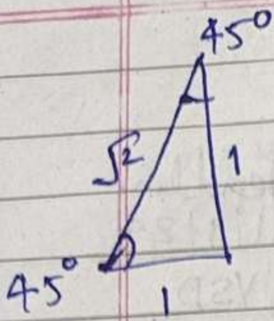
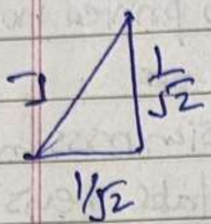
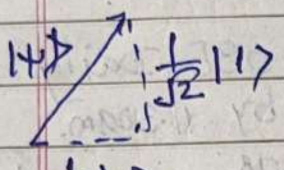
measuring a quantum state.

property of 45-45-90 triangles



divide by $\sqrt{2}$ to fit





$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\left|\left(\frac{1}{\sqrt{2}}\right)^2\right| = \left|\frac{1}{2}\right| = \frac{1}{2}$$

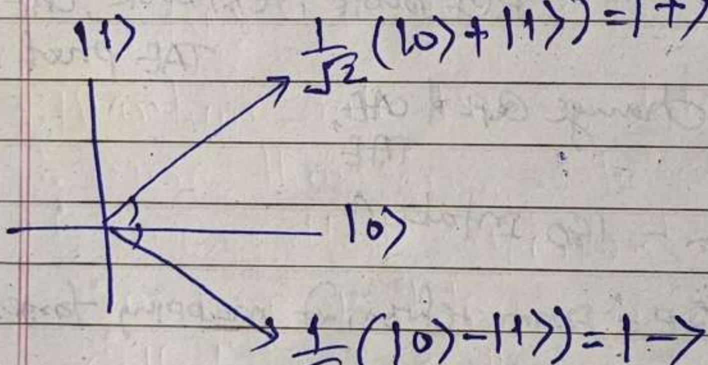
$P(|\psi\rangle = |0\rangle) = |\alpha^2|$

$P(|\psi\rangle = |1\rangle) = |\beta^2|$

$\therefore \alpha^2 + \beta^2 = 1$, because unit circle

$I|0\rangle =$

$I|1\rangle =$



Unit II

Bra ket notation / Applications :-

1) Inner product - Notation - $\langle \, | \, \rangle$
2) outer product : $| \, \rangle \langle \, |$
3) Tensor product : $| \, \rangle \otimes | \, \rangle$
$| \, \rangle | \, \rangle$

$$\left((| \, \rangle)^* \right)^T = \langle \, |$$

Dual vectors.

Example.

$|a\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$ given bra a ?

$\langle a| = [1 \;\; i^*]$

$= [1 \;\; -i]$

calculate Inner product $\langle a|a \rangle$ = number or scalar

outer product $|a\rangle\langle a|$ = matrix.

$| \, \rangle \otimes | \, \rangle = | \, \rangle$

eg: $|a\rangle \otimes |a\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix}$

1 multiply with $\begin{bmatrix} 1 \\ i \end{bmatrix}$

$$= \begin{bmatrix} 1 \begin{bmatrix} 1 \\ i \end{bmatrix} \\ i \begin{bmatrix} 1 \\ i \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ i \\ i^2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ i \\ -1 \end{bmatrix}_{4 \times 1} \text{ order}$$

Ex: 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= a \begin{bmatrix} e & f \\ g & h \end{bmatrix} \; b \begin{bmatrix} e & f \\ g & h \end{bmatrix} \; c \begin{bmatrix} e & f \\ g & h \end{bmatrix} \; d \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

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$$\begin{bmatrix} a\begin{bmatrix} e & f \\ g & h \end{bmatrix} & b\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ c\begin{bmatrix} e & f \\ g & h \end{bmatrix} & d\begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{bmatrix} = \begin{bmatrix} p \\ \\ \\ \end{bmatrix}_{4\times 4}$$

If

$$[\quad]_{m\times n} \otimes [\quad]_{p\times q} = [\quad] = [\quad]_{(m\times p)\times(p\times q)}$$

Sq. matrix

Applications of Tensor products:

- We can deal with multiple systems.

