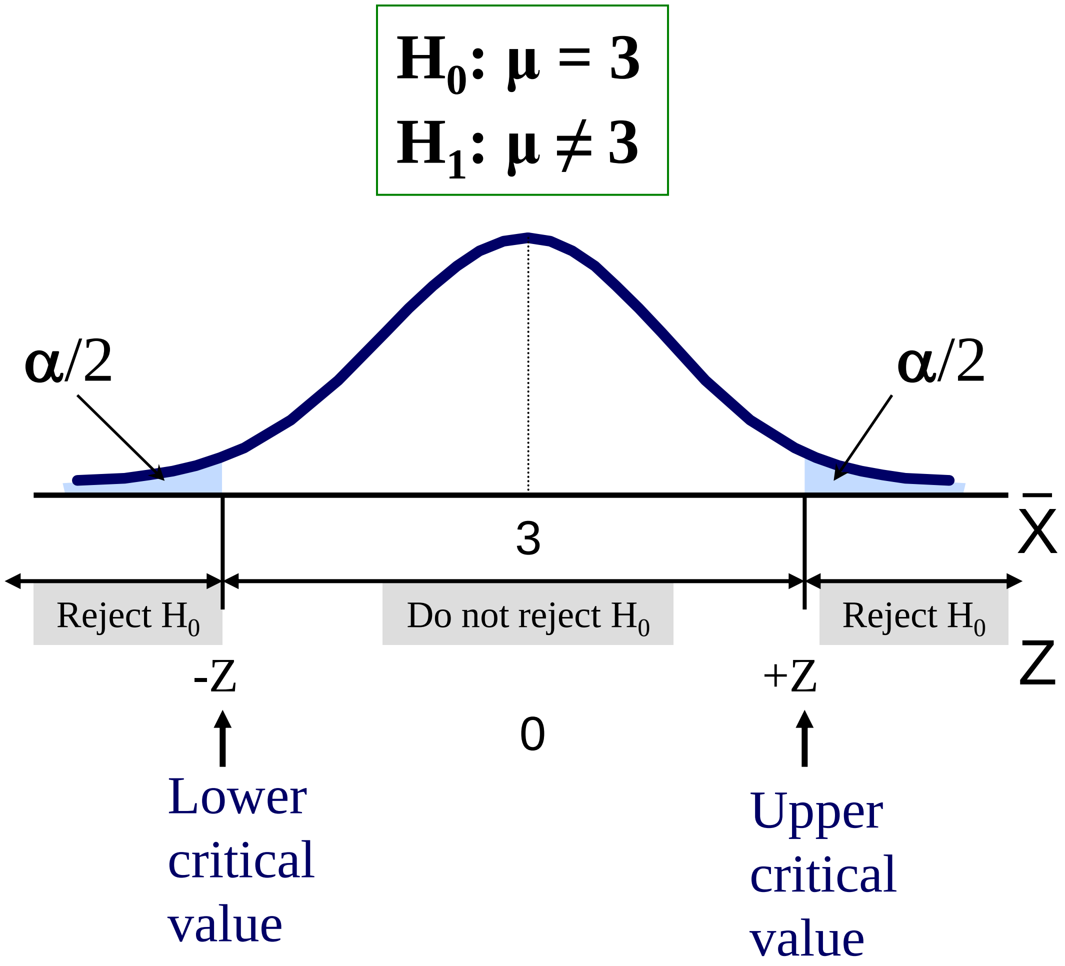
- Convert sample statistic ($\overline{X}$) to test statistic

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

- Determine the critical Z values for a specified level of significance α from a table .
- Decision Rule: If the test statistic falls in the rejection region, reject H_0 ; otherwise do not reject H_0

Hypothesis Testing: σ Known

- There are two cutoff values (critical values), defining the regions of rejection



Hypothesis Testing: σ Known

Example: Test the claim that the true mean weight of chocolate bars manufactured in a factory is 3 ounces.

- State the appropriate null and alternative hypotheses
  - H_0: $\mu = 3$ H_1: $\mu \neq 3$ (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test
- Choose a sample size
  - Suppose a sample of size $n = 100$ is selected

Hypothesis Testing: σ Known

- Determine the appropriate technique
  - σ is known so this is a Z test
- Set up the critical values
  - For $\alpha = .05$ the critical Z values are ± 1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

