MT210 Midterm 1 Sample 1

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February 16, 2011

Question 1. SYSTEMS OF LINEAR EQUATIONS

Determine the values of k such that the linear system

$$\begin{array}{rcrcr} 9x_1 & + & kx_2 & = & 9 \\ kx_1 & + & x_2 & = & -3 \end{array}$$

is consistent.

Answer

We apply row-reduction algorithm to the augmented matrix corresponding to the system given above: Assume that $k \neq 0$, then we get

$$\begin{bmatrix} 9 & k & 9 \\ k & 1 & -3 \end{bmatrix} \xrightarrow{(-k/9)R_1+R_2 \to R_2} \begin{bmatrix} 9 & k & 9 \\ 0 & 1-\frac{k^2}{9} & -3-k \end{bmatrix}.$$

By Theorem 2, we know that the system above is consistent if and only if there is no row of the form $[0\ 0\ 1]$ which implies that either we must have $1-\frac{k^2}{9} \neq 0$ or we must have $1-\frac{k^2}{9} = 0$ and $-3-k = 0$.

We need to examine the case $k = 0$. If $k = 0$, then we have $9x_1 = 9$ or $x_1 = 1$ and $x_2 = -3$. So, the system is consistent. Note that if $k = -3$ the given system is still consistent. Finally, we conclude that the system above is consistent if and only if $k \neq 3$.

Question 2. ROW REDUCTION AND ECHELON FORMS

Determine when the augmented matrix below represents a consistent linear system.

$$\begin{bmatrix} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{bmatrix}$$

Answer

We apply row-reduction algorithm to the augmented matrix corresponding to the system given above:

$$\begin{bmatrix} 1 & 0 & 2 & a \\ 2 & 1 & 5 & b \\ 1 & -1 & 1 & c \end{bmatrix} \xrightarrow[-1R_1+R_3\leftrightarrow R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & -1 & -1 & c-a \end{bmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b-2a \\ 0 & 0 & 0 & b-3a+c \end{bmatrix}.$$

By Theorem 2, we know that the system above is consistent if and only if $b - 3a + c = 0$.

Question 3. VECTOR EQUATIONS

Determine if $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

If $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, express $\mathbf{b}$ as a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Answer

We need to use the fact that $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if and only if the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ has a solution. We need to reduce the augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & -7 \end{bmatrix} \xrightarrow[-3R_3+R_2 \to R_2]{-2R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 0 & 11 & 22 \\ 0 & -1 & -3 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 0 & 11 & 22 \\ 0 & -1 & -3 & -7 \end{bmatrix} \xrightarrow[(1/11)R_3 \to R_3]{R_2 \to R_3} \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & -1 & -3 & -7 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[3R_3+R_2 \to R_2, -R_2 \to R_2]{-9R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

$$G.S. = \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$

Finally, we see that $1 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 + 2 \cdot \mathbf{a}_3 = \mathbf{b}$.

QUESTION 4. THE MATRIX EQUATION A$\mathbf{x}$=$\mathbf{b}$

A. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

ANSWER

We need to reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[-2R_2+R_1 \to R_1]{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that x_1 and x_2 are basic variables and x_3 is a free variable. We rewrite the system, i.e., we get $x_1 + 3x_3 = 0$ and $x_2 - x_3 = 0$ OR $x_1 = -3x_3$ and $x_2 = x_3$.

$$G.S. = \begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_3 \ is \ free. \end{cases}$$

B. Is it possible to solve $A\mathbf{x} = \mathbf{b}$ for **any given** $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ where A is the matrix given in part **A**? Explain.

ANSWER

NO. By Theorem 6, $A\mathbf{x} = \mathbf{b}$ has a solution FOR ANY GIVEN $\mathbf{b}$ if and only if A has 3 pivot positions. As you can see above, A has only 2 pivot positions. As a conclusion, it is not possible to solve $A\mathbf{x} = \mathbf{b}$ for any given $\mathbf{b}$.

C. Describe the set of all $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

ANSWER

We need to reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 1 & 3 & 0 & b_2 \\ 1 & 1 & 2 & b_3 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & -1 & 1 & b_3 - b_1 \end{bmatrix} \xrightarrow[-2R_2+R_1 \to R_1]{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 3 & -2b_2 + 3b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}$$

By Theorem 2, the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $b_3 + b_2 - 2b_1 = 0$.

Question 5. SOLUTION SETS OF LINEAR SYSTEMS

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & -2 & -2 & -2 \\ 3 & -2 & -2 & -2 & -2 \\ -3 & 2 & 1 & 1 & -1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

A. Solve the linear system.
B. Write the general solution in parametric-vector form.
C. Give a particular solution $\mathbf{p}$.
D. Write the solution set for the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Answer

A. We need to reduce the augmented matrix(I'll leave the details to you.)

$$\begin{bmatrix} 1 & -1 & -2 & -2 & -2 & 3 \\ 3 & -2 & -2 & -2 & -2 & -1 \\ -3 & 2 & 1 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & -11 \\ 0 & 1 & 0 & 0 & -8 & -18 \\ 0 & 0 & 1 & 1 & 3 & 2 \end{bmatrix}$$

We see that x_1, x_2 and x_2 are basic variables and x_4 and x_5 are free variables. We rewrite the system, i.e., we get $x_1 - 4x_5 = -11$, $x_2 - 8x_5 = -18$, $x_3 + x_4 + 3x_5 = 2$ OR $x_1 = -11 + 4x_5$, $x_2 = -18 + 8x_5$, $x_3 = 2 - x_4 - 3x_5$. Then we find that

$$G.S. = \begin{cases} x_1 = -11 + 4x_5 \\ x_2 = -18 + 8x_5 \\ x_3 = 2 - x_4 - 3x_5 \\ x_4 \ is \ free \\ x_5 \ is \ free. \end{cases}$$

B. The parametric vector form of the general solution set is

$$G.S. = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{bmatrix} : x_4, x_5 \in \mathbb{R} \right\}$$

C. The particular solution is given as $\mathbf{p} = \begin{bmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

D. The parametric vector form of homogeneous part of the general solution set is

$$v_h = \left\{ x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{bmatrix} : x_4, x_5 \in \mathbb{R} \right\}$$

Question 6. LINEAR INDEPENDENCE

Determine if the following sets of vector are linearly independent. If **not**, write one vector as a linear combination of other vectors in the set.

A.) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Answer

This set is linearly dependent by Theorem 9, i.e., zero vector is in the set.

B.) $\left\{ \begin{bmatrix} -5 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 36 \\ 12 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\}$

Answer

This set is linearly dependent by Theorem 8, i.e., more vectors than the entries in each vectors.

C.) $\left\{ \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$

Answer

This problem requires work. We need to solve the system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$:

$$\begin{bmatrix} -1 & 3 & 0 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow G.S. = \begin{cases} x_1 = 0 \\ x_2 = 0. \end{cases}$$

As a conclusion, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because the system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$ has ONLY the trivial solution.

D.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} \right\}$

Answer

This set is linearly dependent by Theorem 7, i.e., the third vector is the addition of the first two vector.

E.) $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} \right\}$

Answer

This problem requires work. We need to solve the system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$:

$$\begin{bmatrix} 1 & -1 & -4 \\ 0 & 3 & 2 \\ -1 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow G.S. = \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0. \end{cases}$$

As a conclusion, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent because the system $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has ONLY the trivial solution.