Model ESA Paper

1

a. Let A be an $m \times n$ matrix A such that the following hold

* The first row of A is not zero.
* $m > 2$.
* A has rank greater than 1.
* The entries in each column of A are in arithmetic progression. **[2+3+3]**

i. Find the rank of A.

ii. For what b will the system $Ax = b$ be consistent? Justify your answer.

iii. For what value of n does there exist b for which there is a unique solution?

b. **[4+4]**

i. Find the LU-factorization of A when $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$.

ii. Solve $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ using the LU-factorization: That is using solving in two steps $Lc = b$ and $Ux = c$.

c. Consider the following system: **[2+2]**

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

i. For what value of a is there a temporary breakdown in Gaussian elimination?

ii. For what value of a does the Gaussian elimination break down permanently?

2

a. Find the four fundamental subspaces and their dimensions and a basis, given **[12]**

$$A = \begin{pmatrix} 1 & -1 & 2 & -2 & 3 \\ -2 & 2 & 0 & 4 & -2 \\ 0 & 3 & 1 & -1 & 6 \\ -1 & -2 & -3 & 3 & -9 \end{pmatrix}$$

b. **[4+2+2]**

i. Following is a table for an $m \times n$ matrix A with blanks:

	Left-inverse	Right-inverse
$m < n$, rank $A = m$		
$m > n$, rank $A = n$		

Fill in each blank with one of the three items:

$(A^T A)^{-1} A^T$

$A^T (AA^T)^{-1}$

does not exist

ii. The left-inverse of an $m \times n$ matrix A when $m > n$

(a) equals $(A^T A)^{-1} A^T$ if the rank of A is n

(b) equals $A^T (AA^T)^{-1}$ if the rank of A is n

(c) does not exist

iii. Using the above information or otherwise, compute a left-inverse for $\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$

3

a. Let $T : M_{2\times2} \to M_{2\times2}$ be the linear operator given by $T(A) = A \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. What is the matrix of T in the standard ordered basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

b. Which of the following planes/lines through the origin is orthogonal to the intersection of the planes $x - y + z = 0$ and $2x + y - 3z = 0$?

c. Find the best-fitting straight line through the points $(1, 2), (2, 3), (3, 5)$ and $(5, 7)$.

4

a. Which of the following is true about the largest eigenvalue λ_{max} of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$. Use power method with initial vector $x_0 = (1, 0, 0)$ and do iterations until you get the same eigenvector in successive iterations upto 4 decimal places. **[8]**

b. i. What are the eigenvectors for the matrix $A(A^T A)^{-1} A^T$ where A is an $m \times n$ matrix ($m \geq n$) of rank n? **[8]**

ii. Is the matrix above diagonalizable? If not why not? If yes, what would be a valid eigenvector matrix S for the matrix $A(A^T A)^{-1} A^T$ where A is not necessarily square? What is the eigenvalue matrix Λ? **[4]**

5

a. **[4+4+4]**

i. Is the equation $p(x, y, z) = 400000$ solvable in the variables x, y, z where $p(x, y, z) = z^2 - x^2 - y^2 + 2xy - 2xz - 2yz$? Why or why not?

ii. Is the polynomial $p(x, y, z)$ a sum of squares? Why or why not?

iii. What is the least value of a so that the polynomial $a(x^2 + y^2 + z^2) + p(x, y, z)$ is a sum of squares?

b. **[6+2]**

i. Calculate the SVD of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

ii. What are valid SVDs for A^T and $A^T A$?

Answers

Unit I

1a i. Let the j^{th} column begin with a_j and have a common difference r_j. Thus the ij^{th} entry of A is $a_j + (i-1)r_j$. Thus if $v = (a_1, a_2, ..., a_n)$ and $w = (r_1, r_2, ..., r_n)$ then

$$A = \begin{pmatrix} v \\ v + w \\ v + 2w \\ \vdots \\ v + (m-1)w \end{pmatrix}$$

Applying the row operations $R_i \leftarrow R_i - R_1$ for $m \geq i > 1$ we have

$$A = \begin{pmatrix} v \\ w \\ 2w \\ \vdots \\ (m-1)w \end{pmatrix}$$

If $w = 0$, then the matrix has at most one nonzero row, which is possibly v which means A has rank at most 1, contrary to data. Thus $w \neq 0$. So now we use operations $R_i \leftarrow R_i - (i-1)R_2$ for all $m \geq i \geq 2$.

