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Data Structures and Algorithms
CS F211

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Pilani Campus, Pilani

Agenda: Red-Black Trees

Red-Black Trees

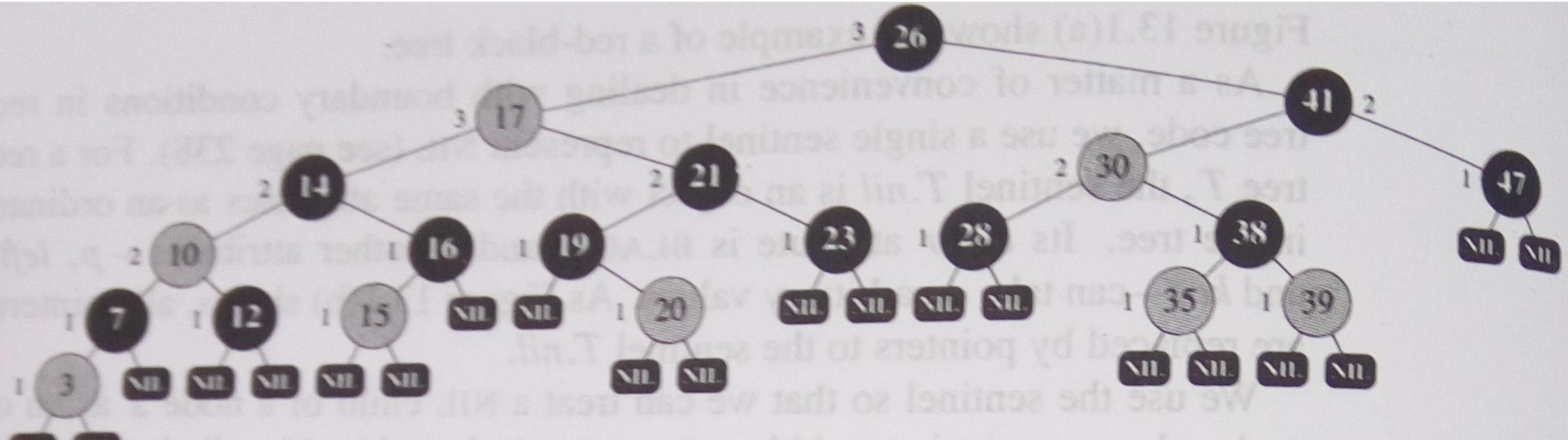
- *Red-black trees*:
  - Binary search trees augmented with node color
  - Operations designed to guarantee that the height $h = O(\lg n)$

Red-Black Properties

- The *red-black properties*:
  1. Every node is either red or black
  2. Every leaf (NULL pointer) is black
     - Note: this means every “real” node has 2 children
  3. If a node is red, both children are black
     - Note: can’t have 2 consecutive reds on a path
  4. Every path from node to descendent leaf contains the same number of black nodes
  5. The root is always black

Red-Black Tree

- Each node has the following attributes:
  - Color
  - Key
  - Left child
  - Right child
  - Parent
- If a child of the parent of a node does not exist, the corresponding pointer attribute of the node contains the value NIL.
- We shall regard these NIL’s as being pointers to leaves (external nodes) of the BST and the normal, key bearing nodes as being internal nodes of the tree.
- By constraining the node colors on any simple path from the root to a leaf, red black trees ensure that no such path is more than twice as long as any other path.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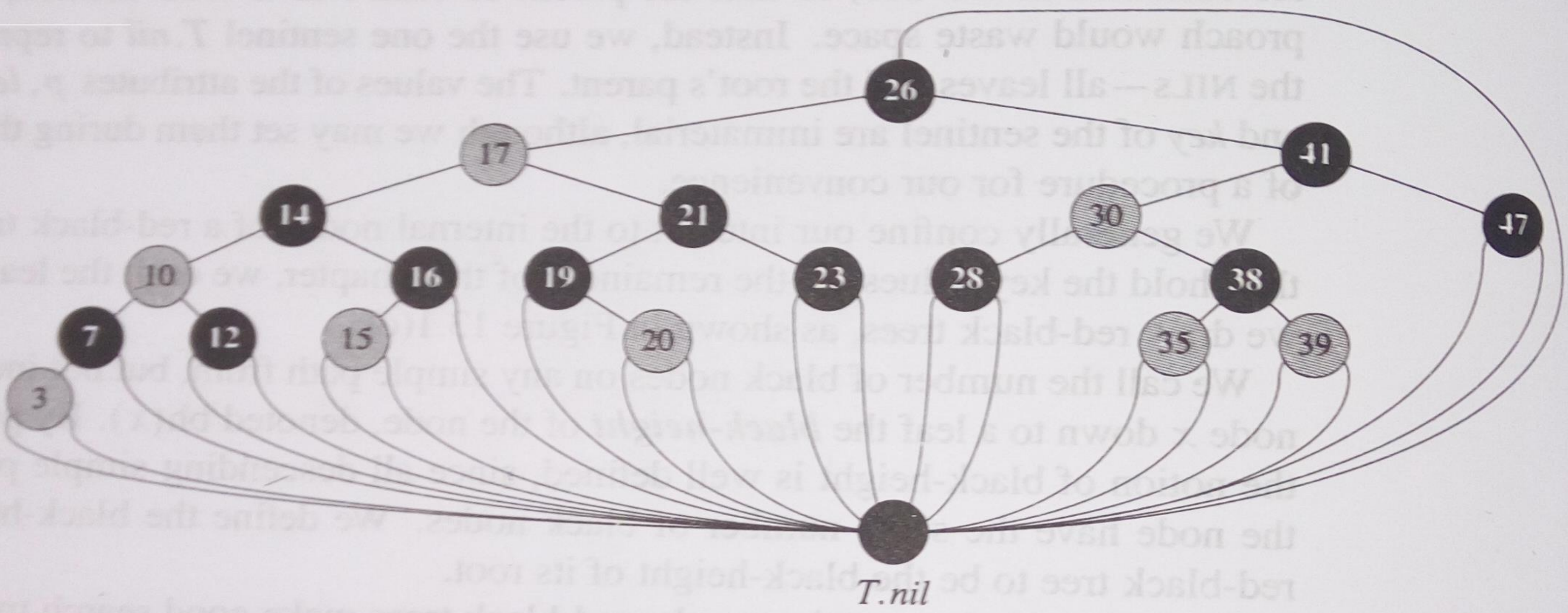
NIL
(a)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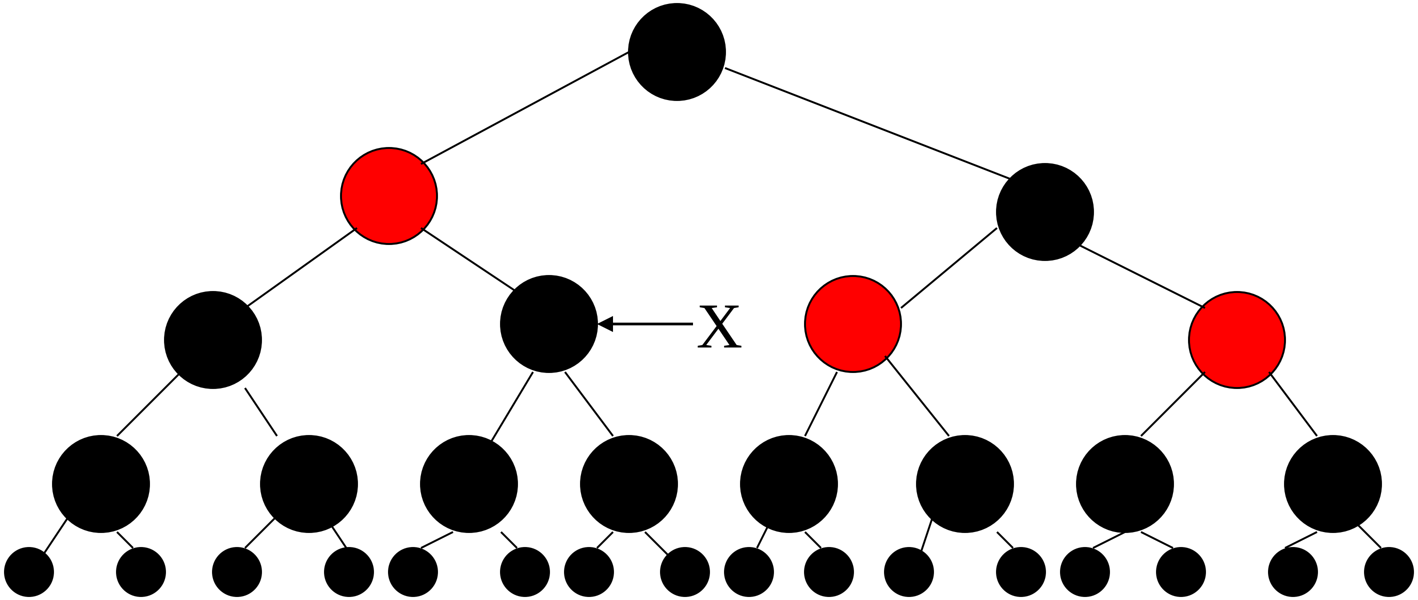
T.nil
(b)