    $n = 100, \quad \bar{X} = 2.84$

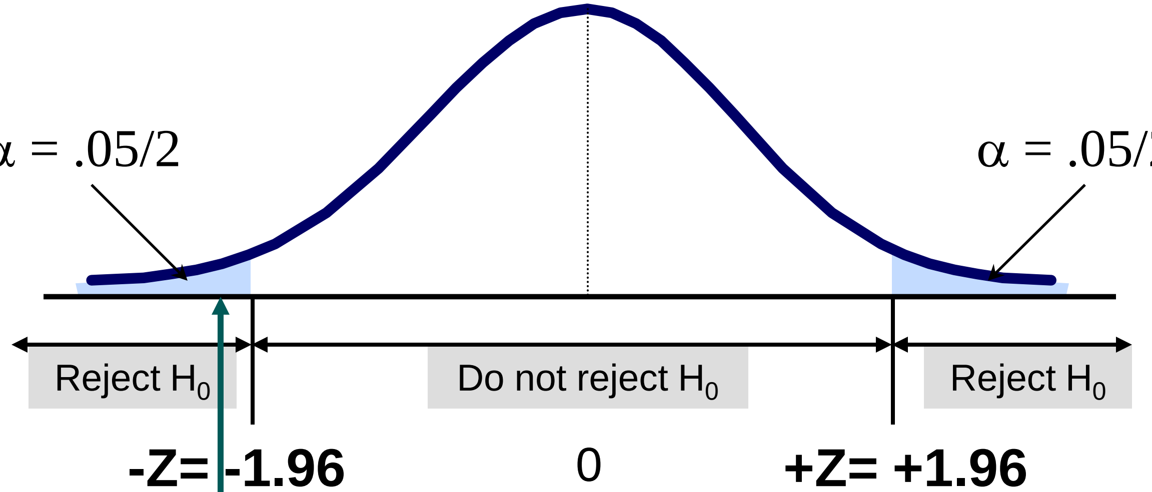
    ($\sigma = 0.8$ is assumed known from past company records)

So the test statistic is:

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing: σ Known

- Is the test statistic in the rejection region?



Reject H_0 if $Z < -1.96$ or $Z > 1.96$; otherwise do not reject H_0

Here, $Z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing: σ Known

- Reach a decision and interpret the result
  - Since $Z = -2.0 < -1.96$, you reject the null hypothesis and conclude that there is sufficient evidence that the mean weight of chocolate bars is not equal to 3.

Hypothesis Testing: σ Known

6 Steps of Hypothesis Testing:

1. State the null hypothesis, H_0 and state the alternative hypotheses, H_1
2. Choose the level of significance, α, and the sample size n.
3. Determine the appropriate statistical technique and the test statistic to use
4. Find the critical values and determine the rejection region(s)

Hypothesis Testing: σ Known

5. Collect data and compute the test statistic from the sample result
6. Compare the test statistic to the critical value to determine whether the test statistic falls in the region of rejection. Make the statistical decision: Reject H_0 if the test statistic falls in the rejection region. Express the decision in the context of the problem

Hypothesis Testing: σ Known p-Value Approach

- The **p-value** is the probability of obtaining a test statistic equal to or more extreme (< or >) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

Hypothesis Testing: σ Known p-Value Approach

- Convert Sample Statistic (ex. $\bar{X}$) to Test Statistic (ex. Z statistic)
- Obtain the p-value from a table.
- Compare the p-value with α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

Hypothesis Testing: σ Known p-Value Approach

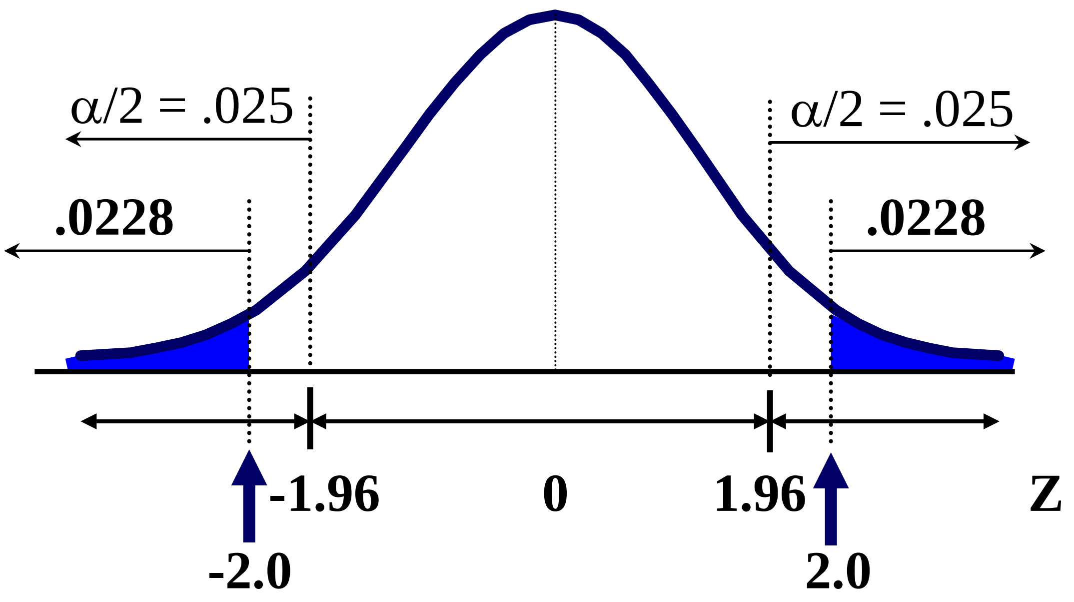
- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