Ex. one quantum state is $|\psi_1\rangle$ } what will be there combination.
another " " is $|\psi_2\rangle$

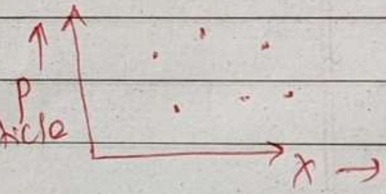
$|\psi_1\rangle|\psi_2\rangle$

in maths term.

$$|\psi_1\rangle \otimes |\psi_2\rangle$$

Superposition :-

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$



quantum state $|\psi\rangle$ can take more than one state at a time called superposition.

particle state depends on position and momentum.

Sometime ψ_1 state or sometimes ψ_2 state shows. Called as Collapse of wave function.

a seems b, sometimes c, " d.

$$\boxed{a} = c_b\boxed{b} + c_c\boxed{c} + c_d\boxed{d}$$

Coefficient ⇒ probability amplitude.

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superposition over a basis state :-

$|\psi\rangle = \quad |\psi_1\rangle + \quad |\psi_2\rangle + \quad + |\psi_3\rangle$

substates should be disjoint

The basis states should be complete.
The basis should span the quantum state.

Called as orthogonality.

* orthoganality :-
It differentiate between the substates.

$\langle\psi_1|\psi_2\rangle = 0$; $\langle\psi_2|\psi_3\rangle = 0$; $\langle\psi_n|\psi_n\rangle = 0$

Inner product of substates = 0

$\langle\psi_1|\psi_1\rangle = 1.$ $\quad \boxed{\langle\psi_i|\psi_j\rangle = \delta_{ij}} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Called as orthonormal states ; orthonormality

when state not same
when states are same.

Completeness
mathematically we can write all basis as below

outer product $\rightarrow$ $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4| = 1.$

$= \sum_i |\psi_i\rangle\langle\psi_i| = 1.$ $\qquad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity matrix.

all basis state's outer product = 1 i.e. identity matrix.
and it forms complete orthonormal basis.

$|\psi\rangle = \sum_i c_i |\psi_i\rangle$

Quantum bits
needs two basis states.

$|\psi\rangle = c_0|\psi_0\rangle + c_1|\psi_1\rangle$

for combine system calculation we use tensor product

prove :- Two quantum systems described with their complete orthonormal basis when combined create a new orthonormal bases.

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→ Mathematically we can write qubit as

$|\psi_{qubit}\rangle = c_0|0\rangle + c_1|1\rangle$ counts how many times state 1.

how many times 0 state.

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $[1] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now check orthonormality & completeness

Orthonormality :

$$\langle i|j\rangle = \delta_{ij} = |\psi_i\rangle\langle\psi_j|$$ inner product =1

$$\langle 0|0\rangle = 1 = \langle 1|1\rangle \quad ; \quad \langle 0|1\rangle = 0 = \langle 1|0\rangle$$

Unit II
Quantum Gates.

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Quantum gates are equivalent ~~to~~ unitary operators.

Basis states

1) Single Qubit Basis $= \{ \underset{\hookrightarrow 0}{|0\rangle} , \underset{\hookrightarrow 1}{|1\rangle} \}$