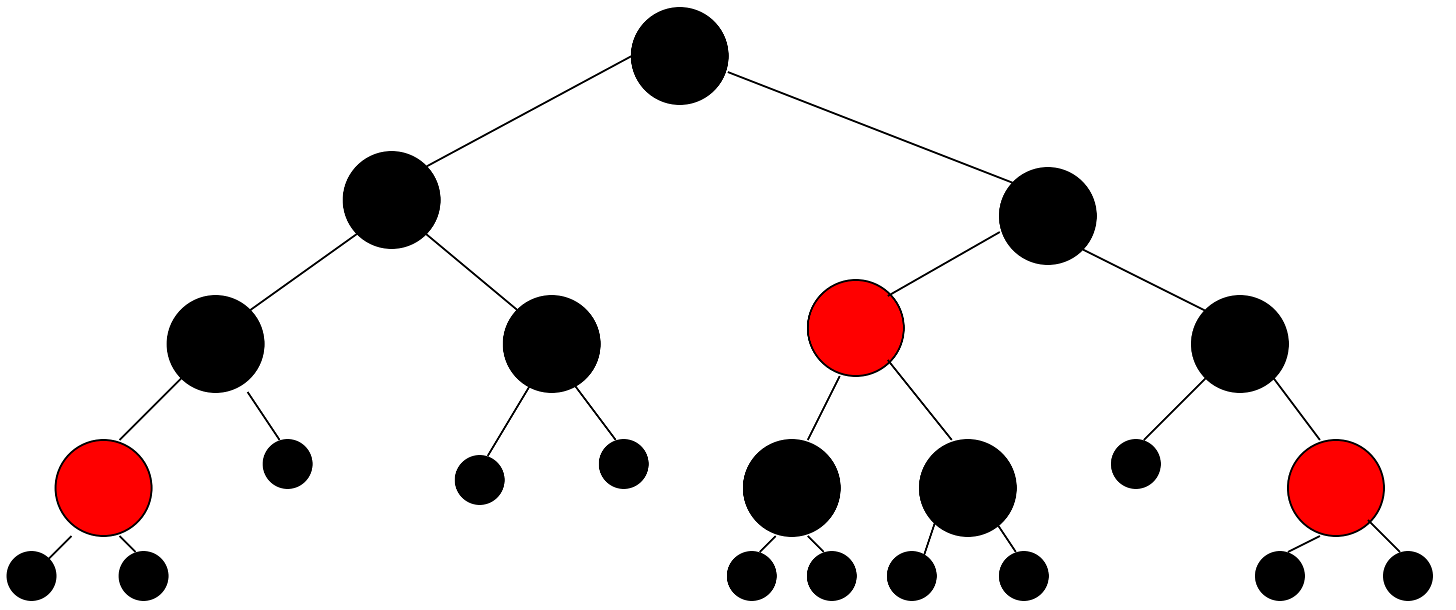
Black-Height

- ***Black-height:*** The black-height of a node, X, in a red-black tree is the number of Black nodes on any path to a NULL (or leaf), not counting X.



A Red-Black Tree with NULLs shown

Black-Height of the tree (the root) = 3
Black-Height of node “X” = 2

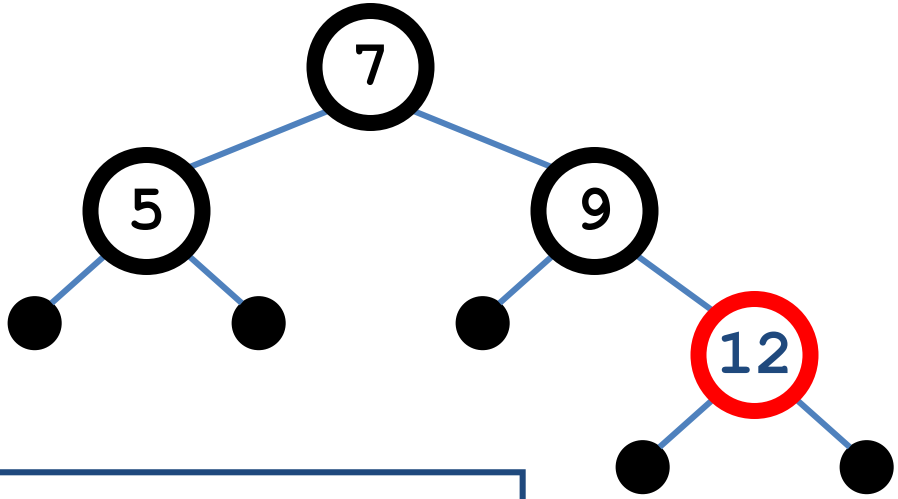


A Red-Black Tree with

Black-Height = _____

Red-Black Trees: An Example

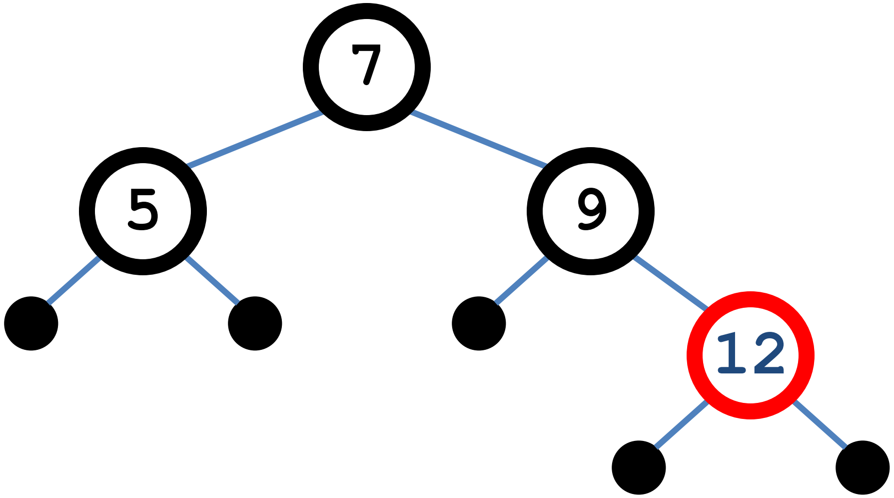
- *Color this tree:*



Red-black properties:
1. Every node is either red or black
2. Every leaf (NULL pointer) is black
3. If a node is red, both children are black
4. Every path from node to descendent leaf contains the same number of black nodes
5. The root is always black

Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*



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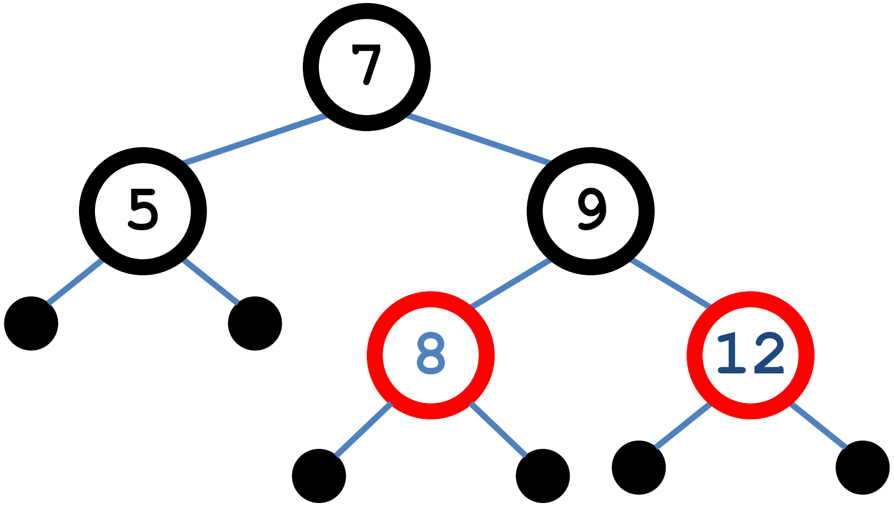
Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*
  - *What color should it be?*



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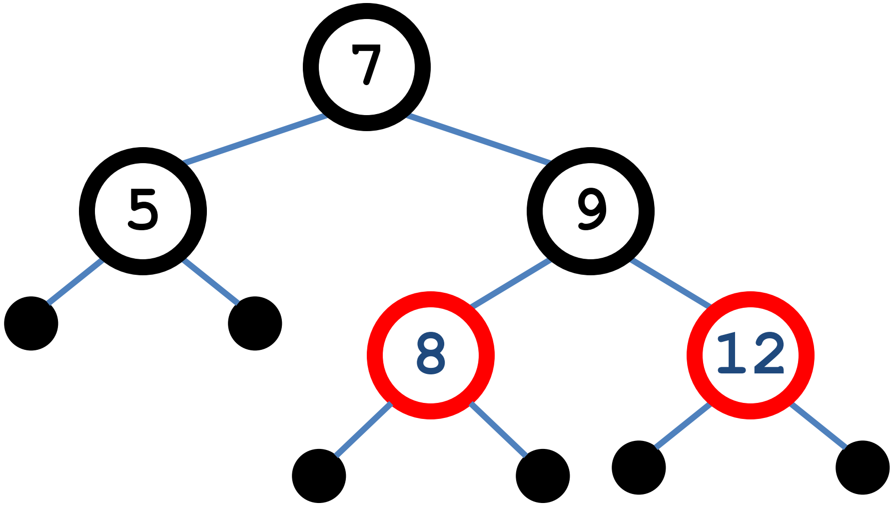

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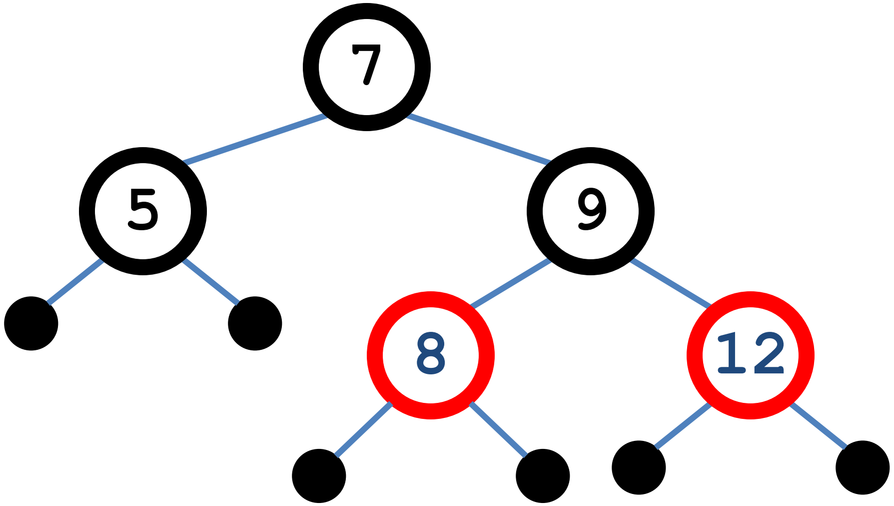
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*



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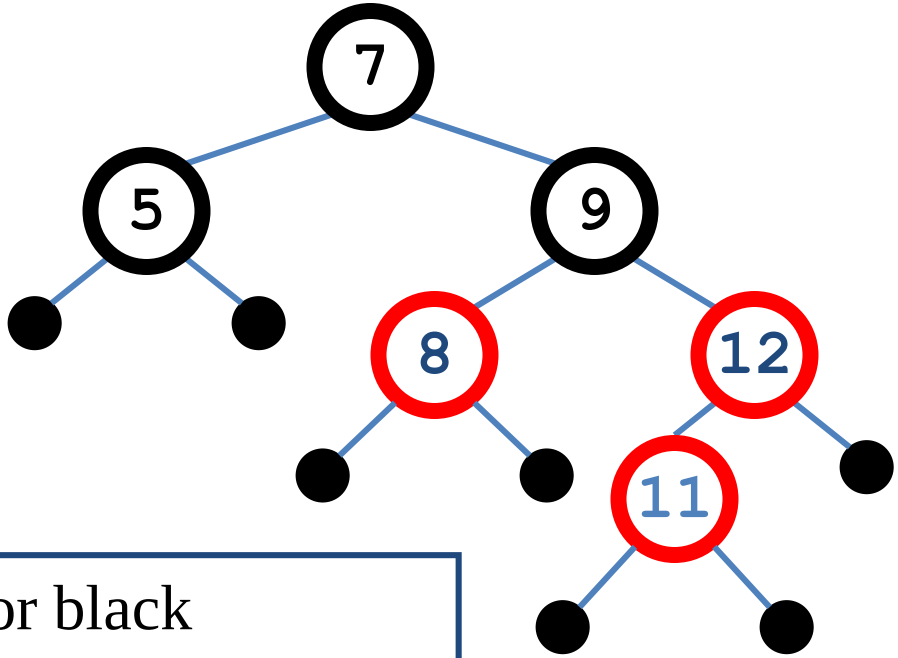
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*