$\overline{X} = 2.84$ is translated to a Z score of Z = -2.0

$$P(Z < -2.0) = .0228$$
$$P(Z > 2.0) = .0228$$

p-value

=.0228 + .0228 = .0456

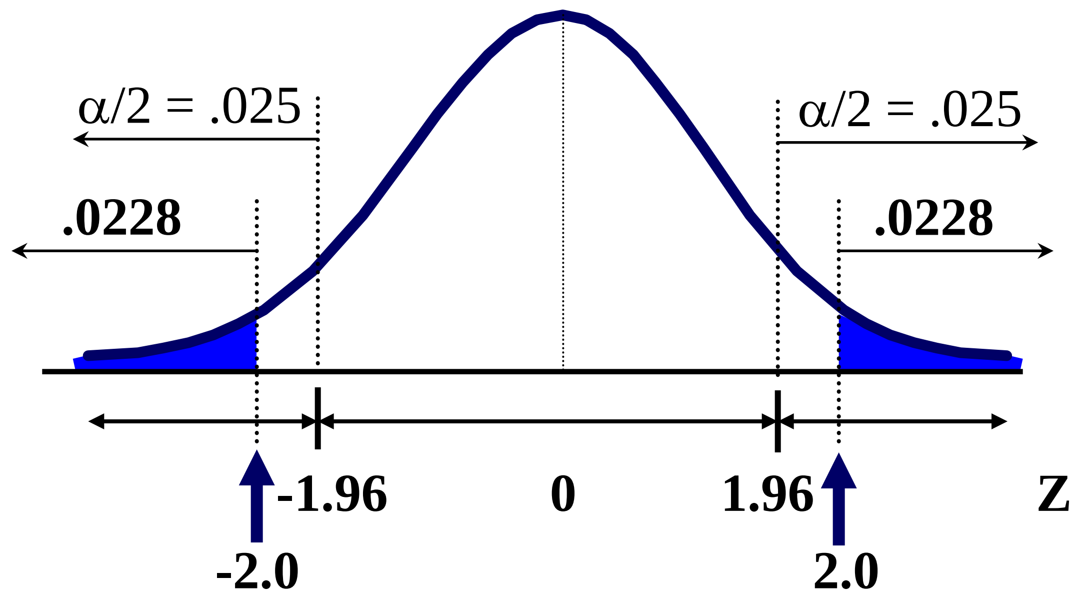


Hypothesis Testing: σ Known p-Value Approach

- Compare the p-value with α
  - If p-value $<$ α, reject H_0
  - If p-value $\geq$ α, do not reject H_0

Here: p-value = .0456
α = .05

Since .0456 < .05, you reject the null hypothesis



Hypothesis Testing: σ Known Confidence Interval Connections

- For $\bar{X} = 2.84$, $\sigma = 0.8$ and $n = 100$, the 95% confidence interval is:

$$2.84 - (1.96)\frac{0.8}{\sqrt{100}} \quad \text{to} \quad 2.84 + (1.96)\frac{0.8}{\sqrt{100}}$$

$$2.6832 \le \mu \le 2.9968$$

- Since this interval does not contain the hypothesized mean (3.0), you reject the null hypothesis at $\alpha = .05$

