

# 13. Reinforcement Learning

---

[Read Chapter 13]  
[Exercises 13.1, 13.2, 13.4]

- Control learning
- Control policies that choose optimal actions
- $Q$  learning
- Convergence

# Control Learning

---

Consider learning to choose actions, e.g.,

- Robot learning to dock on battery charger
- Learning to choose actions to optimize factory output
- Learning to play Backgammon

Note several problem characteristics:

- Delayed reward
- Opportunity for active exploration
- Possibility that state only partially observable
- Possible need to learn multiple tasks with same sensors/actuators

# One Example: TD-Gammon

---

[Tesauro, 1995]

Learn to play Backgammon

Immediate reward

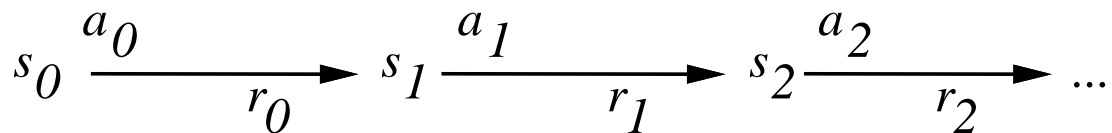
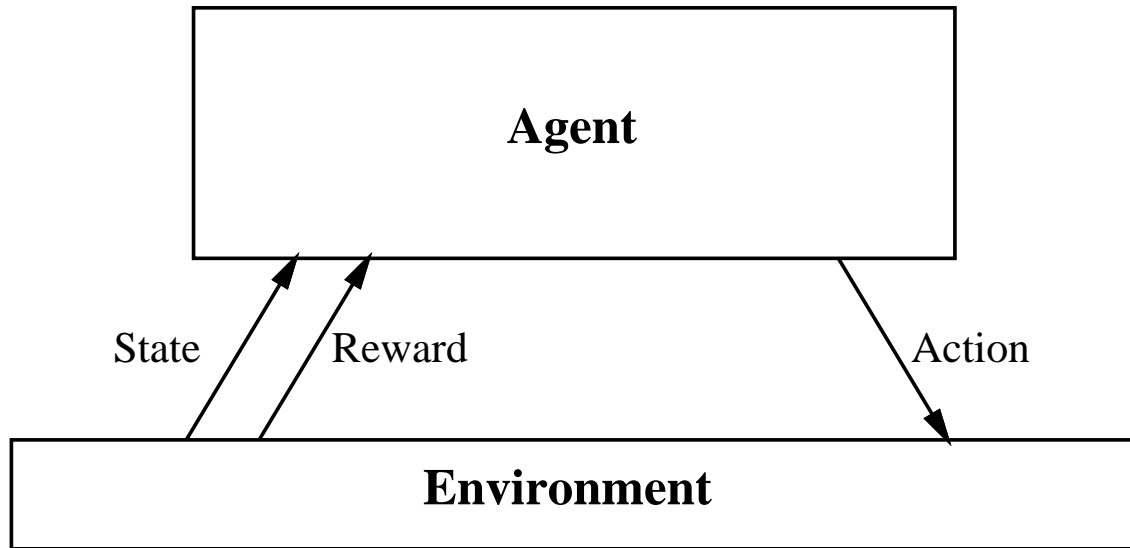
- +100 if win
- -100 if lose
- 0 for all other states

Trained by playing 1.5 million games against itself

Now approximately equal to best human player

# Reinforcement Learning Problem

---



Goal: Learn to choose actions that maximize

$$r_0 + \gamma r_1 + \gamma^2 r_2 + \dots, \text{ where } 0 \leq \gamma < 1$$

# Markov Decision Processes

---

Assume

- finite set of states  $S$
- set of actions  $A$
- at each discrete time agent observes state  $s_t \in S$  and chooses action  $a_t \in A$
- then receives immediate reward  $r_t$
- and state changes to  $s_{t+1}$
- Markov assumption:  $s_{t+1} = \delta(s_t, a_t)$  and  $r_t = r(s_t, a_t)$ 
  - i.e.,  $r_t$  and  $s_{t+1}$  depend only on *current* state and action
  - functions  $\delta$  and  $r$  may be nondeterministic
  - functions  $\delta$  and  $r$  not necessarily known to agent

# Agent's Learning Task

---

Execute actions in environment, observe results,  
and

- learn action policy  $\pi : S \rightarrow A$  that maximizes

$$E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \dots]$$

from any starting state in  $S$

- here  $0 \leq \gamma < 1$  is the discount factor for future rewards

Note something new:

- Target function is  $\pi : S \rightarrow A$
- but we have no training examples of form  $\langle s, a \rangle$
- training examples are of form  $\langle \langle s, a \rangle, r \rangle$

# Value Function

---

To begin, consider deterministic worlds...