$$A = \begin{pmatrix} v \\ w \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Also, since the rank of A exceeds 1, the first two rows are independent. Hence the rank of A is 2.

ii. If the same sequence of row operations is done on $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$

we have $R_i \leftarrow R_i - R_1$ leads to $\begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_1 \\ \vdots \\ b_m - b_1 \end{pmatrix}$

next $R_i \leftarrow R_i - (i-1)R_2$ leads to $\begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_2 + b_1 \\ b_4 - 3b_2 + 2b_1 \\ \vdots \\ b_m - (m-1)b_2 - (m-2)b_1 \end{pmatrix}$. The i^{th} entry of b is $b_i - (i-1)b_2 + (i-2)b_1$ For consistency we must have $b_i = (i-1)b_2 - (i-2)b_1$ for all $i > 3$. Thus $b = \begin{pmatrix} b_1 \\ b_2 \\ 2b_2 - b_1 \\ 3b_2 - 2b_1 \\ \vdots \\ (m-1)b_2 - (m-2)b_1 \end{pmatrix}$ We see an arithmetic progression down the column vector b with initial term b_1 and common difference $b_2 - b_1$.

iii. The system is consistent whenever the rank of A equals the rank of $[A : b]$ which is achievable as seen in part ii above. The uniqueness of the solution comes when the number of unknowns n also equals these ranks. Thus there exists b satisfying uniqueness when $n = 2$.

1b We do row operations as follows: Firstly $R_2 \leftarrow R_2 - 4R_1$, $R_3 \leftarrow R_3 - 7R_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix}$$

Next we do $R_3 \leftarrow R_3 - 2R_2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix} = U$$

Thus $L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$.

Now our system is

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

First solve $Lc = b$ for c that is,

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

We have $c_1 = 1$.
Next, $4c_1 + c_2 = 4 + c_2 = 2$, so $c_2 = -2$.
Third, $7c_1 + 2c_2 + c_3 = 7 - 4 + c_3 = 3$, so $c_3 = 0$ so $c = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

Now solve for $Ux = c$, that is:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

We have $z = 0$.
Next we have $-3y - 6z = -3y = -2$ so $y = 2/3$.
Third we have $x + 2y + 3z = x + (4/3) + 0 = 1$, so $x = -1/3$.

Therefore the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 0 \end{pmatrix}$

1c i. The Gaussian elimination proceeds thus: $R_2 \leftarrow R_2 - 2R_1$ leads to

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & a+2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

Here we observe a breakdown when $a = -2$, since the second pivot then becomes zero. However it is temporary since We can swap the last two rows to repair it and get three pivots.

ii. Let $a \neq -2$ above. We resume at $\begin{pmatrix} 1 & -1 & 1 \\ 0 & a+2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

We do the operation $R_3 \leftarrow (a+2)R_3 - 2R_2$ and get $\begin{pmatrix} 1 & -1 & 1 \\ 0 & a+2 & -1 \\ 0 & 0 & a+4 \end{pmatrix}$. Thus now for the value $a = -4$ we observe a permanent breakdown.

Unit II

2a i. We column-reduce to find which of the columns form a basis for $\mathcal{C}(A)$ as follows:
$C_2 \leftarrow C_2 + C_1 \quad C_3 \leftarrow C_3 - 2C_1 \quad C_4 \leftarrow C_4 + 2C_1 \quad C_5 \leftarrow C_5 - 3C_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 8 & 4 \\ 0 & 3 & 1 & -1 & 6 \\ -1 & -3 & -1 & 1 & -6 \end{pmatrix}$$

$C_4 \leftarrow C_4 - 2C_3 \quad C_5 \leftarrow C_5 - C_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ 0 & 3 & 1 & -3 & 5 \\ -1 & -3 & -1 & 3 & -5 \end{pmatrix}$$

$C_4 \leftarrow C_4 - \frac{1}{3}C_2 \quad C_5 \leftarrow C_5 - \frac{4}{3}C_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ -1 & -3 & -1 & 0 & 0 \end{pmatrix}$$

The leading non-zero entries in each column is in a different row. Thus we can stop the reduction and conclude $\{C_1, C_2, C_3\}$ is a basis. Alternatively using appropriate row operations at the end we can conclude that $\{C_1, C_3, C_4\}$ and $\{C_1, C_3, C_5\}$ are also bases for $\mathcal{C}(A)$. Thus $C(A)$ has dimension 3.