Hypothesis Testing: σ Known One Tail Tests

- In many cases, the alternative hypothesis focuses on a particular direction

$H_0: \mu \geq 3$

$H_1: \mu < 3$

→ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

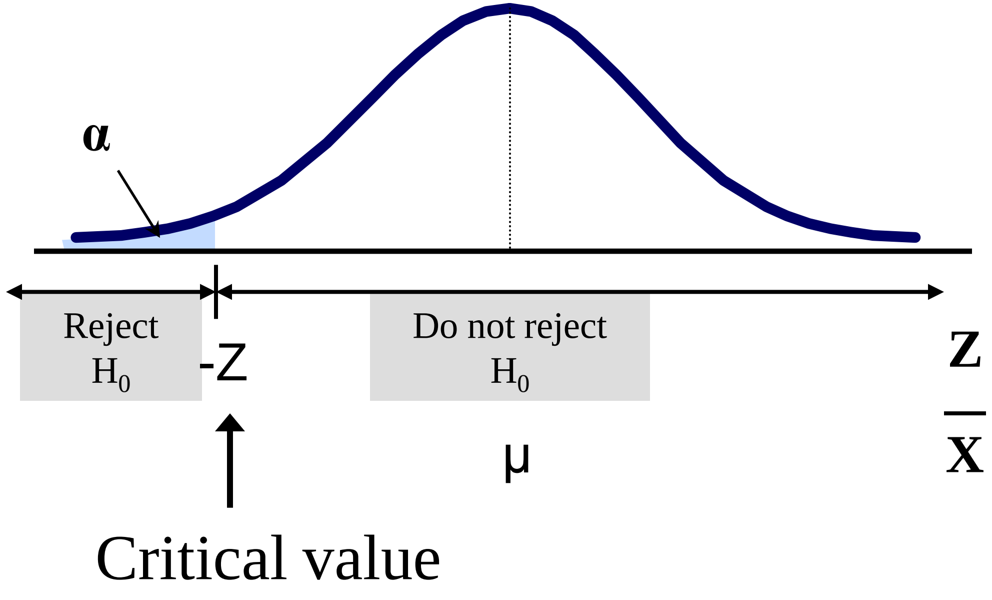
$H_0: \mu \leq 3$

$H_1: \mu > 3$

→ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

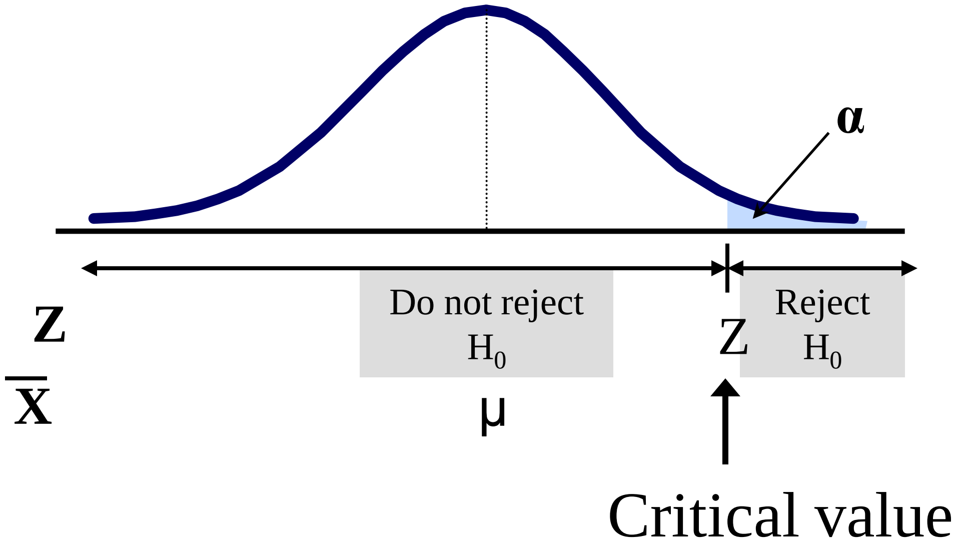
Hypothesis Testing: σ Known Lower Tail Tests

- There is only one critical value, since the rejection area is in only one tail.



Hypothesis Testing: σ Known Upper Tail Tests

- There is only one critical value, since the rejection area is in only one tail.



Hypothesis Testing: σ Known Upper Tail Test Example

A phone industry manager thinks that customer monthly cell phone bills have increased, and now average more than $52 per month. The company wishes to test this claim. Past company records indicate that the standard deviation is about $10.

Form hypothesis test:

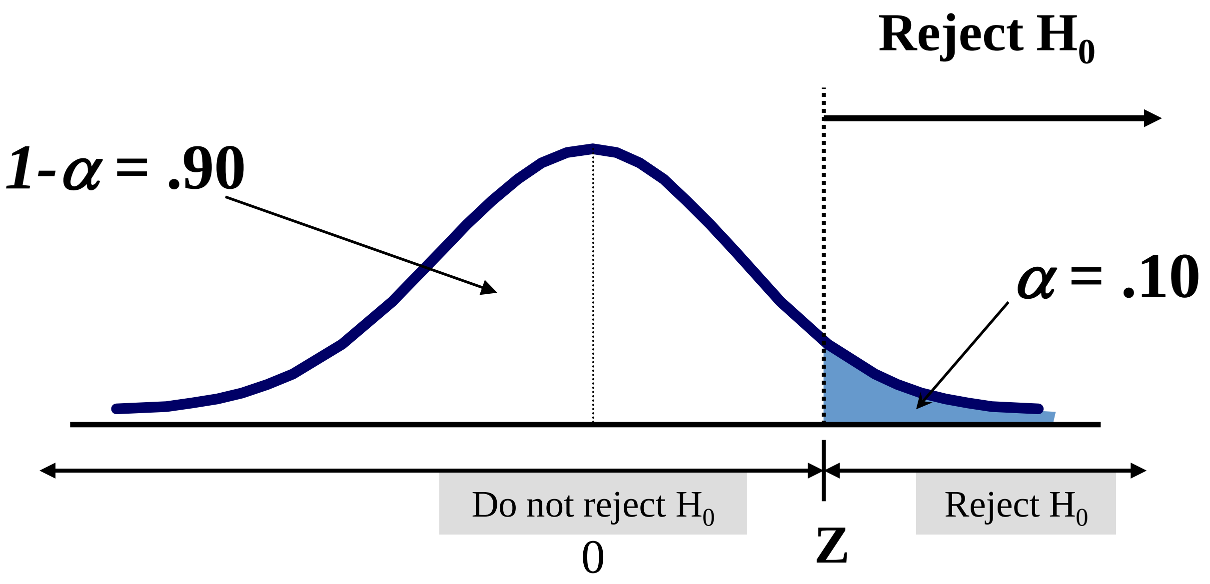
H_0: $\mu \leq 52$ the mean is less than or equal to than $52 per month

