# Discrete Mathematics and Logic (UE18CS205)

# Unit 2 - Sets, Functions and Relations

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# Sets

- A **set** is an unordered collection of objects.
- Sets are discrete structures used to group objects together, often the objects having similar properties.

- The objects in a set are called the **elements** or **members** of the set.
- A set is said to **contain** its elements.
- An element is said to **belong** to the set.

- $\mathbf{a \in S}$ denotes that “a” is an element of the set S.
- $\mathbf{a \notin S}$ denotes that “a” is not an element of the set S.

**Set roster form**: All the members of the set are listed separated by commas enclosed in curly braces.

Eg: A set of natural numbers from 1 to 5.

$S = \{1, 2, 3, 4, 5\}$

Eg: the set of positive integers less than hundred

$S = \{1, 2, 3, \dots, 99\}$

**Set builder form:** Elements in the set are characterized by stating the property or properties they must have to be members.

Eg: the set of positive integers less than hundred

$A = \{ x \in \mathbf{Z^+} \mid x < 100\}$

Eg: set of rational numbers.

$B = \{ p/q \mid p \in \mathbf{Z}; q \in \mathbf{Z}, \text{ and } q \neq 0\}$

Well known number sets:

- **N** = {0, 1, 2, 3, ...}, the set of natural numbers
- **Z** = {..., -2, -1, 0, 1, 2, ...}, the set of integers
- $\mathbf{Z}^+$ = {1, 2, 3, 4, ...}, set of positive integers
- **Q** = {p/q | p $\in$ **Z ;** q $\in$ **Z** and q $\neq$ 0} the set of rational numbers
- **R** the set of real numbers
- **C** the set of complex numbers

**Equality** of two sets
Two sets are equal if and only if they have the same elements.

i.e., if A and B are sets, then A and B are equal if and only if
$\forall x\ (x \in A \leftrightarrow x \in B)$.

if A and B are equal sets, it can be denoted as **A = B**.

Eg: $\{a, b\} = \{a, b\}$

Eg: $\{1,3,5\} = \{3,1,5\}$

Eg: $\{1,3,3,5,5,5,5\} = \{1,3,5\}$

Eg: $\{1\} \neq \{\{1\}\}$

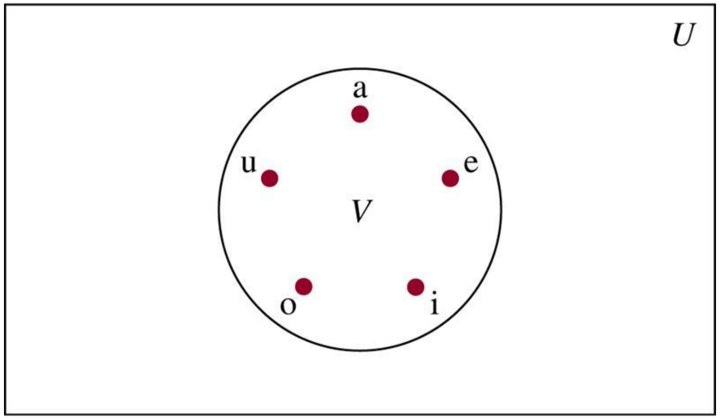
- Empty set / Null set
  $\{ \} = \Phi$

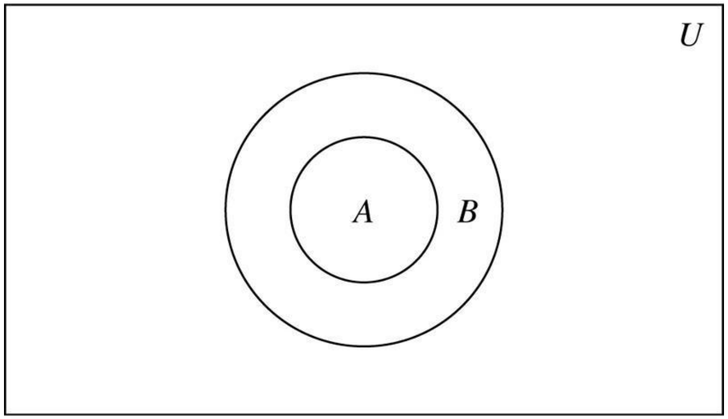
- Singleton set
  $\{ a \}, \{ Z^+ \}$

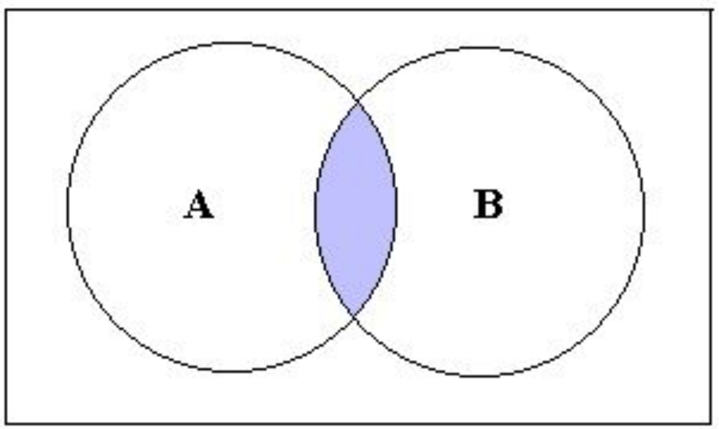
- $\Phi \neq \{\Phi\}$

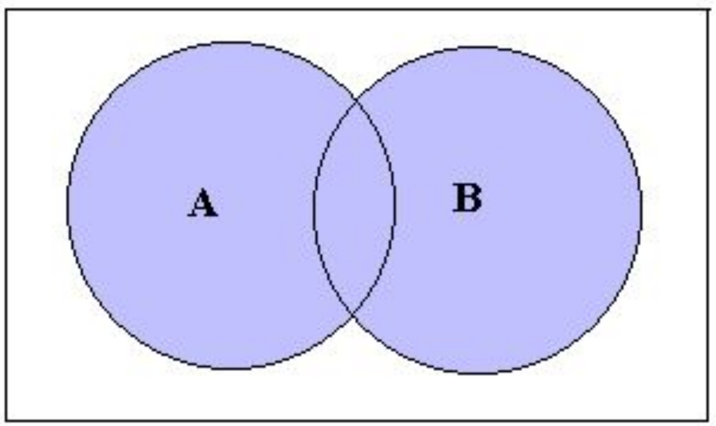
# Venn Diagram

Sets can be represented graphically using Venn diagrams.



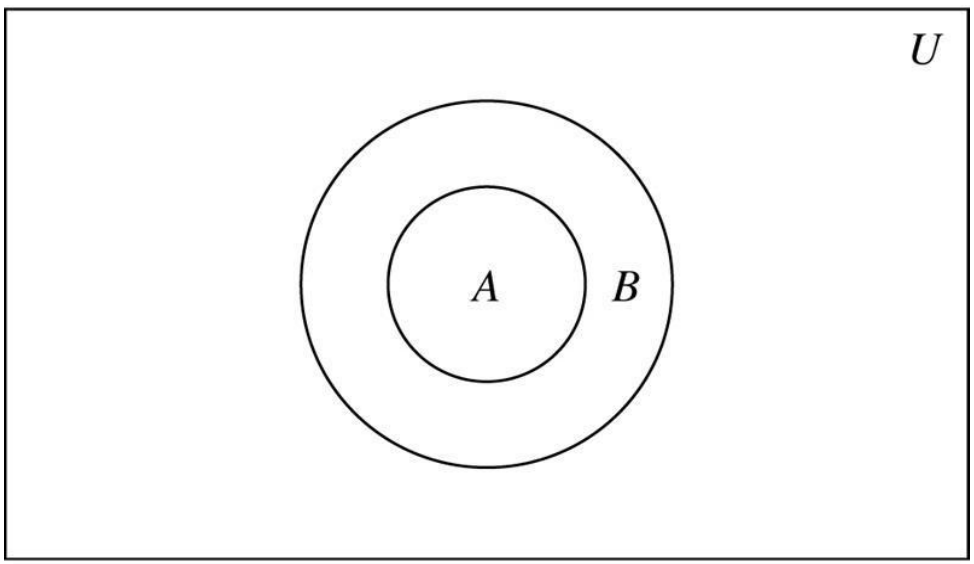



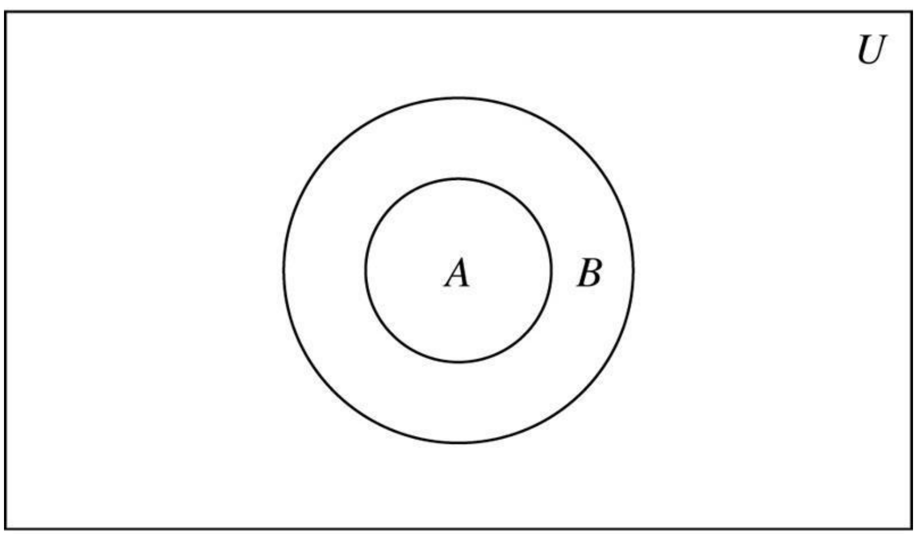
# Subsets:

- The set A is called a subset of set B if and only if every element of A is also an element of B.
- We use the notation A ⊆ B to indicate that A is a subset of set B.
- A ⊆ B if and only if **∀x (x ∈ A** → **x ∈ B)** is true.



# Proper Subset:

- When set A is a subset of set B but $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.
- For $A \subset B$ to be true it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A.
- That is, A is a proper subset of B if
  $\boldsymbol{\forall x\, (x \in A \rightarrow x \in B) \land \exists x\, (x \in B \land x \notin A)}$ is true.



**Theorem:** For every set S,

1. $\varnothing \subseteq S$ 2. $S \subseteq S$

**Proof of $\varnothing \subseteq S$:**

Let S be a set.

$\varnothing \subseteq S$ iff $\forall x(x \in \varnothing \rightarrow x \in S)$.

Because the empty set contains no elements, it follows that $x \in \varnothing$ is always false.

It follows that the conditional statement $x \in \varnothing \rightarrow x \in S$ is always true, because the hypothesis is always false and a conditional statement with a false hypothesis is true.

That is $\forall x(x \in \varnothing \rightarrow x \in S)$ is true.

$\therefore \varnothing \subseteq S$

It's a vacuous proof.

**Cardinality:** If there are exactly **n** distinct elements in S where **n** is a non-negative integer, we say that S is a **finite set** and that **n** is the **cardinality** of S. The cardinality of S is denoted by **|S|**.

Eg: Set S of letters in the English alphabet. Then |S| = 26.

Eg: |∅| = 0 .

A set is said to be **infinite,** if it is not finite.

Eg: The set of positive integers {1, 2, 3, …} is infinite.

**Power Set** of S, P(S), is the set of all subsets of the set S.

If a set has **n** elements, then its power set has $\mathbf{2^n}$ elements.

Eg: P({0,1,2}) = {∅, {0}, {1}, {2}, {0,1}, {1,2}, {0,2}, {0,1,2})

The **Ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element, …, $a_n$ as its $n^{th}$ element.

**Cartesian product** of A and B is

$$\mathbf{A\ X\ B} = \{(a, b) \mid a \in A \wedge b \in B\}$$

Eg: A = {a, b} and B = {1, 2, 3}

A X B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}

B X A = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

$A \times B \neq B \times A$

$|A \times B| = |A| * |B|$

A subset R of AXB is called a **relation** from set A to set B.

The **cartesian product** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i$ belongs to set $A_i$ for $i = 1, 2, \ldots, n$.

That is, $A_1 \times A_2 \times \cdots \times A_n =$

$$\{(a_1, a_2, ..., a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

## Truth Sets

Given Predicate P, and domain D, the truth set of P is the set of elements x in D for which P(x) is true.

The truth set of P(x) is denoted by $\{x \in D \mid P(x)\}$.

Q: What are the truth sets of the predicates P(x), Q(x), and R(x), where the domain is the set of integers and
P(x) is "$|x| = 1$",
Q(x) is "$x^2 = 2$", and
R(x) is "$|x| = x$" ?

$\forall_{x \in S}$ (P(x)) denotes the universal quantification of P(x) over all elements in the set S.

i.e., $\boldsymbol{\forall_{x \in S} (P(x))}$ is shorthand for $\boldsymbol{\forall x\, (x \in S} \rightarrow \boldsymbol{P(x))}$.

$\exists_{x \in S}$ (P(x)) denotes the existential quantification of P(x) over all elements in S.

i.e., $\boldsymbol{\exists_{x \in S} (P(x))}$ is shorthand for $\boldsymbol{\exists x\, (x \in S \wedge P(x))}$.

**Russell's Paradox**

Let the domain be the set of all sets

$S = \{x \mid x \notin x\}$

Is S a member of itself?

Suppose, $S \in S$. Then, the predicate $x \notin x$ is false. Hence, S should not belong to S. It's a contradiction.

Suppose, $S \notin S$. Then, the predicate $x \notin x$ is true. Hence, S should belong to S. It's a contradiction.

Therefore, it is a paradox.

Analogy:

Predicate: I help people who can't help themselves.

Suppose I'm one of those people.

When I'm sick (i.e. I can't help myself), I'm in a paradox.

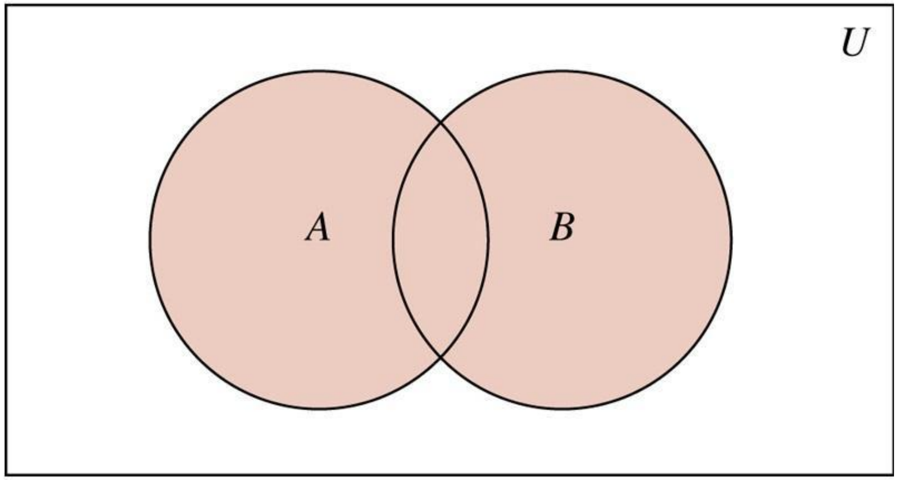
According to the predicate, I should help "me", but I can't do that because I'm sick. If I don't help myself, I'm violating the predicate.