ii. The basis for the row space of A which is also the column space $\mathcal{C}(A^T)$ is found by row-reduction as follows:

$R_2 \leftarrow R_2 + 2R_1 \quad R_4 \leftarrow R_4 + R_1$

$$\begin{pmatrix} 1 & -1 & 2 & -2 & 3 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 3 & 1 & -1 & 6 \\ 0 & -3 & -1 & 1 & -6 \end{pmatrix}$$

$R_4 \leftarrow R_4 + R_3$

$$\begin{pmatrix} 1 & -1 & 2 & -2 & 3 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 3 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The leading entries in every row is in a different entry. Thus we can stop the reduction and conclude that $\{R_1, R_2, R_3\}$ is a basis. Alternatively by appropriate row operations we can also conclude that $\{R_1, R_2, R_4\}$ is a basis.

iii. We compute $\mathcal{N}(A)$ by solving the equation

$$Ax = 0$$

where $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}^T$. In augmented form we have

$$\begin{pmatrix} 1 & -1 & 2 & -2 & 3 & : & 0 \\ -2 & 2 & 0 & 4 & -2 & : & 0 \\ 0 & 3 & 1 & -1 & 6 & : & 0 \\ -1 & -2 & -3 & 3 & -9 & : & 0 \end{pmatrix}$$

Since we need to row-reduce anyway we arrive at the below augmented matrix following the steps in part ii.

$$\begin{pmatrix} 1 & -1 & 2 & -2 & 3 : & 0 \\ 0 & 0 & 4 & 0 & 4 : & 0 \\ 0 & 3 & 1 & -1 & 6 : & 0 \\ 0 & 0 & 0 & 0 & 0 : & 0 \end{pmatrix}$$

$R_1 \leftarrow 3R_1 + R_3$ gives

$$\begin{pmatrix} 3 & 0 & 7 & -7 & 15 : & 0 \\ 0 & 0 & 4 & 0 & 4 : & 0 \\ 0 & 3 & 1 & -1 & 6 : & 0 \\ 0 & 0 & 0 & 0 & 0 : & 0 \end{pmatrix}$$

$R_1 \leftarrow R_1 - \frac{7}{4}R_2$ and $R_3 \leftarrow R_3 - \frac{1}{4}R_2$ gives

$$\begin{pmatrix} 3 & 0 & 0 & -7 & 8: & 0 \\ 0 & 0 & 4 & 0 & 4: & 0 \\ 0 & 3 & 0 & -1 & 5: & 0 \\ 0 & 0 & 0 & 0 & 0: & 0 \end{pmatrix}$$

Swapping rows 2 and 3 and scaling we get

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{3} & \frac{8}{3}: & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{3}: & 0 \\ 0 & 0 & 1 & 0 & 1: & 0 \\ 0 & 0 & 0 & 0 & 0: & 0 \end{pmatrix}$$

Solving we get $x_1 = \frac{7}{3}x_4 - 8\frac{8}{3}x_5 \quad x_2 = \frac{1}{3}x_4 - \frac{5}{3}x_5 \quad x_3 = -x_5$. Now we have, in terms of x_4 and x_5 that

$$x = (\frac{7}{3}x_4 - \frac{8}{3}x_5, \frac{1}{3}x_4 - \frac{5}{3}x_5, -x_5, x_4, x_5)$$

$$x = \frac{x_4}{3}(7, 1, 0, 3, 0) + \frac{x_5}{3}(-8, -5, -3, 0, 3)$$

Thus a basis for $\mathcal{N}(A)$ is $\{(7, 1, 0, 3, 0), (-8, -5, -3, 0, 3)\}$

iv. We compute $\mathcal{N}(A^T)$ by solving $A^T x = 0$. In part i., A has been column-reduced. This is equivalent to row-reducing A^T so transposing the reduced matrix from part i and omitting rows that are all zero, we get the following augmented form.

$$\begin{pmatrix} 1 & 2 & 0 & -1 & : & 0 \\ 0 & 0 & 3 & -3 & : & 0 \\ 0 & 4 & 1 & -1 & : & 0 \end{pmatrix}$$

We reduce to Echelon form as follows:

$R_1 \leftarrow 2R_1 - R_3$

$$\begin{pmatrix} 2 & 0 & -1 & -1 & : & 0 \\ 0 & 0 & 3 & -3 & : & 0 \\ 0 & 4 & 1 & -1 & : & 0 \end{pmatrix}$$