H_1: $\mu > 52$ the mean **is** greater than $52 per month

(i.e., sufficient evidence exists to support the manager's claim)

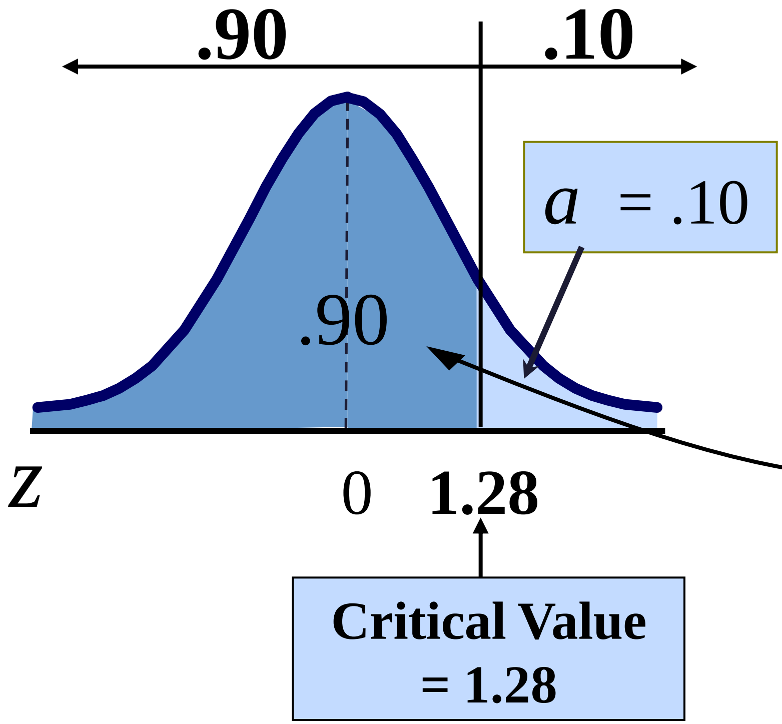
Hypothesis Testing: σ Known Upper Tail Test Example

- Suppose that $\alpha = .10$ is chosen for this test
- Find the rejection region:



Hypothesis Testing: σ Known Upper Tail Test Example

What is Z given *a* = 0.10?



Z	.07	**.08**	.09
1.1	.8790	.8810	.8830
1.2	.8980	.8997	.9015
1.3	.9147	.9162	.9177

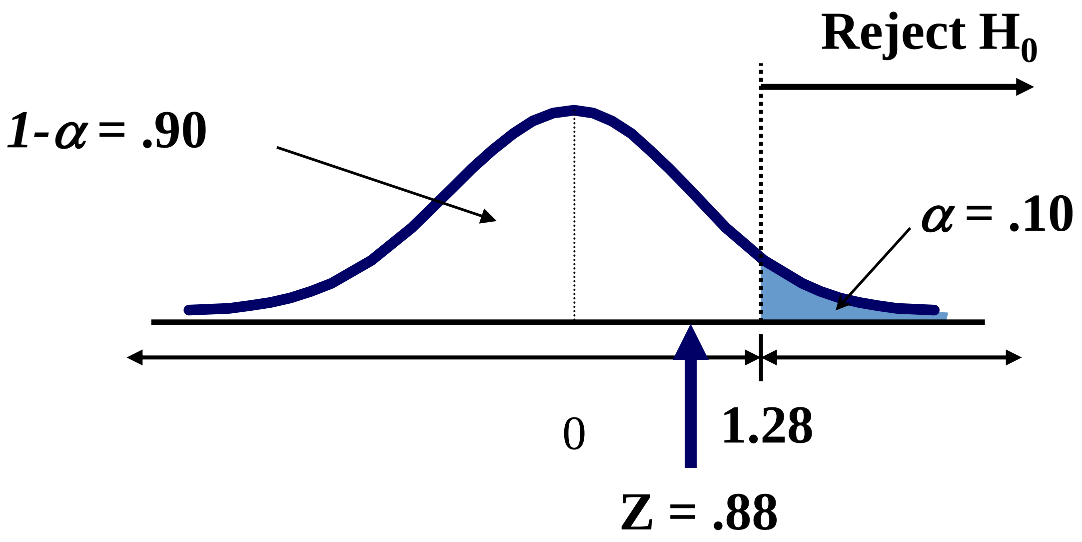
Hypothesis Testing: σ Known Upper Tail Test Example