The Big Bang Theory Scene: Leonard's car. "Play that funky music, white boy" is playing on the stereo.

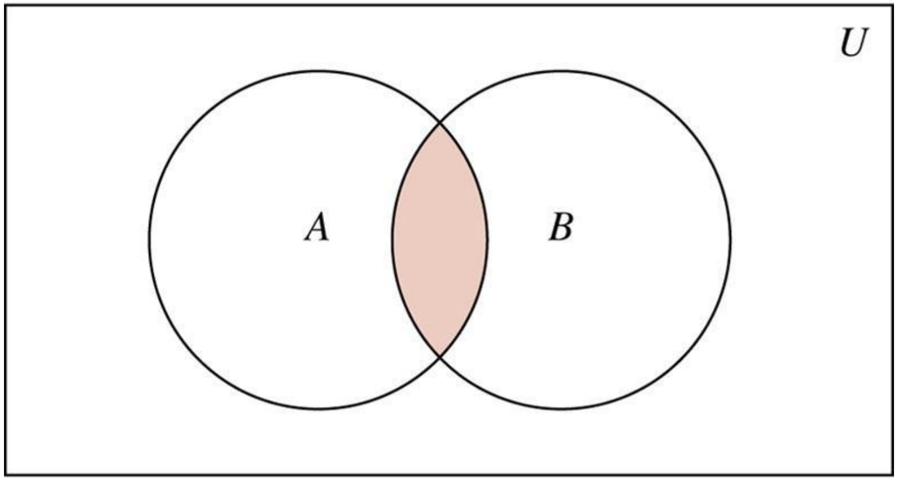
- Sheldon: So they're requesting that the white boy play the funky music, yes?
  - Leonard: Yes.
- Sheldon: And this music we're listening to right now is funky as well?
  - Leonard: Sure.
- Sheldon: Let me ask you this. Do you think this song is the music the white boy ultimately plays?
  - Leonard: It could be.
- Sheldon: So it's like the musical equivalent of **Russell's Paradox**, the question of whether the set of all sets that don't contain themselves as members contains itself?
  - Leonard: Exactly.
- Sheldon: Well then I hate it. Music should just be fun!

# Set Operations

- Union of sets A and B contains those elements in A, B or both.
  - **$A \cup B = \{ x \mid x \in A \vee x \in B \}$**
- Intersection of sets A and B contains those elements in both A and B.
  - **$A \cap B = \{ x \mid x \in A \wedge x \in B \}$**




$A \cup B$ is shaded.




$A \cap B$ is shaded.

Two sets are **disjoint** when their intersection is empty.

$A \cap B = \varnothing$

Eg: $\{1, 2\} \cap \{3, 4\} = \varnothing$

Q: What are the resulting sets of the following.

1. $\{1, 2, 3\} \cap \{1, 2\} = \{1, 2\}$
2. $\{1, 2, 3\} \cap \{R, G, B\} = \varnothing$
3. $\{1, 2, 3\} \cap \varnothing = \varnothing$
4. $\{1, 2, 3\} \cup \{1, 4\} = \{1, 2, 3, 4\}$
5. $\{1, 2, 3\} \cup \{R, G, B\} = \{1, 2, 3, R, G, B\}$
6. $\{1, 2, 3\} \cup \varnothing = \{1, 2, 3\}$
7. $\{1, 2, 3\} \cup \{\ \} = \{1, 2, 3\}$

Q: What are the resulting sets of the following.

1. $\{1, 2, 3\} \cap \{1, 2\} = \{1, 2\}$
2. $\{1, 2, 3\} \cap \{R, G, B\} = \varnothing$
3. $\{1, 2, 3\} \cap \varnothing = \varnothing$
4. $\{1, 2, 3\} \cup \{1, 4\} = \{1, 2, 3, 4\}$
5. $\{1, 2, 3\} \cup \{R, G, B\} = \{1, 2, 3, R, G, B\}$
6. $\{1, 2, 3\} \cup \varnothing = \{1, 2, 3\}$
7. $\{1, 2, 3\} \cup \{\ \} = \{1, 2, 3\}$

Two sets are **disjoint** when their intersection is empty.

$A \cap B = \varnothing$

**Cardinality of union:**

If sets A and B are disjoint, $|A \cup B| = |A| + |B|$

In general, $|A \cup B| = |A| + |B| - |A \cap B|$

Generalization of this result (of n sets) is called the **principle of inclusion-exclusion**.

Eg: $A = \{1, 2, 3\}$, $B = \{3, 4\}$

$|A \cup B| = |A| + |B| - |A \cap B| = 3+2-1 = 4$

Eg: $| \{1, 2, 3\} \cup \{ 2, 3, 4 \} | = 3+3-2 = 4$

## Difference of two sets:

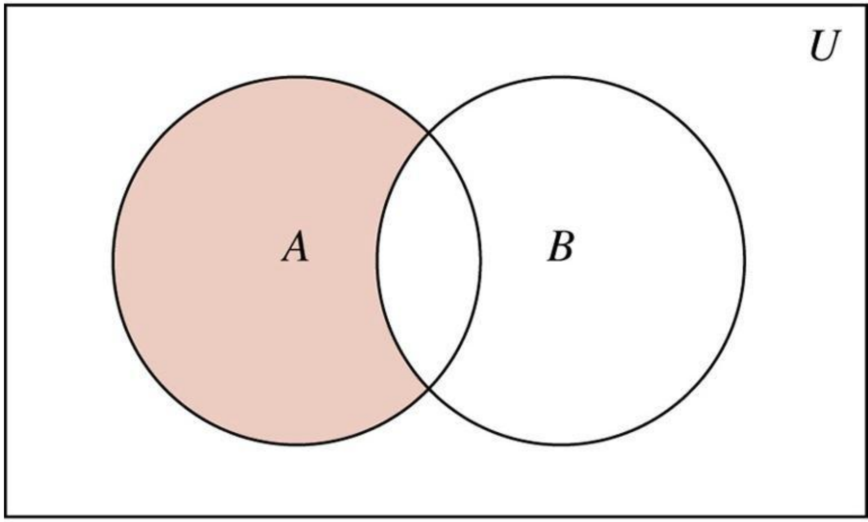
Difference of two sets, A - B, is set containing elements in A but not in B.

**$A - B = \{ x \mid x \in A \land x \notin B\}$**

Eg: {1, 2 } - {3, 4} = {1, 2}

Eg: {1, 2, 3 } - {3, 4} = {1, 2}

Eg: {1, 2, 3} - {1, 2, 3} = ∅




$A - B$ is shaded.

Eg: {1, 2, 3} - { 1, 2 } = {3}

Eg: {1, 2, 3} - { 1, 2, 3 } = { }

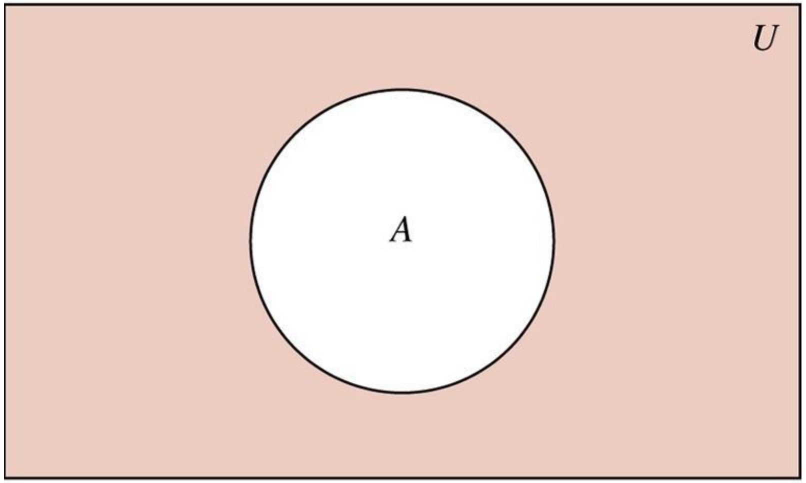
Eg: {1, 2, 3} - ∅ = {1, 2, 3}

**Complement of a set:**

Complement, $\bar{A}$ (A bar) or A`, is the complement with respect to the universal set, U. That is, the difference U - A is the complement of A.

$\bar{A} = \{ x \mid x \in U \land x \notin A\}$

$\bar{A} = \{ x \mid x \notin A\}$



$\bar{A}$ is shaded.

Eg: $U = \{ 1, 2, 3, 4, 5, 6 \}$

$A = \{ 1, 2, 3, 4 \}$

$\bar{A} = \{ 5, 6\}$

Eg: $U = \{a, e, i, o, u \}$

$A = \{a, e, o \}$

$\bar{A} = \{i, u\}$

**TABLE 1 Set Identities.**

| ***Identity*** | ***Name*** |
|---|---|
| $A \cup \varnothing = A$<br>$A \cap U = A$ | Identity laws |
| $A \cup U = U$<br>$A \cap \varnothing = \varnothing$ | Domination laws |
| $A \cup A = A$<br>$A \cap A = A$ | Idempotent laws |
| $\overline{(\overline{A})} = A$ | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Commutative laws |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ | Associative laws |
| $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | Distributive laws |
| $\overline{A \cup B} = \overline{A} \cap \overline{B}$<br>$\overline{A \cap B} = \overline{A} \cup \overline{B}$ | De Morgan's laws |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption laws |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \varnothing$ | Complement laws |

**Membership Tables**
(Observe the similarities with the Truth Tables)

Eg: Use a **membership table** to show
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**TABLE 2** **A Membership Table for the Distributive Property.**

| $A$ | $B$ | $C$ | $B \cup C$ | $A \cap (B \cup C)$ | $A \cap B$ | $A \cap C$ | $(A \cap B) \cup (A \cap C)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Membership Table**

Prove De Morgan’s law (A U B)` = A` ∩ B`

using membership table.

| $A$ | $B$ | $A \cup B$ | $\overline{A \cup B}$ | $\overline{A}$ | $\overline{B}$ | $\overline{A} \cap \overline{B}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |

**Q:** Prove De Morgan’s law (A ∩ B)` = A` U B`

**without** using membership table.

**Q:** Use set builder notation and logical equivalences to establish the De Morgan's law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Soln:

| | |
|---|---|
| $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ | By definition of complement |
| $= \{x \mid \neg(x \in (A \cap B))\}$ | By definition of $\notin$ symbol |
| $= \{x \mid \neg((x \in A) \wedge (x \in B))\}$ | By definition of intersection |
| $= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ | By De Morgan's law of logic |
| $= \{x \mid x \notin A \vee x \notin B$ | By definition of $\notin$ symbol |
| $= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$ | By definition of complement |
| $= \{x \mid x \in \overline{A} \cup \overline{B}\}$ | By definition of union |
| $= \overline{A} \cup \overline{B}$ | By meaning of set builder notation |

Therefore, $\overline{A \cap B} = \overline{A} \cup \overline{B}$

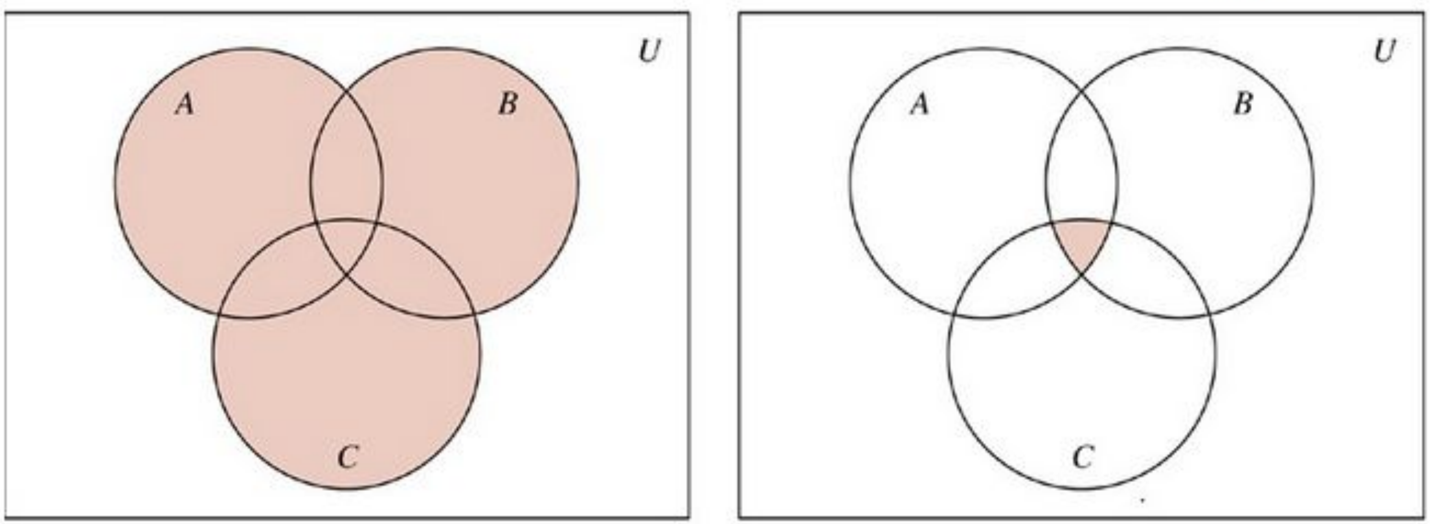
# Generalized Unions and Intersections

## Union:

Union of a collection of sets contains elements that are members of at least one set in the collection. $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

## Intersection

Intersection of a collection of sets contains elements that are members of all sets in the collection. $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=0}^{n} A_i$




(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

# Computer Representation of Sets:

U = { 5, 4, 3, 2, 1, 0 } = 111111

A = { 2, 1, 0 } = 000111

B = { 3, 2 } = 001100

$A \cup B = A \vee B$

$\{2, 1, 0\} \cup \{3, 2\} = \{3, 2, 1, 0\}$

$$\begin{array}{rl} & 000111 \\ \vee & \underline{001100} \\ & 001111 \end{array} \qquad \begin{array}{rl} & \{2, 1, 0\} \\ \cup & \underline{\{3, 2\}} \\ & \{3, 2, 1, 0\} \end{array}$$

