Optimal Filter Banks- A Statistical Perspective

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Optimal Filter Banks from a Statistical Perspective, with an investigation into construction and applications

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Abstract

A core area of research in the field of Multi rate Signal Processing is the development of better Filter banks. Filter banks are commonly used for applications like compression, and an improvement in design of these can improve the gains achieved by compression. We study the methods used to optimally construct such filter banks, specifically looking into the properties and construction of Principal component filter banks. Specifically, we look into the behaviour of the coding gain of a such a filter bank with number of channels for a variety of input signals, to come up with a design criteria to maximize the coding gain. We additionally look into design of Principal component filter banks for 1D signals given constraints on resources and number of subbands.

Index Terms

Multi-rate Signal Processing, Filter Banks, Optimal Filter banks, Principal component filter banks, Coding gain

I. Background

Multi rate signal processing involves analyzing a signal by breaking it into several subbands using carefully designed filter banks. There of immense use in applications ranging from Compression, efficient communication, denoising, and extracting useful features out of signals in several domains including bio-medical signals. Designing such filter banks is an important research problem, which has been studied in great depth. Several filter banks, ranging from the ones based on the simplest Haar wavelet based to more complex ones like the Daubechies family of wavelets [1] and Ricker wavelets have been proposed for different applications. But the answer to if it is possible for we to design possibly better filter banks with better subbands for these applications and the way to design filter banks for several other applications is an active research area.

The design of such filter banks is done to optimize certain metrics, which are relevant for the particular application. For example, for one such metric to be optimized is the coding gain, maximizing which is useful for applications in compression and efficient communication. Additionally, another factor taken care of during the design of such filter banks is the amount of resources (adders, multipliers, memory) used up by such a filter. Taking in to consideration this trade-off between performance and resources is of utmost importance in such a design.

For such a design problem, it is important to know the absolute best design one can construct given a number of resources, the optimal filter. The optimal filter, would be dependent on the input power spectral density (as it will allow us to tailor the subbands). It has been proved in [2] that such a filter bank, given that the input PSD is known, is the the principal component filter bank. We study methods to design such an optimal filter (PCFB) for different input PSDs. We also study how a commonly used metric for a filter's efficiency, the coding gains varies with the number of subbands used in the PCFB, to get a finer look at the resource vs performance trade-off in such a filter bank.

We also study extending techniques made to design realistic optimal filters for 1 dimensional signals, like AEVD [3] and SBR2 [4] to non separable multi dimensional signals. Extending these algorithms to

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non separable multi dimensional signals requires analysis of application of these techniques to a single dimension and analyzing the sequence of operations of these techniques in multiple dimensions.

II. Literature Survey

We give a brief overview of some of the most important references we have studied in our survey of the field.

A. Theory of Optimal Orthonormal Subband Coders

In [2] the author has tried to extend the theory of orthogonal coder and methods for its optimal design, specifically they have derived a set of necessary and sufficient conditions for the coding-gain optimality of an orthonormal subband coder for given input statistics. Additionally they have proposed a step by step method to construct a sequence of optimal compaction filters one at a time that satisfy these conditions. These filters then can be used along with the signal statistics to compute variance per band and hence the coding gain. We use these proposed techniques to analyze the behaviour of coding gain, it's properties and and analyse constraints on the input PSD so that it increases monotonically with the increase in number of subbands.

B. Variation of Coding Gain With Number of Channels in PCFB

The work in [5] proposes to solve the problem of determining the number of channels for a PCFB in advance. Here the authors analyse the change in coding gain of 1-D PCFB with the change in number of channels. Specifically they show results for 3 different type of PSDs (Square, Smooth, Triangle), where they derive the expression for coding gain and show the conditions under which a monotonic increase in coding gain is obtained with increase with the increase in the number of channels. This paper also tries to identify PSDs or a class of PSDs where the coding gain increases monotonically with increase in the number of bands. For this they give an theoretical proof for the Triangular, Inverted triangular and exponential Decay PSDs. We have tried to extend this idea to Beta, Gaussian and cosine and sinusoid PSDs.

C. Design of Optimal Non-separable Filter Banks For Multiple Dimensions

This work [6] delves into the problem of designing optimal filters for multi-dimensional signals. Design of optimal filter banks for 1D signals has been studied in detail in literature [2], but very few works examine design of filter banks for multi dimensional signals, like images. For certain special signals which are separable in dimension, techniques for designing optimal filter banks for 1D signals can directly be applied to each separable dimension. This paper deals with designing optimal filter banks for the general non separable multi dimensional signals. It proposes extensions to the AEVD [3] and SBR2 [4] schemes, which are designed for making optimal filters for 1D signals, to multiple dimensions. The AEVD technique (Approximate eigenvalue decomposition) involves applying a degree-1 FIR PU transformation at each iteration such that the zero order diagonal energy of the resultant PH system is decreasing. For extension to multiple dimensions, this transform is applied independently to the different dimensions. The paper explains about the order (sequence) in which the iterations are to be done on each dimension. It also looks into extending the SBR2C algorithm to a multi dimensional signal.

III. Theory of Optimal filters

The optimality of a filter bank can be measured in terms of several metrics, a prominent one of which is the coding gain.

A. Coding gain

The coding gain achieved by a filter bank quantifies the efficiency of compression achieved by it. It is formally defined as the ratio of the error due to quantization if the filter was not used to the quantization error when the filter is used. So, the coding gain (CG) can be written as-

$$CG = \frac{\epsilon_{direct}}{\epsilon_{SBC}}$$

where ϵ_{direct} is the quantization error if no filter is used, while ϵ_{SBC} is the quantization error if the filter bank is used (subband coder).

This expression for the coding gain of a filter bank can be relaxed under the assumption that the input is a weak sense stationary process. In [2], it is shown that under this assumption, the expression for coding gain of a filter bank can be further simplified as-

$$CG = \frac{AM(\sigma_k^2)}{GM(\sigma_k^2)} = \frac{\Sigma_{k=1}^{M} \sigma_k^2}{(\Pi_{k=1}^{M} \sigma_k^2)^{\frac{1}{M}}\, M} \tag{1}$$

where $\sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} S_{xx} H_k^2 d\omega$ are the subband variances of each of the filter banks subbands.

B. Constructing filters with maximal coding gain

The maximum coding gain for a filter bank with M number of bands can be obtained if the bands follow the following properties (and the PSD is known)-

- The subbands satisfy the majorization property, which means that subbands having a higher subband variance should be point wise (in frequency domain) greater than the other subbands having a lower subband variance.
- The subband outputs after decimation should be uncorrelated.

These 2 conditions are proven to be necessary and sufficient for a filter bank to be optimal in terms of coding gain in [2]. A class of filter banks which satisfy these conditions are the Principal component filter banks (PCFBs).

A PCFB can be constructed for a given input PSD. PCFBs follow a motivation similar to the dimensionality reduction technique PCA, designing an M-band PCFB involving dividing the signal into M bands with decreasing subband variances. The following steps can be followed to derive the M PCFB subbands given the signal PSD :

1) for designing the first band, we need to choose a compaction filter for the PSD. A compaction filter is defined as follows:
   - for each frequency ω_0 in $0 \leq \omega \leq \frac{2\pi}{M}$, define alias frequency $\omega_k = \omega_0 + \frac{2\pi k}{M}$, where $0 \leq k \leq M-1$
   - Compare the values of PSD at these M alias frequencies.
   - Define, $H_0(e^{j(\omega_0 + \frac{2\pi(k+L)}{M})}) = \sqrt{M}$ where L is the smallest integer such that PSD is maximum in that alias set.
2) Now to design the next subband filter remove the part of PSD where the first subband filterH_0 is non-zero.
3) Now construct the compaction filter for the second band following the same procedure as above. A pictorial representation of constructing filters is given in Fig 1.
4) Repeat this process till M filters are obtained.

IV. Analysis

In this section, we present the results of our analysis of the coding gain of a PCFB. We firstly discuss some properties of the coding gain along with their proofs. We then, using some of these properties and some insights gained from these present our attempts to derive conditions on an input PSD such that their coding gain is monotonic with respect to M.

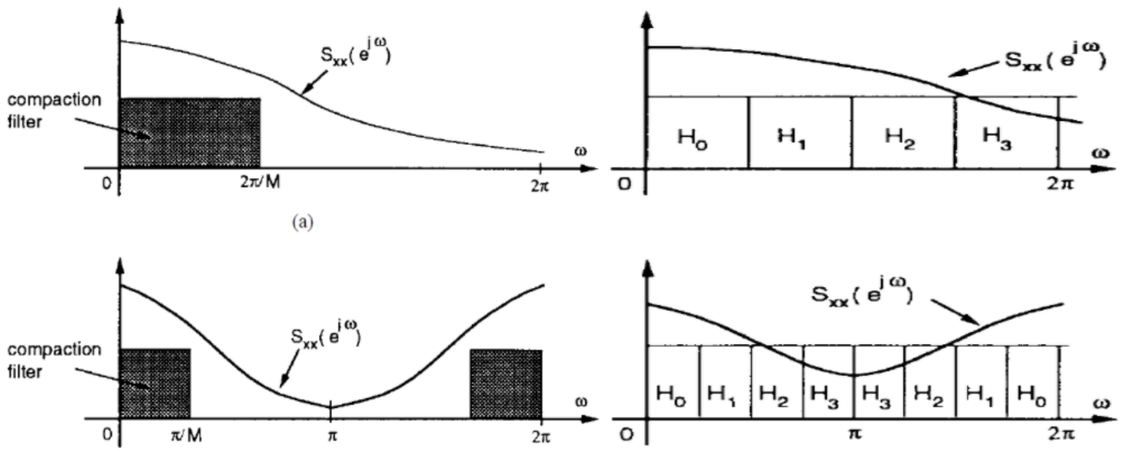

Fig. 1. Left: construction of first compaction filter/principle component filter; Right: Final M constructed filters [2]

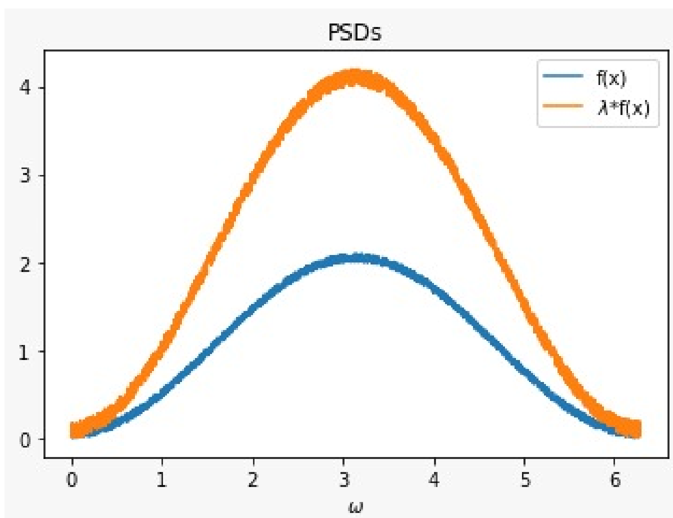

Fig. 2. A plot depicting a general PSD $f(x)$ and it's scaled version $\lambda f(x)$, which we prove to have the same coding gain

A. Properties of coding gain of PCFBs

1) Property 1: Scaling of the PSD has no impact on its coding gain.