- Obtain sample and compute the test statistic.
- Suppose a sample is taken with the following results: $n = 64$, $\bar{X} = 53.1$ (σ=10 was assumed known from past company records)
  - Then the test statistic is:

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Hypothesis Testing: σ Known Upper Tail Test Example

- Reach a decision and interpret the result:

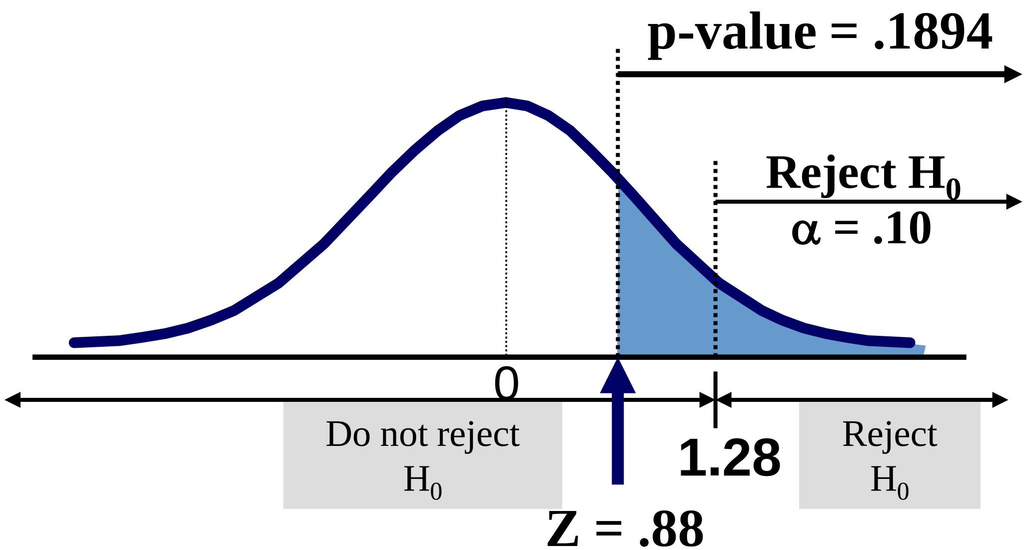


Do not reject H_0 since $Z = 0.88 \leq 1.28$

i.e.: there is not sufficient evidence that the mean bill is greater than $52

Hypothesis Testing: σ Known Upper Tail Test Example

- Calculate the p-value and compare to **α**



$$P(\bar{X} \geq 53.1)$$

$$= P\left(Z \geq \frac{53.1 - 52.0}{10/\sqrt{64}} \right)$$

$$= P(Z \geq 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > α = .10

Hypothesis Testing: σ Unknown

- If the population standard deviation is unknown, you instead use the sample standard deviation S.
- Because of this change, you use the t distribution instead of the Z distribution to test the null hypothesis about the mean.
- All other steps, concepts, and conclusions are the same.

Hypothesis Testing: σ Unknown

- Recall that the t test statistic with n-1 degrees of freedom is:

$$t_{n-1} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

Hypothesis Testing: σ Unknown Example

The mean cost of a hotel room in New York is said to be \$168 per night. A random sample of 25 hotels resulted in $\bar{X}$ = \$172.50 and S = 15.40. Test at the $\alpha = 0.05$ level.

(A stem-and-leaf display and a normal probability plot indicate the data are approximately normally distributed)