Basis $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

pauli Gates

σ_x	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	1 ✓
$\langle 1\|$	1	0

change bra becomes ket vice versa

outer product eqn

~~$X|0\rangle$~~

NOT Gate
Quantum

$$\boxed{X = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

$$X|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle$$

$$X|0\rangle = |1\rangle$$

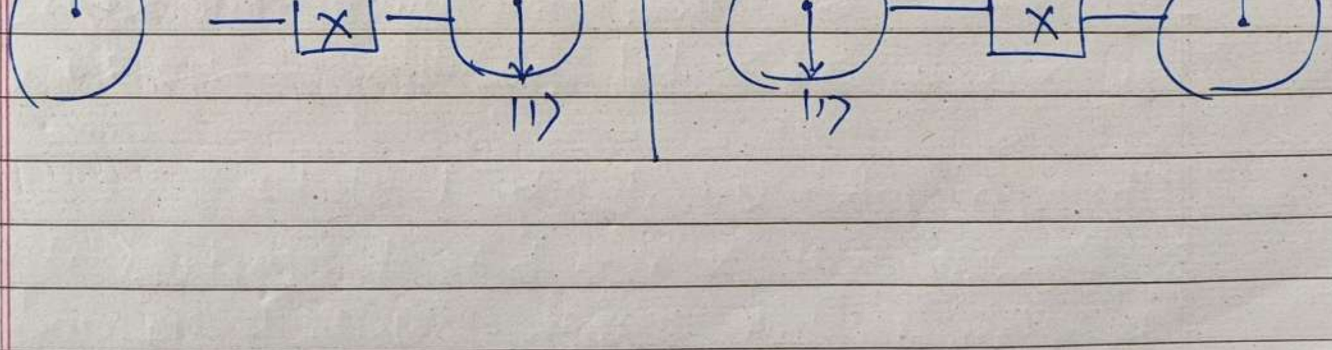
$$|0\rangle \text{---} \boxed{X} \text{---} |1\rangle$$

same 1
diff 0.

$$X|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle$$

$$X|1\rangle = |0\rangle$$

$$|1\rangle \text{---} \boxed{X} \text{---} |0\rangle$$



②

σ_y	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	$-i$
$\langle 1\|$	i	0

$$Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\boxed{Y|0\rangle = i|1\rangle}$$

$$\boxed{Y|1\rangle = -i|0\rangle}$$

$|0\rangle$ — $\boxed{Y}$ — $i|1\rangle$ $\qquad$ $|1\rangle$ — $\boxed{Y}$ — $-i|0\rangle$

③

σ_z	$\|0\rangle$	$\|1\rangle$
$\langle 0$	1	0
$\langle 1$	0	-1

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$Z|0\rangle = |0\rangle \qquad \cancel{Z|1\rangle} =$$

diff.o

$$Z|1\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle$$

$$= \cancel{|0\rangle} - |1\rangle$$

$|0\rangle$ — $\boxed{Z}$ — $|0\rangle$ $\qquad$ $|1\rangle$ — $\boxed{Z}$ — $-|1\rangle$

↳ phase change

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④ Hadmard Gate

H	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\langle 1\|$	$\frac{1}{\sqrt{2}}$	$-1/\sqrt{2}$

$$H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right].$$

Overview of single system:

- Quantum states are represented by column vectors
  - entries are complex numbers
  - Euclidean norm equal to 1
- standard basis measurements
  - outcomes are classical states
  - probabilities are the absolute values squared of the entries.
- operations are represented by unitary matrices.

Inqueue -
Dequeue - dele

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Unit III

IBM

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single system :-
1. classical information
2. Quantum information
   a. Quantum state vectors
   b. standard basis measurements
   c. Unitary operations

Quantum information :-
- simpler
- Quantum states represented by vectors; operations are represented by unitary matrices.
- Quantum states represented by density matrices allows for a more general class of measurements and operations.

Classical information:
Consider a physical system that stores information: let us call it x.

Assume x can be in one of a finite number of classical states at each moment.

Denote this classical state set by Σ.

Examples:
- If x is a bit, then its classical state set is $\Sigma = \{0,1\}$
- If x is a six-sided die, then $\Sigma = \{1,2,3,4,5,6\}$.

There may be uncertainty of states.
for example, if x is a bit, then perhaps it is in the classical state 0 with probability 3/4 and in the classical state 1 with probability 1/4. This is a

probabilistic state of X.

$$P_r(X=0)=\frac{3}{4} \quad \text{and} \quad P_r(X=1)=\frac{1}{4}.$$

way to represent this probabistic state is by a column vector ;

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} \begin{matrix} \leftarrow \text{entry corresponding to } 0 \\ \leftarrow \text{entry corresponding to } 1. \end{matrix}$$

This vector is a probability vector:
- All entries are nonnegative numbers.
- The sum of the entries is 1.

Dirac notation (first part)

Let Σ be any classical state set and assume the elements of Σ have been placed in correspondance with the integers $1, \dots |\Sigma|$.

We denote by $|a\rangle$ the column vector having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Example 1:

If $\Sigma = \{0, 1\}$, then

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

vectors of this form are called standard basis vectors. Each vector can be expressed uniquely as a linear combination of standard basic vectors.

