Unit - 4
Orthogonalization, Eigenvalues and Eigenvectors

Orthogonal bases

A set of vectors $q_1, q_2 \ldots q_n$ is called orthonormal if

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

A matrix that has ~~orthogonal~~ orthonormal columns will be written as Q. For such a matrix, $Q^T Q = I$

i.e. Q^T is a left inverse of Q. In particular, if Q is a square then Q is called an orthogonal matrix.

In this case, $Q^T = Q^{-1}$

Example I: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ dot product b/w col is 0
unit length of column.

Q is orthogonal.
So that $Q^T Q = I$ $\quad Q^T = Q^{-1}$

Example 2: $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ Q is orthogonal
$Q^T = Q^{-1}$. So that $Q^T Q = I$

Example 3: All permutation matrices are orthogonal

$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is such that $P_{23}^T = P_{23}^{-1}$
So that $P_{23}^T P_{23} = I$

Example 4: $Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q^T is a left inverse of Q.

* You can never have a Q matrix of size $m \times n$ where $n > m$.

Advantages of Q matrix:

(1) Q preserves norm.

Proof: $\|Qx\|^2 = (Qx)^T (Qx)$

$= x^T Q^T Q x$

$= x^T I x$

$= x^T x$

$= \|x\|^2$.

(2) Q preserves angle b/w two vectors

Proof: $(Qx)^T (Qy) = x^T Q^T Q y$

$= x^T y$.

$\therefore \cos\theta = \dfrac{x^T y}{\|x\| \, \|y\|} = \dfrac{(Qx)^T (Qy)}{\|Qx\| \, \|Qy\|}$

(3) If $v_1, v_2 \ldots v_n$ is a basis for a v/s V then any $b \in V$ is a linear combination of the v_is.

$\therefore b = c_1 v_1 + c_2 v_2 + \ldots \ldots c_n v_n$

If $q_1, q_2 \ldots q_n$ is an orthogonal basis for V. then

$b = x_1 q_1 + x_2 q_2 + \ldots \ldots x_n v_n$.

$q_1^T b = x_1$

$q_2^T b = x_2$

$q_n^T b = x_n$

$\therefore$ The equation $b = x_1 q_1 + x_2 q_2 + \ldots\ldots + x_n q_n$

$= x_1 \begin{bmatrix} \uparrow \\ q_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ q_2 \\ \downarrow \end{bmatrix} + x_3 \begin{bmatrix} \uparrow \\ q_n \\ \downarrow \end{bmatrix}$

can be written as $\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ q_1 & q_2 & q_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Qx = b$

i.e. $Qx = b$

The solution of the above system of eq^n

$$x = Q^{-1}b = Q^Tb$$

(4) $b = x_1 q_1 + x_2 q_2 + \dots\dots + x_n q_n$

where any x_i can be solved as $q_i^T b$.

$\therefore b = (q_1^T b) q_1 + (q_2^T b) q_2 + \dots\dots + (q_n^T b) q_n$

$$= \frac{q_1^T b}{q_1^T q_1} q_1 + \frac{q_2^T b}{q_2^T q_2} q_2 + \dots + \frac{q_n^T b}{q_n^T q_n} q_n$$

i.e. b = projection of b onto line through q_1
+ projection of b onto line through q_2
⋮
+ projection of b onto line through q_n

$\therefore b$ = sum of projections of b onto ~~plane~~ q_i's.

(5) If the columns of a square matrix Q are orthonormal, then its rows are automatically orthonormal.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

(6) Consider a system of equation, $Qx = b$.
If Q is square, then $x = Q^{-1}b$. or $x = Q^T b$.

If Q is a matrix of size $m \times n$, with $m > n$. Then $Qx = b$ is solvable if and only if b belongs to column space of Q $\quad b \in C(Q)$.
If not, we solve the system by the method of least squares.
The normal equation is: $Q^T Q \hat{x} = Q^T b$ $\quad \therefore \hat{x} = Q^T b$

The point of projection is

$$p = Q\hat{x}$$

$$p = QQ^T b$$

$QQ^T \neq I$

(Because Q is a tall matrix it will never have a right inverse)

$$P = Q(Q^TQ)^{-1}Q^T$$

$$P = QQ^T$$

Gram-Schimdt Probcess of orthogonalization

To construct an orthonormal set of vectors q_1 q_2 q_3 from a set of linearly independent vectors a, b, c

1) The first vector q_1 can go in any of the 3 directions (a, b, c). Let us choose the direction of q_1 along a.

$$\therefore q_1 = \frac{\vec{a}}{\|a\|}$$

So that $\|q_1\| = 1$

2) The second vector b is independent of a. If it is already orthogonal to a: we can choose q_2 in this direction and write:

$$q_2 = \frac{b}{\|b\|}$$

so that $\|q_2\| = 1$.

If b is not orthogonal to a:

b has components in the direction of q_1 and in $\perp$'r direction.

If we want q_2 to go in this $\perp$'r direction then, we need to subtract the component of b in the direction of q_1 from b.

Consider $B = b - \frac{(q_1^T b)\, q_1}{\underbrace{(q_1^T q_1)}_{\to 1}}$

Now, $q_2 = \frac{B}{\|B\|}$, so that $\|q_2\| = 1$

3) The third vector c is independent of both a and b. Hence, it is not in the plane of a and b. If it is already orthogonal, then we can choose q_3 in this direction and write $q_3 = \frac{c}{\|c\|}$, so that $\|q_3\| = 1$.