H_0: $\mu = 168$

H_1: $\mu \neq 168$

Hypothesis Testing: σ Unknown Example

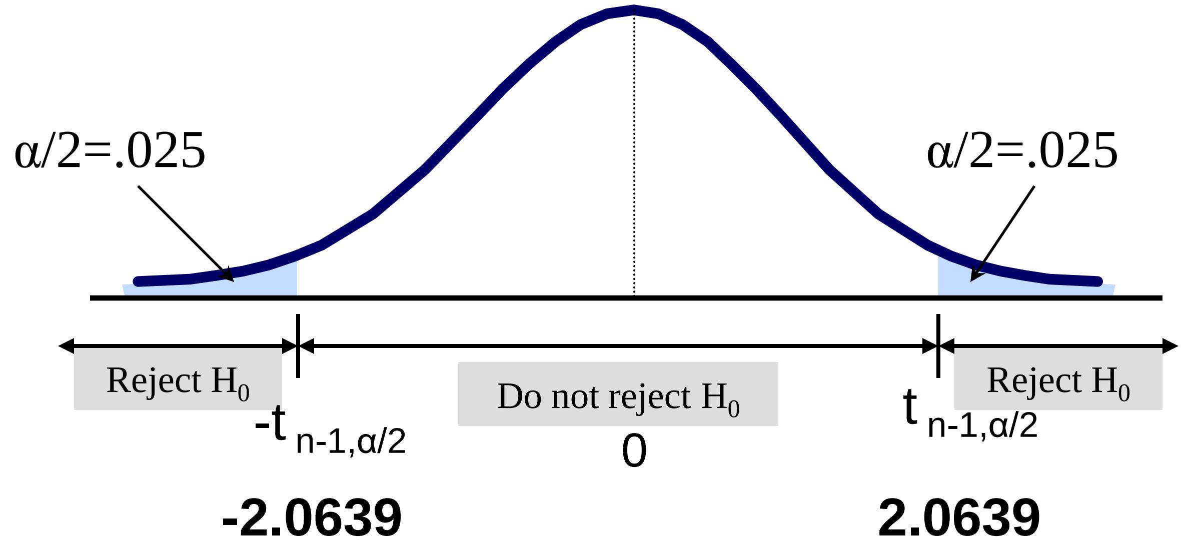
H_0: $\mu = 168$

H_1: $\mu \neq 168$

- **α = 0.05**
- **n = 25**
- **σ is unknown, so use a t statistic**
- **Critical Value:**

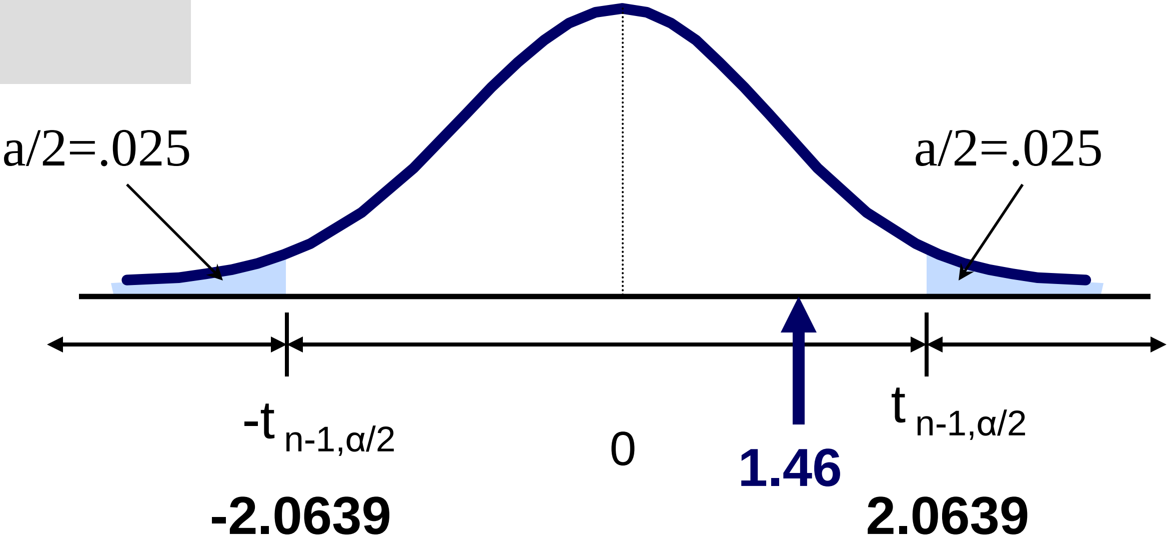
  $t_{24} = \pm 2.0639$

Determine the regions of rejection



Hypothesis Testing: σ Unknown Example

$$t_{n-1} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$



Do not reject H_0: not sufficient evidence that true mean cost is different from $168

Hypothesis Testing: Connection to Confidence Intervals

- For $\overline{X} = 172.5$, $S = 15.40$ and $n = 25$, the 95% confidence interval is:

$$172.5 - (2.0639)\frac{15.4}{\sqrt{25}} \quad \text{to} \quad 172.5 + (2.0639)\frac{15.4}{\sqrt{25}}$$

$$166.14 \leq \mu \leq 178.86$$

- Since this interval contains the hypothesized mean (168), you do not reject the null hypothesis at $\alpha = .05$

Hypothesis Testing: σ Unknown

- Recall that you assume that the sample statistic comes from a random sample from a normal distribution.
- If the sample size is small (< 30), you should use a box-and-whisker plot or a normal probability plot to assess whether the assumption of normality is valid.
- If the sample size is large, the central limit theorem applies and the sampling distribution of the mean will be normal.