Example

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$$

Measuring probabilistic states:

What happens if we measure a system X while it is in some probabilistic state?

We see a classical state, chosen at random according to the probabilities.

Suppose classical state $a \in \Sigma$.

$$P_r(X=a) = 1$$

This probabilistic state is represented by the vector $|a\rangle$.

Example.

Consider the probabilistic state of a bit X where $Pr(X=0) = \frac{3}{4}$ and $P_r(X=1) = \frac{1}{4}$.

Measuring X selects a transition, chosen at random:

$$\begin{tikzcd} & \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle \arrow[dl, "\text{probability } 3/4"'] \arrow[dr, "\text{Probability } 1/4"] & \\ |0\rangle & & |1\rangle \end{tikzcd}$$

Deterministic operations:

no uncertainty involved.

- Every function $f : \Sigma \rightarrow \Sigma$ describes a deterministic operation that transforms the classical state a into $f(a)$, for each $a \in \Sigma$.

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Given any function $f : \Sigma \to \Sigma$, there is a (unique) matrix M satisfying

$$M|a\rangle = |f(a)\rangle \quad (\text{for every } a \in \Sigma)$$

This matrix has exactly one 1 in each column, and 0 for all other entries:

$$M(b,a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$$

The action of this operation is described by matrix vector multiplication:

$$v \longmapsto Mv$$

Example:

for $\Sigma = \{0,1\}$, there are four functions of the form $f : \Sigma \to \Sigma$:

Input is I. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a	$f_1(a)$
0	0
1	0

a	$f_2(a)$
0	0
1	1

a	$f_3(a)$
0	1
1	0

a	$f_4(a)$
0	1
1	1

Constant function for 0
$f_1(0) = 0$
$f_1(1) = 0$

I
$f_2(0) = 0$
$f_2(1) = 1$

Not
$f_3(0) = 1$
$f_3(1) = 0$

Constant function for 1.
$f_4(0) = 1$, $f_4(1) = 1$.

$$M(b,a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$$

row Corresponding to classical state b

column Corresponding to classical state b

The action of this operation is described by matrix vector multiplication

$$\boxed{v \longmapsto Mv}$$

V = probabilistic vector

$\mapsto$ - mathematical notation to show how one thing gets map to another.

$\cancel{f_1|0\rangle} =$

$\cancel{f_1|1\rangle} =$

M represent a given function.

$f: \Sigma \to \Sigma$ which satisfies.

$M|a\rangle = |f(a)\rangle$ for every $a \in \Sigma$ such as matrix always exist and is unique.

M_1, M_2, M_3, M_4 Corresponds to function $f_1\ f_2\ f_3\ f_4$.

$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

Matrices with exactly '1' in one column and rest of others are zero.

$\Sigma = \{0, 1\}$ Bra $\langle 0| = (1\ 0)$

$\langle 1| = (0\ 1)$

$n = 2 \therefore 2^n = 2^2 = 4$ Combination

$|0\rangle\langle 1| = \cancel{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0\ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

formula $M(b,a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$

from above function with 1 bit i/p of a

a	$f_1(a)$	a	$f_2(a)$	a	$f_3(a)$	a	$f_1(a)$
0	0	0	0	0	1	0	1
1	0	1	1	1	0	1	1

$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$M_1\ M_2\ M_3\ M_4$ Corresponds to function $f_1\ f_2\ f_3\ f_4$

$f_1|0\rangle = 1|0\rangle + 0\cdot|1\rangle = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\times\begin{pmatrix}0\\1\end{pmatrix} = \begin{matrix}1\\0\end{matrix}$

$f_1|1\rangle = 1|0\rangle + 0\cdot|1\rangle = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\times\begin{pmatrix}0\\1\end{pmatrix} = \begin{matrix}1\\0\end{matrix}$

$$M_1 = \overset{m_1}{\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}}\overset{f_1}{\begin{pmatrix}0\\0\end{pmatrix}} = \begin{pmatrix}0\\0\end{pmatrix}$$ same.

$$M_2 = \overset{m_2}{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}\overset{f_2}{\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}0\\1\end{pmatrix}$$