Proof: Consider a PSD $f(x)$, then its coding gain, as described in section III, is

$$CG = \frac{AM(\sigma_k^2)}{GM(\sigma_k^2)} = \frac{1}{M}\frac{\sum_{k=1}^{M}\sigma_k^2}{\left[\prod_{k=1}^{M}\sigma_k^2\right]^{\frac{1}{M}}} \tag{2}$$

$$where,\ \ \sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,H_k^2\,dx \tag{3}$$

Let the PSD be scaled by a factor of λ, i.e. the new PSD now is $\lambda f(x)$. The analysis filter bank H_k will not change since their construction depends on the relative values of the PSD are, and in this case the 'relative' values, even after scaling, remain the same. Therefore, the equation (3) now changes to,

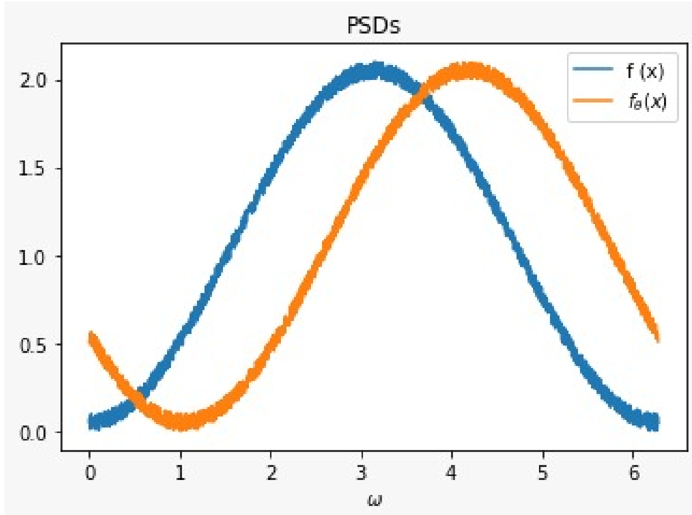

Fig. 3. A plot depicting an example of general PSD $f(x)$ and it's cyclically shifted version $f_\theta(x)$, which we prove to have the same coding gain

$$\begin{aligned}\sigma^2_{knew} &= \frac{1}{2\pi}\int_0^{2\pi} \lambda f(x)\, H_k^2\, dx \\ &= \lambda\sigma_k^2\end{aligned} \tag{4}$$

Substituting, the new variance from equation (4) in equation (2),

$$\begin{aligned}CG_{new} = \frac{AM(\,\sigma^2_{knew}\,)}{GM(\,\sigma^2_{knew}\,)} &= \frac{1}{M}\frac{\sum_{k=1}^{M}\sigma^2_{knew}}{\left[\prod_{k=1}^{M}\sigma^2_{knew}\right]^{\frac{1}{M}}} \\ &= \frac{1}{M}\frac{\sum_{k=1}^{M}\lambda\sigma_k^2}{\left[\prod_{k=1}^{M}\lambda\sigma_k^2\right]^{\frac{1}{M}}} \\ &= \frac{1}{M}\frac{\lambda\sum_{k=1}^{M}\sigma_k^2}{\lambda\left[\prod_{k=1}^{M}\lambda\sigma_k^2\right]^{\frac{1}{M}}} \\ &= CG \qquad\qquad (from\ equation\ (2))\end{aligned} \tag{5}$$

Hence Proved. ■

2) Property 2: Shifting the PSD cyclically has no impact on its coding gain.

Proof: Consider a PSD $f(x)$ which is cyclically shifted by θ. Let the shifted PSD be called as $f_\theta(x)$. $f(x)$ and $f_\theta(x)$ are related by -

$$f_{theta}(x) = f(x +_{2\pi} \theta) \tag{6}$$

where $+_{2\pi}$ represents modulo 2π addition to enable the cyclic shifting. Next, we see that for constructing a M band PCFB for $f(x)$, we need to know the alias frequencies for every ω and their relative order of magnitude. Let any such alias frequency be ω_0. Let the other alias frequencies be defined as $\omega_k =$

$\omega_0 +_{2\pi} \frac{2\pi k}{M}$. Now let their arrangement in decreasing order of magnitude at $f(\omega_k)$ be represented by the series $\omega_0^{'}, \omega_1^{1}...\omega'_{M-1}$. The PCFB for $f(x)$ having subbands $H_0, H_1...H_{M-1}$ would be constructed by setting-

$$H_k(\omega'_k) = \sqrt{M}$$

where $\omega_0 \in [0, \frac{2\pi}{M}]$ and ω'_k is constructed using all such $\omega_0 \in [0, \frac{2\pi}{M}]$ for all $k \in Z, [0, M-1]$ and $H_k(\omega) = 0$ for all other values of ω where it is not defined by this equation. Now similarly constructing a PCFB for $f_\theta(x)$, we similarly choose an $\omega_{0,\theta} \in [0, \frac{2\pi}{M}]$ and define the other alias frequencies similarly. Now when we arrange them in their decreasing order of magnitude, we observe that any of the $\omega'_{k,theta}$ can be related to the ω'_k by the following equation-

$$\omega'_{k,theta} = \omega'_k +_{2\pi} \theta$$

So, the PCFB subbands are also effectively shifted by the same cyclical shift θ and are given by-

$$H_{k,\theta}(\omega'_k +_{2\pi} \theta) = \sqrt{M}$$

Such a shift in the subbands will not cause a change in their subband variances which are given by

$$\sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} S_{xx} H_k^2 d\omega$$

both the subband filters, and the PSD (S_{xx}) get shifted by the same amount. As the subband variance remains the same, the Coding gain for the PCFBs for $f(x)$ and $f_\theta(x)$ are also the same.

Hence Proved. ■

3) Property 3: Adding a positive constant to a PSD decreases it's coding gain, other factors remaining the same.

Proof: Consider the subband variances of the PCFB designed for $f(x)$ to be $a_1, a_2...a_M$. Now the coding gain for such the PCFB designed for $f(x)$ is

$$CG = \frac{\sum_{k=1}^{M} \frac{a_k}{M}}{(\Pi_{k=1}^{M} a_k)^{\frac{1}{M}}} \tag{7}$$

Now, if a positive constant, say λ is added to $f(x)$, the resulting PSD $f(x) + \lambda$ will have the same PCFBs designed fpr it, as the relative magnitude at each alias frequency remains the same. This can be checked using the process described in section III-B. Now, the subband variances for this can be calculated using the formula-

$$\begin{aligned}
\sigma_k^2 &= \frac{1}{2\pi}\int_0^{2\pi} (f(x) + \lambda) H_k^2 dx \\
\sigma_k^2 &= \frac{1}{2\pi}(\int_0^{2\pi} f(x) H_k^2 dx + \int_0^{2\pi} \lambda H_k^2 dx) \\
\sigma_k^2 &= \frac{1}{2\pi}(2\pi \times a_k + \int_0^{2\pi} \lambda M dx) \\
\sigma_k^2 &= a_k + \frac{1}{2\pi}\frac{2\pi}{M}\lambda M dx \\
&\quad as\ any\ subband\ has\ width\ \frac{2\pi}{M} \\
\sigma_k^2 &= a_k + \lambda
\end{aligned} \tag{8}$$

Hence, the subband variances for this are λ greater than those for $f(x)$. Now the coding gain can be calculated as the ratio of the AM to the GM of the subband variances-

$$CG = \frac{\sum_{k=1}^{M} \frac{a_k+\lambda}{M}}{(\Pi_{k=1}^{M}(a_k+\lambda))^{\frac{1}{M}}}$$
$$CG = \frac{\sum_{k=1}^{M}(\frac{a_k}{M}) \quad + \quad \lambda}{(\Pi_{k=1}^{M}(a_k+\lambda))^{\frac{1}{M}}} \tag{9}$$

Now analyzing this expression, which can be written as being of the form $\frac{a+c}{b+d}$ This expression is lesser than the fraction $\frac{a}{b}$ if $\frac{c}{d} < \frac{a}{b}$. Now $\frac{a}{b}$, here originally the coding gain of $f(x)$ is the greater than 1 as it is a ratio of the AM to the GM. Now we go on to prove that the ratio of the term $\frac{c}{d}$ is lesser than 1 to prove that the coding gain of $f(x)+\lambda$ is less than that of $f(x)$. Now, the numerator of the coding gain is already of the required form $= \sum_{k=1}^{M}(\frac{a_k}{M}) \quad + \quad \lambda$. So, we need to prove that the denominator satisfies-

$$\left(\Pi_{k=1}^{M}(a_k+\lambda)\right)^{\frac{1}{M}} > \left(\Pi_{k=1}^{M}a_k\right)^{\frac{1}{M}} + \lambda$$

So, now we calculate the value of (by raising both sides of the above equation to M)-

$$\left(\Pi_{k=1}^{M}(a_k+\lambda)\right) - (\left(\Pi_{k=1}^{M}a_k\right)^{\frac{1}{M}} + \lambda)^M \tag{10}$$

We need to prove this term 10 to be greater than or equal to 0 for the condition on the denominator to be satisfied. So, now expanding the products, we analyze the coefficients of λ^k in this expression

$$=S_{M-k} - C_k^M(\Pi_{k=1}^{M}a_k)^{\frac{M-k}{M}}$$
$$here\ the\ term\ S_{M-k}\ is\ the\ sum\ of\ all\ the\ possible\ products\ of\ m-k\ subband\ variances\ a_1, a_2..a_M$$
$$=C_k^M\left(\frac{S_{M-k}}{C_k^M} - (\Pi_{k=1}^{M}a_k)^{\frac{M-k}{M}}\right)$$
$$=C_k^M\left(\frac{S_{M-k}}{C_k^M} - (\Pi_{k=1}^{M}a_k^{C_k^{M-1}})^{\frac{1}{C_k^M}}\right)$$
$$This\ is\ because\ \frac{C_k^{M-1}}{C_k^M} = \frac{M-k}{M}$$
$$=C_k^M\left(\frac{S_{M-k}}{C_k^M} - (\Pi_{k=1}^{M}a_k^{C_k^{M-1}})^{\frac{1}{C_k^M}}\right) \geq 0 \tag{11}$$

This is because the term $\frac{S_{M-k}}{C_k^M}$ is an AM while $(\Pi_{k=1}^{M}a_k^{C_k^{M-1}})^{\frac{1}{C_k^M}}$ is the GM of the same terms, and as $AM \geq GM$, the expression is greater than 0.

Hence Proved. ■

4) Property 4: Let f and g be two PSDs with the same PCFB, then $CG(f+g) < 2\alpha$, where $\alpha = max\{CG(f), CG(g)\}$.

Proof: The coding gain for the PSD $f(\omega)$ is given as,

$$CG = \frac{AM(\sigma_k^2)}{GM(\sigma_k^2)} = \frac{1}{M}\frac{\sum_{k=1}^{M}\sigma_k^2}{\left[\prod_{k=1}^{M}\sigma_k^2\right]^{\frac{1}{M}}}$$
$$where,\ \ \sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)\, H_k^2\, d\omega$$

Let,

$$CG_f = \frac{1}{M} \frac{\sum_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega}{\left[\prod_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega\right]^{\frac{1}{M}}}$$
$$CG_g = \frac{1}{M} \frac{\sum_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega}{\left[\prod_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega\right]^{\frac{1}{M}}}$$

Now let,

$$a = \frac{1}{M} \sum_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega$$
$$b = \left[\prod_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega\right]^{\frac{1}{M}}$$
$$c = \frac{1}{M} \sum_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega$$
$$d = \left[\prod_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega\right]^{\frac{1}{M}}$$

Then,

$$CG_f = \frac{a}{b}$$
$$CG_g = \frac{c}{d}$$

Now lets consider the coding gain of $f + g$,

$$\begin{aligned}
CG_{f+g} &= \frac{1}{M} \frac{\sum_{k=1}^{M} \int_0^{2\pi} (f+g)(\omega)\, H_k^2 \, d\omega}{\left[\prod_{k=1}^{M} \int_0^{2\pi} (f+g)(\omega)\, H_k^2 \, d\omega\right]^{\frac{1}{M}}} \\
&= \frac{\frac{1}{M} \sum_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega + \frac{1}{M} \sum_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega}{\left[\prod_{k=1}^{M} \left(\left(\int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega\right) + \left(\int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega\right)\right)\right]^{\frac{1}{M}}} \\
&= \frac{a+c}{\left[\left(\left(\int_0^{2\pi} \ f(\omega)\, H_1^2 \, d\omega\right) + \left(\int_0^{2\pi} \ g(\omega)\, H_1^2 \, d\omega\right)\right) \ldots \left(\left(\int_0^{2\pi} \ f(\omega)\, H_M^2 \, d\omega\right) + \left(\int_0^{2\pi} \ g(\omega)\, H_M^2 \, d\omega\right)\right)\right]^{\frac{1}{M}}} \\
&= \frac{a+c}{\left[\left(\int_0^{2\pi} \ f(\omega)\, H_1^2 \, d\omega\right) \ldots \left(\int_0^{2\pi} \ f(\omega)\, H_M^2 \, d\omega\right) + \left(\int_0^{2\pi} \ g(\omega)\, H_1^2 \, d\omega\right) \ldots \left(\int_0^{2\pi} \ g(\omega)\, H_M^2 \, d\omega\right) + \ldots\right]^{\frac{1}{M}}} \\
&< \frac{a+c}{\left[\left(\int_0^{2\pi} \ f(\omega)\, H_1^2 \, d\omega\right) \ldots \left(\int_0^{2\pi} \ f(\omega)\, H_M^2 \, d\omega\right) + \left(\int_0^{2\pi} \ g(\omega)\, H_1^2 \, d\omega\right) \ldots \left(\int_0^{2\pi} \ g(\omega)\, H_M^2 \, d\omega\right)\right]^{\frac{1}{M}}} \\
&= \frac{a+c}{\left[\left(\prod_{k=1}^{M} \int_0^{2\pi} \ f(\omega)\, H_k^2 \, d\omega\right) + \left(\prod_{k=1}^{M} \int_0^{2\pi} \ g(\omega)\, H_k^2 \, d\omega\right)\right]^{\frac{1}{M}}} \\
&= \frac{a+c}{(b^M + d^M)^{\frac{1}{M}}}
\end{aligned} \tag{12}$$

