MT210 Midterm 1 Sample 2

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February 19, 2011

Question 1. SYSTEMS OF LINEAR EQUATIONS

The augmented matrix of a linear system has the form

$$\begin{bmatrix} a & 1 & 1 \\ 2 & a-1 & 1 \end{bmatrix}$$

Determine the values of a for which the linear system is consistent.

Answer

We apply row-reduction algorithm to the augmented matrix corresponding to the system given above: Assume that $a \neq 0$, then we get

$$\begin{bmatrix} a & 1 & 1 \\ 2 & a-1 & 1 \end{bmatrix} \xrightarrow{(-2/a)R_1+R_2 \to R_2} \begin{bmatrix} a & 1 & 1 \\ 0 & a-1-\frac{2}{a} & 1-\frac{2}{a} \end{bmatrix}.$$

By Theorem 2, we know that the system above is consistent if and only if there is no row of the form [0 0 1]. Therefore, we must have either $a-1-\frac{2}{a} \neq 0$ or we must have $a-1-\frac{2}{a} = 0$ and $1-\frac{2}{a} = 0$. Let us solve the equation $a-1-\frac{2}{a} = 0$ or $(a+1)(a-2) = 0$ or $a = -1$ or $a = 2$.

We need to examine the case $a = 0$. If $a = 0$, then we have $x_2 = 1$ and $x_1 = 1$. So, the system is consistent. Note that the case $a = 2$ also gives a consistent system. Finally, we conclude that the system above is consistent if and only if $a \neq -1$.

Question 2. ROW REDUCTION AND ECHELON FORMS

Write the augmented matrix corresponding the system below:

$$\begin{array}{rcrcrcr} x_1 & - & 6x_2 & - & 4x_3 & = & -5 \\ 2x_1 & - & 10x_2 & - & 9x_3 & = & -4 \\ -x_1 & + & 6x_2 & + & 5x_3 & = & 3. \end{array}$$

Solve the system by applying the row reduction algorithm. If the system is consistent, find the general solution set.

Answer

The augmented matrix corresponding to the given system is

$$\begin{bmatrix} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{bmatrix}.$$

We need to reduce the augmented matrix

$$\begin{bmatrix} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -6 & -4 & -5 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[R_3+R_2 \leftrightarrow R_2]{3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -7 & 13 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & 13 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[(1/2)R_2 \leftrightarrow R_2]{7R_3+R_1 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$G.S. = \begin{cases} x_1 = -1 \\ x_2 = \ \ 2 \\ x_3 = -2 \end{cases}$$

Question 3. VECTOR EQUATIONS

Determine if $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ where

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 6 \\ -1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -17 \\ 17 \\ 7 \end{bmatrix}.$$

If $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, express $\mathbf{b}$ as a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_1$, and $\mathbf{a}_1$.

Answer

We need to solve the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$:

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -3 & 6 & -1 & -17 \\ 4 & -1 & 2 & 17 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[-4R_4+R_3\to R_3]{-2R_4+R_1\to R_1, 3R_4+R_2\to R_2} \begin{bmatrix} 0 & -3 & -7 & -11 \\ 0 & 12 & 8 & 4 \\ 0 & -9 & -10 & -11 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[4R_1+R_2\leftrightarrow R_2]{-3R_1+R_3\to R_3} \begin{bmatrix} 0 & -3 & -7 & -11 \\ 0 & 0 & -20 & -40 \\ 0 & 0 & 11 & 22 \\ 1 & 2 & 3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & -7 & -11 \\ 0 & 0 & -20 & -40 \\ 0 & 0 & 11 & 22 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[(-1/2)R_3\to R_3]{R_2+R_1\to R_1, (-1/2)R_2\to R_2} \begin{bmatrix} 0 & -3 & -27 & -51 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[-R_3+R_2\leftrightarrow R_2]{(-1/3)R_1\to R_1} \begin{bmatrix} 0 & 1 & 9 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 9 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[-9R_3+R_1\to R_1]{-2R_1+R_4\to R_1} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{bmatrix} \xrightarrow[-R_3+R_2\to R_2]{-2R_1+R_4\to R_4, -3R_3+R_4\to R_4} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

$$G.S. = \begin{cases} x_1 = 3 \\ x_2 = -1 \\ x_3 = 2 \end{cases}$$

Finally, we see that $3 \cdot \mathbf{a}_1 - 1 \cdot \mathbf{a}_2 + 2 \cdot \mathbf{a}_3 = \mathbf{b}$.

Question 4. THE MATRIX EQUATION A$\mathbf{x}$=$\mathbf{b}$

A.) Write the given matrix equation below as system of linear equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}.$$

Answer

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & - & x_2 & - & 2x_3 & = & -5 \\ 2x_1 & + & & - & 4x_3 & = & 5 \end{array}$$

B.) Solve the system and write the general solution.

Answer

We need to reduce the augmented matrix that represents the given system (I'll leave the details to you)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & -5 \\ 2 & 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & 15/2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

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$$G.S. = \begin{cases} x_1 = -7/2 \\ x_2 = 15/2 \\ x_3 = -3 \end{cases}$$

QUESTION 5. SOLUTION SETS OF LINEAR SYSTEMS

A. Solve the nonhomogeneous system $A\mathbf{x}=\mathbf{b}$ and write the solution in parametric vector form where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & 2 & -4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

ANSWER

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & -3 & 0 \\ -1 & 2 & -4 & 0 \end{bmatrix} \xrightarrow[-2R_2+R_1\to R_1]{R_2+R_3\to R_3} \begin{bmatrix} 0 & -3 & 5 & -1 \\ 1 & 2 & -3 & 0 \\ 0 & 4 & -7 & 0 \end{bmatrix} \xrightarrow[3R_3+R_1\leftrightarrow R_1]{R_1+R_3\to R_3} \begin{bmatrix} 0 & 0 & -1 & -4 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & -4 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix} \xrightarrow[-2R_1+R_3\leftrightarrow R_3, -R_1+R_2\to R_2]{-2R_3+R_2\to R_2} \begin{bmatrix} 0 & 0 & -1 & -4 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$$

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$$G.S. = \begin{cases} x_1 = -2 \\ x_2 = 7 \\ x_3 = 4 \end{cases}$$

B. Using the parametric vector form of the solution set in part **A.**, determine a particular solution $\mathbf{p}$.

ANSWER

We see that $\mathbf{p} = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$ is a particular solution.

C. Write the general solution for the system $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

ANSWER

The parametric vector form of homogeneous part of the general solution set is

$$v_h = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

QUESTION 6. LINEAR INDEPENDENCE

Find the value(s) of h for which the following set of vectors

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix} \right\}$$

is **linearly independent**.

ANSWER

We need to solve the homogeneous system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$:

$$\begin{bmatrix} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & -h & 3h+1 & 0 \end{bmatrix} \xrightarrow{hR_2+R_3 \to R_3} \begin{bmatrix} 1 & h & 1 & 0 \\ 0 & 1 & 2h & 0 \\ 0 & 0 & 2h^2+3h+1 & 0 \end{bmatrix}$$

Since we want the given vectors to be linearly independent, we have to have ONLY the trivial solution. In other words, we have to have $2h^2 + 3h + 1 \neq 0$. Let us solve the equation $2h^2 + 3h + 1 = 0$ or $(2h+1)(h+1) = 0$. We get $h = -1/2$ or $h = -1$. As a conclusion, , $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \}$ is linearly independent if and only if $h \neq -1/2$ and $h \neq -1$.