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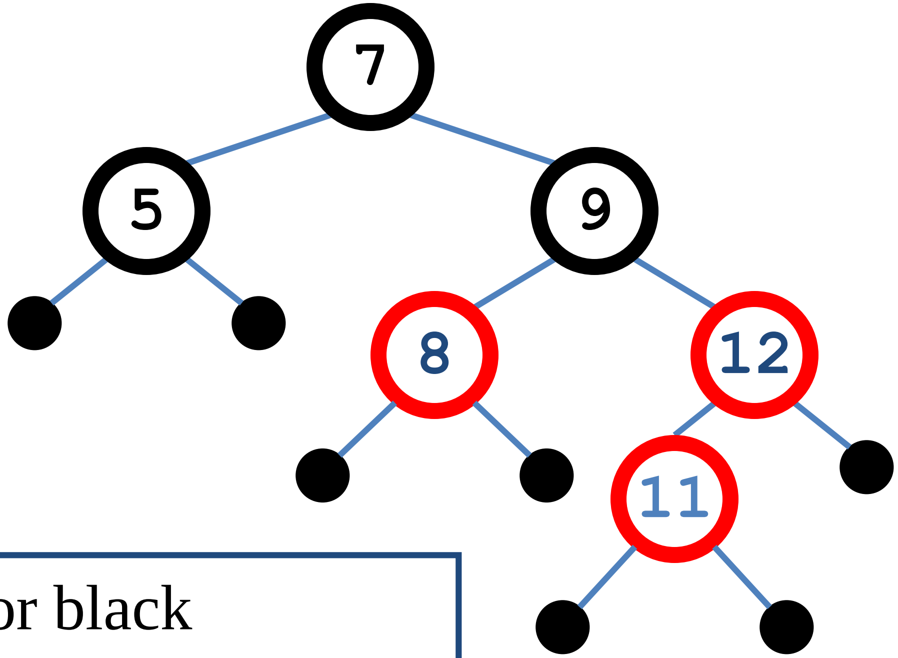
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#3)



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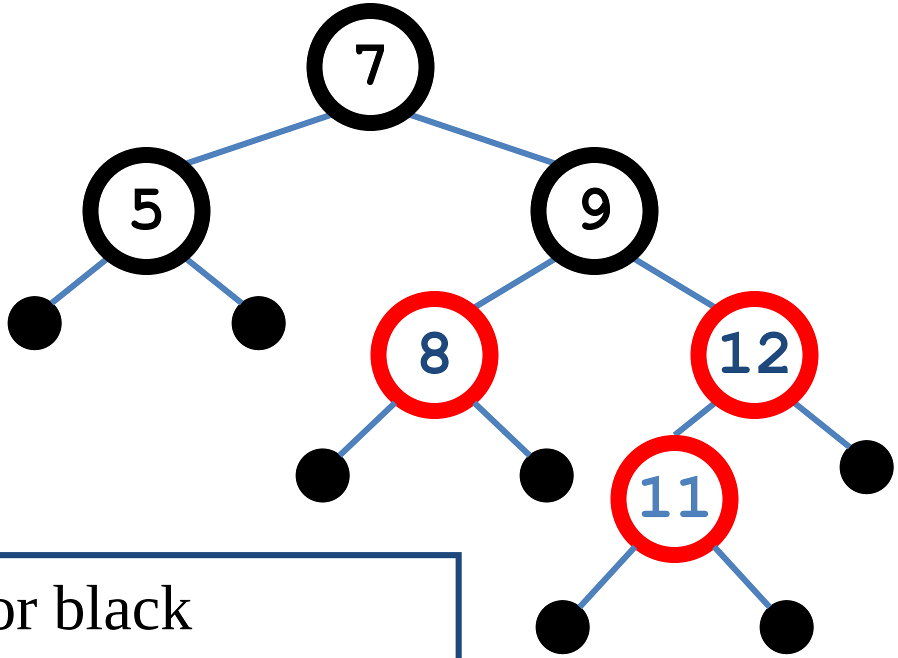
5. The root is always black

Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#3)
    - Can't be black! (#4)



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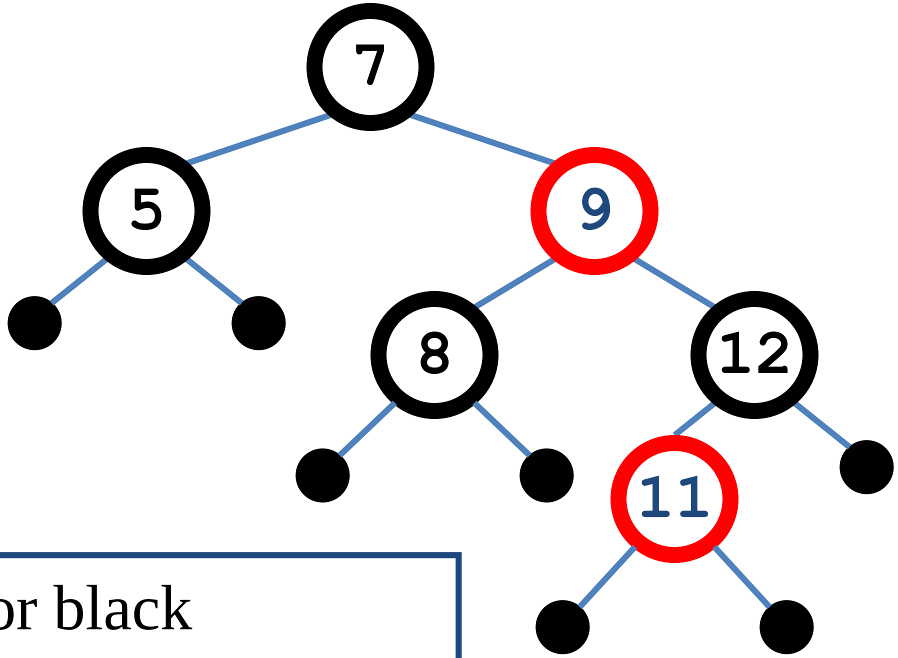
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Solution: recolor the tree