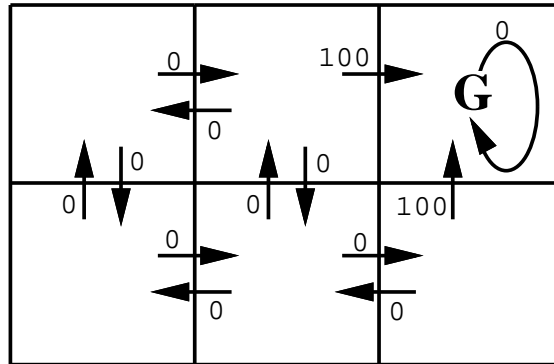
For each possible policy  $\pi$  the agent might adopt, we can define an evaluation function over states

$$\begin{aligned} V^\pi(s) &\equiv r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \dots \\ &\equiv \sum_{i=0}^{\infty} \gamma^i r_{t+i} \end{aligned}$$

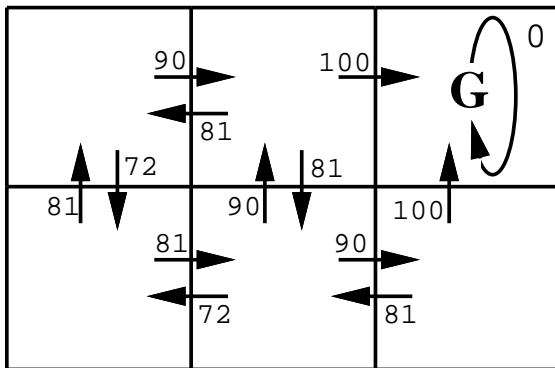
where  $r_t, r_{t+1}, \dots$  are generated by following policy  $\pi$  starting at state  $s$

Restated, the task is to learn the optimal policy  $\pi^*$

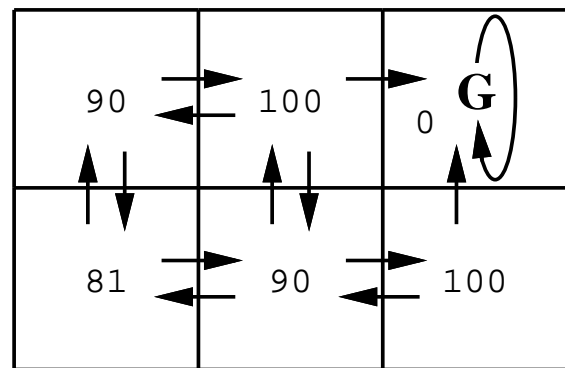
$$\pi^* \equiv \operatorname{argmax}_{\pi} V^\pi(s), (\forall s)$$



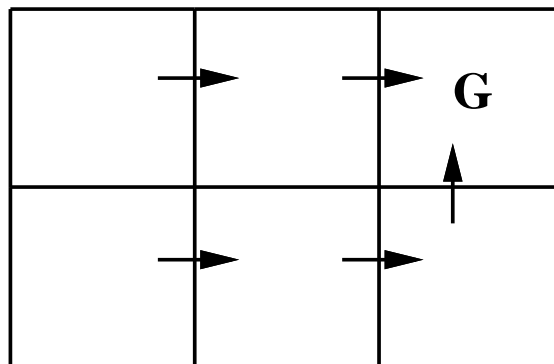
$r(s, a)$  (immediate reward) values



$Q(s, a)$  values



$V^*(s)$  values



One optimal policy



# What to Learn

---

We might try to have agent learn the evaluation function  $V^{\pi^*}$  (which we write as  $V^*$ )

It could then do a lookahead search to choose best action from any state  $s$  because

$$\pi^*(s) = \operatorname{argmax}_a [r(s, a) + \gamma V^*(\delta(s, a))]$$

A problem:

- This works well if agent knows  $\delta : S \times A \rightarrow S$ , and  $r : S \times A \rightarrow \mathbb{R}$
- But when it doesn't, it can't choose actions this way

## $Q$ Function

---

Define new function very similar to  $V^*$

$$Q(s, a) \equiv r(s, a) + \gamma V^*(\delta(s, a))$$

If agent learns  $Q$ , it can choose optimal action even without knowing  $\delta$ !

$$\pi^*(s) = \operatorname{argmax}_a [r(s, a) + \gamma V^*(\delta(s, a))]$$

$$\pi^*(s) = \operatorname{argmax}_a Q(s, a)$$

$Q$  is the evaluation function the agent will learn

# Training Rule to Learn $Q$

---

Note  $Q$  and  $V^*$  closely related:

$$V^*(s) = \max_{a'} Q(s, a')$$

Which allows us to write  $Q$  recursively as

$$\begin{aligned} Q(s_t, a_t) &= r(s_t, a_t) + \gamma V^*(\delta(s_t, a_t)) \\ &= r(s_t, a_t) + \gamma \max_{a'} Q(s_{t+1}, a') \end{aligned}$$

Nice! Let  $\hat{Q}$  denote learner's current approximation to  $Q$ . Consider training rule

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

where  $s'$  is the state resulting from applying action  $a$  in state  $s$

# $Q$ Learning for Deterministic Worlds

---

For each  $s, a$  initialize table entry  $\hat{Q}(s, a) \leftarrow 0$

Observe current state  $s$

Do forever:

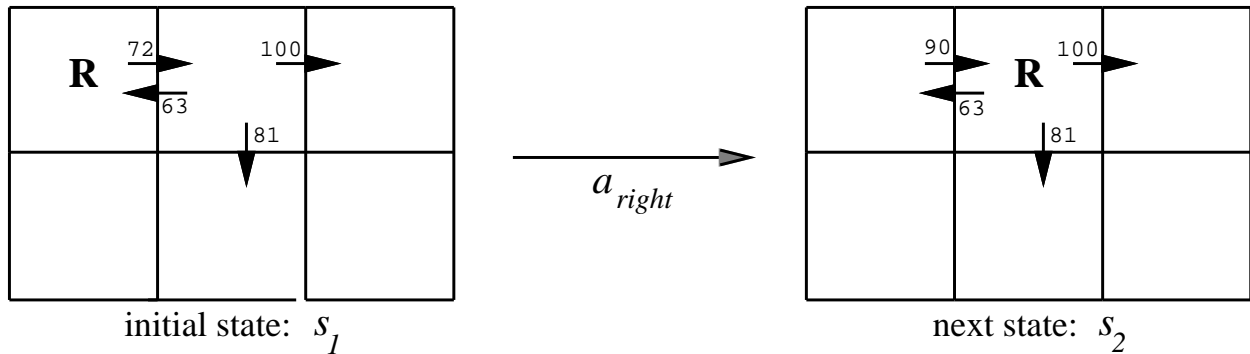
- Select an action  $a$  and execute it
- Receive immediate reward  $r$
- Observe the new state  $s'$
- Update the table entry for  $\hat{Q}(s, a)$  as follows:

$$\hat{Q}(s, a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s', a')$$

- $s \leftarrow s'$

# Updating $\hat{Q}$

---



$$\begin{aligned}
 \hat{Q}(s_1, a_{right}) &\leftarrow r + \gamma \max_{a'} \hat{Q}(s_2, a') \\
 &\leftarrow 0 + 0.9 \max\{63, 81, 100\} \\
 &\leftarrow 90
 \end{aligned}$$

notice if rewards non-negative, then

$$(\forall s, a, n) \quad \hat{Q}_{n+1}(s, a) \geq \hat{Q}_n(s, a)$$

and

$$(\forall s, a, n) \quad 0 \leq \hat{Q}_n(s, a) \leq Q(s, a)$$

$\hat{Q}$  converges to  $Q$ . Consider case of deterministic world where see each  $\langle s, a \rangle$  visited infinitely often.