Now taking a small detour,
consider a function $h(x) = x^{1/n}$ (n is a positive integer), then $h(x)$ will always be concave down ($h" < 0$). Due to this shape we can always say that for x_1 and x_2 (both positive),

$$\begin{aligned}
h\left(\frac{x_1+x_2}{2}\right) &> \frac{h(x_1)+h(x_2)}{2} \\
\left(\frac{x_1+x_2}{2}\right)^{1/n} &> \frac{(x_1)^{1/n}+(x_2)^{1/n}}{2} \\
\frac{(x_1+x_2)^{1/n}}{2^{1/n}} &> \frac{(x_1)^{1/n}+(x_2)^{1/n}}{2} \\
(x_1+x_2)^{1/n} > \frac{(x_1+x_2)^{1/n}}{2^{1/n}} &> \frac{(x_1)^{1/n}+(x_2)^{1/n}}{2} \\
(x_1+x_2)^{1/n} &> \frac{(x_1)^{1/n}+(x_2)^{1/n}}{2} \qquad (13)
\end{aligned}$$

Using equation (13) in (12),

$$\begin{aligned}
CG_{f+g} &< \frac{a+c}{\frac{\left((b^M)^{\frac{1}{M}}+(d^M)^{\frac{1}{M}}\right)}{2}} \\
&= 2\left(\frac{a+c}{b+d}\right) \qquad (14)
\end{aligned}$$

Now, without loss of generality we assume the coding gain of f is more than that of g, i.e. $\frac{a}{b} > \frac{c}{d}$. Now lets find the upper bound of $\frac{a+c}{b+d}$,

$$\begin{aligned}
\frac{a}{b} &> \frac{c}{d} \\
ad &> bc \\
ad+ab &> bc+ab \\
a(b+d) &> b(a+c) \\
\frac{a}{b} &> \frac{a+c}{b+d} \qquad (15)
\end{aligned}$$

Now substituting the upper bound of $\frac{a+c}{b+d}$ from (15) to (14),

$$\begin{aligned}
CG_{f+g} &< 2\left(\frac{a+c}{b+d}\right) \\
&< 2\left(\frac{a}{b}\right) \\
&= 2\ max\{CG(f), CG(g)\} \qquad (16)
\end{aligned}$$

Hence Proved. ■

B. Approximating the coding gain to analyze monotonicity

This attempt focuses on analyzing, imposing and assuming certain conditions on the PSD so as to obtain some sufficient criterion to guarantee monotonicity of the PSDs coding gain. This attempt assumes the PSD to be a monotonically decreasing function. Let this PSD be $f(\omega)$ We first construct a PCFB using the algorithm described in [2]. The filter bank is described by the following equation for $k = 0$ to $M - 1$:

$$H_k\left(e^{j\omega}\right) = \begin{cases} \sqrt{M}, & \text{when } \omega \in \left(\frac{2k\pi}{M}, \frac{2(k+1)\pi}{M}\right) \\ 0, & \text{otherwise} \end{cases}$$

Then the subband variances are calculated,

$$\sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)H_k^2\ d\omega = \frac{1}{2\pi}M\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} f(\omega)\ d\omega$$

Now the coding gain is given by,

$$\begin{aligned} CG = \frac{AM(\sigma_k^2)}{GM(\sigma_k^2)} &= \frac{\Sigma_{k=0}^{M-1}\sigma_k^2}{M\left(\Pi_{k=0}^{M-1}\sigma_k^2\right)^{\frac{1}{M}}} \\ &= \frac{\Sigma_{k=0}^{M-1}\frac{1}{2\pi}\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} Mf(\omega)\ d\omega}{\frac{1}{2\pi}M\left(\Pi_{k=0}^{M-1}\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} Mf(\omega)\ d\omega\right)^{\frac{1}{M}}} \\ &= \frac{\frac{M}{2\pi}\Sigma_{k=0}^{M-1}\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} f(\omega)\ d\omega}{\frac{M^2}{2\pi}\left(\Pi_{k=0}^{M-1}\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} f(\omega)\ d\omega\right)^{\frac{1}{M}}} \\ &= \frac{\int_0^{2\pi} f(\omega)\ d\omega}{M\left(\Pi_{k=0}^{M-1}\int_{\frac{2k\pi}{M}}^{\frac{2(k+1)\pi}{M}} f(\omega)\ d\omega\right)^{\frac{1}{M}}} \end{aligned} \tag{17}$$

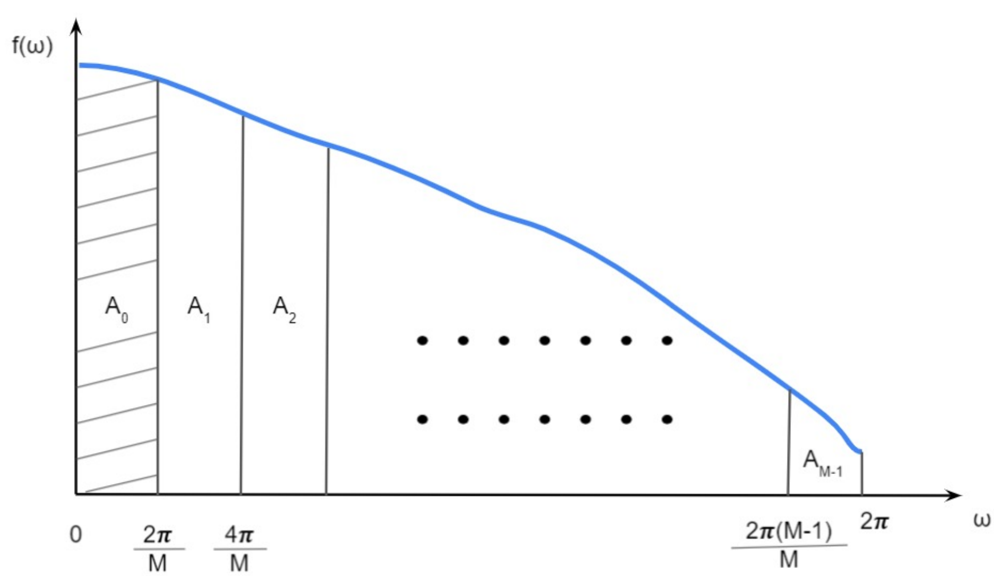

Fig. 4. Area partitioning in the PSD $f(\omega)$

In equation (17) the integrals are basically the areas A_0, A_1, ... A_{M-1}, as shown in figure 4. The numerator there, the total area of the PSD, is a constant, i.e. does not change with number of bands M; call it λ. Equation (17) now becomes,

$$CG = \frac{\lambda}{M\left(\Pi_{k=0}^{M-1} A_k\right)^{\frac{1}{M}}} \tag{18}$$

Equation (18) is the equation at M number of bands, and so lets call it CG_M. For coding gain to be monotonically increasing with M, $\frac{CG_{M+1}}{CG_M}$ should be greater than 1. From equation (18),

$$\frac{CG_{M+1}}{CG_M} = \frac{M\left(\Pi_{k=0}^{M-1} A_k\right)^{\frac{1}{M}}}{(M+1)\left(\Pi_{k=0}^{M} A'_k\right)^{\frac{1}{M+1}}} \tag{19}$$

In equation (23) A'_k are the corresponding areas for the case of $M+1$ bands. Now we will try to find a minimum bound on this expression, and impose conditions for this minimum bound to be greater than 1. If this happens, then, with the imposed conditions, $\frac{CG_{M+1}}{CG_M}$ would always be greater than 1, leading to monotonicity of the coding gain. Call this ratio α, for convenience
For finding the minimum value of the ratio in equation (23), we'll maximize the denominator and minimize the numerator. For this maximization and minimization we will do upper and lower bounding of the areas in question. This will be done by constructing rectangles above and below the area (similar to Riemann sums). Lets analyze the the PSD according to their second derivative, because which areas (numerator or denominator) are to be upper bounded or lower bound will depend on how the decreasing PSD orients itself (convex or concave). Let us proceed with assuming that $f(\omega)$ is convex ($f" > 0$)

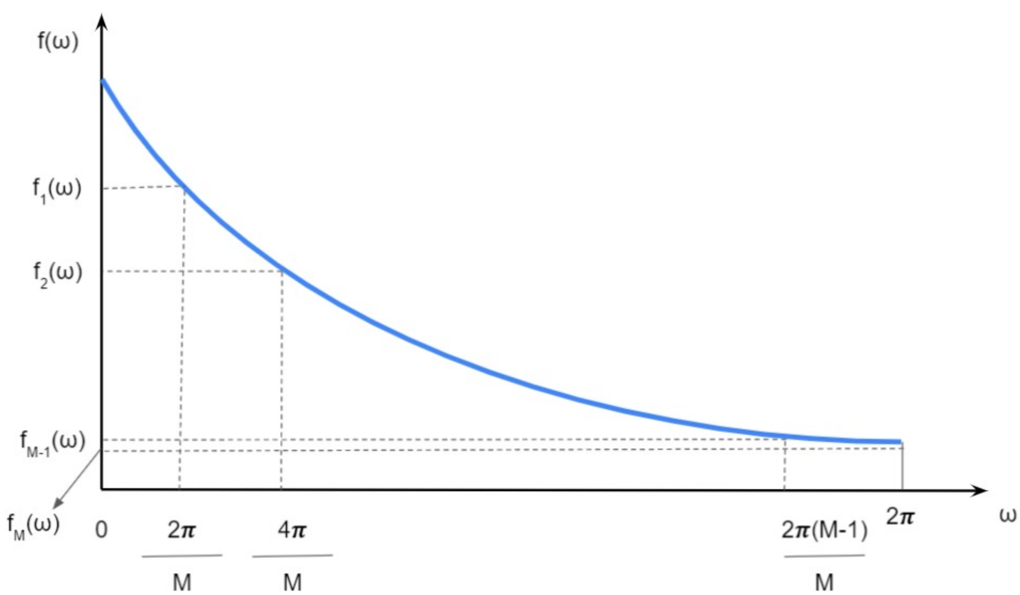

Fig. 5. Boundaries in PSD $f(\omega)$ for the case of M channels

Upper-bounds for the areas in denominator of α and the lower bounds of the areas in its numerator can be calculated by looking at figures 5 and 6.
A_k's lower-bound $= f_{k+1}\,\frac{2\pi}{M}$

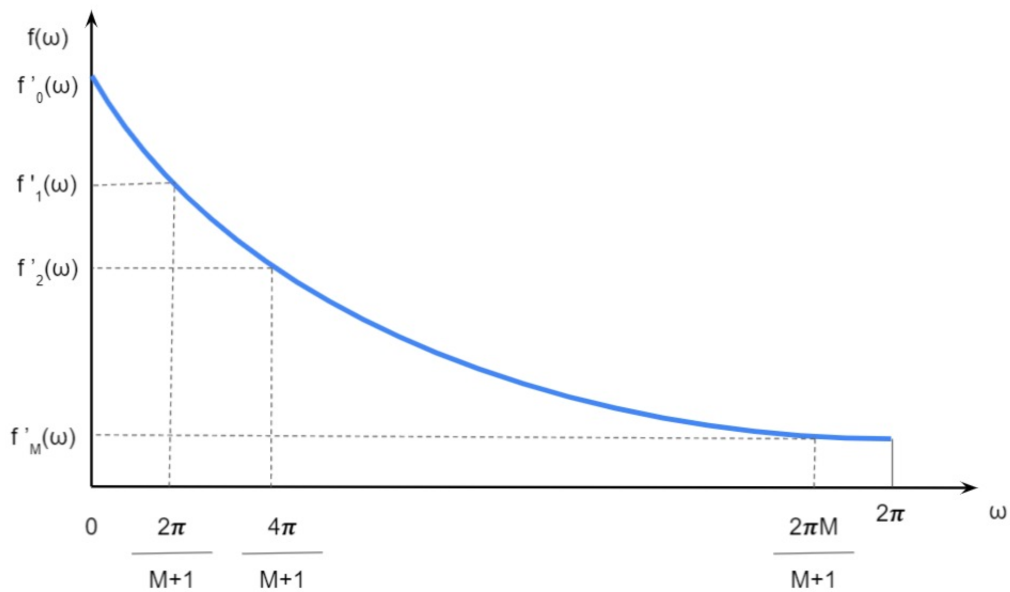

Fig. 6. Boundaries in PSD $f(\omega)$ for the case of $M+1$ channels

A'_k's upper-bound $= f'_{k+1}\,\frac{2\pi}{M+1}$
The lower-bound (LB) for α now becomes,

$$\begin{aligned}\alpha_{LB} &= \frac{M\left(\Pi_{k=0}^{M-1} f_{k+1}\,\frac{2\pi}{M}\right)^{\frac{1}{M}}}{(M+1)\left(\Pi_{k=0}^{M} f'_k\,\frac{2\pi}{M+1}\right)^{\frac{1}{M+1}}} \\ &= \frac{\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M} f'_k\right)^{\frac{1}{M+1}}} \end{aligned} \tag{20}$$