$f_2|0\rangle = 1\cdot|0\rangle + \underbrace{0\cdot|1\rangle}_{0} = \begin{pmatrix}1\\0\end{pmatrix}$

$f_2|1\rangle = \cancel{1|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}}\ \ 0|0\rangle + 1|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$M_3 = \overset{m_3}{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}}\overset{f_3}{\begin{pmatrix}1\\0\end{pmatrix}} = \begin{pmatrix}0\\1\end{pmatrix}$$ Complement

$\cancel{f_3|0\rangle = 0\cdot|0\rangle + 0|1\rangle = |1\rangle}\ \ 0\cdot|0\rangle + 0\cdot|1\rangle = |1\rangle$

$\cancel{f_3|1\rangle = 0\cdot|0\rangle + 0|1\rangle =}\ \ 1|0\rangle + 0|1\rangle = |0\rangle$

$$M_4 = \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}\overset{f_4}{\begin{pmatrix}1\\1\end{pmatrix}} = \begin{pmatrix}0\\1\end{pmatrix}$$ same

Dirac notation (second part)

Let Σ be any classical state set, and assume the elements of Σ have been placed in correspondence with the integers $1, \ldots, |\Sigma|$.

We denote by $\langle a|$ the row vector having a 1 in the entry corresponding to $a \in \Sigma$, with 0 for all other entries.

Example:

If $\Sigma = \{0, 1\}$ then

$\langle 0| = (1 \ \ 0)$ and $\langle 1| = (0 \ \ 1)$

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if we multiply a row vector to column vector yields a scalar:

$$\begin{pmatrix} * & * & * \end{pmatrix}\begin{pmatrix} * \\ \vdots \end{pmatrix} = \cancel{(* \ * \ *)} \ (*)$$

$$(0 \ 1 \ 0 \cdots 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

~~Inner~~ product $\langle a|b\rangle = \begin{cases} 1 & a=b \\ 0 & a\neq b \end{cases}$

multiplying a column vector to a row vector yields a matrix:

$$\begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix}(* \ * \ * \cdots *) = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & & & \vdots \\ * & * & \cdots & * \end{pmatrix}$$

Example

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \ 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Deterministic operations.

Every function $f: \Sigma \to \Sigma$ describes a deterministic operation that transforms the classical state a into $f(a)$, for each $a \in \Sigma$.

Given any function $f: \Sigma \to \Sigma$, there is a (unique) matrix M satisfying

$$M|a\rangle = |f(a)\rangle \quad (\text{for every } a \in \Sigma)$$

This matrix may be expressed as

$$M = \sum_{b\in\Sigma} |f(b)\rangle\langle b|$$

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Its action on standard basis vectors works as required:

$$M|a\rangle = \Big(\underbrace{\sum_{b\in\Sigma} |f(b)\rangle\langle b|}_{\text{put in value}}\Big)|a\rangle = \sum |f(b)\rangle\underbrace{\langle b|a\rangle}_{=1}$$

$$= |f(a)\rangle. \quad \text{if } b=a$$

probabilistic Operations:

probabilistic Operations are classical operations that may introduce randomness or uncertainty.

Example:

Here is a probabilistic operation on a bit:
If the classical state is 0, then do nothing
If the classical state is 1, then flip the bit with probabity $\frac{1}{2}$.

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

probabilistic operations are described by stochastic matrices:
- All entries are nonnegative real numbers
- The entries in every column sum to 1.

Composing probabilistic operations:

suppose X is a system and $M_1 \dots M_n$ are stochastic matrices representing probabilistic operations on X.

Applying the first probabilistic operation to the probability vector v, then applying the second probabilistic operation to the result yields this vector:

$$M_2(M_1 v) = (M_2 M_1) v \qquad \text{associative} \quad \text{change parenthesis}$$

The probabilistic operation obtained by composing the first and second probabilistic operations is represented by the matrix product M_2M_1.

Inquene - insertion operation
Dequeue - deletion operation

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Composing the probabilistic operations represented by the matrices $M_1 \dots M_n$ (in that order) is represented by this matrix product:

$$M_n \cdots M_1.$$

The order is important: matrix multiplication is not commutative.