*Proof:* Define a full interval to be an interval during which each  $\langle s, a \rangle$  is visited. During each full interval the largest error in  $\hat{Q}$  table is reduced by factor of  $\gamma$

Let  $\hat{Q}_n$  be table after  $n$  updates, and  $\Delta_n$  be the maximum error in  $\hat{Q}_n$ ; that is

$$\Delta_n = \max_{s,a} |\hat{Q}_n(s, a) - Q(s, a)|$$

For any table entry  $\hat{Q}_n(s, a)$  updated on iteration  $n + 1$ , the error in the revised estimate  $\hat{Q}_{n+1}(s, a)$  is

$$\begin{aligned} |\hat{Q}_{n+1}(s, a) - Q(s, a)| &= |(r + \gamma \max_{a'} \hat{Q}_n(s', a')) \\ &\quad - (r + \gamma \max_{a'} Q(s', a'))| \\ &= \gamma |\max_{a'} \hat{Q}_n(s', a') - \max_{a'} Q(s', a')| \\ &\leq \gamma \max_{a'} |\hat{Q}_n(s', a') - Q(s', a')| \\ &\leq \gamma \max_{s'', a'} |\hat{Q}_n(s'', a') - Q(s'', a')| \\ |\hat{Q}_{n+1}(s, a) - Q(s, a)| &\leq \gamma \Delta_n \end{aligned}$$

Note we used general fact that

$$|\max_a f_1(a) - \max_a f_2(a)| \leq \max_a |f_1(a) - f_2(a)|$$

# Nondeterministic Case

---

What if reward and next state are non-deterministic?

We redefine  $V, Q$  by taking expected values

$$\begin{aligned} V^\pi(s) &\equiv E[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \dots] \\ &\equiv E\left[\sum_{i=0}^{\infty} \gamma^i r_{t+i}\right] \end{aligned}$$

$$Q(s, a) \equiv E[r(s, a) + \gamma V^*(\delta(s, a))]$$



# Nondeterministic Case

---

$Q$  learning generalizes to nondeterministic worlds

Alter training rule to

$$\hat{Q}_n(s, a) \leftarrow (1 - \alpha_n)\hat{Q}_{n-1}(s, a) + \alpha_n[r + \max_{a'} \hat{Q}_{n-1}(s', a')]$$

where

$$\alpha_n = \frac{1}{1 + \text{visits}_n(s, a)}$$

Can still prove convergence of  $\hat{Q}$  to  $Q$  [Watkins and Dayan, 1992]

# Temporal Difference Learning

---

$Q$  learning: reduce discrepancy between successive  $Q$  estimates

One step time difference:

$$Q^{(1)}(s_t, a_t) \equiv r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

Why not two steps?

$$Q^{(2)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \gamma^2 \max_a \hat{Q}(s_{t+2}, a)$$

Or  $n$ ?

$$Q^{(n)}(s_t, a_t) \equiv r_t + \gamma r_{t+1} + \cdots + \gamma^{(n-1)} r_{t+n-1} + \gamma^n \max_a \hat{Q}(s_{t+n}, a)$$

Blend all of these:

$$Q^\lambda(s_t, a_t) \equiv (1-\lambda) \left[ Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \cdots \right]$$

# Temporal Difference Learning

---

$$Q^\lambda(s_t, a_t) \equiv (1-\lambda) [Q^{(1)}(s_t, a_t) + \lambda Q^{(2)}(s_t, a_t) + \lambda^2 Q^{(3)}(s_t, a_t) + \dots]$$

Equivalent expression:

$$Q^\lambda(s_t, a_t) = r_t + \gamma [ (1 - \lambda) \max_a \hat{Q}(s_t, a) + \lambda Q^\lambda(s_{t+1}, a_{t+1}) ]$$

TD( $\lambda$ ) algorithm uses above training rule

- Sometimes converges faster than  $Q$  learning
- converges for learning  $V^*$  for any  $0 \leq \lambda \leq 1$  (Dayan, 1992)
- Tesauro's TD-Gammon uses this algorithm

# Subtleties and Ongoing Research

---

- Replace  $\hat{Q}$  table with neural net or other generalizer
- Handle case where state only partially observable
- Design optimal exploration strategies
- Extend to continuous action, state
- Learn and use  $\hat{\delta} : S \times A \rightarrow S$
- Relationship to dynamic programming