Let $a = \frac{2\pi}{M}$, then lets check when is $k\frac{2\pi}{M}$ less than $(k+1)\frac{2\pi}{M}$

$$\begin{aligned} k\frac{2\pi}{M} &< (k+1)\frac{2\pi}{M} \\ ka &< (k+1)\frac{aM}{M+1} \\ k &< M \end{aligned}$$

Hence, $k\frac{2\pi}{M}$ less than $(k+1)\frac{2\pi}{M}$ when $k < M$, which is always for in this case! Since $f(\omega)$ is a monotonically decreasing function,
$f(k\frac{2\pi}{M}) > f((k+1)\frac{2\pi}{M})$. This would then mean, $f_1 > f'_2$, $f_2 > f'_3$, ... $f_{M-1} > f'_M$. But we actually need,

$$\left(f\left(k\frac{2\pi}{M}\right)\right)^{\frac{1}{M}} > \left(f\left((k+1)\frac{2\pi}{M}\right)\right)^{\frac{1}{M+1}} \tag{21}$$

Equation (21), can be imposed as a condition $\forall\ M$. After imposing this condition equation (23) reduces to the following,

$$\begin{aligned}\alpha_{LB} &> \frac{(f_M)^{\frac{1}{M}}}{(f'_0 f'_1)^{\frac{1}{M+1}}} \\ &= \frac{(f(2\pi))^{\frac{1}{M}}}{\left(f(0)f(2\pi\frac{M}{M+1})\right)^{\frac{1}{M+1}}} \end{aligned} \tag{22}$$

However, it is not possible to easily simplify or estimate this condition. Therefore, we need the approximation to be more stringent. A similar situation occurs, with a concave PSD.

To make the bounds more stringent we try to do lower/upper-bounding by making a trapezium-based approximation instead of a rectangular one. This should tighten the bound which we obtained before.
For now assume that the PSD $f(\omega)$ is convex. In this case we would use the trapezium-based approximation for upper-bounding the area of the segments in the denominator. The lower bounding of the area in the numerator will be done by the rectangular-based approximation as before.
The area of the trapezium is half of the distance between its parallel sides multiplied by the sum of the parallel sides. The upper-bound of the area A'_k now becomes $\frac{\frac{2\pi}{M+1}}{2}(f'_k + f'_{k+1})$. The process of upper-bounding can be seen in figure 7.

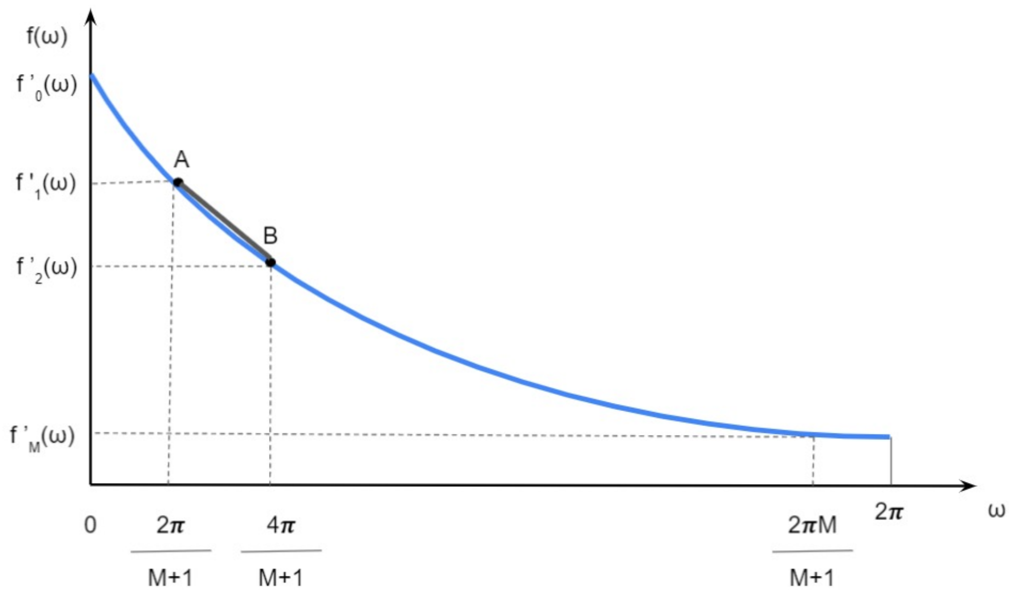

Fig. 7. Trapezium-based upper-bounding

α_{LB} now becomes

$$\begin{aligned}\alpha_{LB} &= \frac{M\left(\Pi_{k=0}^{M-1} f_{k+1}\,\frac{2\pi}{M}\right)^{\frac{1}{M}}}{(M+1)\left(\Pi_{k=0}^{M}\frac{\frac{2\pi}{M+1}}{2}(f'_k + f'_{k+1})\right)^{\frac{1}{M+1}}} \\ &= \frac{2\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M}(f'_k + f'_{k+1})\right)^{\frac{1}{M+1}}} \end{aligned} \tag{23}$$

Now replace $f'_k + f'_{k+1}$ by $\beta_k f'_{k+1}$ after finding the appropriate β_k such that $f'_k + f'_{k+1} < \beta_k f'_{k+1}$. This will further decrease the lower bound (call it α'_{LB}),

$$
\begin{aligned}
\alpha'_{LB} &= \frac{2\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M}(\beta_k f'_{k+1})\right)^{\frac{1}{M+1}}} \\
&= \frac{2\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M}(\beta_k)\right)^{\frac{1}{M+1}}\left(\Pi_{k=0}^{M}(\beta_k f'_{k+1})\right)^{\frac{1}{M+1}}} \\
&= \frac{2}{\left(\Pi_{k=0}^{M}(\beta_k)\right)^{\frac{1}{M+1}}}\frac{\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M}(f'_{k+1})\right)^{\frac{1}{M+1}}} \\
&= \gamma\frac{\left(\Pi_{k=0}^{M-1} f_{k+1}\right)^{\frac{1}{M}}}{\left(\Pi_{k=0}^{M}(f'_{k+1})\right)^{\frac{1}{M+1}}} \qquad where\ \gamma = \frac{2}{\left(\Pi_{k=0}^{M}(\beta_k)\right)^{\frac{1}{M+1}}}
\end{aligned} \tag{24}
$$

Now imposing condition in equation (21), $\forall\ M$, in equation (24) we get,

$$
\alpha'_{LB} = \gamma\frac{(f_M)^{\frac{1}{M}}}{(f'_1 f'_M)^{\frac{1}{M+1}}}
$$

We run into the same trouble as we did in the previous case when both upper and lower bounds was achieved by rectangular approximation. A similar situation arises in the case of a purely concave monotonically decreasing PSD as well.

C. Extending the coding gain to the real domain for analysis

Similar to the attempt to approximate the coding gain, we try to analyze the factors affecting the monotonicity of the coding gain of a given PSD with respect to the number of subbands M by extending the definition of the coding gain from integer M to real M. Let the PSD, which is assumed to be monotonically decreasing be $f(x)$. Also, we assume that $f(x)$ is continuous. Now the coding gain for such a PSD is given by the expression-

$$
CG = \frac{AM(\sigma_k^2)}{GM(\sigma_k^2)} = \frac{\Sigma_{k=1}^{M}\sigma_k^2}{(\Pi_{k=1}^{M}\sigma_k^2)^{\frac{1}{M}}\ M} \tag{25}
$$

where

$$
\sigma_k^2 = \frac{1}{2\pi}\int_0^{2\pi} f(x)H_k^2\ d\omega
$$

Here H_k is the k^{th} subband of the PCFB designed for $f(x)$ given by-

$$
H_k(\omega) = \begin{cases} \sqrt{M} & when\ \frac{2\pi(k-1)}{M} \le \omega \le \frac{2\pi(k)}{M} \\ 0 & otherwise \end{cases} \quad k = [1, M],\ k \in \mathbb{Z} \tag{26}
$$

This can be verified using the process described in section III-B. Now substituting the expression of H_k in equation 25, we get the expression of the coding gain as-

$$
C.G.(f(x), M) = \frac{\int_0^{2\pi} f(y)dy}{\left(\prod_{k=1}^{k=M}(M\int_{\frac{2\pi(k-1)}{M}}^{\frac{2\pi k}{M}} f(y)dy)\right)^{\frac{1}{M}}}
$$

We observe that the numerator of $CG(f(x), m)$ is independent of M (derived from the AM of the subband variances), while the denominator is the only part which depends on M. So, analyzing the denominator independently and taking a natural log to simplify it , it becomes-

$$g(M) = \frac{\sum_{k=1}^{M} ln(M \int_{\frac{2\pi(k-1)}{M}}^{\frac{2\pi k}{M}} f(y)dy)}{M}$$

Here $g(M)$ represents the natural log of the denominator of $C.G.(f(x), M)$. If $g(M)$ is monotonically decreasing, $C.G.(f(x), M)$ will be monotonically decreasing wrt to M. So, for simplicity , we analyze the function $g(M)$ in place of $C.G.(f(x), M)$.

Now, to analyze the trend of variation of $g(M)$ with M, we see that M being a discrete variable makes this kind of analysis difficult. Our attempts to analyze $g(M)$ with a discrete M were not very fruitful, one of the reasons being lack of good enough approximations for the integrals in $g(M)$. Instead, we propose to analyse a continuous extension of $g(M)$ called $g(x)$ defined as-

$$g(x) = \frac{\sum_{k=1}^{ceil(x)} ln(x \int_{\frac{2\pi(k-1)}{M}}^{min(\frac{2\pi k}{M}, 2\pi} f(y)dy) z_k}{x}$$

$$where\ z_k = \begin{cases} = 1 & k = [1, floor(x)],\ k \in \mathbb{Z} \\ = (x - floor(x)) & k = ceil(x) \end{cases}$$

This continuous extension is motivated by the fact that arithmetic mean can be replace by the idea of weighted mean, where the weights of individual terms can be non integers. We observe that $g(x)$ is continuous at integer points, as can be seen by the values of the right and left hand limits at integer points M-

$$Left\ hand\ limit-$$

$$\lim_{x \to M^-} g(x) = \frac{\sum_{k=1}^{M} ln(x \int_{\frac{2\pi(k-1)}{x}}^{min(\frac{2\pi k}{x}, M)} f(y)dy)}{x}$$

$$\lim_{x \to M^-} g(x) = \frac{\sum_{k=1}^{M} ln(M \int_{\frac{2\pi(k-1)}{x}}^{min(\frac{2\pi k}{x}, M)} f(y)dy)}{M} = g(M)$$

$$Right\ hand\ limit-$$

$$\lim_{x \to M^+} g(x) = \frac{\sum_{k=1}^{M} ln(x \int_{\frac{2\pi(k-1)}{x}}^{min(\frac{2\pi k}{x}, M)} f(y)dy)}{x}$$

$$\lim_{x \to M^+} g(x) = \frac{\sum_{k=1}^{M} ln(M \int_{\frac{2\pi(k-1)}{x}}^{min(\frac{2\pi k}{x}, M)} f(y)dy) + (x - M) ln(M \int_{\frac{2\pi(M)}{x}}^{2\pi} f(y)dy)}{M}$$

$$\lim_{x \to M^+} g(x) = \frac{\sum_{k=1}^{M} ln(M \int_{\frac{2\pi(k-1)}{x}}^{\frac{2\pi k}{M}} f(y)dy) + 0}{M} = g(M)$$

So, $g(x)$ is a continuous extension of $g(M)$, with continuity as proved above at integer M. We can evaluate the derivative of $g(x)$ and try to analyze it, to reveal insights into $g(M)$. If $g(x)$ is monotonically decreasing, then g(M) is also monotonically decreasing. The derivative of $g(x)$ is calculated as-