$A \cap B = A \wedge B$

$\{2, 1, 0\} \cap \{3, 2\} = \{2\}$

$$\begin{array}{rl} & 000111 \\ \wedge & \underline{001100} \\ & 000100 \end{array} \qquad \begin{array}{rl} & \{2, 1, 0\} \\ \cap & \underline{\{3, 2\}} \\ & \{2\} \end{array}$$

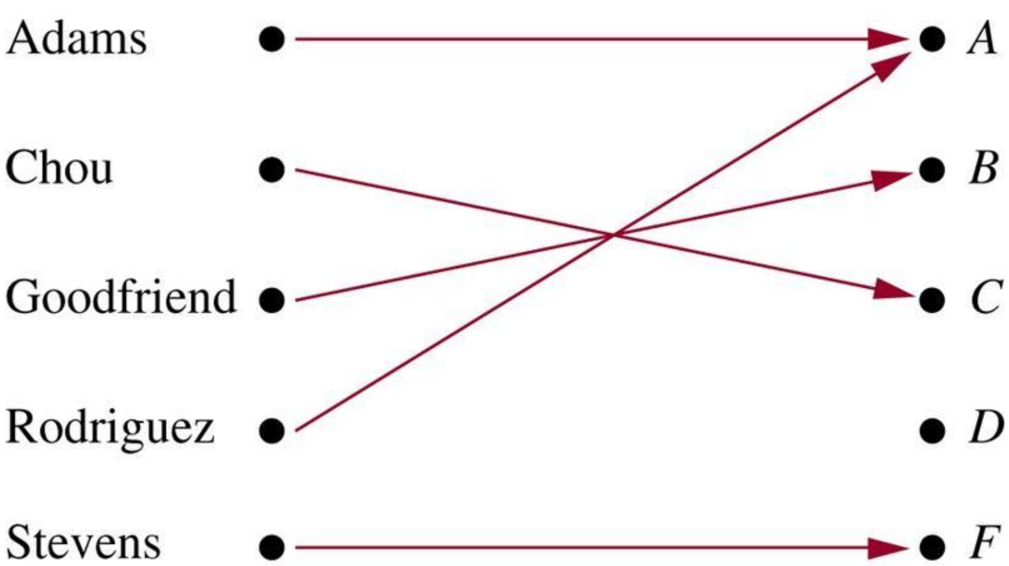
# Functions

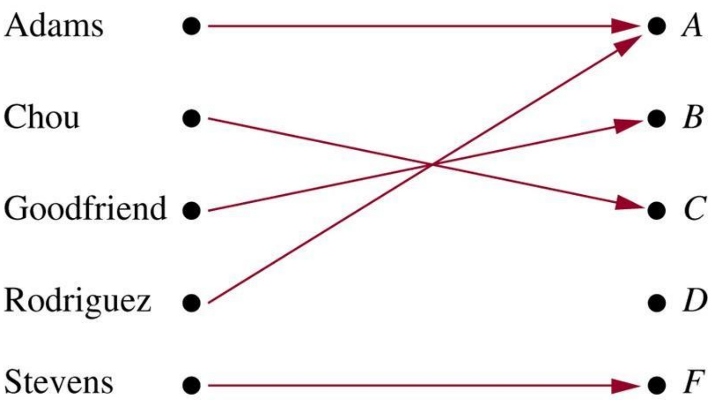
**Function f** from A to B is assignment of **exactly one element of B** to **each element of A**. f(a) = b where 'b' is an element of B assigned by 'f' to the element 'a' of A.

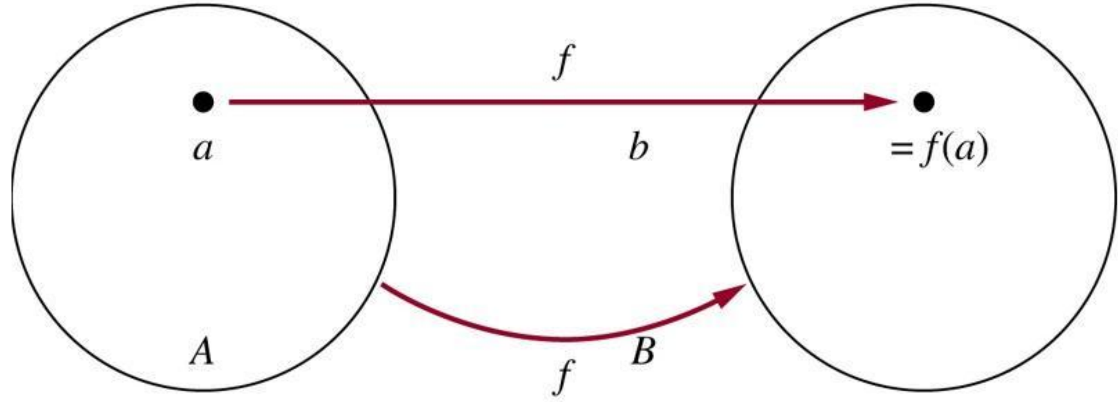
$f : A \rightarrow B$

Functions also called **mappings** or **transformations**.

Eg: **g : Students → Grades**






**Domain** of the function ‘g’ is the set of Students.

**Co-domain** of the function ‘g’ is the set of Grades.

**Image** of g(Goodfriend) is B.

**Range** of the function ‘g’ is {A, B, C, F}

Domain of $f : A \rightarrow B$ is A

Co-domain of $f : A \rightarrow B$ is B

Image $f( a ) = b$ is b

Preimage $f( a ) = b$ is a

Range of $f : A \rightarrow B$ is set of all images of elements of A

Two real-valued functions with the same domain can be added and multiplied.

Let $f_1 : A \to R$ and $f_2 : A \to R$

Then, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 \bullet f_2)(x) = f_1(x) \bullet f_2(x)$

Eg:

$f_1 : R \to R$

$f_2 : R \to R$

$f_1(x) = x^2$

$f_2(x) = x - x^2$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$

$(f_1\ f_2)(x) = f_1(x) \bullet f_2(x) = x^2 \bullet (x - x^2) = x^3 - x^4$

**Image of a subset of the domain:**

$f : A \longrightarrow B$

$C \subseteq A$

Image of C under f is a subset of B.

$f( C ) \subseteq B$

$f( C ) = \{ x \mid \exists c \in C ( x = f( c ) ) \}$

## Types of functions

- One-to-One (Injective) function
- Onto (Surjective) function
- One-to-One Correspondence (Bijective) function

**One-to-One (Injective) functions:**

A function is said to be One-to-one or injective, if and only if f(a)=f(b) implies that a=b for all a and b in the domain of f.

That is, **f is one-to-one iff $\forall a \forall b \, ( f(a) = f(b) \rightarrow a = b )$**

That is, **f is one-to-one iff $\forall a \forall b \, ( a \neq b \rightarrow f(a) \neq f(b) )$**

Eg: A = { a, b, c, d },

B = { 1, 2, 3, 4, 5 }

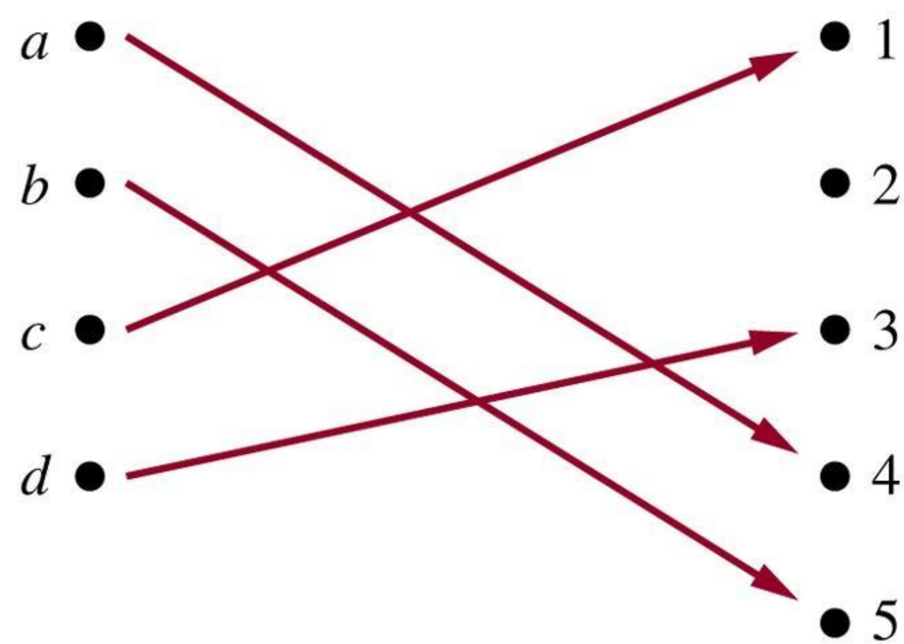
$g : A \rightarrow B$

g(a) = 4,

g(b) = 5,

g(c) = 1,

g(d) = 3



Let f be a function. Which of the following defines one-to-one function f?

1. $\forall a \forall b \, ( f(a) = f(b) \leftrightarrow a = b )$
2. $\forall a \forall b \, ( a = b \rightarrow f(a) = f(b) )$
3. $\forall a \forall b \, ( f(a) = f(b) \rightarrow a = b )$
4. $\forall a \forall b \, ( a \neq b \rightarrow f(a) \neq f(b) )$

**Increasing/Decreasing functions:**

Consider a function f whose domain and codomain are subsets of the set of real numbers.

Function f is **increasing** if $f(x) \leq f(y)$ for real $x < y$

That is, $\forall x \forall y( x < y \rightarrow f(x) \leq f(y) )$

Function f is **strictly increasing** if $f(x) < f(y)$ for real $x < y$.

That is, $\forall x \forall y( x < y \rightarrow f(x) < f(y) )$

Strictly increasing functions must be one-to-one.

Function f is **decreasing** if $f(x) \geq f(y)$ for real $x < y$.

That is, $\forall x \forall y( x < y \rightarrow f(x) \geq f(y) )$

Function f is **strictly decreasing** if $f(x) > f(y)$ for real $x < y$.

That is, $\forall x \forall y( x < y \rightarrow f(x) > f(y) )$

Strictly decreasing functions must be one-to-one.

**Onto (Surjective) functions:**

A function f from A to B is called onto or surjective, if and only if every element b $\in$ B there is an element a $\in$ A with f(a) = b.

i.e., f is Onto function iff every b $\in$ B has a $\in$ A with f(a) = b.

In short, $\boldsymbol{\forall y \exists x\, (f(x) = y)}$

Eg: A = { a, b, c, d },

B = { 1, 2, 3 }

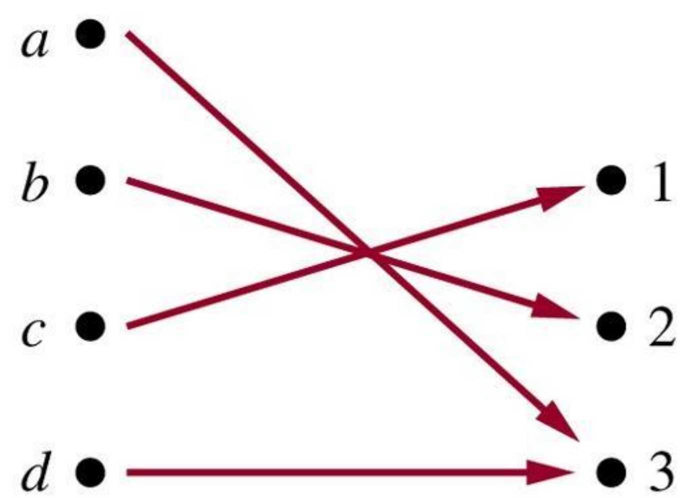
G : A $\rightarrow$ B

G(a) = 3,

G(b) = 2,

G(c) = 1,

G(d) = 3

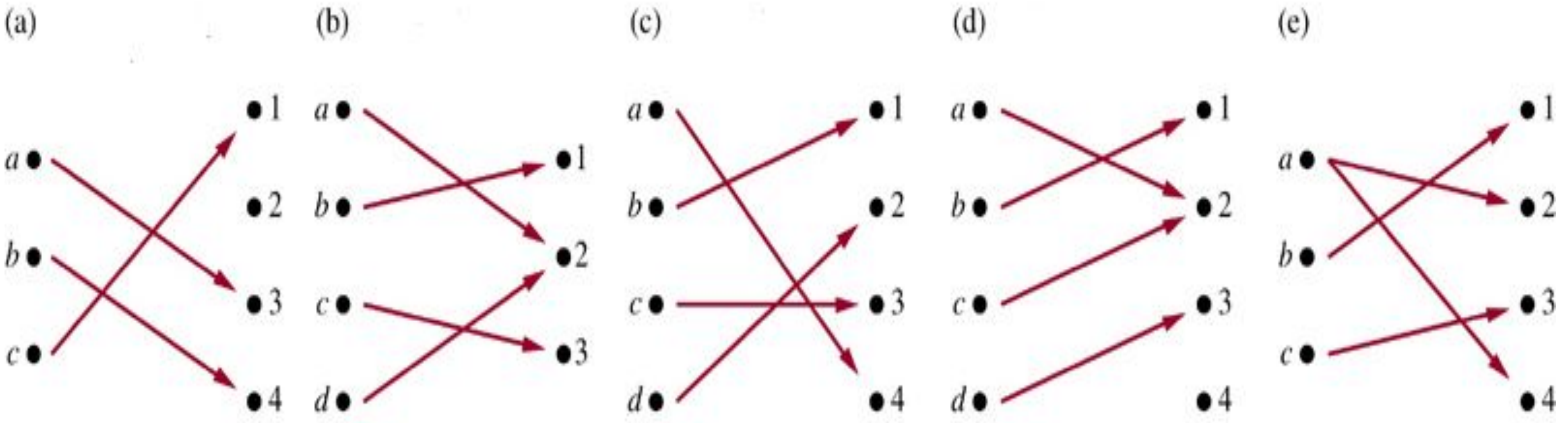


**One-to-One Correspondence (Bijection) functions:**

A function is One-to-one correspondence or bijective function if only if it is one-to-one and onto.

Eg: What kind relations are these?

1. One-to-one correspondence
2. One-to-one but not onto
3. Onto but not one-to-one
4. Neither one-to-one nor onto



Q: Diagram the following functions and mention whether they are one-to-one, onto or one-to-one correspondence:

1. $f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$
   $f(a) = 1$
   $f(b) = 2$
   $f(c) = 3$
   $f(d) = 4$

1. $g : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$
   $g(a) = 1$
   $g(b) = 1$
   $g(c) = 4$
   $g(d) = 4$

## Inverse Function:

Inverse of a function **f** from A to B such that

**$f^{-1}$**(b) = a when f(a) = b.

Function f is **invertible** when **f** is one-to-one correspondence (i.e. one-to-one and onto), otherwise inverse function of **f** does not exists.