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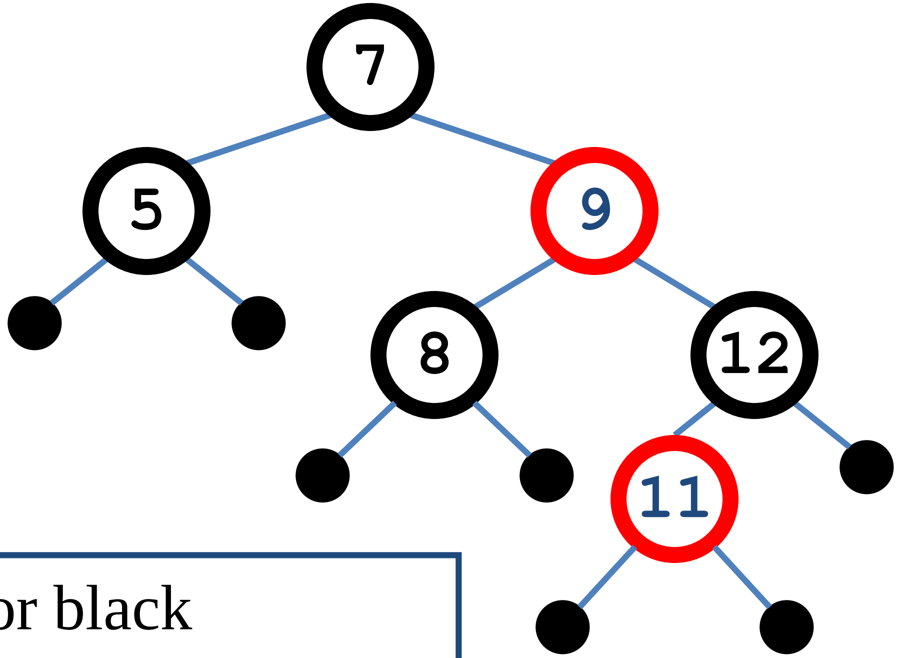
3. If a node is red, both children are black
4. Every path from node to descendent leaf contains the same number of black nodes
5. The root is always black

Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*



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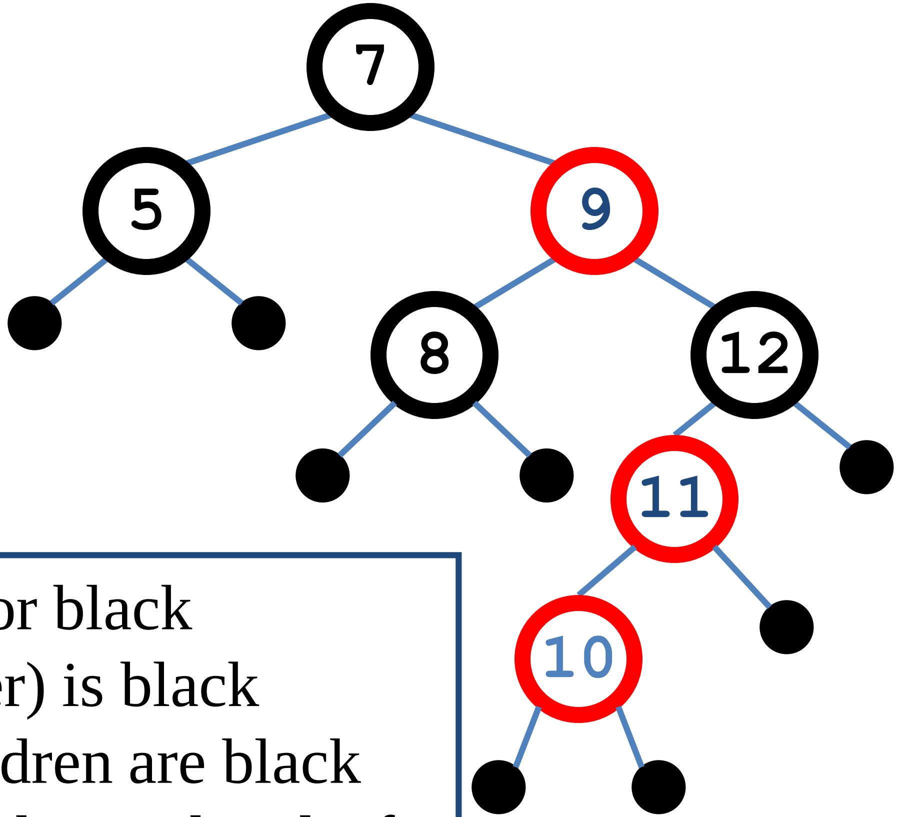
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*



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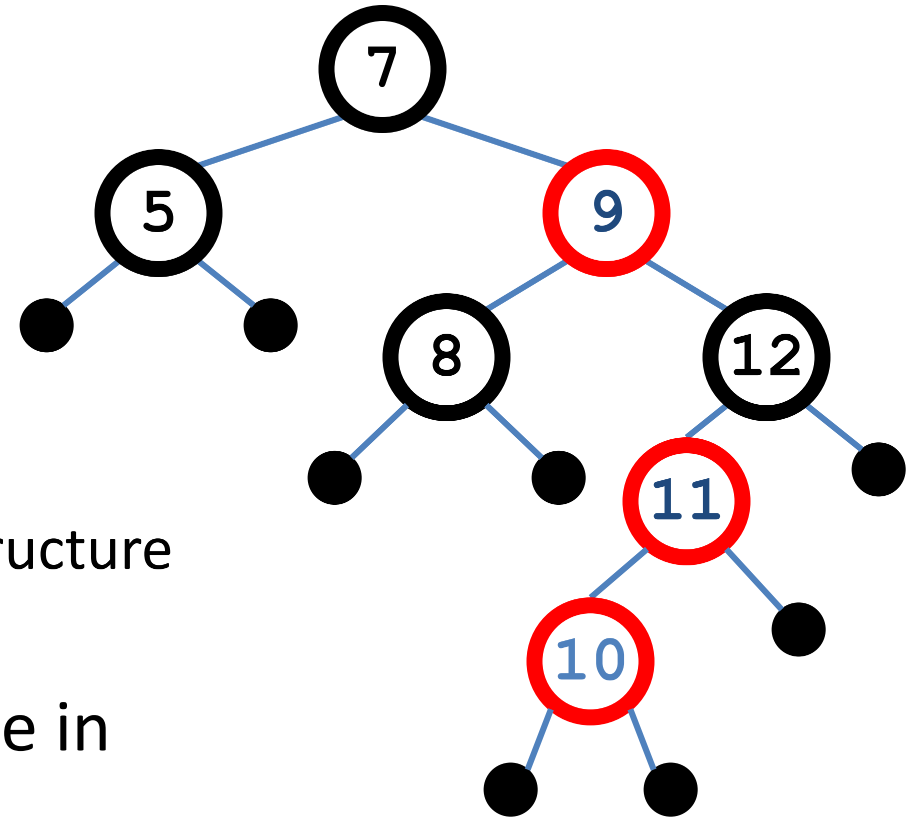
3. If a node is red, both children are black
4. Every path from node to descendent leaf contains the same number of black nodes
5. The root is always black