If c is not orthogonal to them:

If c has components in the direction of q_1, q_2 and in the $\perp$'r direction. If we want q_3 to go in this $\perp$'r direction, then we need to subtract the components of c in q_1 and q_2 directions from c.

Consider $C = c - \frac{(q_1^T c)\, q_1}{\underbrace{q_1^T q_1}_{\to 1}} - \frac{(q_2^T c)\, q_2}{\underbrace{q_2^T q_2}_{\to 1}}$

Now, $q_3 = \frac{C}{\|C\|}$, so that $\|q_3\| = 1$

Proof:

$$q_1^T B = q_1^T \left(b - (q_1^T b) q_1 \right)$$
$$= q_1^T b - (q_1^T b)\, q_1^T q_1$$
$$= q_1^T b - q_1^T b$$
$$= 0$$

This implies that they are orthogonal. (B is orthogonal to q_1)

$$q_1^T c = q_1^T\left(c - (q_1^T c) q_1 - (q_2^T c) q_2\right)$$
$$= q_1^T c - (q_1^T c)(q_1^T q_1) - 0$$
$$= q_1^T c - q_1^T c$$
$$= 0$$

This implies that q_1 is orthogonal to c.

Similarly, q_2 is orthogonal to c.

$A = QR$ Factorisation

Given a set of linearly independent vectors a, b, c we construct a set of orthonormal vectors q_1, q_2, q_3 using the GS process. If A is a matrix whose columns are a, b, c and Q is the matrix whose col are q_1, q_2, q_3 we now find a relation b/w A & Q. To do this we express a, b, c as linear combination of q_1, q_2, q_3

a = projection of a onto the line through q_1
$= (q_1^T a) q_1$

b = sum of projection of b onto lines through q_1 and q_2
$= (q_1^T b) q_1 + (q_2^T b) q_2$

c = sum of projection of b onto lines through q_1, q_2 and q_3
$= (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$.

$$\therefore A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$\boxed{A = QR}$$

A and Q will be equivalent (If A is □ Q is also □.
If A is ▯ Q is also ▯)

R is always square matrix. (and upper triangular in nature)

When the system $Ax = b$ is inconsistent we solve it by the method of least squares. The normal equations are $A^T A \hat{x} = A^T b$

DATE: PAGE:

$$(QR)^T QR\,\hat{x} = (QR)^T b$$

$$R^T Q^T Q R\,\hat{x} = R^T Q^T b$$

$$R^T R\,\hat{x} = R^T Q^T b$$

$\boxed{R\hat{x} = Q^T b}$ this is of the form: $\boxed{Ux = c}$

The above system can be solved by back substitution, since R is upper triangular

Q1. Find a 3rd column so that the matrix $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix}$ is orthogonal

$$1/\sqrt{3}\,x + 1/\sqrt{3}\,y + 1/\sqrt{3}\,z = 0$$

$$x^2 + y^2 + z^2 = 1$$

$$1/\sqrt{14}\,x + 2/\sqrt{14}\,y + -3/\sqrt{14}\,z = 0.$$

$x + y + z = 0$

$x + 2y - 3z = 0$

$x + y + 1 = 0$

$x + 2y - 3 = 0$

$x + y = -1$

$x + 2y = 3$

$y = 4$ $\quad x = -5$

$x, y, z = (-5, 4, 1)$

$\dfrac{-5}{\sqrt{42}} \quad \dfrac{4}{\sqrt{42}} \quad \dfrac{1}{\sqrt{42}}$

$$(x, y, z) = \left(\frac{-5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right) \text{ or } \left(\frac{5}{\sqrt{42}}, \frac{-4}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$$

Q2. $Q = \cancel{1/\sqrt{2} \quad 0} \quad \cancel{2/}$ $\begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}_{3\times 2}$, $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}_{2\times 1}$ and $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$

verify that : (i) $Q^T Q = I$

(ii) $\|Qx\| = \|x\|$ and $\|Qy\| = \|y\|$

(iii) $(Qx)^T Qy = x^T y$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\|x\|$ $\sqrt{11}$ $\qquad$ $\|y\|$ $\sqrt{54}$

$$\|Qx\| \quad \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} \cdot \sqrt{54}$$

$$\|Qy\| = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \sqrt{11}$$

$$(Qx)^T (Qy) = \underset{1\times 3}{\begin{bmatrix} 3 & -1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}} = \quad 3+7+2 = 12.$$

$$x^T y = \begin{bmatrix} \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = -6+18 = 12.$$

Hence verified.

Q3. If w is a subspace spanned by the orthogonal vectors $(2,5,-1)$ and $(-2,1,1)$. Find the point in w that is closest to $(1,2,3)$.

$$w = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}$$

$$p = A\hat{x}$$

$$\hat{x} = \frac{Q^T b}{Q^T Q}$$

$$\hat{x} = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix} \cdot \hat{x}$$

$$p = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -12/30 \\ 1 \\ 6/30 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = (-2/5, 2, 1/5)$$

$$p = \frac{a_1^T b}{a_1^T a_1} a_1 + \frac{a_2^T b}{a_2^T a_2} a_2$$

$$p = \begin{bmatrix} 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{9}{30} a_1 +$$

[2

$$\frac{3}{10}(2, 5, -1) + \frac{1}{2}(2, 1, 1)$$

$$\begin{bmatrix} -2, & 1, & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{3}{6} a_2$$

$$\left(\frac{3}{5} + 1\right), \left(\frac{3}{2} + \frac{1}{2}\right), \left(\frac{-3}{10} + \frac{1}{2}\right)$$

$$\left(\frac{2}{5}, 2, \frac{1}{5}\right)$$