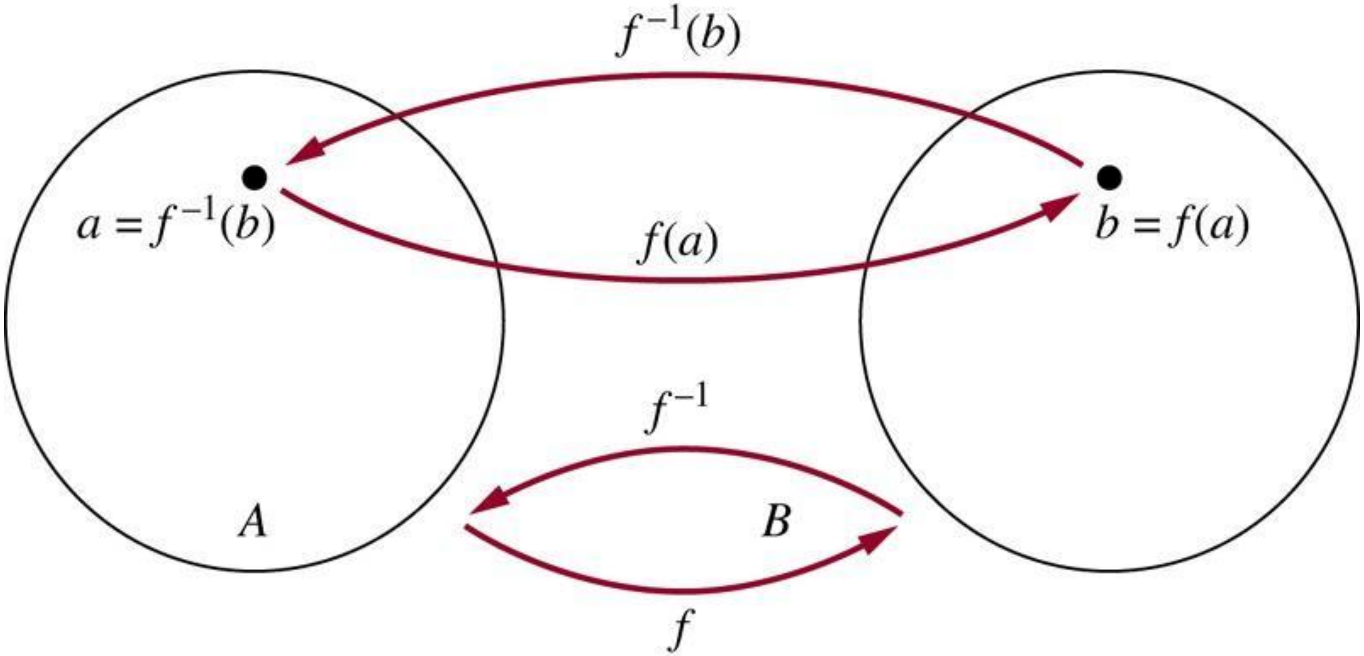
Eg: f : $\mathbf{R^+} \rightarrow \mathbf{R^+}$

$f(x) = x^2$

$f^{-1}(y) = y^{1/2}$

$f(3) = 3^2 = 9$

$f^{-1}(9) = 9^{1/2} = 3$




Eg: f : $\mathbf{Z} \rightarrow \mathbf{Z}$

$f(x) = x + 3$

$f^{-1}(y) = y - 3$

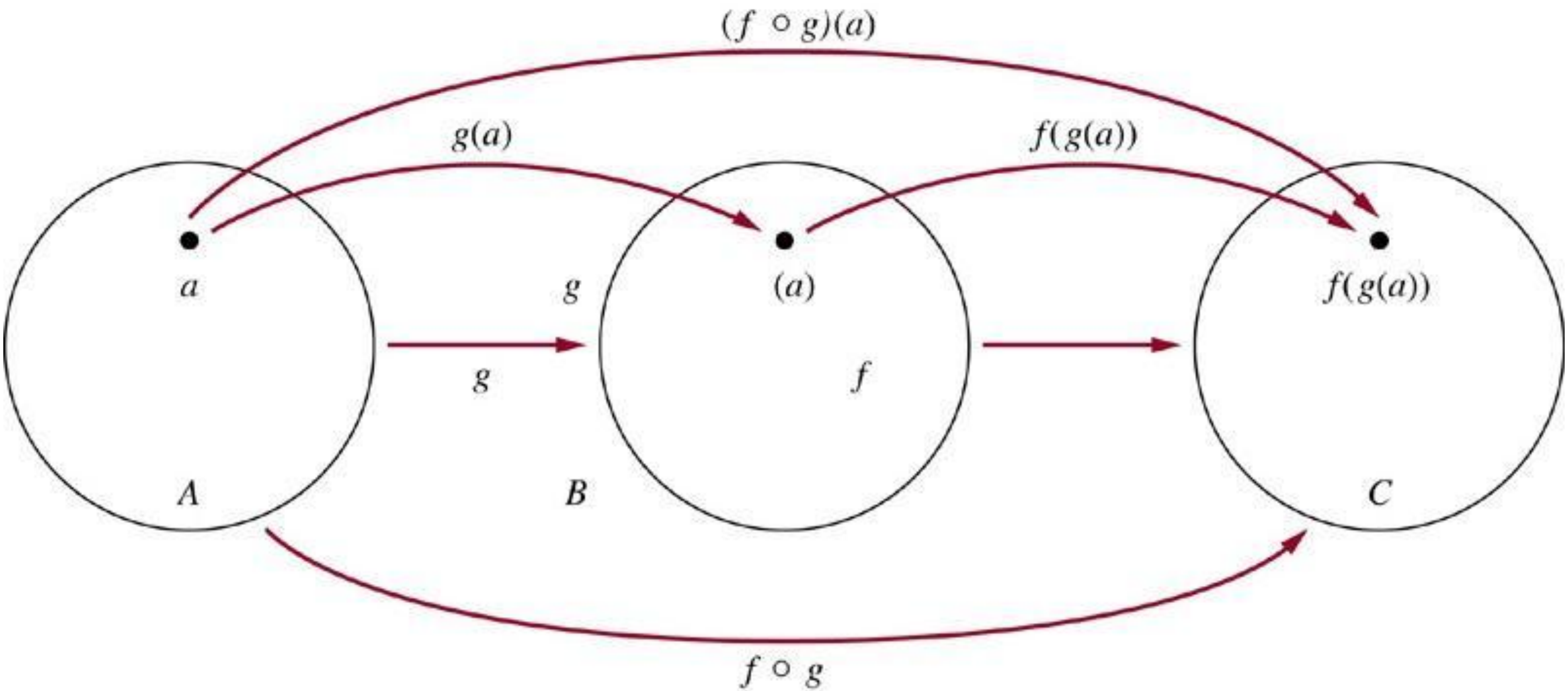
$f(20) = 23$

$f^{-1}(23) = 20$

## Composition of Functions:

Composition of function g from A to B and function f from B to C.

$(f \circ g)(a) = f(g(a))$



$g: \{a, b, c\} \rightarrow \{X, Y, Z\}$

$g(a) = X$

$g(b) = Y$

$g(c) = Z$

$f: \{X, Y, Z\} \rightarrow \{1, 2, 3\}$

$f(X) = 1$

$f(Y) = 2$

$f(Z) = 3$

$(f \circ g)(a) = f(g(a)) = f(X) = 1$

$(f \circ g)(b) = f(g(b)) = f(Y) = 2$

$(f \circ g)(c) = f(g(c)) = f(Z) = 3$

$$\begin{tikzcd}
a \arrow[r, "g"] & X \arrow[r, "f"] & 1 \\
b \arrow[r] & Y \arrow[r] & 2 \\
c \arrow[r] & Z \arrow[r] & 3
\end{tikzcd}$$

$g \circ f$ is not defined because *f* range, $\{1, 2, 3\}$, is not a subset of *g* domain, $\{a, b, c\}$

$(g \circ f)(X) = g(f(X)) = g(1)$ is not defined.

Eg:

$f(x) = 5x + 7$

$g(x) = 3x + 2$

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 5(3x+2) + 7 = 15x + 17$

$(g \circ f)(x) = g(f(x)) = g(5x+7) = 3(5x+7) + 2 = 15x + 23$

$(f \circ g)(x) \neq (g \circ f)(x)$

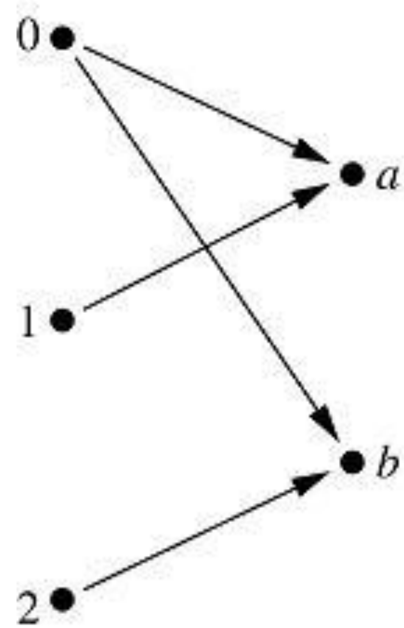
## Relations

A **binary relation** from A to B is a subset of AXB.

Element **a** is related to **b** by **R** is denoted by **aRb**.

aRb denotes (a, b) $\in$ R

$a \not{R} b$ denotes (a, b) $\notin$ R




| $R$ | $a$ | $b$ |
|---|---|---|
| 0 | × | × |
| 1 | × | |
| 2 | | × |

Eg:

A = {0, 1, 2}

B = {a, b}

A x B = { (0, a), (1, a), (0, b), (1, b), (2, a), (2, b) }

R = { (0, a), (1, a), (0, b), (2, b) } $\subseteq$ A x B

## Matrix Representation of Relations:

2-dimensional 0-1 matrix is used for binary relations.

One row for each element of A

One column for each element of B

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \\ 0 \; if \; (a_i, b_j) \notin R \end{cases}$$

## Directed Graph (Digraph) Representation of Relations:

Set V of vertices (nodes) representing elements of the sets.

Set E ordered pairs of elements of V called edges (arcs).

Vertex a is initial vertex and vertex b is terminal vertex of edge (a, b).

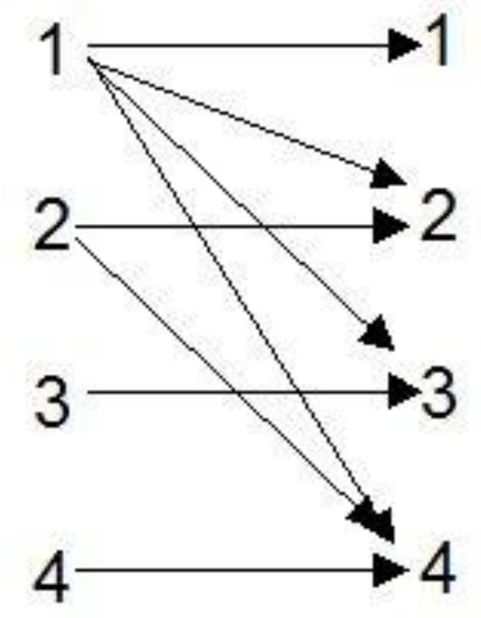
Edge (a, a) is a loop.

**Relations on a Set:**

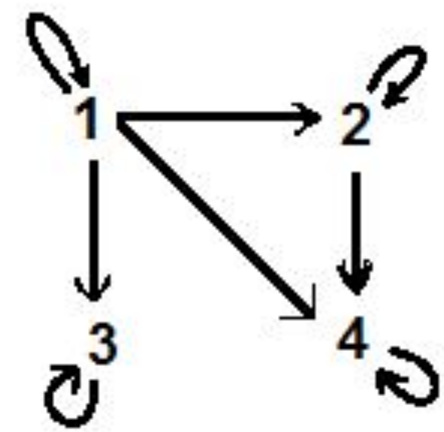
Relation on the set A is a relation from A to A.

Eg: A = {1,2,3,4}

A x A = { (1,1), (1,2), (1,3), (1,4),
(2,1), (2,2), (2,3), (2,4),
(3,1), (3,2), (3,3), (3,4),
(4,1), (4,2), (4,3), (4,4) }




R = { (a, b) | a divides b}
= {(1,1), (1,2), (1,3), (1,4),
(2,2), (2,4), (3,3), (4,4)}



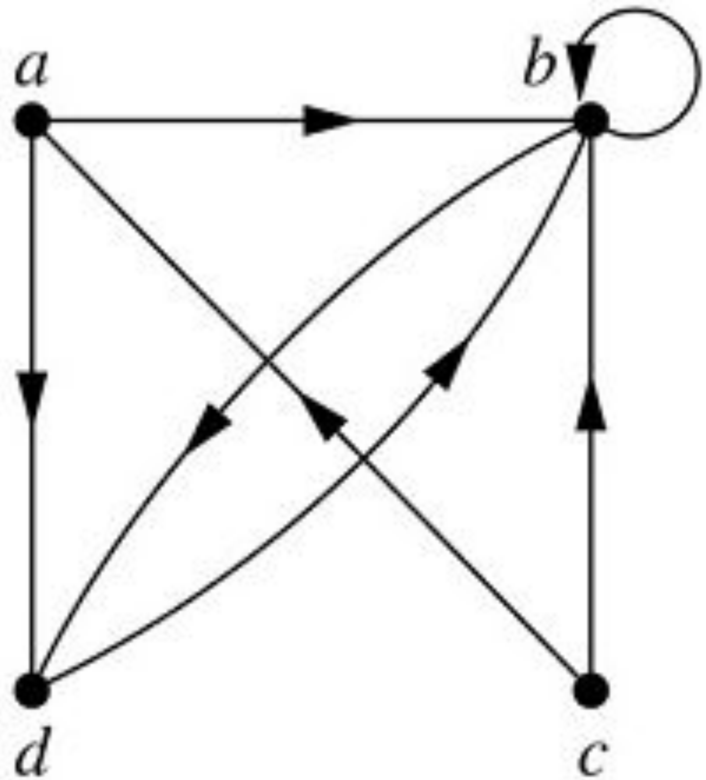

R1 = {(a, b) | a >= b}

R2 = {(a, b) | a = b}

Eg: Set A = {a, b, c, d}

Relation R = {(a, d), (a, b), (b, b), (b, d), (c, b), (c, a), (d, b)}

| | a | b | c | d |
|---|---|---|---|---|
| a | 0 | 1 | 0 | 1 |
| b | 0 | 1 | 0 | 1 |
| c | 1 | 1 | 0 | 0 |
| d | 0 | 1 | 0 | 0 |



Eg: Set $A = \{1, 2, 3, 4\}$

Show the matrix and digraph representation of the relation

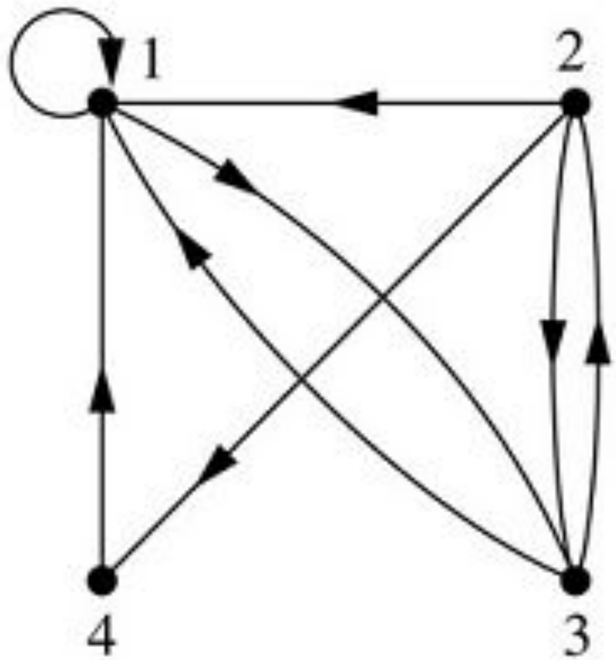
$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$

Eg: Set A = {1, 2, 3, 4}

Show the matrix and digraph representation of the relation

R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 2 | 1 | 0 | 1 | 1 |
| 3 | 1 | 1 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 |



**Relations:**

- Let A = {1, 2, 3, 4}. Relation R on set A be {(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)}.
- Let S = {a,b,c,d,e,f}. Relation R on set S be {(a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)}
- Relation R on the set of integers such that aRb if and only if a = b or a = -b.
- Relation R on the set of real numbers such that aRb if and only if a - b is an integer.
- Relation R = {(a, b) | a ≡ b (mod 10)}.
- Relation R = {(a, b) | a ≡ b (mod m)}, where $m \in Z^+ \land m>1$
- Relation R on the set of strings of English letters such that aRb if and only if Length(a) = Length(b).