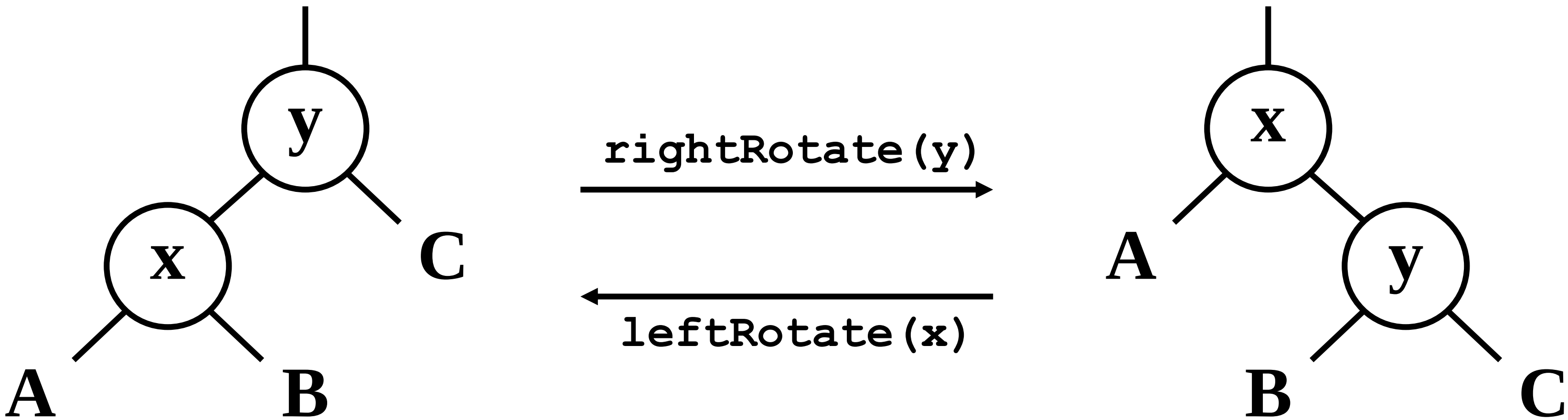
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*
    - A: no color! Tree is too imbalanced
    - Must change tree structure to allow recoloring
  - Goal: restructure tree in O(lg *n*) time

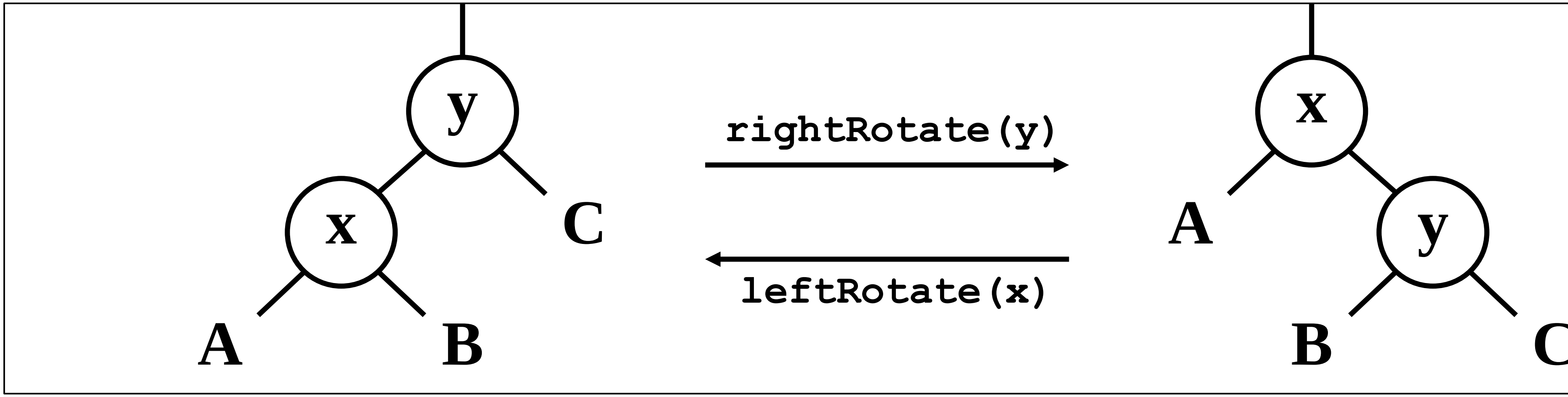


RB Trees: Rotation

- Our basic operation for changing tree structure is called *rotation*:

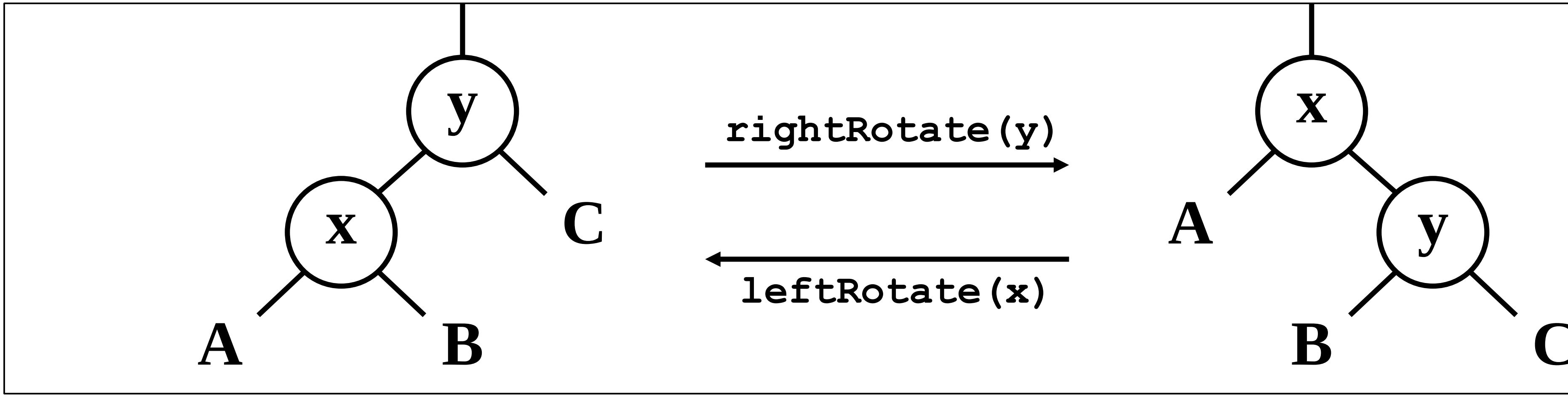


- *Does rotation preserve inorder key ordering?*



Left-Rotate(T,x)

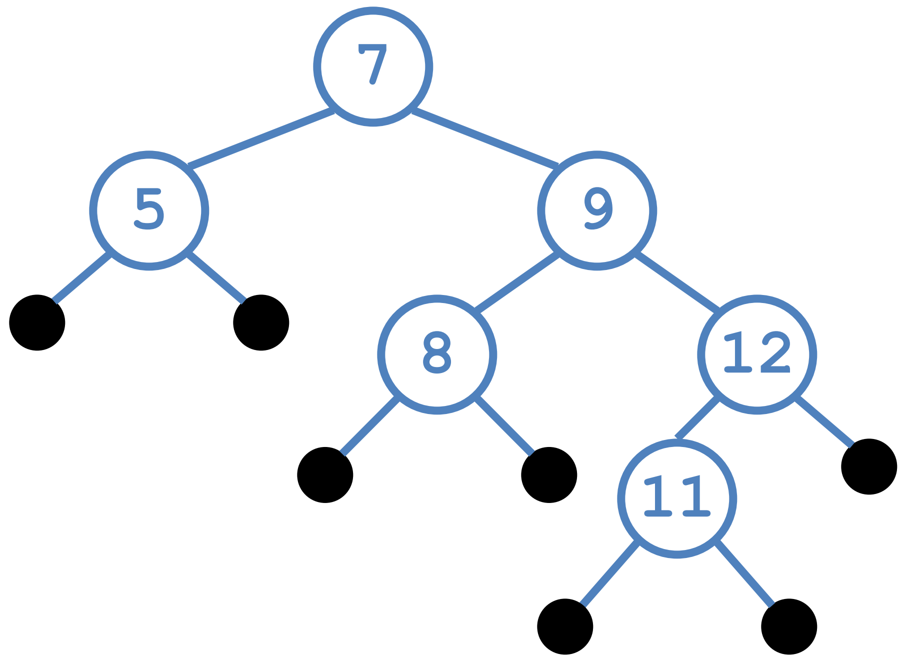
```
y = right(x)                    ; assume right(x) <> NIL
right(x) = left(y)              ; move y's child over
if left(y) <> NIL
then parent(left(y)) = x
parent(y) = parent(x)           ; move y up to x's position
if parent(x) = NIL
then root(T) = y
else if x = left(parent(x))
    then left(parent(x)) = y
    else right(parent(x)) = y
left(y) = x                     ; move x down
parent(x) = y
```




```
Right-Rotate(T,y)
    x = left(y)          ; assume left(y) <> NIL
    left(y) = right(x)
    if right(x) <> NIL
    then parent(right(x)) = y
    parent(x) = parent(y)
    if parent(y) = NIL
    then root(T) = x
    else if y = left(parent(y))
        then left(parent(y)) = x
        else right(parent(y)) = x
    right(x) = y
    parent(y) = x
```