1) for the case $x \notin Z$

$$g'(x) = \frac{x(g'_{num}(x)) - g_{num}(x)}{x^2}$$

where $g_{num}(x)$ represents the numerator of $g(x)$. Now $g'_{num}(x)$ can be given as -

$$\sum_{k=1}^{floor(x)} \frac{1}{xI_k} \times \left(x \left(f\left(\frac{2\pi k}{x}\right) \times -\frac{2\pi k}{x^2} - f\left(\frac{2\pi(k-1)}{x}\right) \times -\frac{2\pi(k-1)}{x^2} \right) + I_k \right)$$
$$+(x - floor(x))\frac{1}{xI_k} \times \left(x \left(-f\left(\frac{2\pi(ceil(x)-1)}{x}\right) \times -\frac{2\pi(ceil(x)-1)}{x^2} + I_k \right) \right)$$
$$+ln(x \int_{\frac{2\pi(ceil(x)-1)}{x}}^{2\pi} f(y)dy)$$

2) for case $x \in Z$

In such a case, we observe that ceil(x) and floor(x) are discontinuous and non differentiable at integer points, so we must manually evaluate the derivative for these components of the derivative. We first calculate the left hand derivative if $g(x)$ in such a case-

$$g'(x) = \frac{x(g'_{num}(x)) - g_{num}(x)}{x^2}$$

where $g'_{num}(x)$ in this case is given by -

$$g'_{num}(x) = \sum_{k=1}^{x-1} \frac{1}{xI_k} \times \left(x \left(f\left(\frac{2\pi k}{x}\right) \times -\frac{2\pi k}{x^2} - f\left(\frac{2\pi(k-1)}{x}\right) \times -\frac{2\pi(k-1)}{x^2} \right) + I_k \right)$$
$$+\frac{1}{xI_k} \times \left(x \left(-f\left(\frac{2\pi(x-1)}{x}\right) \times -\frac{2\pi(x-1)}{x^2} \right) + I_k \right)$$
$$+ln(x \int_{\frac{2\pi(x-1)}{x}}^{2\pi} f(y)dy)$$

In the case of the right hand derivative, $g'_{num}(x)$ is given by the expression (the form of the rest of the derivative being the same as the left hand derivative) -

$$g'_{num}(x) = \sum_{k=1}^{x-1} \frac{1}{xI_k} \times (x \left(f(\frac{2\pi k}{x}) \times -\frac{2\pi k}{x^2} - f(\frac{2\pi(k-1)}{x}) \times -\frac{2\pi(k-1)}{x^2} \right) + I_k)$$
$$+\frac{1}{xI_k} \times \left(x(f(2\pi) \times \frac{2\pi}{x} - f(\frac{2\pi(x-1)}{x}) \times -\frac{2\pi(x-1)}{x^2}) + I_k) \right)$$
$$+ln(x \int_{\frac{2\pi(x-1)}{x}}^{2\pi} f(y)dy)$$

The Right hand and left hand derivatives are unequal and differ by 1 term. The right hand derivative is smaller than the left hand derivative. So, it suffices to require that the left hand derivative and the the derivative at non integral points is negative for $g(x)$ to be decreasing. Also, the left hand derivative shares it's functional form with the derivative at other points. So, now in g'_x we observe that

- The denominator $= x^2$ will not affect the sign.
- Using property IV-A1 , scaling the PSD will not change the trend of the coding gain.
- We need to make the summation in $g'_{num}(x)$ negative, so we impose the following condition on f-

$$f(\frac{2\pi k}{x})\frac{2\pi}{x} > f(\frac{2\pi(k-1)}{x})\frac{2\pi(k-1)}{x}$$

for all $k \in Z, [0, floor(x)]$ This condition IV-C is satisfied by any PSD $f(x)$ which also satisfies -

$$\begin{aligned}(xf(x))' > 0 \\ => f(x) + xf'(x) > 0\end{aligned} \quad (27)$$

- We also want the rest of the terms in the numerator of $g(x)$ to add up to a negative value, imposing which we get a condition as follows-

$$x - \sum_{k=1}^{ceil(x)} ln(\frac{\int_{\frac{2\pi(k-1)}{x}}^{min(\frac{2\pi k}{x},2\pi)} f(y)dy}{\int_{\frac{2\pi floor(x)}{x}}^{2\pi} f(y)dy}) \times z_k$$
$$+\frac{x(x-floor(x))}{I_{ceil(x)}} f(\frac{2\pi(ceil(x)-1)}{x}) \frac{2\pi(ceil(x)-1)}{x^2} - \frac{x}{I_1} f(\frac{2\pi}{x}) \frac{2\pi}{x^2}$$

D. Proof of monotonicity for sinusoidal PSDs

We will first prove the monotnonicty of coding gain of the PSD, $S_{xx} = sin(\omega/2)$, for $\omega = 0$ to $\omega = 2\pi$. As derived before,

$$\begin{aligned} CG &= \frac{AM}{GM} \\ &= \frac{1}{2M\, sin\left(\frac{\pi}{4M}\right)\left(\prod_{k=0}^{M-1} sin\left(\frac{(2k+1)\pi}{4M}\right)\right)^{\frac{1}{M}}} \end{aligned} \tag{28}$$

$$CG = \frac{1}{2M\, sin\left(\frac{\pi}{4M}\right)\left(\prod_{k=0}^{M-1} sin\left(\frac{(2k+1)\pi}{4M}\right)\right)^{\frac{1}{M}}} \tag{29}$$

Let,

$$D = 2M\, sin\left(\frac{\pi}{4M}\right)\left(\prod_{k=0}^{M-1} sin\left(\frac{(2k+1)\pi}{4M}\right)\right)^{\frac{1}{M}} \tag{30}$$

and now let,

$$P = \prod_{k=0}^{M-1} sin\left(\frac{(2k+1)\pi}{4M}\right) \tag{31}$$

$$\begin{aligned} &= \prod_{k=0}^{M-1}\left(\frac{e^{j\frac{(2k+1)\pi}{4M}} - e^{-j\frac{(2k+1)\pi}{4M}}}{2j}\right) \\ &= \left(\frac{1}{2j}\right)^M \left(\prod_{k=0}^{M-1} e^{j\frac{(2k+1)\pi}{4M}}\left(1-e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \\ &= \left(\frac{1}{2j}\right)^M \left(\prod_{k=0}^{M-1} e^{j\frac{(2k+1)\pi}{4M}}\right)\left(\prod_{k=0}^{M-1}\left(1-e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \end{aligned} \tag{32}$$

From equation (31) it is clear that P is real. P is also always positive because all the individual terms in the product are also positive (each term is $sine$ of an angle which is greater than 0 and less than $\frac{\pi}{2}$). Since, P is both real and positive, $P = |P|$, where $|\cdot|$ is the modulus operator. Therefore, from equation (32),

$$
\begin{aligned}
P = |P| &= \left| \left(\frac{1}{2j}\right)^M \left(\prod_{k=0}^{M-1} e^{j\frac{(2k+1)\pi}{4M}}\right) \left(\prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \right| \\
&= \left| \left(\frac{1}{2j}\right)^M \right| \left| \left(\prod_{k=0}^{M-1} e^{j\frac{(2k+1)\pi}{4M}}\right) \right| \left| \left(\prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \right| \\
&= \left(\frac{1}{2^M}\right) \left(\prod_{k=0}^{M-1} \left| e^{j\frac{(2k+1)\pi}{4M}} \right| \right) \left| \left(\prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \right| \\
&= \left(\frac{1}{2^M}\right) \left(\prod_{k=0}^{M-1} 1 \right) \left| \left(\prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \right| \\
&= \left(\frac{1}{2^M}\right) \left| \left(\prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)\right) \right|
\end{aligned} \tag{33}
$$

In equation (33) let, $A = \prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right)$, giving us,

$$
P = \left(\frac{1}{2^M}\right) |A| \tag{34}
$$

Now lets focus on A,

$$
\begin{aligned}
A &= \prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k+1)\pi}{2M}}\right) \\
&= \prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k)\pi}{2M}} e^{-j\frac{\pi}{2M}}\right) \\
&= \prod_{k=0}^{M-1} \left(1 - e^{-j\frac{(2k)\pi}{2M}} a\right) \quad where,\ a = e^{-j\frac{\pi}{2M}} \\
&= a^M \prod_{k=0}^{M-1} \left(\frac{1}{a} - e^{-j2\pi\left(\frac{k}{2M}\right)}\right)
\end{aligned} \tag{35}
$$

Take complex conjugate on both sides in equation (35),

$$
\bar{A} = (\bar{a})^M \prod_{k=0}^{M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)}\right) \tag{36}
$$

Now replace the product variable k in equation (35) by $k = 2M - (l+1)$. The limits on l become, from $l = M$ to $l = 2M - 1$. The equation ((35) now becomes,

$$
\begin{aligned}
A &= a^M \prod_{l=M}^{2M-1} \left(\frac{1}{a} - e^{-j2\pi\left(\frac{(2M-(l+1))}{2M}\right)} \right) \\
&= a^M \prod_{l=M}^{2M-1} \left(\frac{1}{a} - e^{-j2\pi\left(\frac{2M}{2M}\right)} e^{j2\pi\left(\frac{(l+1)}{2M}\right)} \right) \\
&= a^M \prod_{l=M}^{2M-1} \left(\frac{1}{a} - (1) e^{j2\pi\left(\frac{l}{2M}\right)} e^{j2\pi\left(\frac{1}{2M}\right)} \right) \\
&= a^M \prod_{l=M}^{2M-1} \left(\frac{1}{a} - e^{j2\pi\left(\frac{l}{2M}\right)} \frac{1}{a^2} \right) \qquad \left(\because e^{j2\pi\left(\frac{1}{2M}\right)} = \frac{1}{a^2} \right) \\
&= a^M \left(\frac{1}{a^2} \right)^M \left[\prod_{l=M}^{2M-1} \left(a - e^{j2\pi\left(\frac{l}{2M}\right)} \right) \right] \qquad \left(taking\ \frac{1}{a^2}\ common\ from\ each\ term \right) \\
&= \left(\frac{1}{a} \right)^M \left[\prod_{l=M}^{2M-1} \left(a - e^{j2\pi\left(\frac{l}{2M}\right)} \right) \right] \\
&= (\bar{a})^M \left[\prod_{l=M}^{2M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{l}{2M}\right)} \right) \right] \qquad \left(\because \frac{1}{a} = \bar{a} \right) \\
&= (\bar{a})^M \left[\prod_{k=M}^{2M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \qquad (replacing\ the\ product\ variable\ l\ by\ k) \qquad (37)
\end{aligned}
$$

Now, from the theory of complex numbers, $|A|^2 = A\bar{A}$. Substituting the expressions of A (from equation (37)) and $\bar{A}$ (from equation (36)) we get,

$$
\begin{aligned}
|A|^2 &= A\bar{A} \\
&= \left((\bar{a})^M \left[\prod_{k=M}^{2M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \right) \left((\bar{a})^M \left[\prod_{k=0}^{M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \right) \\
&= (\bar{a})^{2M} \left[\prod_{k=M}^{2M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \left[\prod_{k=0}^{M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \\
&= (\bar{a})^{2M} \left[\prod_{k=0}^{2M-1} \left(\frac{1}{(\bar{a})} - e^{j2\pi\left(\frac{k}{2M}\right)} \right) \right] \qquad (38)
\end{aligned}
$$

Now, as a slight detour, consider the n^{th} roots of unity α_0, α_1,..., α_{n-1}, given by,

$$
\alpha_k = e^{j2\pi\frac{k}{n}} \qquad where,\ 0 \le k \le n-1,\ k \in \mathbb{Z}
$$

Therefore,

$$
\begin{aligned}
(x^n - 1) &= (x-\alpha_0)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1}) \\
&= \prod_{k=0}^{n-1} (x - \alpha_k) \qquad (39)
\end{aligned}
$$

Now looking at equation (38) substitute $\frac{x}{(\bar{a})}$ in place of x in equation (39),

$$\begin{aligned}\left(\left(\frac{x}{(\bar{a})}\right)^n - 1\right) &= \left(\frac{x}{(\bar{a})} - \alpha_0\right)\left(\frac{x}{(\bar{a})} - \alpha_1\right)\left(\frac{x}{(\bar{a})} - \alpha_2\right)...\left(\frac{x}{(\bar{a})} - \alpha_{n-1}\right) \\ &= \prod_{k=0}^{n-1}\left(\frac{x}{(\bar{a})} - \alpha_k\right) \end{aligned} \tag{40}$$