## Properties of Relations on a set:

1. Reflexive
2. Symmetric
3. Antisymmetric
4. Transitive

## Reflexive Relations:

A relation R on a set A is reflexive iff

- $(a,a) \in R$ for every element $a \in A$.
- **$\forall a \in A \; ((a, a) \in R)$.**

Eg: $A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$ is reflexive

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 1 | 1 | 0 | 0 |
| **2** | 0 | 1 | 0 | 0 |
| **3** | 0 | 0 | 1 | 0 |
| **4** | 0 | 0 | 0 | 1 |

## Symmetric Relations:

A relation R on a set A is symmetric iff

- $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$.
- $\forall a \in A\ \forall b \in A\ ((a, b) \in R \rightarrow (b, a) \in R)$
- $\forall a \in A\ \forall b \in A\ ((a, b) \notin R \rightarrow (b, a) \notin R)$

Eg: Let set A = {1, 2, 3, 4}

R = {(1, 2), (2, 1), (1, 4), (4, 1), (3, 3)} symmetric

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 0 | 1 | 0 | 1 |
| **2** | 1 | 0 | 0 | 0 |
| **3** | 0 | 0 | 1 | 0 |
| **4** | 1 | 0 | 0 | 0 |

## Antisymmetric Relations:

A relation R on a set A is antisymmetric iff

- $a=b$ whenever $(a,b) \in R$ and $(b,a) \in R$, for all $a,b \in R$.
- $\forall a \in A\ \forall b \in A\ ((a,b) \in R \land (b,a) \in R \rightarrow a=b)$
- $\forall a \in A\ \forall b \in A\ ((a \neq b) \rightarrow (a, b) \notin R \lor (b, a) \notin R)$

Eg: Let set A = {1, 2, 3, 4}

R = {(1, 2), (3, 3), (4, 1)} antisymmetric

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 0 | 1 | 0 | 0 |
| **2** | 0 | 0 | 0 | 0 |
| **3** | 0 | 0 | 1 | 0 |
| **4** | 1 | 0 | 0 | 0 |

## Symmetric? Antisymmetric?

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 1 | 1 |
| **2** | 1 | 0 | 0 |
| **3** | 1 | 0 | 1 |

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 1 | 1 |
| **2** | 0 | 0 | 0 |
| **3** | 1 | 0 | 1 |

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 1 | 1 |
| **2** | 0 | 0 | 0 |
| **3** | 0 | 1 | 1 |

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 0 | 0 |
| **2** | 0 | 0 | 0 |
| **3** | 0 | 0 | 1 |

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 1 | 0 |
| **2** | 0 | 0 | 0 |
| **3** | 0 | 0 | 1 |

| | **1** | **2** | **3** |
|---|---|---|---|
| **1** | 0 | 0 | 0 |
| **2** | 0 | 0 | 0 |
| **3** | 0 | 0 | 0 |

# Symmetric? Antisymmetric?

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 1 | 1 | | |
| **2** | 1 | 0 | 0 | | |
| **3** | 1 | 0 | 1 | **Y** | **N** |

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 1 | 1 | | |
| **2** | 0 | 0 | 0 | | |
| **3** | 1 | 0 | 1 | **N** | **N** |

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 1 | 1 | | |
| **2** | 0 | 0 | 0 | | |
| **3** | 0 | 1 | 1 | **N** | **Y** |

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | | |
| **2** | 0 | 0 | 0 | | |
| **3** | 0 | 0 | 1 | **Y** | **Y** |

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 1 | 0 | | |
| **2** | 0 | 0 | 0 | | |
| **3** | 0 | 0 | 1 | **N** | **Y** |

| | 1 | 2 | 3 | | |
|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | | |
| **2** | 0 | 0 | 0 | | |
| **3** | 0 | 0 | 0 | **Y** | **Y** |

## Transitive Relations:

A relation R on a set A is transitive iff

- $(a,c)\in R$ whenever $(a,b)\in R$ and $(b,c)\in R$, for all $a,b,c\in R$.
- $\forall a\in A\ \forall b\in A\ \forall c\in A\ ((a,b)\in R \land (b,c)\in R \rightarrow (a,c)\in R)$

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 0 | 1 | 1 | 1 |
| **2** | 0 | 0 | 1 | 1 |
| **3** | 0 | 0 | 1 | 0 |
| **4** | 0 | 0 | 1 | 0 |

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 1 | 0 | 1 | 0 |
| **2** | 0 | 0 | 0 | 0 |
| **3** | 1 | 0 | 1 | 0 |
| **4** | 0 | 0 | 0 | 0 |

*Reflexive* relation R on A, if $\forall a \in A, (a, a) \in R$.

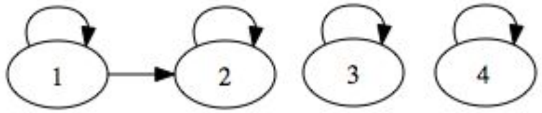
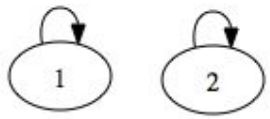
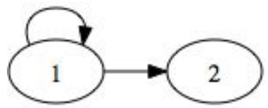
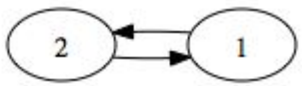
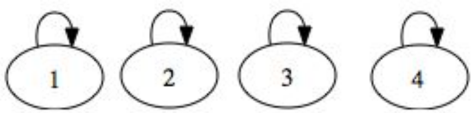
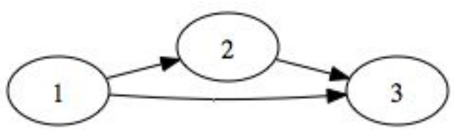
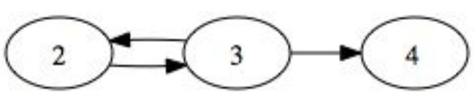
*Symmetric* relation R on A, if $\forall a \forall b \in A, (a, b) \in R \rightarrow (b, a) \in R$.

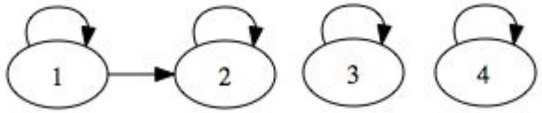
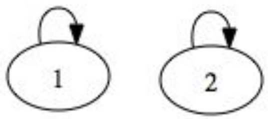
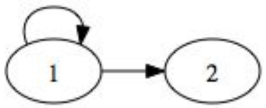
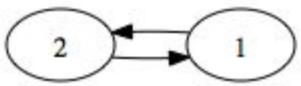
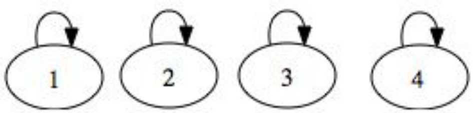
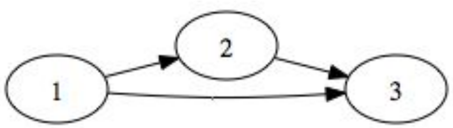
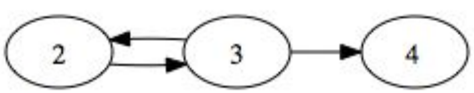
*Antisymmetric* relation R on A, if $\forall a \forall b \in A, (a, b) \in R \wedge (b, a) \in R \rightarrow a=b$.

*Transitive* relation R on A, if $\forall a \forall b \forall c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$.

Relations on the set $A = \{1, 2, 3\}$

| | *reflexive* | *symmetric* | *antisymmetric* | *transitive* |
|---|---|---|---|---|
| $R_0 = \{(1,1), (2,2), (3,3)\}$ | Yes | Yes | Yes | Yes |
| $R_1 = \{(2,2), (2,3), (3,2)\}$ | No | Yes | No (2,3) (3,2) | No (3,3) |
| $R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$ | Yes | Yes | No (1,2) (2,1) | Yes |
| $R_3 = \{(2,3), (3,2)\}$ | No | Yes | No (2,3) (3,2) | No (2,2) (3,3) |
| $R_4 = \{(1,2), (2,3), (1,3)\}$ | No | No | Yes | Yes |

| Relation | Reflexive | Symmetric | Antisymmetric | Transitive |
|---|---|---|---|---|
|   | | | | |
|   | | | | |
|   | | | | |
|   | | | | |
|   | | | | |
|   | | | | |
|   | | | | |
|   | | | | |

| Relation | Reflexive | Symmetric | Antisymmetric | Transitive |
|---|---|---|---|---|
|  | Y | N | Y | Y |
|  | Y | Y | Y | Y |
|  | N | N | Y | Y |
|  | N | Y | N | N |
|  | Y | Y | Y | Y |
|  | N | N | Y | Y |
|  | N | N | Y | N |
|  | N | N | N | N |

Q: Mention the properties (reflexive, symmetric, antisymmetric and transitive properties) of the relations.

1. Relation of $\{a, b, c\}$,
   $R = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$
   Ans: …

2. Relation of $\{a, b, c, d\}$,
   $S = \{(a, c), (c, a), (a, d), (d, a), (d, d), (a, b), (b, a), (b, b)\}$
   Ans: …

Q: Mention the properties (reflexive, symmetric, antisymmetric and transitive properties) of the relations.

1. Relation of {a, b, c},
   R = {(a, a), (b, b), (c, c), (b, c), (c, b)}
   Ans: Y, Y, N, Y

2. Relation of {a, b, c, d},
   S = {(a, c), (c, a), (a, d), (d, a), (d, d), (a, b), (b, a), (b, b)}
   Ans: N, Y, N, N

**Combining Relations:**

Relations from A to B are subsets of A x B and can be combined in any way two sets can be combined.

Eg: A = {0, 1, 2}, B = {a, b}

A x B = { (0, a), (1, a), (0, b), (1, b), (2, a), (2, b) }

R1 = {(0, a), (0, b)}, R2 = {(0, b), (1, a), (1, b)}

R1 ∩ R2 = {(0, b)}

R1 U R2 = {(0, a), (0, b), (1, a), (1, b)}

R1 - R2 = {(0, a)}

R2 - R1 = {(1, a), (1, b)}

R1 ⊕ R2 = R1 U R2 - R1 ∩ R2

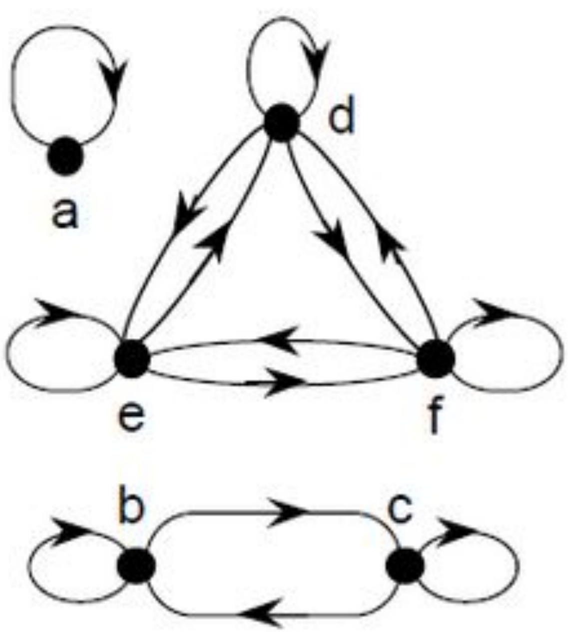
= {(0, a), (0, b), (1, a), (1, b)} - {(0,b)}

= {(0, a), (1, a), (1, b)}

## Equivalence Relations:

A relation R on a set A is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.

Eg: Let S = {a,b,c,d,e,f}. Relation R on set S be {(a,a),(b,b),(b,c),(c,b),(c,c),(d,d),(d,e),(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)}




| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| a | 1 | | | | | |
| b | | 1 | 1 | | | |
| c | | 1 | 1 | | | |
| d | | | | 1 | 1 | 1 |
| e | | | | 1 | 1 | 1 |
| f | | | | 1 | 1 | 1 |

Q: Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$

be a relation on A. Verify that R is an equivalence relation.

Soln:

R is reflexive since it contains (1,1), (2,2), (3,3) and (4,4).

That is, $\forall x\, (x,x) \in R$

R is symmetric since it contains (1,2), (2,1), (3,4), (4,3) and no (a,b) where (b,a) is not in R.

That is, $\forall x \forall y\, ((x,y) \in R \rightarrow (y,x) \in R)$

R is transitive since for every pair of (x,y) and (y,z), there is (x,z) in R.

That is, $\forall x \forall y \forall z\, ((x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R)$

Therefore, R is an equivalence relation.

**Q:** Let R be a relation on the set of real numbers such that aRb iff a-b is an integer. Prove whether R is an equivalence relation.

**Soln:** …

**Q:** Let R be a relation on the set of real numbers such that aRb iff a-b is an integer. Prove whether R is an equivalence relation.

**Soln:** a-a=0 and $0 \in Z$

That is, $\forall$a (aRa). $\therefore$R is reflexive.

Let a-b = k be an integer.

Then, b-a = -k, which is also an integer.

That is, if aRb, then bRa. $\therefore$R is symmetric.

Let a-b=k and b-c=m where k and m are integers.

Then, a-c = (a-b)-(c-b) = k-(-m), which is an integer.