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\text{Not}}$$

$$M_2 M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad M_1 M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Quantum information:

A quantum state of a system is represented by a column vector whose indices are placed in correspondence with the classical states of that system:

- The entries are complex numbers
- The sum of the absolute values squared of the entries must equal 1.

Definition:

The euclidean norm for vectors with complex number entries is defined like this:

$$v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \Rightarrow \|v\| = \sqrt{\sum_{k=1}^{n} |\alpha_k|^2} = 1.$$

sum of squared absolute value vector

Quantum state vectors are therefore unit vectors with respect to this norm

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Examples of qubit states:

- standard basis states: $|0\rangle$ and $|1\rangle$
- plus/minus states:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

- A state without a special name:

$$|\psi\rangle = \underset{\alpha}{\frac{1+2i}{3}}|0\rangle - \underset{\beta}{\frac{2}{3}}|1\rangle = \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix}.$$

$$\left(\frac{1+2i}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 = 1.$$

The notation $|\psi\rangle$ is commonly used to refer to an arbitrary vector.

for any column vector $|\psi\rangle$, the row vector $\langle\psi|$ is the conjugate transpose of $|\psi\rangle$:

$$\langle\psi| = |\psi|^{\dagger} \leftarrow \text{dagger}$$

$$\langle\psi| = \frac{1-2i}{3}\langle 0| - \frac{2}{3}\langle 1| = \left(\frac{1-2i}{3}, -\frac{2}{3}\right)$$

Measuring quantum states:

There are different notation for measurement

1) std basis measurement
2) projective "

→ Here standard basis measurements:

- The possible outcomes are the classical states.
- The probability for each classical state to be the outcome is the absolute value squared of the corresponding quantum state vector entry.

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Example 1:

measuring the quantum state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

yields an outcome as follows:

$$Pr(\text{outcome is } 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \quad Pr(\text{outcome is } 1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Example 2.

measuring the quantum state

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

yields an outcome as follows:

$$Pr(\text{outcome is } 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \quad Pr(\text{outcome is } 1) = \left|\frac{-1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Example 3:

measuring the quantum state

$$\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

yields an outcome as follows:

$$Pr(\text{outcome is } 0) = \left|\frac{1+2i}{3}\right|^2 = \frac{5}{9} \quad Pr(\text{outcome is } 1) = \left|\frac{-2}{3}\right|^2 = \frac{4}{9}$$

Example 4:

measuring the quantum state $|0\rangle$ gives the outcome 0 with certainty and measuring the quantum state $|1\rangle$ gives the outcome 1 with certainty because absolute value square is equal to 1.

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measuring a system changes its quantum state: if we obtain the classical state a, the new quantum state becomes $|a\rangle$.

$$\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

probability $\frac{5}{9}$ → $|0\rangle$

Probability $\frac{4}{9}$ → $|1\rangle$

Unitary operations: describes how quantum states of a system can change.

The set of allowable operations that can be performed on a quantum state is different than it is for classical information.

operation on quantum state vectors are represented by unitary ~~operations~~. matrices.

Definition:

A square matrix U having complex number entries is unitary if it satisfies the equalities

$$U^{\dagger}U = I = UU^{\dagger}$$

where $U^{\dagger}$ is the conjugate transpose of U and I is the identity matrix.

$UU^{-1}=I$ Both equalities are equivalent to $U^{-1}=U^{\dagger}$.

The condition that $n \times n$ matrix U is unitary is equivalent to

$$\|Uv\| = \|v\|$$

for every n-dimensional column vector v with complex number entries.

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Qubit Unitary operations:

1. Pauli operations

Pauli operations are ones represented by the pauli matrices:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Common alternative notations: $X = \sigma_x,\ Y = \sigma_y,\ Z = \sigma_z.$

The operation σ_x is also called a <u>bit flip</u> (or a NOT operation) and σ_z operation is called a phase flip:

$$\sigma_x |0\rangle = |1\rangle \qquad \sigma_z |0\rangle = |0\rangle$$
$$\sigma_x |1\rangle = |0\rangle \qquad \sigma_z |1\rangle = -|1\rangle$$

2. Hadamard operation:

represented by this matrix:

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix}$$

Checking that H is unitary is a straightforward calculation:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix}^{\dagger} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

H is its own Conjugate

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3. phase operations

A phase operation is one described the matrix

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

for any choice of a real number θ.