Now, to solve the product term in equation (38) put $x = 1$ in equation (40),

$$\left(\left(\frac{1}{(\bar{a})}\right)^n - 1\right) = \prod_{k=0}^{n-1}\left(\frac{1}{(\bar{a})} - \alpha_k\right) \tag{41}$$

The product term in equation (38) is exactly the same as equation (41) when $n = 2M$, $i.e.$, when we are considering $(2M)^{th}$ roots of unity. Substituting the $L.H.S$ from equation (41) in equation (38),

$$\begin{aligned}|A|^2 &= (\bar{a})^{2M}\left(\left(\frac{1}{(\bar{a})}\right)^{2M} - 1\right) \\ &= (\bar{a})^{2M}\left(\frac{1}{(\bar{a})^{2M}} - 1\right) \\ &= \left(1 - (\bar{a})^{2M}\right) \\ &= \left(1 - \left(e^{j\frac{\pi}{2M}}\right)^{2M}\right) \quad (\because a = e^{-j\frac{\pi}{2M}}) \\ &= \left(1 - \left(e^{j\pi}\right)\right) \\ &= (1 - (-1)) \\ &= 2 \\ \therefore |A| &= \sqrt{2} \quad (\because |A| \; is \; positive) \end{aligned} \tag{42}$$

From equation (34),

$$\begin{aligned}P &= \left(\frac{1}{2^M}\right)|A| \\ &= \left(\frac{1}{2^M}\right)(\sqrt{2}) \quad (from \; equation \; (42)) \\ \therefore P &= \frac{\sqrt{2}}{2^M} \end{aligned} \tag{43}$$

From equation (30) and (31),

$$
\begin{aligned}
D &= 2M\,sin\left(\frac{\pi}{4M}\right)(P)^{\frac{1}{M}} \\
&= 2M\,sin\left(\frac{\pi}{4M}\right)\left(\frac{\sqrt{2}}{2^M}\right)^{\frac{1}{M}} \qquad (from\ equation\ (33)) \\
&= 2M\,sin\left(\frac{\pi}{4M}\right)\left(\frac{\sqrt{2}}{2^M}\right)^{\frac{1}{M}} \\
&= M\,2^{\frac{1}{2M}}sin\left(\frac{\pi}{4M}\right) \qquad (44)
\end{aligned}
$$

We will now prove the monotonically decreasing nature of D with an increase in M which would imply a monotonically increasing nature of CG with an increase in M, from equations (29) and (30).

Consider a function $f(x)$ as,

$$f(x) = x\,2^{\frac{1}{2x}}sin\left(\frac{\pi}{4x}\right)\ ,\ x \in \mathbb{R} \qquad (45)$$

If we can prove that $f(x)$ is a decreasing function $\forall\ x \geq 1$, then we can conclusively say that the same trend will be observed for D with an increase in M, $M \in \mathbb{Z}$, $M \geq 1$. We thus try to prove $\frac{d(f(x))}{dx} < 0\ \forall\ x \geq 1$, $x \in \mathbb{R}$.
See $f(x)$ is positive $\forall\ x \geq 1$, and so we can take ln on both sides in equation (45),

$$
\begin{aligned}
ln(f(x)) &= ln\left(x\,2^{\frac{1}{2x}}sin\left(\frac{\pi}{4x}\right)\right) \\
&= ln(x) + \frac{1}{2x}ln(2) + ln\left(sin\left(\frac{\pi}{4x}\right)\right) \qquad (46)
\end{aligned}
$$

Differentiate equation (46),

$$
\begin{aligned}
\left(\frac{1}{f(x)}\right)\frac{d(f(x))}{dx} &= \frac{1}{x} - \frac{1}{2x^2}ln(2) + \frac{1}{\left(sin\left(\frac{\pi}{4x}\right)\right)}\cdot cos\left(\frac{\pi}{4x}\right)\cdot\left(-\frac{\pi}{4x^2}\right) \\
\frac{d(f(x))}{dx} &= f(x)\left(\frac{1}{x} - \frac{ln(2)}{2x^2} + \frac{1}{\left(sin\left(\frac{\pi}{4x}\right)\right)}\cdot cos\left(\frac{\pi}{4x}\right)\cdot\left(-\frac{\pi}{4x^2}\right)\right) \\
&= 2^{\frac{1}{2x}}\left(sin\left(\frac{\pi}{4x}\right) - \frac{ln(2)}{2x}sin\left(\frac{\pi}{4x}\right) - \frac{\pi}{4x}cos\left(\frac{\pi}{4x}\right)\right) \\
&= 2^{\frac{1}{2x}}g(x) \quad where,\ g(x) = \left(sin\left(\frac{\pi}{4x}\right) - \frac{ln(2)}{2x}sin\left(\frac{\pi}{4x}\right) - \frac{\pi}{4x}cos\left(\frac{\pi}{4x}\right)\right) \qquad (47)
\end{aligned}
$$

We know $2^{\frac{1}{2x}}$ is always positive, so to prove $\frac{d(f(x))}{dx} < 0\ \forall\ x \geq 1$, $x \in \mathbb{R}$ we will need to prove that $g(x) < 0\ \forall\ x \geq 1$, $x \in \mathbb{R}$.
We see,

$$
\begin{aligned}
g(1) &= \frac{1}{\sqrt{2}} - \frac{ln(2)}{2}\frac{1}{\sqrt{2}} - \frac{\pi}{4}\frac{1}{\sqrt{2}} < 0 \quad and, \qquad (48) \\
\lim_{x\to\infty} g(x) &= 0 \qquad (49)
\end{aligned}
$$

Looking at equation (48) and (49) if we can prove that the nature of $g(x)$ is monotonically increasing $\forall$ $x \geq 1$, it can be concluded that $g(x) < 0 \ \forall \ x \geq 1$, which would then mean that $\frac{d(f(x))}{dx} < 0 \ \forall \ x \geq 1$, which is what we set out to prove.
$\therefore$ To prove $g(x)$ is monotonically increasing $\forall \ x \geq 1$:
Consider $\frac{d(g(x))}{dx}$,

$$\begin{aligned}\frac{d(g(x))}{dx} &= \frac{ln(2)}{2x^2} sin\left(\frac{\pi}{4x}\right) + \frac{\pi ln(2)}{8x^3} cos\left(\frac{\pi}{4x}\right) - \frac{\pi^2}{16x^3} sin\left(\frac{\pi}{4x}\right) \\ &= \frac{1}{2x^2}\left(ln(2) - \frac{\pi^2}{8x}\right) sin\left(\frac{\pi}{4x}\right) + \frac{\pi ln(2)}{8x^3} cos\left(\frac{\pi}{4x}\right) \end{aligned} \tag{50}$$

Now for $g(x)$ to be monotonically increasing $\forall \ x \geq 1$, $\frac{d(g(x))}{dx} > 0 \ \forall \ x \geq 1$. To prove the same consider the two terms in equation (50). Let,

$$\begin{aligned} T_1 &= \frac{1}{2x^2}\left(ln(2) - \frac{\pi^2}{8x}\right) sin\left(\frac{\pi}{4x}\right) \\ T_2 &= \frac{\pi ln(2)}{8x^3} cos\left(\frac{\pi}{4x}\right) \end{aligned}$$

We see that $T_2 > 0 \ \forall \ x \geq 1$, while T_1 can be $\geq$ or < 0

- Case 1: $T_1 \geq 0$
  When $T_1 \geq 0$, then $T_1 + T_2$, $i.e.$, $\frac{d(g(x))}{dx} > 0$. Let's see when this happens.

$$\begin{aligned} T_1 &\geq 0 \\ \frac{1}{2x^2}\left(ln(2) - \frac{\pi^2}{8x}\right) sin\left(\frac{\pi}{4x}\right) &\geq 0 \\ \therefore ln(2) &\geq \frac{\pi^2}{8x} \quad \left(\because \frac{1}{x^2} \ and \ sin\left(\frac{\pi}{4x}\right) are \ > 0 \ for \ x \geq 1\right) \\ \therefore x &\geq \frac{\pi^2}{8ln(2)} = 1.77985... \end{aligned} \tag{51}$$

  Therefore $\forall \ x \geq \frac{\pi^2}{8ln(2)}$, $\frac{d(g(x))}{dx} > 0$.

- Case 2: $T_1 < 0$
  From Case 1 we see that $T_1 < 0$ when $x < \frac{\pi^2}{8ln(2)}$. Our considerations are anyway for $x \geq 1$, so we need to find out what is happening in $1 \leq x < \frac{\pi^2}{8ln(2)}$.
  For $T_1 + T_2$ to be positive, $|T_1|$ should be less than $|T_2|$. Let's compare T_1 and T_2 term by term: for $1 \leq x < \frac{\pi^2}{8ln(2)}$, $sin\left(\frac{\pi}{4x}\right) \geq cos\left(\frac{\pi}{4x}\right)$ and each of $sin\left(\frac{\pi}{4x}\right)$ and $cos\left(\frac{\pi}{4x}\right)$ are positive. Therefore, what we need is, $\left|\frac{1}{2x^2}\left(ln(2) - \frac{\pi^2}{8x}\right)\right| < \left|\frac{\pi ln(2)}{8x^3}\right|$. Let's investigate when this would happen,

$$\begin{aligned}
\left|\frac{1}{2x^2}\left(ln(2)-\frac{\pi^2}{8x}\right)\right| &< \left|\frac{\pi ln(2)}{8x^3}\right| \\
\frac{1}{2x^2}\left(\frac{\pi^2}{8x}-ln(2)\right) &< \frac{\pi ln(2)}{8x^3} \\
x > \frac{\left(\frac{\pi^2}{2}-\pi ln(2)\right)}{4ln(2)} &= 0.9944...
\end{aligned} \tag{52}$$

Therefore, from equation (52) we can conclusively say that in $1 \leq x < \frac{\pi^2}{8ln(2)}$, $T_1 + T_2 > 0$

Combining both the cases $\frac{d(g(x))}{dx} = T_1 + T_2 > 0$, for $x \geq 1$, which is what we wanted to prove. This shows that $f(x)$ is a monotonically decreasing function $\forall\ x \geq 1$, $x \in \mathbb{R}$. This would mean the same trend will be followed for D in equation (35), because D is just an integer (≥ 1) sampled version of $f(x)$. Now if D is monotonically decreasing function for increasing M, it would imply that the coding gain CG is a monotonically increasing function with increasing M, because CG is just the inverse of CG from equations (29) and (30).
$\therefore$ the CG of $sin(\frac{\omega}{2})$ is monotonically increasing with M.
Now, we can extend this idea to any sinusoid of the form $sin(\frac{\omega}{2}+\phi)$, where ϕ is a phase shift in $[0, 2\pi]$ using the property IV-A2. Hence any such sinusoid will have it's coding gain increase with M.
This proof can be easily extended to $|sin(\omega)|$. This consists of two lobes instead of the one in $|sin(\frac{\omega}{2})|$. However, in this case we need to consider the case of M being even and odd separately. In both cases we end up with similar product of sinusoids, which can solve exactly the way it has been done in the case of $sin(\frac{\omega}{2})$. This can then be extended to $|sin(\omega+\phi)|$ by property IV-A2.
Hence proved. ■

E. Analysis of coding gain for resource constrained scenario with simulations

In this section we discuss effect of resources and number of subbands constraints on achieved coding gain for 4 different PSDs. We run simulations in MATLAB implementing the AEVD technique for designing realistic optimal filters. The main goal of performing these experiments is to find trends if any in the achieved coding by varying the number of subbands and the resources. Fig 8 - Fig 11 show the PSDs used in the simulation. These PSDs correspond to Auto Regressive(AR) processes with 4 different parameter settings, in all of which we use 4 poles to define the AR process. Fig 12 - Fig 19 then plot the CG achieved vs McMillan (resources) and #subbands. It can be inferred from the plots that increasing both the resources and #subbands leads to increase in the achieved coding gain. The first result that increasing resources leads to increases in the coding gain is quite intuitive. the second result is bit more surprising as we first hypothesised that increasing the bands keeping the number of resources constant will first increase and then decrease after a peak point. But the simulations show that's not the case. We also increased the subbands to 30 and 100 and observed that the difference between the achieved coding gain and max achievable coding gain kept decreasing. There are some instances where we do see the coding gain drop by increasing the #subbands as in Fig 13 and Fig 19, but in general we see that the trend is of an increasing nature across all PSDs. It is import to note here that we only ran simulations for four randomly picked AR processes PSDs, more experiments can be done taking different PSDs to observe the natire of coding gain. A theoretical proof for any AR process PSD is still needed to explain the observations.But the experiments performed do present some interesting results.