That is, if aRb and bRc, then aRc. $\therefore$R is transitive.

Because R is reflexive, symmetric and transitive,

R is an equivalence relation.

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

If $a \equiv b \pmod{m}$, then “$m \mid (a - b)$” …… (read: m divides a-b)

…

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

If $a \equiv b \pmod{m}$, then “$m \mid (a - b)$” …… (read: m divides a-b)

$a \equiv a \pmod{m}$ because $m|a-a$, which is same as $m|0$.

$\therefore$R is reflexive.

Let $a \equiv b \pmod{m}$

i.e., $m \mid (a-b)$

$mk = a-b$, where k is an integer

$m(-k) = b-a$

i.e., $b \equiv a \pmod{m}$ because $-k$ is also an integer

$\therefore$ R is symmetric.

Q: Let ‘a’, ‘b’ and ‘m’ are integers with $m > 1$. Show that the relation $R = \{ (a, b) \mid a \equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

Soln:

…

Let $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

i.e., $mk = a-b$ and $ml = b-c$, where k and l are integers.

$mk+ml = a-b+b-c$

$m(k+l) = a-c$

i.e., $a \equiv c \pmod{m}$

∴ R is transitive.

R is reflexive, symmetric and transitive.

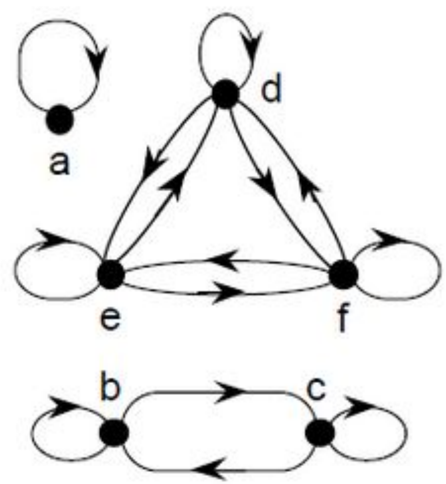
∴ R is an equivalence relation.

**Equivalence Classes:**

Let R be an equivalence relation on a set A. The set of all the elements that are related to an element ‘a’ of A is called equivalence class of ‘a’ denoted by $[a]_R$ or $[a]$ when the relation is implicit.

i.e. $[a]_R = \{s \mid (a,s) \in R\}$

Elements of $[a]_R$ are also known as representatives of $[a]_R$ .




| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| a | 1 | | | | | |
| b | | 1 | 1 | | | |
| c | | 1 | 1 | | | |
| d | | | | 1 | 1 | 1 |
| e | | | | 1 | 1 | 1 |
| f | | | | 1 | 1 | 1 |

Q: What are the equivalence classes of 0 and 1 for the congruence modulo 10?

Soln:

The equivalence class of 0 contains all integers ‘a’ such that $a \equiv 0 \pmod{10}$.

$[0] = \{ \dots, -20, -10, 0, 10, 20, \dots \}$

Similarly, the equivalence class of 1 contains all integers ‘a’ such that $a \equiv 1 \pmod{10}$.

$[1] = \{ \dots, -19, -9, 1, 11, 21, \dots \}$

**Congruence classes modulo m**: are the equivalence classes of the relation congruence modulo m.

$[a]_m = \{ \dots, a-2m, a-m, \mathbf{a}, a+m, a+2m, \dots \}$

Q: What is the equivalence class of an integer ‘a’ for the equivalence relation R defined by
aRb iff a = b or a = -b?

Soln:
aRb iff a = b or a = -b means a related itself and negative of itself.

That is, $[a]_R = \{a, -a\}$

For example, $[10]_R = \{10, -10\}$
$[-100]_R = \{100, -100\}$
$[0]_R = \{0\}$

**Theorem**: Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) $aRb$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Note:

1. Two equivalence classes are either disjoint or identical.

2. Let R be an equivalence relation on a set A and let $a,b \in A$. $[a] \neq [b]$ iff $[a] \cap [b] = \emptyset$.

3. For $a,b \in A$, if $b \in [a]$ then $[a] = [b]$.

**Partition of a Set:**

Let A be a nonempty set. Let P be a set of nonempty subsets $A_1, A_2, \ldots, A_n$ of the set A such that

$A_i \cap A_j = \varnothing$ for $i \neq j$ … Mutually Exclusive

$A_1 \cup A_2 \cup \ldots \cup A_n = A$ … Collectively exhaustive

The set $P = \{A_1, A_2, \ldots, A_n\}$ is called the partition of A.

# Partial Orderings

**Partial Order:**

A relation R on the set S is called a partial order/ordering if it is **reflexive**, **antisymmetric** and **transitive**.

**Poset (S, R):**

Relation R is a partial ordering on set S.

Eg: $(Z, \leq)$ is a Poset.

Eg: $(Z^+, |)$ is a Poset.

**(S, $\preccurlyeq$):**

(S, $\preccurlyeq$) is notation for poset where relation $\preccurlyeq$ is a partial ordering on set S.

**Q:** Show that ($Z^+$, |) is a Poset. (It's the divisibility relation)

Soln: a|a for every integer a.

∴ | is Reflexive

Whenever a≠b, at least one of a|b or b|a is false.

∴ | is antisymmetric.

Whenever a|b and b|c, a|c.

∴ | is transitive.

∴ ($Z^+$, |) is a Poset.

Q: Show that $(P(S), \subseteq)$ is a Poset.

Soln: …

**Comparable and Incomparable:**
Elements a and b are incomparable when they are elements of a poset $(S, \leqslant)$ such that neither $a \leqslant b$ nor $b \leqslant a$.

Eg: In poset $(Z^+, \leq)$, for every pair (a,b) either aRb or bRa. For instance, $10 \leq 20$. That is, every pair (a,b) is comparable.

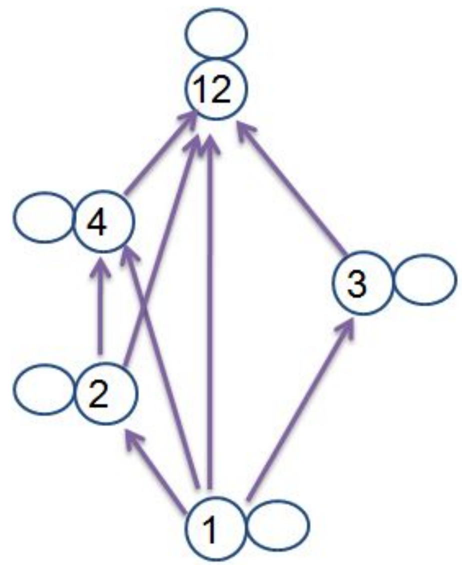
Eg: In poset $(Z^+, |)$, 5 and 7 are incomparable because $5|7$ is false and $7|5$ is false. Whereas 6 and 18 are comparable because 6 divides 18.

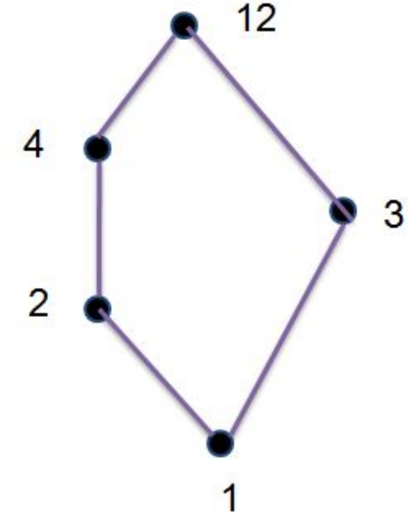
**Total Ordering (Linear Ordering):**
Every pair of elements in S are comparable.
Eg: $(Z, \leq)$

**Hasse Diagram:**

In a digraph of a Partial Order,

1. **remove self-loops** because we know the partial order is **reflexive**.




2. **remove direction marks** from edges because we know that the edges always point upwards as the relation is **antisymmetric**.

1. **remove transitive edges** because we know the relation is **transitive**. If there are edges (a,b) and (b,c), remove (a,c).

# Eg: Course prerequisites

a: ICS

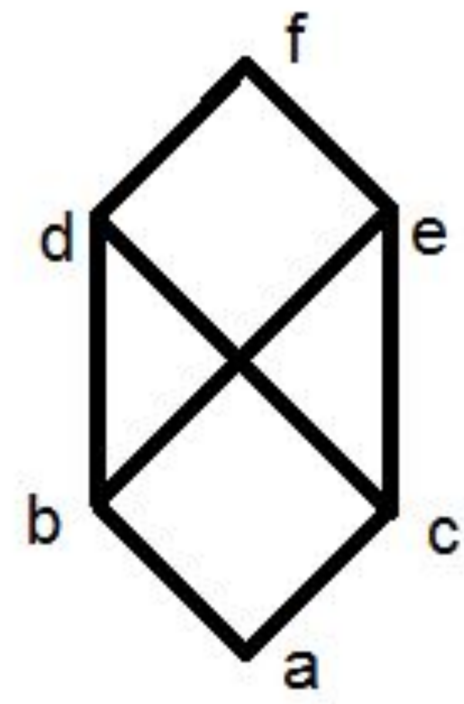
b: DML (requires ICS)

c: DS (requires ICS)

d: DAA (requires DS and DML)

e: DBMS (requires DS and DML)

f: AppDev (requires DAA and DBMS)



**Maximal and Minimal elements:**

'a' is a maximal in the poset (S, $\leqslant$) if there is no $b \in S$ such that $a < b$.

'a' is a minimal in the poset (S, $\leqslant$) if there is no $b \in S$ such that $b < a$.

**Greatest and Least elements:**

'a' is the greatest element of the poset (S, $\leqslant$) if $b \leqslant a$ for all $b \in S$.

'a' is the least element of the poset (S, $\leqslant$) if $a \leqslant b$ for all $b \in S$.

**Upper Bound and Lower Bound elements:**

If 'u' is an element of S such that $a \leqslant u$ for all elements $a \in A$, then u is an upper bound of A.

If 'l' is an element of S such that $l \leqslant a$ for all elements $a \in A$, then l is a lower bound of A.

**Least Upper Bound and Greatest Lower Bound elements:**

'x' is an upper bound that is less than every other upper bound of A.

'l' is a lower bound that is greater than every other lower bound of A.

Maximal elements:12

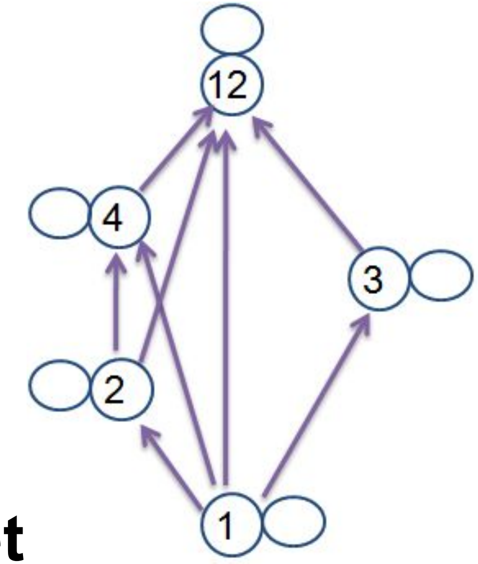
Minimal elements: 1

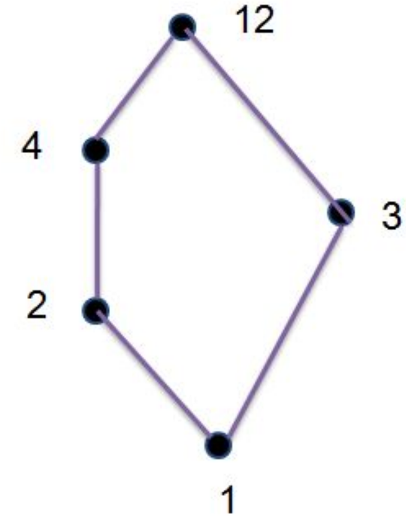
Greatest element: 12

Least element: 1

**These above ones**

**are defined on the poset**




**The following are defined on a subset of the poset.**

Let the subset be A= {2,3}

Upper bound of A: 12

Lower bound of A: 1

Least Upper bound of A: 12

Greatest Lower bound of A: 1

Eg: Poset ({1,2,...,24}, |). It's a "divides" relation.

Maximal elements:

Minimal elements:

Greatest element:

Least element:

For the subset A = {2,3},

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

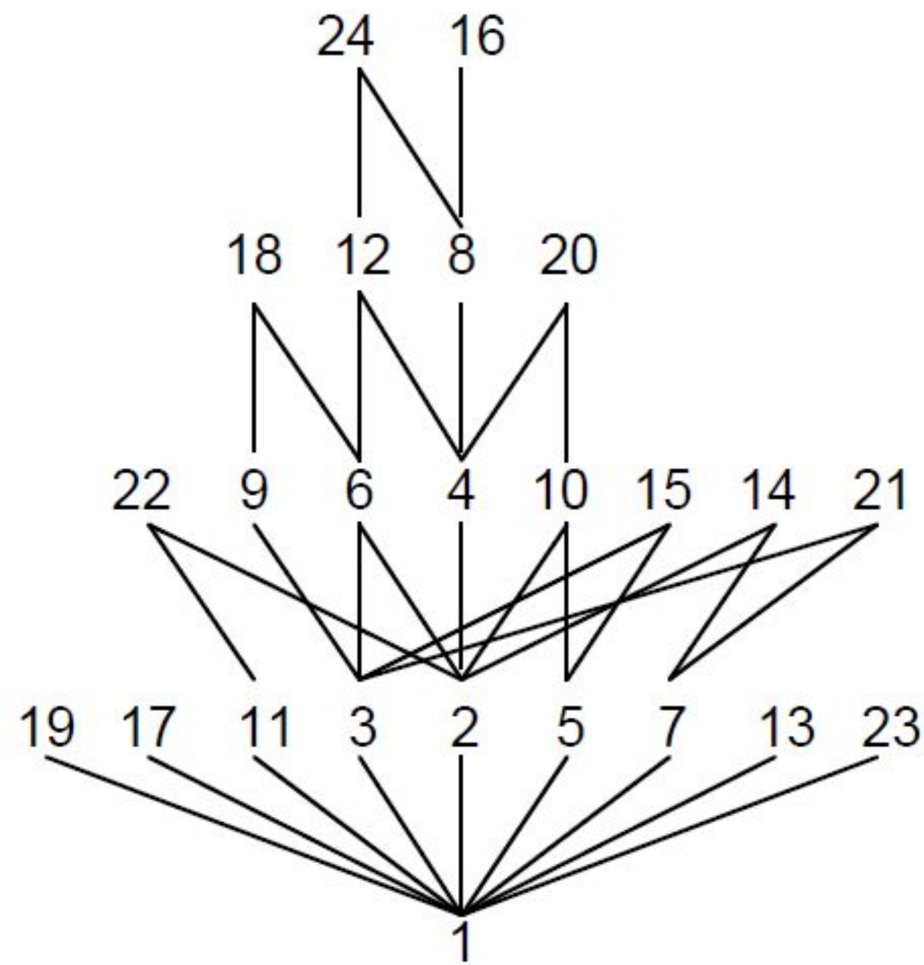
For the subset A = {6,10}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Eg: Poset ({1,2,...,24}, |). It's a "divides" relation.