$R_1 \leftarrow 3R_1 + R_2 \quad R_3 \leftarrow R_3 - \frac{1}{3}R_2$

$$\begin{pmatrix} 6 & 0 & 0 & -6 & : & 0 \\ 0 & 0 & 3 & -3 & : & 0 \\ 0 & 4 & 0 & 0 & : & 0 \end{pmatrix}$$

Scaling we get

$$\begin{pmatrix} 1 & 0 & 0 & -1 & : & 0 \\ 0 & 0 & 1 & -1 & : & 0 \\ 0 & 1 & 0 & 0 & : & 0 \end{pmatrix}$$

Thus we get $x_1 = x_4 \quad x_3 = x_4 \quad x_2 = 0$. So $(x_1, x_2, x_3, x_4) = x_4(1, 0, 1, 1)$. Hence a basis for $\mathcal{N}(A^T)$ is $\{(1, 0, 1, 1)\}$.

		Left-inverse	Right-inverse
2b i.	$m < n$, rank $A = m$	does not exist	$A^T(AA^T)^{-1}$
	$m > n$, rank $A = n$	$(A^TA)^{-1}A^T$	does not exist

Unit III

3a We calculate T on each basis vector and express in terms of all the basis vectors as follows:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&a\\0&1\end{pmatrix}=\begin{pmatrix}1&a\\0&0\end{pmatrix}=1\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+a\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&a\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}=0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&a\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\1&a\end{pmatrix}=0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+1\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+a\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1&a\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}=0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

The matrix of T is $\begin{pmatrix}1&0&0&0\\a&1&0&0\\0&0&1&0\\0&0&a&1\end{pmatrix}$ by definition of the matrix of a linear transformation.

3b The intersection of the planes is given (in matrix form) by the set of column vectors $\begin{pmatrix}x\\y\\z\end{pmatrix}\in N(A)$ where $A=\begin{pmatrix}1&-1&1\\2&1&-3\end{pmatrix}$. The row-echelon form for A is $\begin{pmatrix}1&0&-\frac{2}{3}\\0&1&-\frac{5}{3}\end{pmatrix}$.
Thus $N(A)$ is spanned by the single special solution $(2,5,3)$.
The question asks for the orthogonal complement, which is the plane $2x+5y+3z=0$.
Hence the answer choice.

3c We need to find the best possible $\begin{pmatrix}C\\D\end{pmatrix}$ that can approximate the equality below so that $y = C + Dx$ is the line of best fit:

$$\begin{pmatrix}1&x_1\\1&x_2\\1&x_3\\1&x_4\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}=\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$

In the given problem the system is

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$. Thus,

$$A^T A = \begin{pmatrix} 4 & 11 \\ 11 & 39 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 17 \\ 58 \end{pmatrix}$$

The system $A^T A\hat{x} = A^T b$ (in augmented form) is therefore

$$\hat{x} = \begin{pmatrix} 4 & 11 \\ 11 & 39 \end{pmatrix}^{-1} \begin{pmatrix} 17 \\ 58 \end{pmatrix} = \begin{pmatrix} 5/7 \\ 9/7 \end{pmatrix}.$$

Thus the line of best fit is

$$y = \frac{5}{7} + \frac{9x}{7}$$

Unit IV

4a We begin with the given initial vector

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Ax_0 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 1/7 \\ 4/7 \\ 1 \end{pmatrix} = \lambda_1 x_1$$

$$Ax_1 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1/7 \\ 4/7 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.2857 \\ 9.4286 \\ 14.5714 \end{pmatrix} = 14.5714 \begin{pmatrix} 0.2941 \\ 0.6471 \\ 1 \end{pmatrix} = \lambda_2 x_2$$

$$Ax_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0.2941 \\ 0.6471 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.5882 \\ 10.4118 \\ 16.2353 \end{pmatrix} = 16.2353 \begin{pmatrix} 0.2826 \\ 0.6413 \\ 1 \end{pmatrix} = \lambda_3 x_3$$

$$Ax_3 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0.2826 \\ 0.6413 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.5652 \\ 10.3370 \\ 16.1087 \end{pmatrix} = 16.1087 \begin{pmatrix} 0.2834 \\ 0.6417 \\ 1 \end{pmatrix} = \lambda_4 x_4$$