F. Analysis of PCFBs formed for some special classes of PSDs (midsem)

Here we summarize the results we obtained for the 3 different PSDs classes, Beta, Gaussian and Cosine. We first construct the optimal filters as described in [2] and then find the expression of variance per band

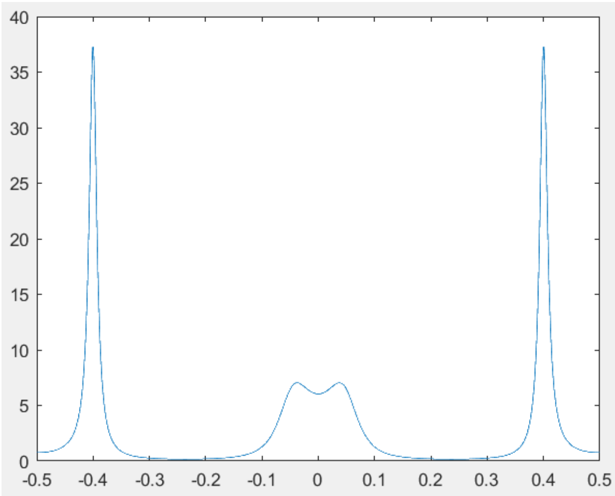

Fig. 8. PSD 1 used in experiments in Section D

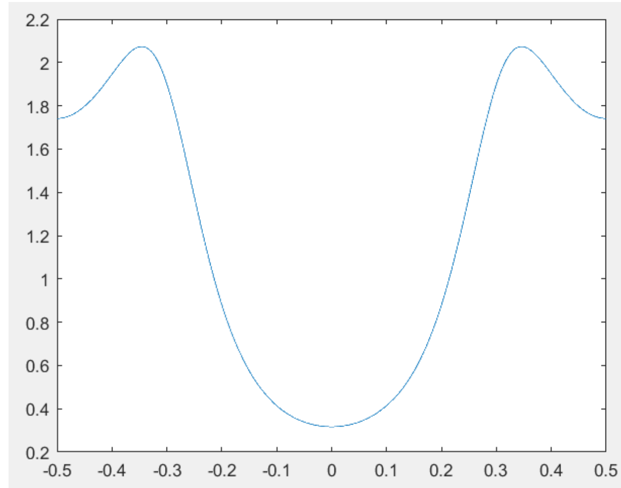

Fig. 9. PSD 2 used in experiments in Section D

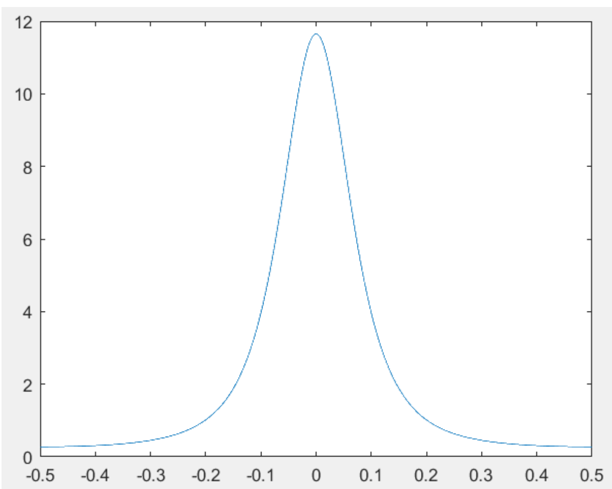

Fig. 10. PSD 3 used in experiments in Section D

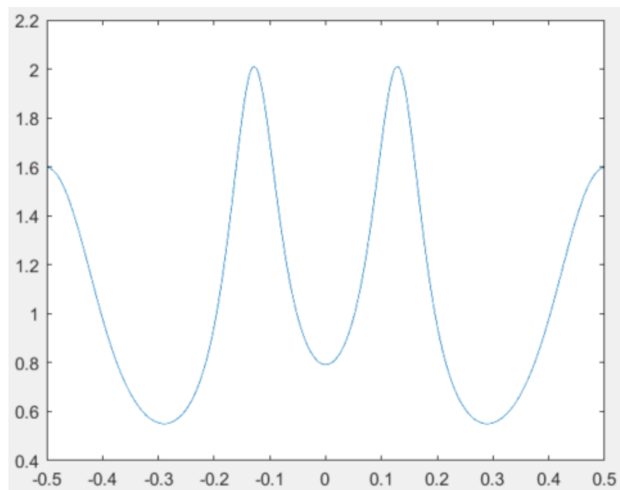

Fig. 11. PSD 4 used in experiments in Section D

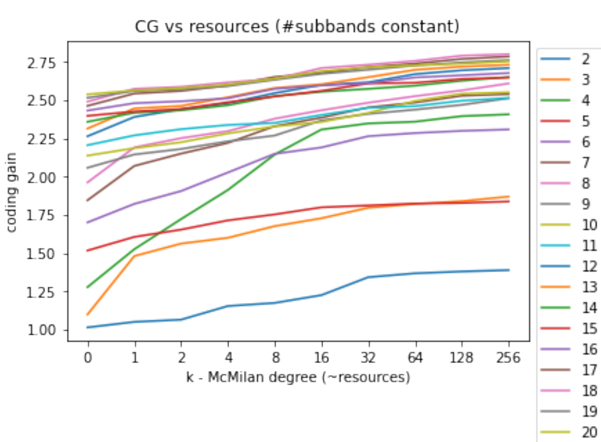

Fig. 12. PSD 1: Coding gain vs McMillan Degree(resourses)

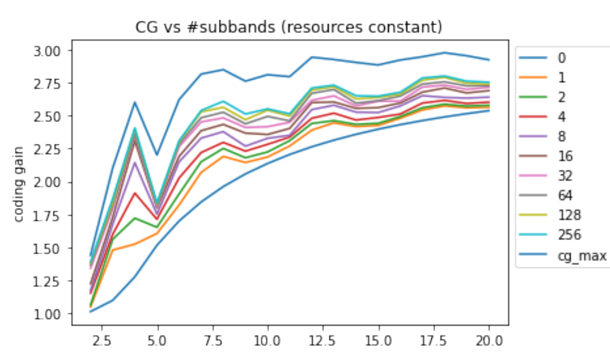

Fig. 13. PSD 1: Coding gain vs #subbands used

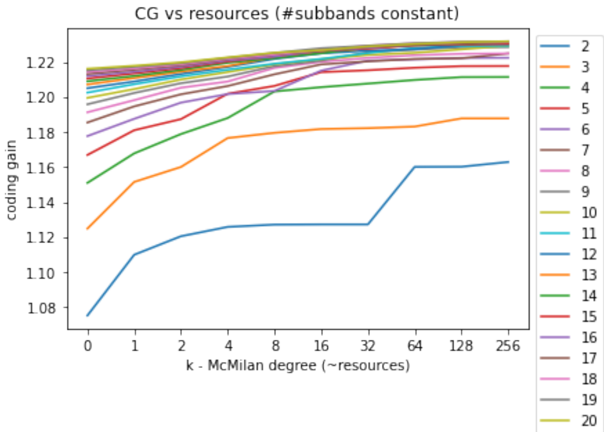

Fig. 14. PSD 2: Coding gain vs McMillan Degree(resourses)

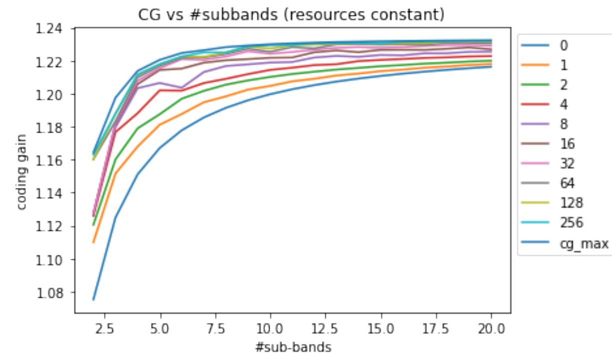

Fig. 15. PSD 2: Coding gain vs #subbands used

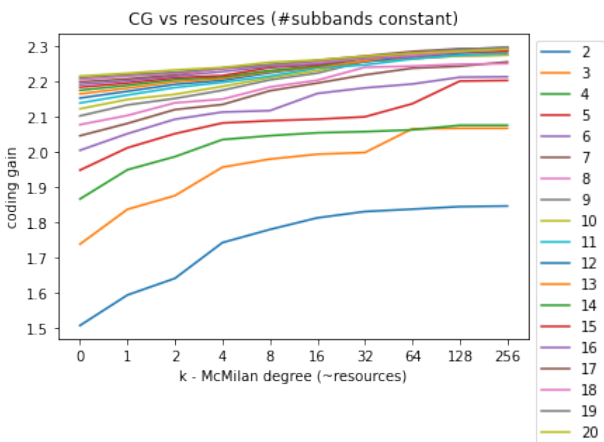

Fig. 16. PSD 3: Coding gain vs McMillan Degree(resourses)

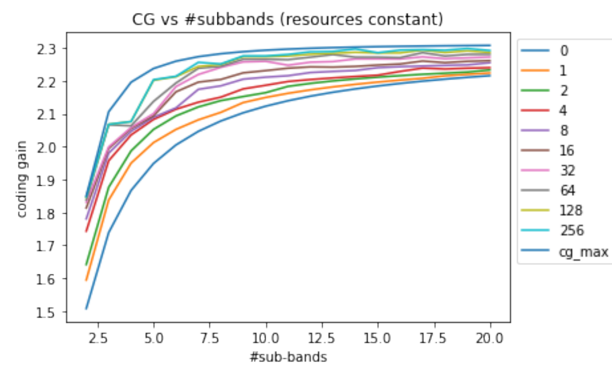

Fig. 17. PSD 3: Coding gain vs #subbands used

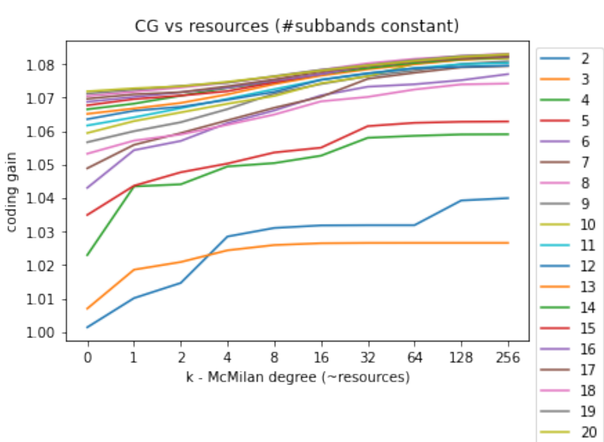

Fig. 18. PSD 4: Coding gain vs McMillan Degree(resources)

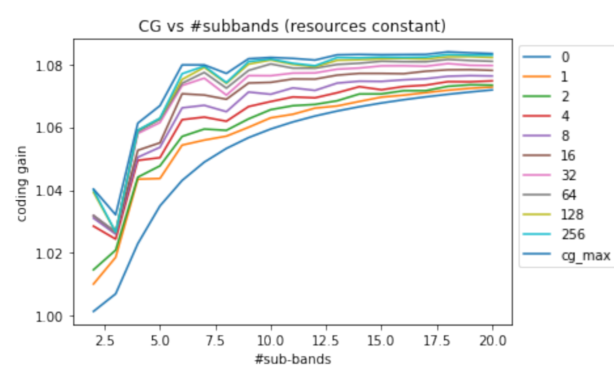

Fig. 19. PSD 4: Coding gain vs #subbands used

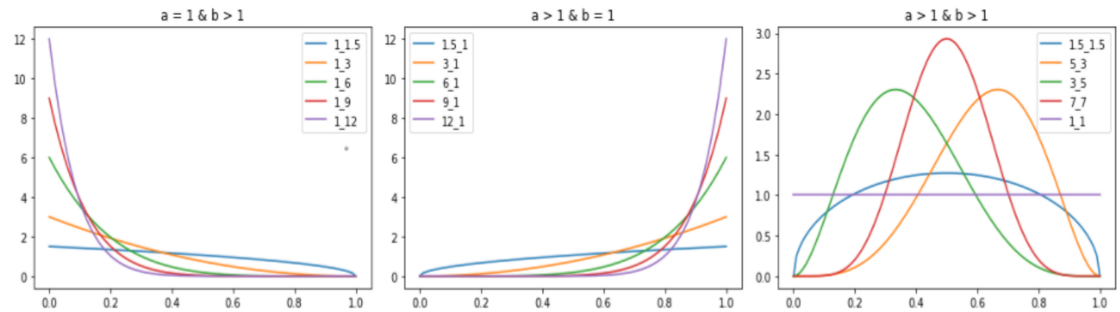

Fig. 20. Beta Distribution for different parameters

for each class. These variance values are then used to derive the coding gain. We have empirically shown that the derived expressions for coding gain in fact increases monotonically with increase in the number of channels. The expressions for variance(1) and coding gain(2) are as follows:

$$\begin{aligned} \sigma_k^2 &= \frac{1}{2\pi}\int_0^{2\pi} S_{xx}\, H_k^2\, d\omega \\ CG &= \frac{AM(\,\sigma_k^2\,)}{GM(\,\sigma_k^2\,)} = \frac{1}{M}\frac{\sum_{k=1}^{M}\sigma_k^2}{\left[\prod_{k=1}^{M}\sigma_k^2\right]^{\frac{1}{M}}} \end{aligned} \tag{53}$$

where S_{xx}, H_k^2 are the signal PSD and the filter response respectively, m is the number of channels

1) Beta PSD: In this section we derive the coding gain expression for the beta distribution. Before we start the derivation lets first have a look at the beta distribution expression and check for any assumptions that need to be made here.