Hypothesis Testing Proportions

- Involves categorical variables
- Two possible outcomes
  - "Success" (possesses a certain characteristic)
  - "Failure" (does not possesses that characteristic)
- Fraction or proportion of the population in the "success" category is denoted by π

Hypothesis Testing Proportions

- Sample proportion in the success category is denoted by p

$$p = \frac{X}{n} = \frac{\text{number of successesin sample}}{\text{sample size}}$$

- When both $n\pi$ and $n(1-\pi)$ are at least 10, p can be approximated by a normal distribution with mean and standard deviation

$$\mu_p = \pi$$

$$\sigma_p = \sqrt{\frac{\pi(1-\pi)}{n}}$$

Hypothesis Testing Proportions

- The sampling distribution of p is approximately normal, so the test statistic is a Z value:

$$Z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}}$$

Hypothesis Testing Proportions Example

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 30 responses. Test at the α = .05 significance level.

First, check:

$n\,\pi = (500)(.08) = 40$

$n(1-\pi) = (500)(.92) = 460$

Hypothesis Testing Proportions Example

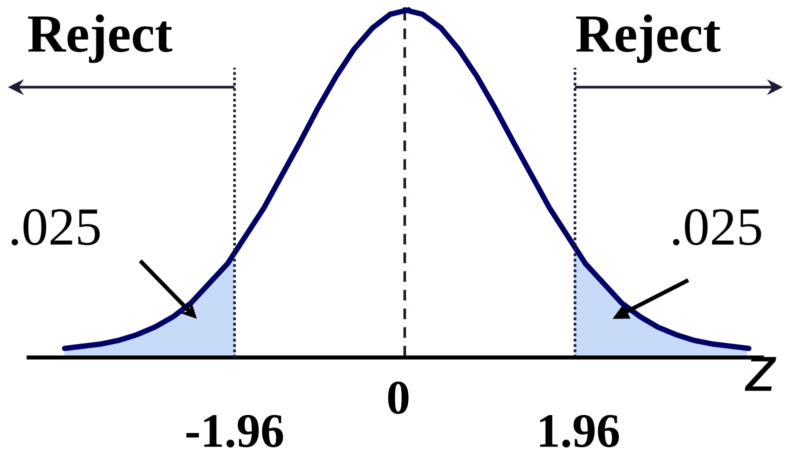
H_0: $\pi = .08$ H_1: $\pi \neq .08$

$\alpha = .05$

$n = 500$, $p = .06$

Critical Values: $\pm$ 1.96

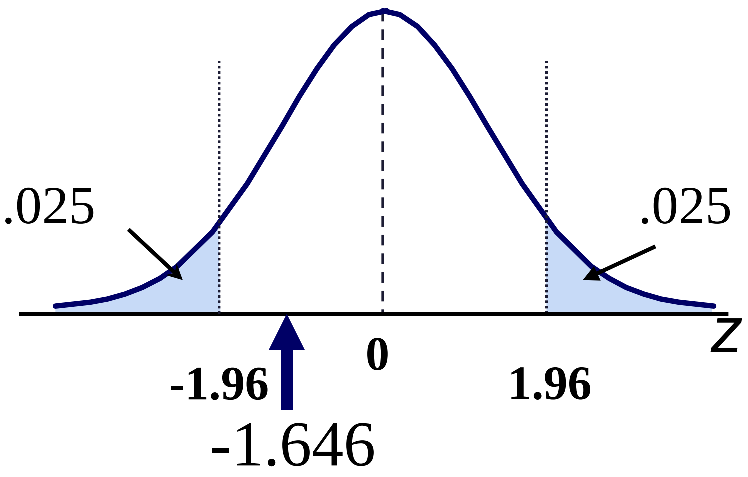
Determine region of rejection



Hypothesis Testing Proportions Example

Test Statistic:

$$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} = \frac{.06-.08}{\sqrt{\frac{.08(1-.08)}{500}}} = -1.648$$



Decision:

Do not reject H_0 at $\alpha = .05$

Conclusion:

There isn't sufficient evidence to reject the company's claim of 8% response rate.

Potential Pitfalls and Ethical Considerations

- Use randomly collected data to reduce selection biases
- Do not use human subjects without informed consent
- Choose the level of significance, α, before data collection

Chapter Summary

In this chapter, we have

- Addressed hypothesis testing methodology
- Performed Z Test for the mean (σ known)
- Discussed critical value and p–value approaches to hypothesis testing
- Performed one-tail and two-tail tests
- Performed t test for the mean (σ unknown)
- Performed Z test for the proportion
- Discussed pitfalls and ethical issues