$$Ax_4 = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}0.2834\\ 0.6417\\ 1\end{pmatrix} = \begin{pmatrix}4.5668\\ 10.3421\\ 16.1174\end{pmatrix} = 16.1174\begin{pmatrix}0.2833\\ 0.6417\\ 1\end{pmatrix} = \lambda_5 x_5$$

$$Ax_5 = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}0.2834\\ 0.6417\\ 1\end{pmatrix} = \begin{pmatrix}4.5667\\ 10.3417\\ 16.1168\end{pmatrix} = 16.1168\begin{pmatrix}0.2833\\ 0.6417\\ 1\end{pmatrix} = \lambda_6 x_6$$

Since the eigenvectors x_5 and x_6 are approximately the same, we can stop the process here and conclude that the largest eigenvalue is approximately 16.1168.

4b. i. The matrix $A(A^TA)^{-1}A^T$ has size $m \times m$.
If $v \in C(A)$, then $v = Ax$ for some x. Thus $A(A^TA)^{-1}A^Tv = A(A^TA)^{-1}A^TAx = Ax = v$, since $(A^TA)^{-1}A^TA = I$.
On the other hand, if $v \in C(A)^\perp = N(A^T)$, then $A(A^TA)^{-1}A^Tv = 0$ since $A^Tv = 0$ by assumption.

This means that the vectors belonging to either $C(A)$ and the vectors belonging to $N(A^T)$ are eigenvectors.

ii. $N(A^T)$ has dimension $m - r$ where r is the rank of A which is also the dimension of $C(A)$. The dimensions add up to m and thus they are a basis for $\mathbb{R}^m$ composed of eigenvectors. Thus the matrix is diagonalizable.

Since $r = n$ here, we have that any n independent columns $C(A)$ followed by the $m - n$ special solutions (as column vectors) for $A^Tx = 0$ form the eigenvector matrix S. The eigenvalue matrix is the diagonal matrix Λ whose first n diagonal entries are 1s and the remaining diagonal entries are zero.

Unit 5

5a. i. In matrix form $p(x,y,z) = v^TAv$ where $v = \begin{pmatrix}x & y & z\end{pmatrix}^T$ and

$$A = \begin{pmatrix}-1 & 1 & -1\\ 1 & -1 & -1\\ -1 & -1 & 1\end{pmatrix}$$

This has characteristic equation

$$\begin{vmatrix}-1-\lambda & 1 & -1\\ 1 & -1-\lambda & -1\\ -1 & -1 & 1-\lambda\end{vmatrix} = 0$$

Solving this we get three roots $\lambda = -2, -1, 2$
Picking v to be an eigenvector corresponding to the eigenvalue 2, we get $v^TAv = 2v^Tv = 400000$. Thus $v^Tv = 200000$. We only need to scale v so that its length is $\sqrt{200000}$. Hence the equation $p(x,y,z) = 400000$ is solvable.

ii. The eigenvalues are not all positive, thus $p(x, y, z)$ is NOT a sum of squares.

iii. The matrix B such that $a(x^2+y^2+z^2)+p(x, y, z)$ is simply $A+aI$ where A is the matrix in part i since the coefficients of x^2, y^2, z^2 are just the diagonal entries. The eigenvalues of $B = A+aI$ are all required to be positive. Thus $-2+a, -1+a, 2+a$ are all positive. Thus the smallest value of a such that $a(x^2+y^2+z^2)+p(x, y, z)$ is a sum of squares is 2.

5b. i. Firstly, $AA^T = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. The characteristic polynomial for AA^T is found to be $\lambda^2 - 9\lambda + 14 = 0$. Thus the eigenvalues are $\lambda = 7, 2$. The eigenvector matrix $U = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$. The columns of V are obtained as follows: $v_i = A^T u_i/\sigma_i$. For $i = 1$ we have

$$v_1 = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{7}} \begin{pmatrix} \frac{3}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \\ 0 \end{pmatrix}$$

By orthogonality of columns of V we have,

$$v_3 = \begin{pmatrix} \frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{pmatrix}$$

Thus an SVD of A is

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{35}} & \frac{5}{\sqrt{35}} \\ -\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} \end{pmatrix}$$

ii. The SVD for A^T that can be derived from above at once is

$$A^T = V\Sigma^T U^T$$

The SVD for $A^T A$ is

$$A^T A = V(\Sigma^T \Sigma)V^T$$