Live Session notes :-

16/3/20

Eigen Values and Eigenvectors :-

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Topics :-

1) What are eigen values and vectors?
2) Geometrical meaning.
3) Why "eigen" ..
4) Properties
5) Largest eigen value.
6) Applications — Diagonalization. ($A = S \Delta S^{-1}$) — powers and Products.

Later, $A = U\Sigma V^T \rightarrow$ SVD. (Unit - 5)

$$A_{n\times n}\, x_{n\times 1} = b_{n\times 1}$$

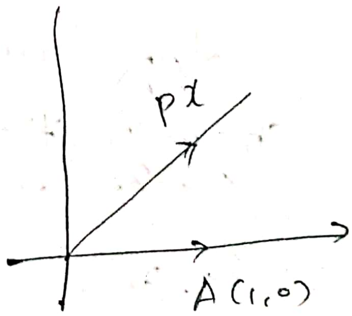
$$= \lambda x$$

$\Rightarrow \quad Ax = \lambda x, \quad \lambda$ = eigenvalue

x = eigen vector.

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$



If $\lambda = 0 \Rightarrow b = 0$

$\lambda = 1 \Rightarrow b = x$.

λ = scale factor.

eigen - Specific, characteristic.

If $\lambda_i > 0 \Rightarrow |A| > 0$.

If $\lambda_i < 0 \Rightarrow |A| < 0$.

$$Ax = \lambda X$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$(A - \lambda I)x = 0. \Rightarrow |A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (-1)^n \lambda^n + C_1 \lambda^{n-1} + C_2 \lambda^{n-2} + \cdots + C_n = 0. \quad (1)$$

LHS is called Characteristic polynomial.

Solving (1), we get $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called characteristic roots or latent roots or eigenvalues.

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum a_{ii}$ = Trace of the matrix.

$\lambda_1 . \lambda_2 \ldots \lambda_n = |A|$.

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Find the eigen values and eigenvectors of A

Solution:

The characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda) - 16\right] + 6\left[-6(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0.$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0.$$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0.$$

$$\Rightarrow \lambda = 0, 3, 15.$$

Sum of the eigen values $= 0 + 3 + 15 = 18$

Sum of the diagonal elements $= 8 + 7 + 3 = 18$.

Case(i) $\lambda = 0$.

$$Ax = \lambda x \Rightarrow Ax = 0.$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the method of cross-multiplication,
Consider first two rows (independent)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{24-14} = \frac{-y}{-32+12} = \frac{z}{56-36}$$

$$\frac{x}{10} = \frac{-y}{-20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Eigen vector corresponding to $\lambda = 0$ is
$K(1, 2, 2)'$, where $K \neq 0$.

$|||^{ly}$ For $\lambda = 3$, $Ax = 3x$

$$\Rightarrow (A - 3I)(x) = 0$$

$$\Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, we get the eigen vector is
$K(2, 1, -2)$.

Also for $\lambda = 15$, the eigen vector is
$K(2, -2, 1)$.

Solving, we get the eigenvector

$x(2, 1, -2)$

Also for $\lambda = 15$, the eigen vector is

$k(2, -2, 1)$.

Note :

For a given eigen value, there are infinite eigen vectors.

Problem : 2 (Repeated eigen values)

Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

Solving, we get $\lambda = 1, 1, 5$ (eigen values)

$(\because 5 + 1 + 1 = 2 + 3 + 2)$.

For $\lambda = 1$, $(A - \lambda I)X = 0$

$$\Rightarrow (A - I)X = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

The eigenvectors are,

$K(-2, 1, 0)$; $K(-1, 0, 1)$.

For $\lambda = 5$, $K(1, 1, 1)$. where $K \neq 0$.