Maximal elements: **24, 16, 18, 20, 22, 15, 14, 21, 19, 17, 13, 23**

Minimal elements: **1**

Greatest element: **None**

Least element: **1**

For the subset A = {2,3},

Upper bound of A: **6, 12, 18, 24**

Lower bound of A: **1**

Least Upper bound of A: **6**

Greatest Lower bound of A: **1**

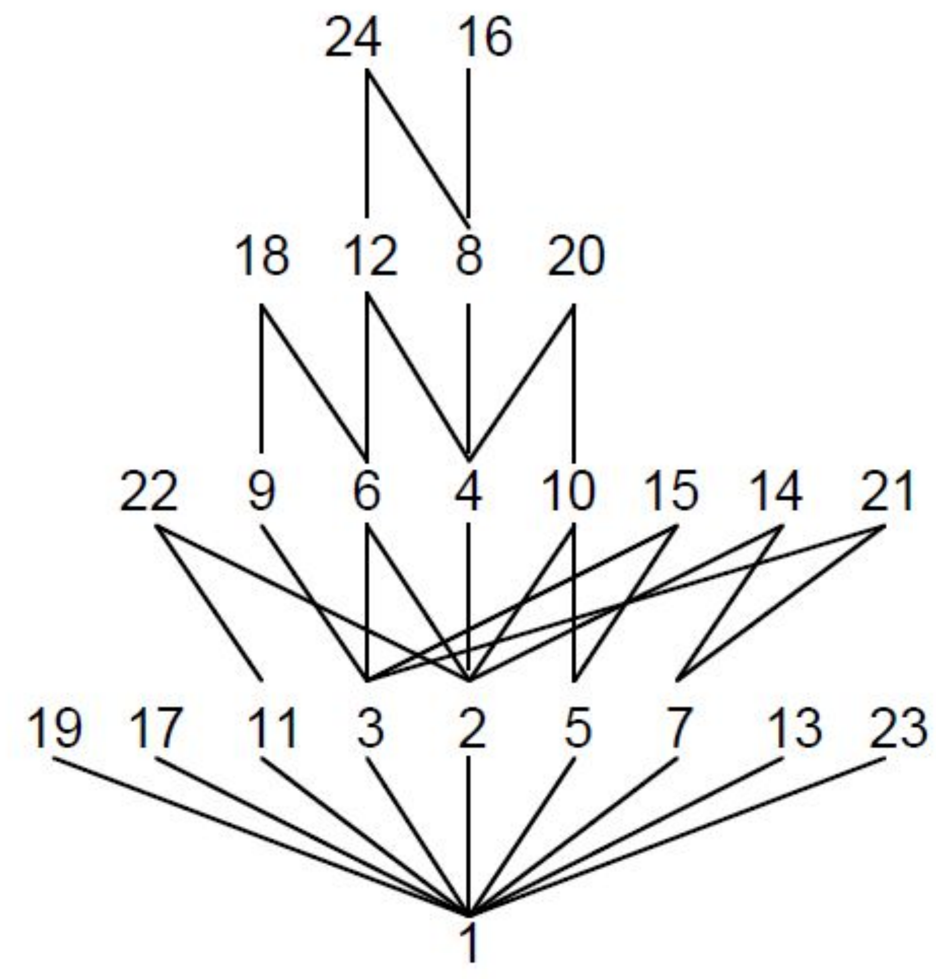
For the subset A = {6,10}

Upper bound of A: **None**

Lower bound of A: **2, 1**

Least Upper bound of A: **None**

Greatest Lower bound of A: **2**



Eg: Let S = Power set of {a,b,c}. Poset (S, $\subseteq$).

Maximal elements:

Minimal elements:

Greatest element:

Least element:

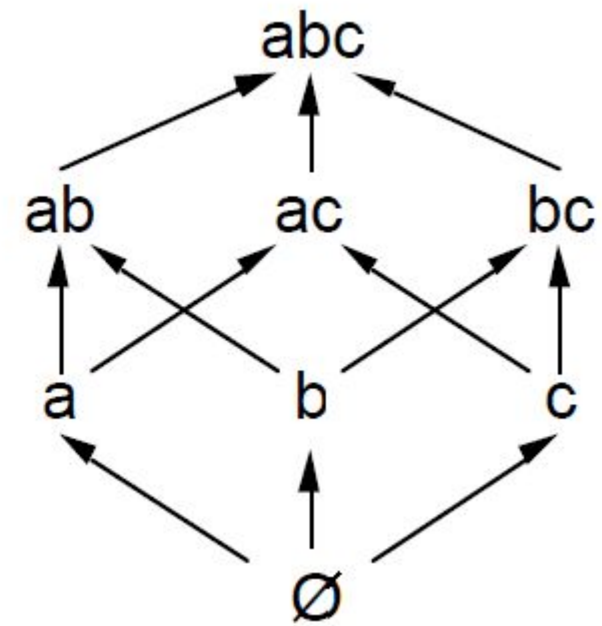
For the subset A = **{ab,b},**

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Eg: Let S = Power set of {a,b,c}. Poset (S, $\subseteq$).

Maximal elements: **abc**

Minimal elements: **ϕ**

Greatest element: **abc**

Least element: **ϕ**

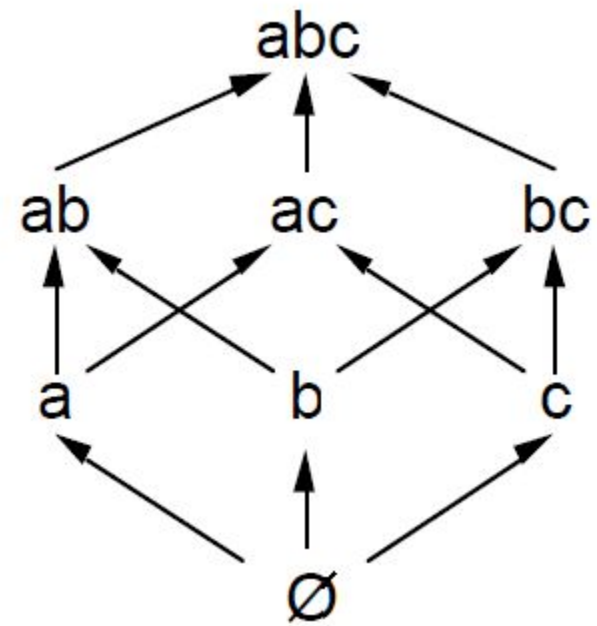
For the subset A = **{ab,b},**

Upper bound of A: **ab, abc**

Lower bound of A: **b, ϕ**

Least Upper bound of A: **ab**

Greatest Lower bound of A: **b**

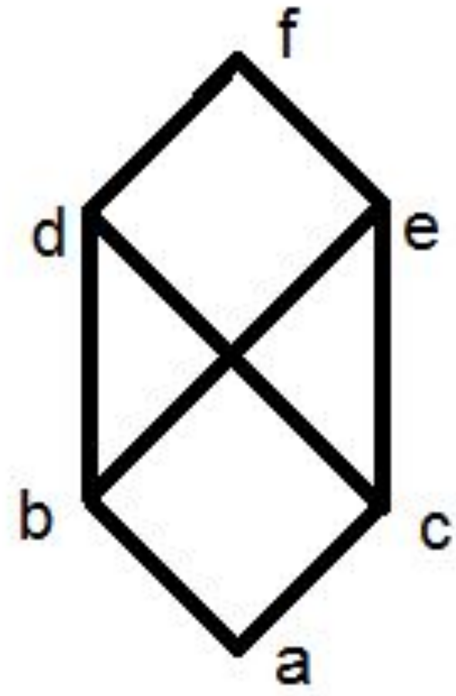


Maximal elements: **f**

Minimal elements: **a**

Greatest element: **f**

Least element: **a**




For the subset A={b, c}

Upper bound of A: **d,e,f**

Lower bound of A: **a**

Least Upper bound of A: **None**

Greatest Lower bound of A: **a**

Note: The above example demonstrates, having multiple upper bounds for a pair of elements doesn’t guarantee to have a least upper bound.

Maximal elements:

Minimal elements:

Greatest element:

Least element:

For the subset A={d,e,f}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

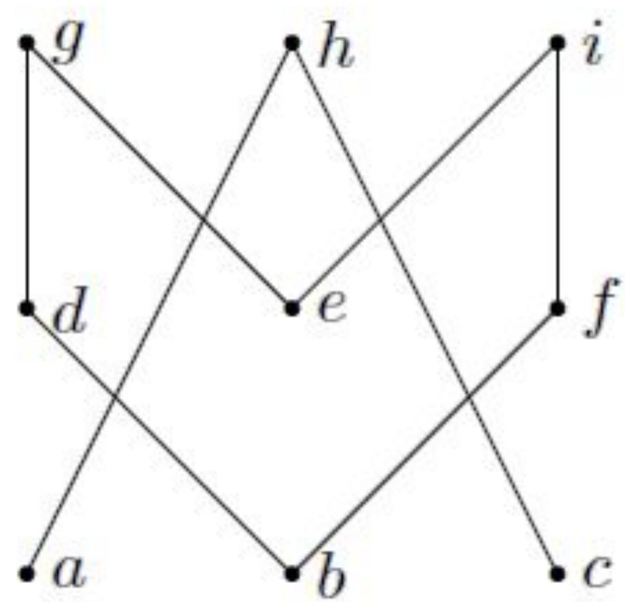
For the subset A={b,d}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Maximal elements: **g,h,i**

Minimal elements: **a,b,c,e**

Greatest element: **None**

Least element: **None**

For the subset A={d,e,f}

Upper bound of A: **None**

Lower bound of A: **None**

Least Upper bound of A: **None**

Greatest Lower bound of A: **None**

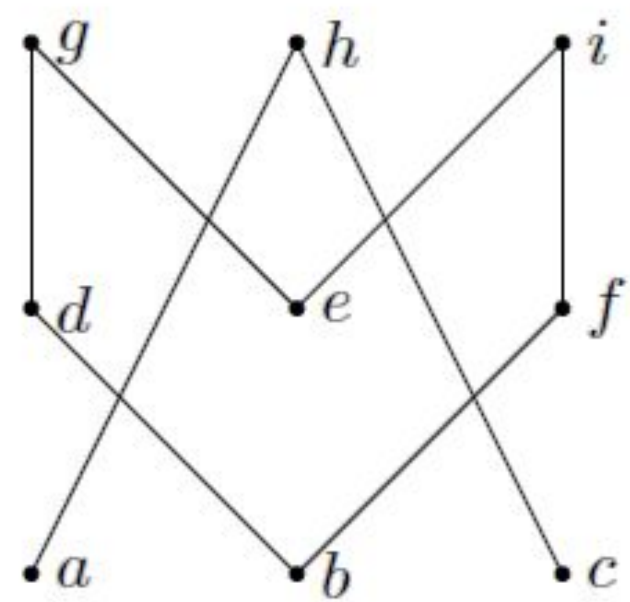
For the subset A={b,d}

Upper bound of A: **d,g**

Lower bound of A: **b**

Least Upper bound of A: **d**

Greatest Lower bound of A: **b**



Maximal elements:

Minimal elements:

Greatest element:

Least element:

For the subset A={c,e}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:

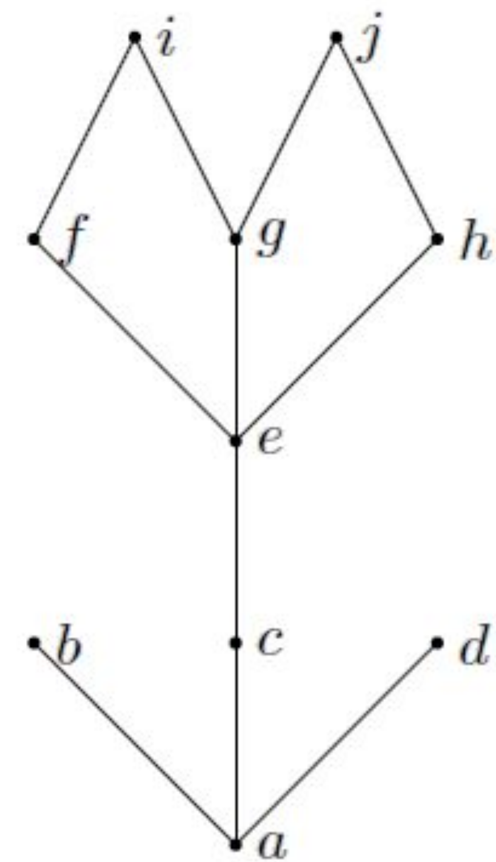
For the subset be A={b,i}

Upper bound of A:

Lower bound of A:

Least Upper bound of A:

Greatest Lower bound of A:



Maximal elements: **b,d,i,j**

Minimal elements: **a**

Greatest element: **None**

Least element: **a**

For the subset A={c,e}

Upper bound of A: **e,f,g,h,i,j**

Lower bound of A: **c,a**

Least Upper bound of A: **e**

Greatest Lower bound of A: **c**

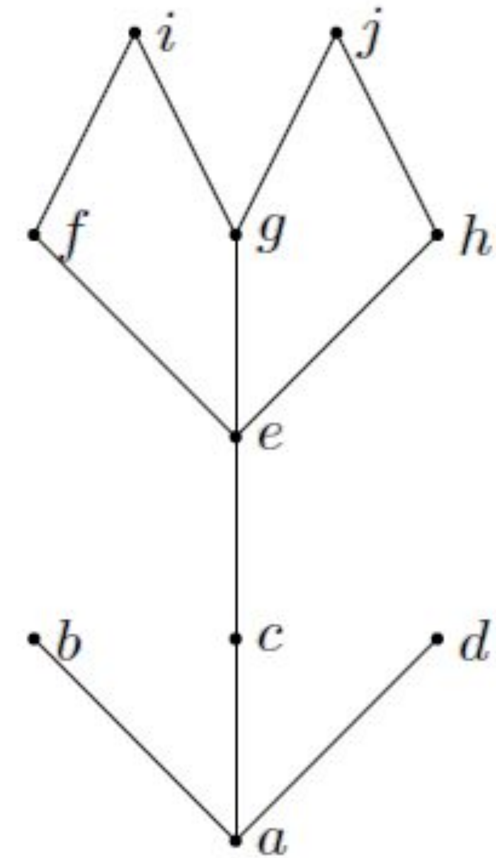
For the subset be A={b,i}

Upper bound of A: **None**

Lower bound of A: **a**

Least Upper bound of A: **None**

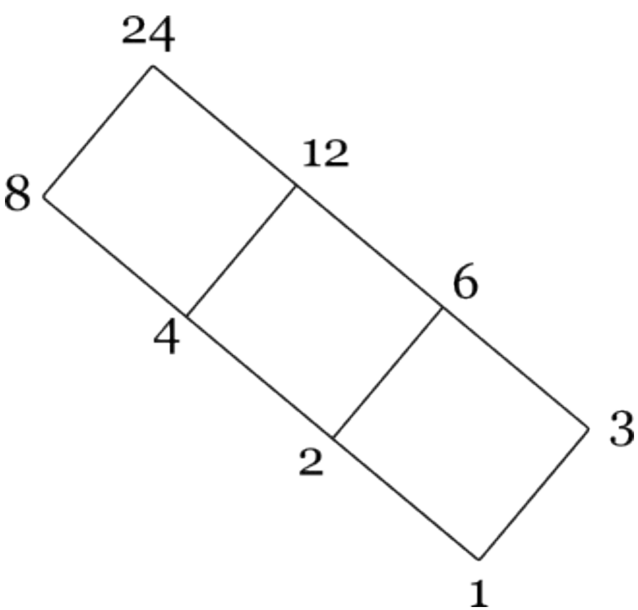
Greatest Lower bound of A: **a**

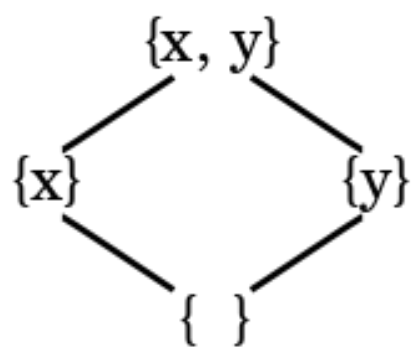


**Lattice:**

A partially ordered set in which every pair of elements has both a **least upper bound** and **greatest lower bound** is called a **lattice**.