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad where \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{54}$$

We can see that beta is a parametric distribution parametrized by a, b. In this work we only consider PSDs that are uni-modal and finite for which will we have to place some constraints on a and b. After applying these constrains we are left with four possible combinations of a and b as shown in Fig 20

- Uni-modality : a and b cannot be smaller than 0 simultaneously
- Finite : a and $b \geq 1$

So now we have four cases:

- Case A : $a = 1, b = 1$
- Case B : $a = 1, b \geq 1$
- Case C : $a \geq 1, b = 1$
- Case D : $a \geq 1, b \geq 1$

Here we will be deriving the equation for case B,C and a special sub-case for D ($a = b$), case A being trivial is not discussed. Now that the setup is done we start with the derivations. We have the signal PSD as define below:

$$S_{xx} = \frac{\left[\frac{\omega}{2\pi}\right]^{a-1}\left[1-\frac{\omega}{2\pi}\right]^{b-1}}{B(a,b)} \tag{55}$$

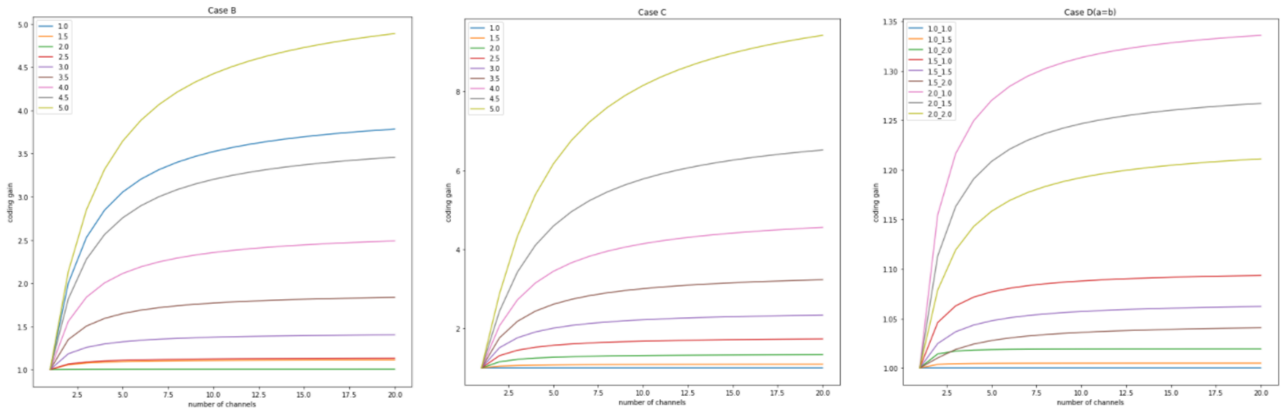

Fig. 21. Coding gain plots for Case B, C and D for different parameter values

We obtain the following equations for coding gain in the three cases using the expression for PSD , steps of deriving the filter as given in theory and equation for subband variance.
Case B:

$$CG_{B(M)} = \frac{1}{M\left[\prod_{k=0}^{M-1}(1-\frac{k}{M})^{b-1}-(1-\frac{k+1}{M})^{b-1}\right]^{\frac{1}{M}}} \tag{56}$$

Case C:

$$CG_{C(M)} = \frac{1}{M\left[\prod_{k=0}^{M-1}(\frac{M-k}{M})^{a}-(\frac{M-k-1}{M})^{a}\right]^{\frac{1}{M}}} \tag{57}$$

Case D(a==b):

$$CG_{D(a=b)(M)} = \frac{I_{\frac{1}{2}}(a,b)}{M\left[\prod_{k=0}^{M-1} I_{\frac{k+1}{2M}}(a,b)-I_{\frac{k}{2M}}(a,b)\right]^{\frac{1}{M}}} \tag{58}$$

Here $I_x(a,b)$ is the incomplete beta function which has the following properties.

$$I_x(a,b) = \int_0^x \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$$
$$I_x(1,b) = 1-(1-x)^{b-1}$$
$$I_x(a,1) = x^a$$
$$I_0(a,b) = 0 \qquad I_1(a,b) = 1$$

For all the 3 cases we show we empirically(Fig 21) that the coding gain increases monotonically with increase in number of channels as in Fig. 2. Also M being even or odd does not affect the calculations in any case.

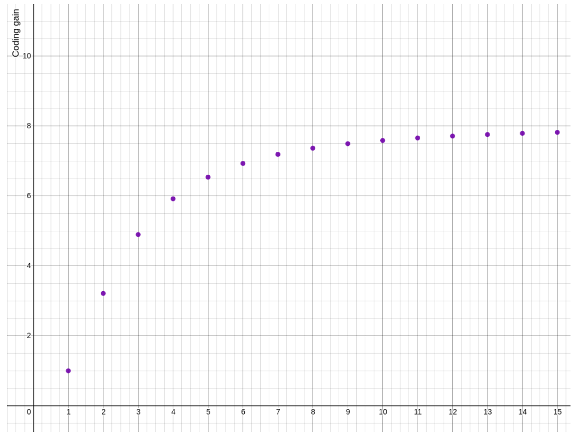

Fig. 22. A plot of the variation of coding gain vs the number of subbands M for an optimal PCFB filter designed for the Gaussian PSD

2) Gaussian PSD: In this section, we analyze the variation in coding gain for a PCFB with the number of subbands when the input PSD is a gaussian. A gaussian is uni-modal, but we will need to restrict it's domain between 0 and 2π for our analysis. Also, for ease of analysis, we take the mean of the gaussian $\mu_g = \pi$ without loss of generality. So, the expression for the input PSD is-

$$S_{xx} = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(\frac{x-\pi}{\sigma_g})^2}{2}}$$

Next, we construct the optimal filter for such an input PSD considering we have M subbands using the process described in theory. Then we obtain the following equations for coding gain using the expression for PSD and equation for subband variance.

$$CG_M = \frac{\frac{1}{2*\pi}\left(erf\left(\frac{\pi}{\sqrt{2}\sigma_g}\right)\right)}{\left(\prod_{k=1}^{M}\left(\frac{M}{2\pi}\left(erf\left(\frac{\pi(k+1)}{\sqrt{2}M\sigma_g}\right) - erf\left(\frac{\pi k}{\sqrt{2}M\sigma_g}\right)\right)\right)\right)^{\frac{1}{M}}} \tag{59}$$

Now, we empirically verify that this expression of coding gain is a monotonic function, by plotting it as a function of M for various values of σ_g. A plot obtained for a particular value of σ_g is shown in Figure 22

3) Cosine PSD: In this section, we analyze the variation in coding gain for a PCFB with the number of subbands when the input PSD is a cosine wave. To make this input PSD uni-modal in a frequency span of 2π we consider the curve $cos(\omega/2)$ for $\omega = -\pi$ to $\omega = \pi$. However, for consistency in our analysis we choose the domain of $\omega = 0$ to $\omega = 2\pi$ for the above mentioned PSD. Hence the input PSD is:

$$S_{xx} = |cos(\omega/2)|$$

Following equation is obtained for coding gain:

$$CG_M = \frac{1}{2M\, sin\left(\frac{\pi}{4M}\right)\left(\prod_{k=0}^{M-1} cos\left(\frac{(2k+1)\pi}{4M}\right)\right)^{\frac{1}{M}}} \tag{60}$$

We have empirically verified the monotonicity of the expression in above equation, by plotting CG_M as a function of M. This plot is shown in figure 23.

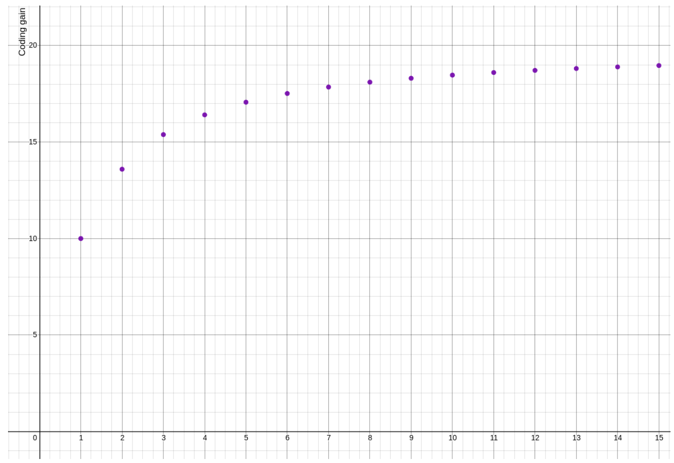

Fig. 23. A plot of the variation of 100 times coding gain vs the number of subbands M for an optimal PCFB filter designed for the Cosine PSD

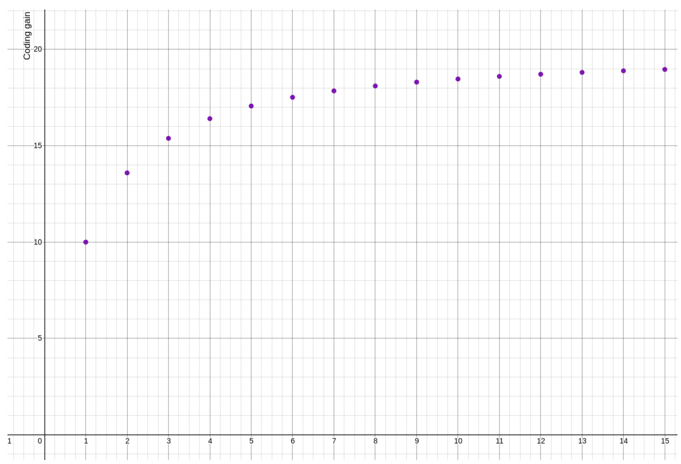

Fig. 24. A plot of the variation of 100 times coding gain vs the number of subbands M for an optimal PCFB filter designed for the Sine PSD

4) Sine PSD: In this section, we analyze the variation in coding gain for a PCFB with the number of subbands when the input PSD is a sine wave. To make this input PSD uni-modal in a frequency span of 2π we consider the curve $sin(\omega/2)$ for $\omega = 0$ to $\omega = 2\pi$. Hence the input PSD is:

$$S_{xx} = sin(\omega/2)$$

Following equation is obtained for coding gain:

$$CG_M = \frac{1}{2M\,sin\left(\frac{\pi}{4M}\right)\left(\prod_{k=0}^{M-1} sin\left(\frac{(2k+1)\pi}{4M}\right)\right)^{\frac{1}{M}}} \tag{61}$$

We have empirically verified the monotonicity of the expression in above equation, by plotting CG_M as a function of M. This plot is shown in figure 24.

V. Conclusion

In this work we present a detailed study analyzing the construction on optimal filter banks. We focus on designing filter banks using coding gain as a metric, explaining in detail the methods used to construct filter banks which maximize it. We then proceed to analyze properties of the coding gain of such filter banks, called Principal component filter banks under various conditions, including scaling, phase shifts, addition of constants and addition of different PSDs. Next we briefly summarize our attempts to derive sufficient conditions on a PSD which can enable the coding gain to be monotonic in terms of the number of subbands M, thus helping simplify the design choices made in such a filter bank. We provide a theoretical proof that the coding gain of a general sinusoid of the form $sin(\frac{\omega}{2} + \phi)$ and $|sin(\omega + \phi)|$ is monotonic to increase in the subbands, thereby providing a valuable insight into the design of PCFBs for one of the most basic PSDs. We also analyze the variation of coding gain under a resource constrained scenario via simulations, providing interesting trends, including the fact that an increase in subbands possibly has a dominant effect on the coding gain. We also analyse empirically the variation of coding gain with M of PCFBs designed for the beta, Gaussian and sinusoidal PSDs by constructing PCFBs for them.

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