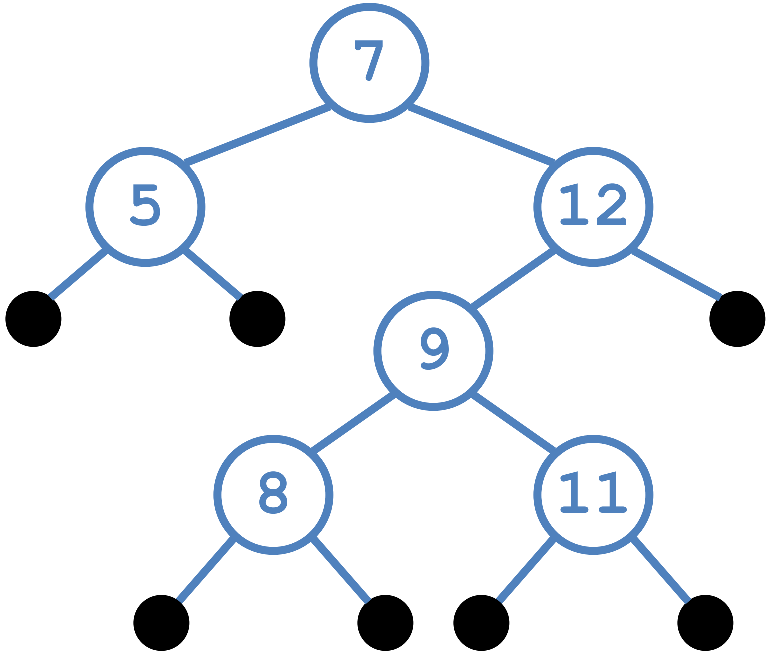
Rotation Example

- Rotate left about 9:



Rotation Example

- Rotate left about 9:



Red-Black Trees: Insertion

- Insertion: the basic idea
  - Insert *x* into tree, color *x* red
  - Which of the red-black properties might be violated?
    - Root is always black
    - Red node cannot have a red child

  - Fix the violated properties.

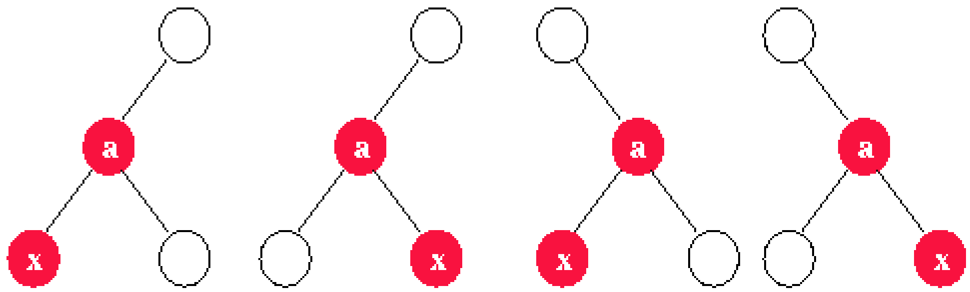
Red-Black Trees: Insertion

Insertion

1. Insert node into tree using BST Insert(T,x) and color node Red
2. Fix violated RBT properties
    1. Root is always black
    2. Red node cannot have a red child
3. Color root Black

Red-Black Trees: Insertion

- If parent node `a' was Black, then no changes are necessary.
- If not, then there are following cases to consider for each of the orientations below.



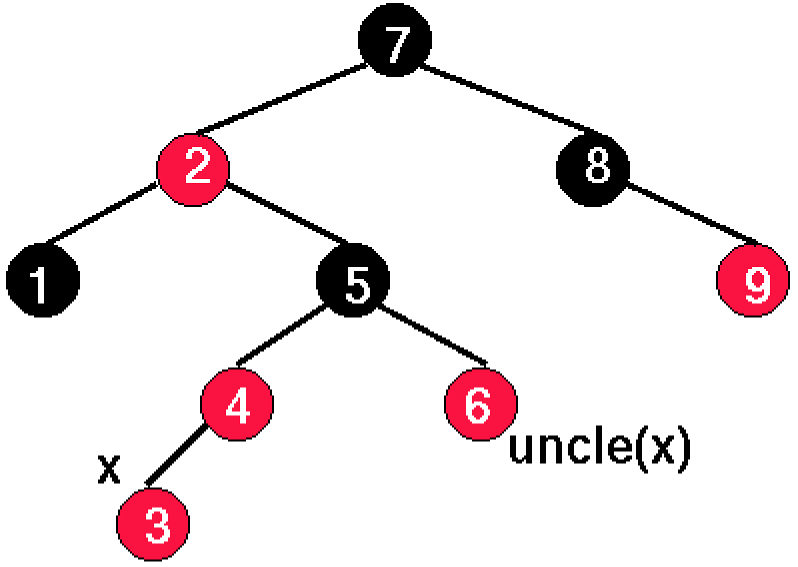
- Move up the tree until there are no violations or we are at the root.
- In the following discussion we will assume the parent is a left child (if the parent is a right child perform the same steps swapping ``right'' and ``left'')

Red-Black Trees: Insertion

RB-Insert(T,x)

Case I: x's uncle is Red

- Change x's grandparent to Red
- Change x's uncle and parent to Black
- Change x to x's grandparent



How to get uncle (x)

if parent(x) = left(parent(parent(x)))
  then uncle(x) = right(parent(parent(x)))
  else uncle(x) = left(parent(parent(x)))

Red-Black Trees: Insertion

RB-Insert(T,x)

Case II: x's uncle is Black, x is the right child of its parent

- Change x to x's parent
- Rotate x's parent (now x) left to make Case III
- Case II is now Case III