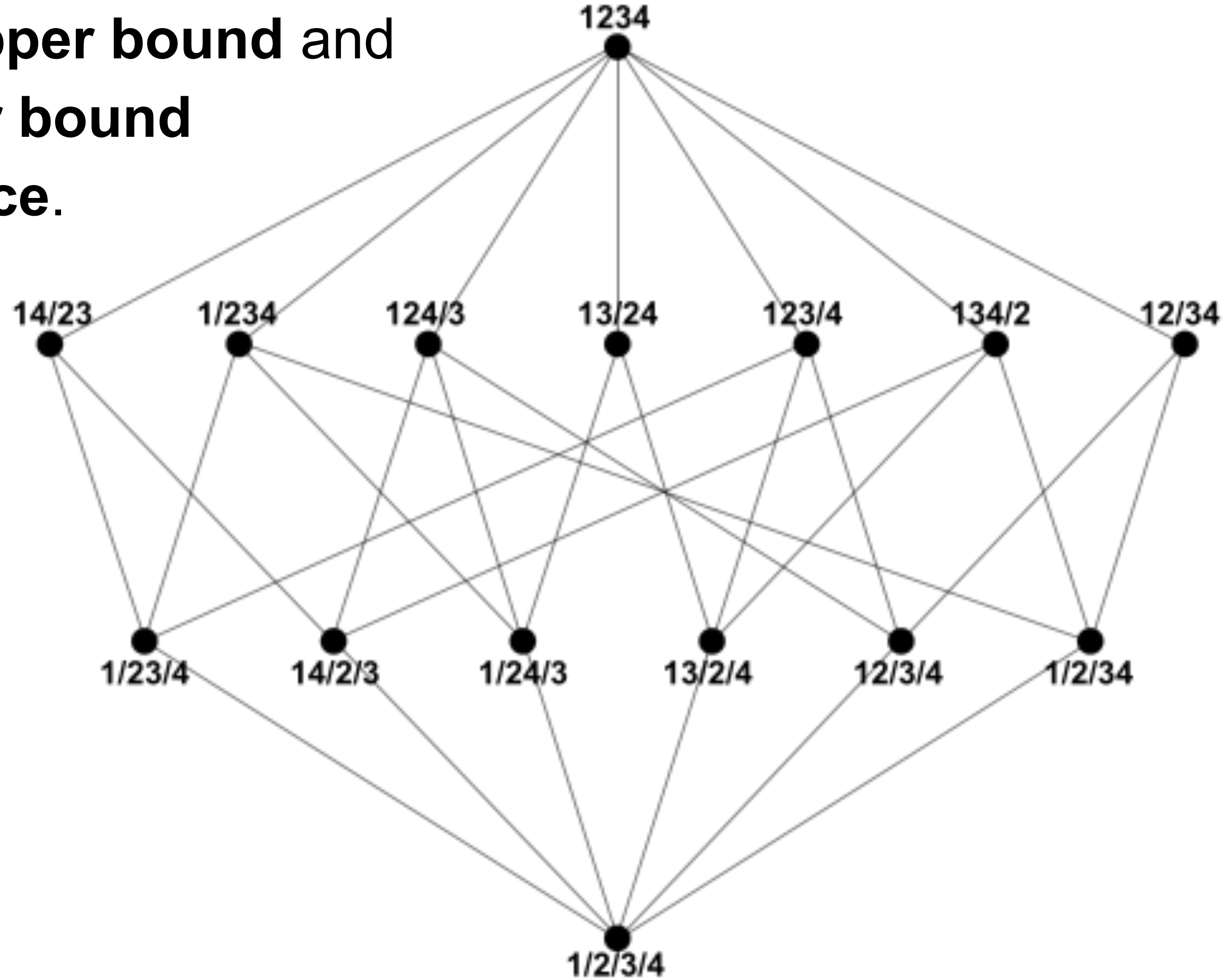
Eg: Poset $(Z^+, |)$



**Lattice:**

A partially ordered set in which every pair of elements has both a **least upper bound** and **greatest lower bound** is called a **lattice**.

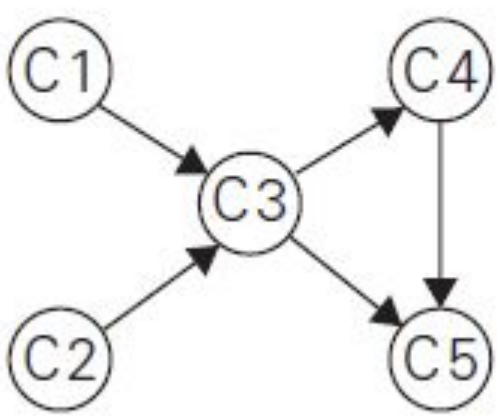


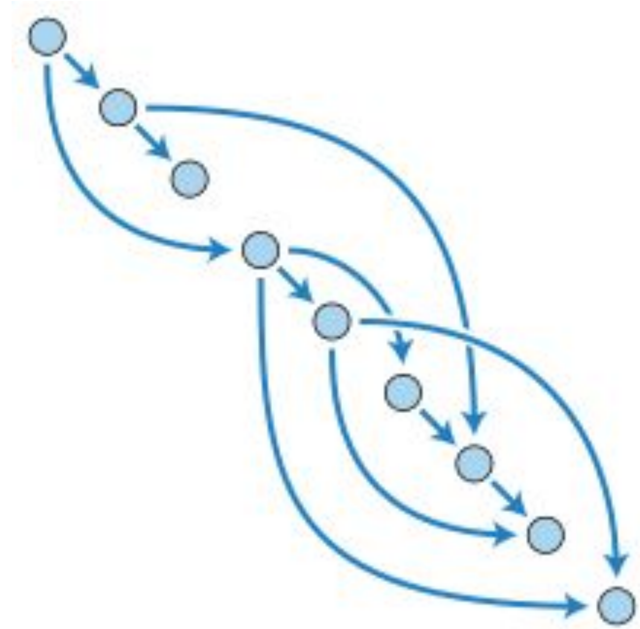
**Topological Sorting:** Constructing a **compatible total ordering** from a partial ordering is called **topological sorting**.

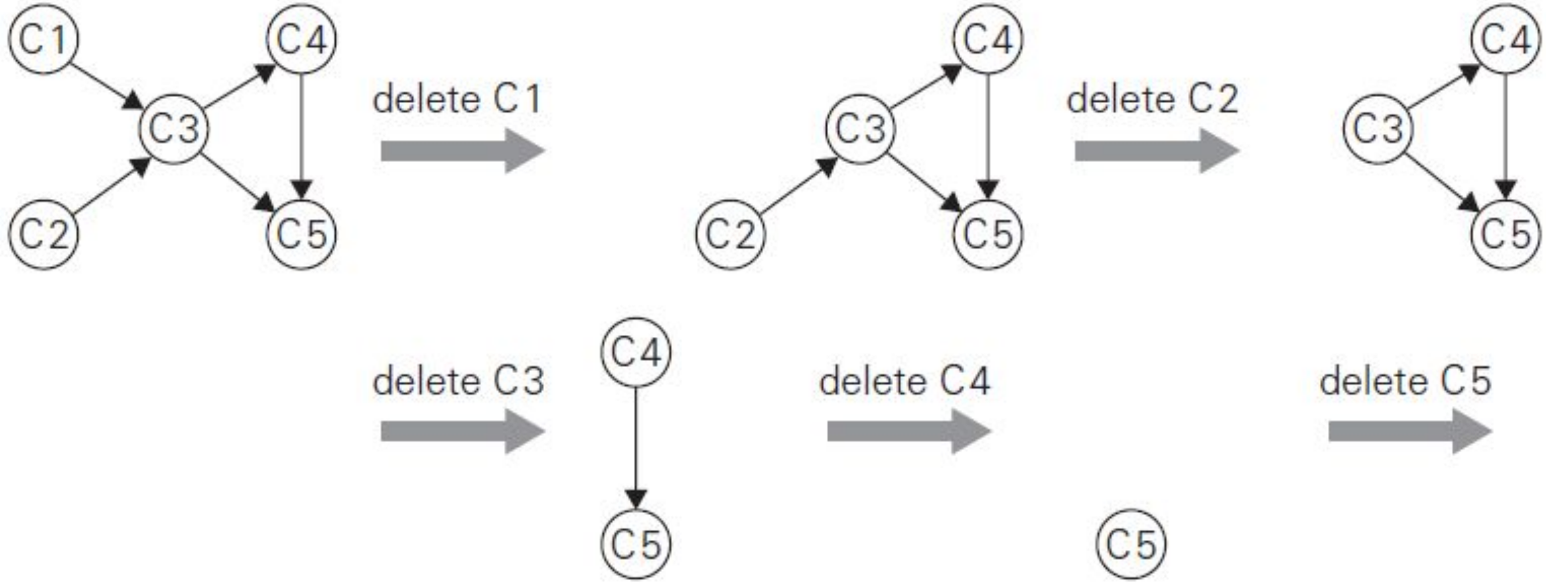
What makes the total ordering compatible with a partial ordering?

If $(a,b) \in R$, then $a \leqslant b$ is in the total ordering.

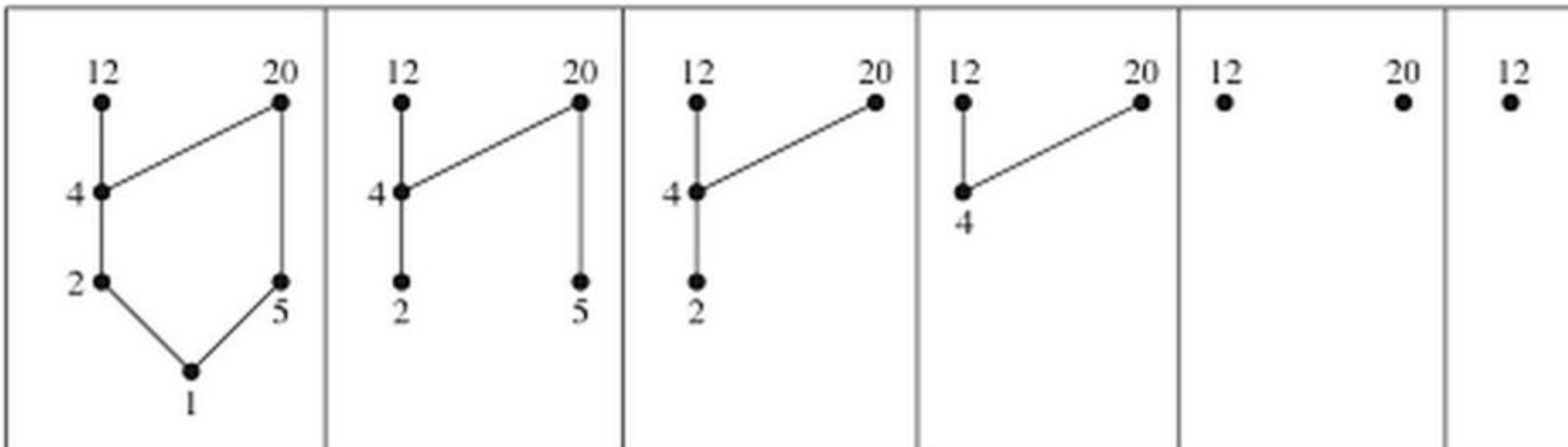
If a and b are not comparable, then it is either $a \leqslant b$ or $b \leqslant a$ in the total ordering.






The solution obtained is C1, C2, C3, C4, C5




| | | | | | | |
|---|---|---|---|---|---|---|
| Minimal element chosen | 1 | 5 | 2 | 4 | 20 | 12 |

**Lemma:** Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

```
Algorithm SourceRemoval_Toposort(V, E)
L ← Empty list that will contain the sorted vertices
S ← Set of all vertices with no incoming edges
while S is non-empty do
    remove a vertex v from S
    add v to tail of L
    for each vertex m with an edge e from v to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S
return L (a topologically sorted order)
```

**Q:** Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Soln: …

**Q:** Let S be Power set of $\{a,b,c\}$.
Find a compatible total ordering for the poset $(S, \subseteq)$.

Soln: …

**Eg:** How many relations are there from a set with **m** elements to a set with **n** elements?

**Ans:**

**Eg:** How many functions are there from a set with **m** elements to a set with **n** elements?

**Ans:**

**Eg:** How many **one-to-one** functions are there from a set with **m** elements to a set with **n** elements?

**Ans:**

**Eg:** How many relations are there from a set with **m** elements to a set with **n** elements?
**Ans: $2^{mn}$**

**Eg:** How many functions are there from a set with **m** elements to a set with **n** elements?
**Ans: $n * n * \ldots n$ (m times) $= n^m$**

**Eg:** How many **one-to-one** functions are there from a set with **m** elements to a set with **n** elements?
**Ans: $n * (n - 1) * (n - 2) * \ldots * (n - m + 1)$, where $m \leq n$**
**$= {}^nP_m$**

**Eg:** How many **onto** functions … (**$n \leq m$**)
**Ans: …**

**<End of Set Theory />**