Case III: x's uncle is Black, x is the left child of its parent

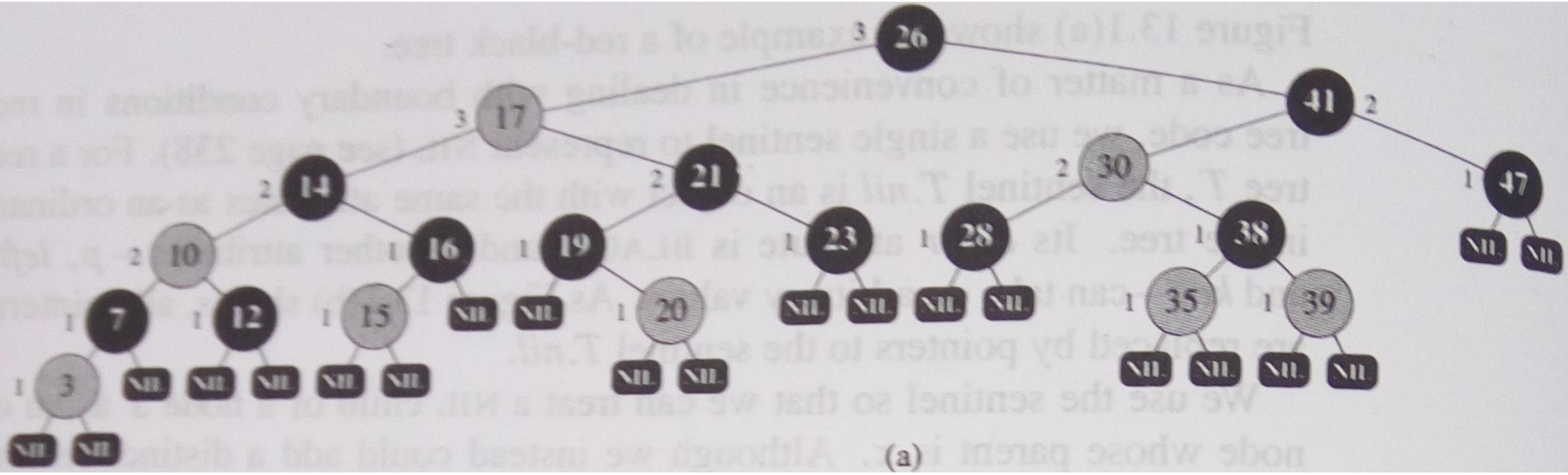
- Set x's parent to Black
- Set x's grandparent to Red
- Rotate x's grandparent right

Red-Black Trees: Insertion

Example:

Insert the following keys in a Red Black Tree in order:

3 2 5 6 9 4 7 8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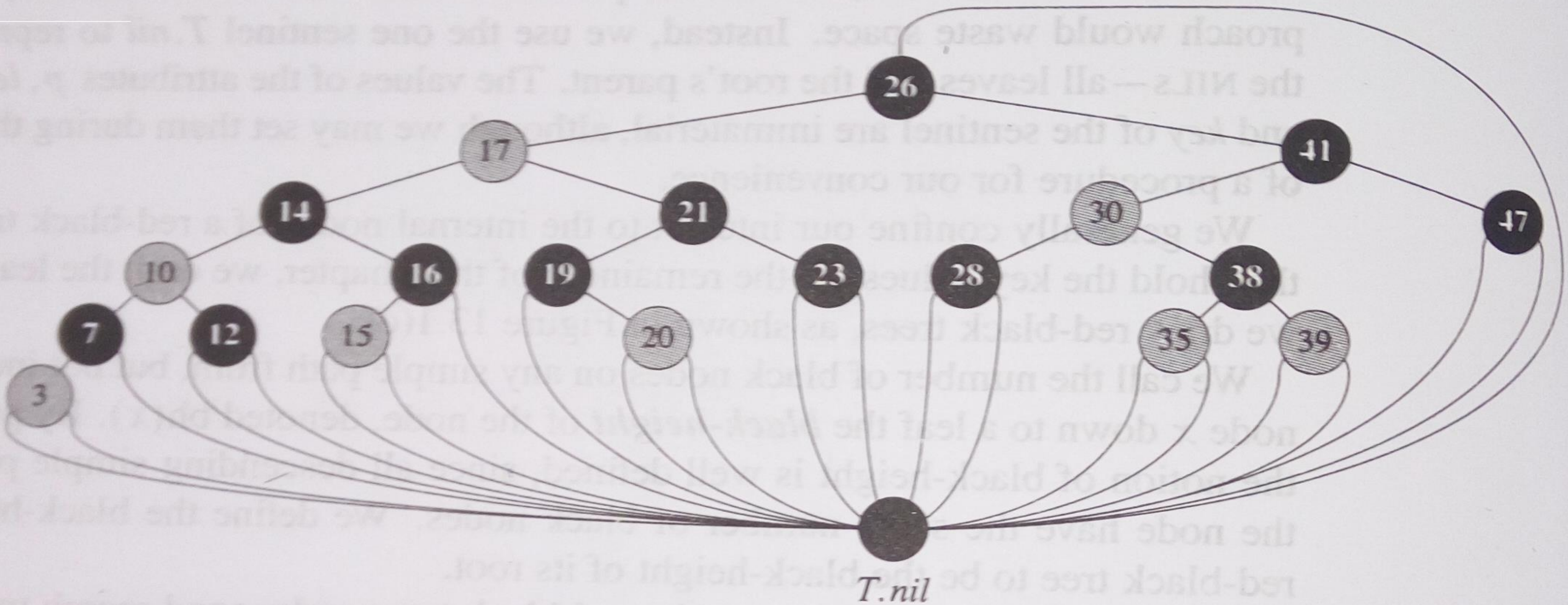
(a)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3
T.nil
(b)

Theorem 1 – In a red-black tree, any subtree rooted at x contains atleast $(\mathbf{2^{bh(x)} - 1})$ internal nodes, where bh(x) is the black height of node x.

Proof: by induction on height of x.

In a red-black tree, at least half the nodes on any path from the root to a NULL (i.e. leaf) must be Black.

Proof – If there is a Red node on the path, there must be a corresponding Black node.

Algebraically this theorem means

$$bh(\ x\ ) \geq h/2$$

where x is the root node.

In a red-black tree, no path from any node, X, to a NULL (i.e. leaf) is more than twice as long as any other path from X to any other NULL (i.e. leaf).

Proof: By definition, every path from a node to any NULL contains the same number of Black nodes. By Theorem 2, atleast ½ the nodes on any such path are Black. Therefore, there can be no more than twice as many nodes on any path from X to a NULL as on any other path. Therefore the length of every path is no more than twice as long as any other path.

Theorem 4 – A red-black tree with ***n*** internal nodes has height **$h \le 2 \lg(n + 1)$**.

Proof: Let h be the height of the red-black tree with root x. By Theorem 2,

$$bh(x) \ge h/2$$

From Theorem 1, $n \ge 2^{bh(x)} - 1$

Therefore $n \ge 2^{h/2} - 1$

$$n + 1 \ge 2^{h/2}$$

$$\lg(n + 1) \ge h/2$$

$$2\lg(n + 1) \ge h$$

RB Trees: Worst-Case Time

- So we've proved that a red-black tree has O(lg *n*) height
- Corollary: These operations take O(lg *n*) time:
  - Minimum(), Maximum()
  - Successor(), Predecessor()
  - Search()
- Insert() and Delete():
  - Will also take O(lg *n*) time
  - But will need special care since they modify tree

Red-Black Trees: Deletion

- As insert had three cases, delete has four different cases.
- Do it yourself.

innovate achieve lead

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Thank You