The operations

$$S = P_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ and } T = P_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

are important examples.

Example 1:

$$H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} = |+\rangle$$

$$H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{pmatrix} = |-\rangle$$

$$H|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$H|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$$
$$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$$

$$H\left(\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1+2i}{3} \\ -2/3 \end{pmatrix} = \begin{pmatrix} \frac{-1+2i}{3\sqrt{2}} \\ \frac{3+2i}{3\sqrt{2}} \end{pmatrix}$$

$$= \frac{-1+2i}{3\sqrt{2}}|0\rangle + \frac{3+2i}{3\sqrt{2}}|1\rangle$$

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Example 2:

$T|0\rangle = |0\rangle$ and $T|1\rangle = \frac{1+i}{\sqrt{2}}|1\rangle$

$$T|+\rangle = T\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}T|0\rangle + \frac{1}{\sqrt{2}}T|1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

$$HT|+\rangle = H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}H|0\rangle + \frac{1+i}{2}H|1\rangle$$

$$= \frac{1}{\sqrt{2}}|+\rangle + \frac{1+i}{2}|-\rangle$$

expand +state & - state

$$\boxed{\begin{array}{l} H|0\rangle = |+\rangle \\ H|1\rangle = |-\rangle \end{array}}$$

$$= \left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + \left(\frac{1+i}{2\sqrt{2}}|0\rangle - \frac{1+i}{2\sqrt{2}}|1\rangle\right)$$

$$= \left(\frac{1}{2} + \frac{1+i}{2\sqrt{2}}\right)|0\rangle + \left(\frac{1}{2} - \frac{1+i}{2\sqrt{2}}\right)|1\rangle$$

Composing unitary operations.

Compositions of unitary operations are represented by matrix multiplications

Example: square root of NOT

Applying a Hadamard operation, followed by the phase operation S, followed by another Hadamard operation yields this operation:

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$$HSH = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

Applying this unitary operation twice yields a NOT operation:

$$(HSH)^2 = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_x .$$

~~MULTIPLE SYSTEMS~~

Quantum Gates : ≡ Unitary operators

① single qubit Basis = $\{|0\rangle, |1\rangle\}$

Basis $\left\{ \underset{\begin{bmatrix}1\\0\end{bmatrix}}{|0\rangle} \; \underset{\begin{bmatrix}0\\1\end{bmatrix}}{|1\rangle} \right\}$

∴ Basis Column $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② 2 qubit basis states

$\{ |0\rangle|0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle \}$

$|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \to |0\rangle$ $\begin{bmatrix}0\\1\\0\\0\end{bmatrix} |1\rangle$ $|2\rangle \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $|3\rangle \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

entire ain

$$|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Basis states as column combine we will get I Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D[1\ 1\ 1\ 1] = I_4 .$$

③ 3 qubit ~~gate~~ basis states.

$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

$\hookrightarrow 0 \quad \hookrightarrow 1 \quad \hookrightarrow 2 \quad \hookrightarrow 3 \quad \hookrightarrow 4 \quad \hookrightarrow 5 \quad \hookrightarrow 6 \quad \hookrightarrow 7$

$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\.\end{bmatrix}$

$UU^{\dagger} = I \quad , \quad U^{-1} = U^{\dagger} \quad |U| = 1$

column satisfy orthonormal condition.

~~Overview of single system~~ (on previous pages)

Multiple system | **Unit IV**

- Classical states

Suppose that we have two systems:

- X is a system having classical state set Σ
- Y is a system having " " " Γ.

Imagine that X and Y are placed side-by-side, with X on the left and Y on the right, and viewed together as if they form a single system.

We denote this new Compound System by (X,Y) or XY.

Question

What are the classical states of (X,Y)?

The classical state set of (X,Y) is the Cartesian product.

$$\Sigma \times \Gamma = \{(a,b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Q. What are the classical states of (X, Y)?

The classical state set of (X, Y) is the cartesian product.

$$\Sigma \times \Gamma = \{(a,b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

classical states:

This description generalizes to more than two systems in a natural way.

Suppose $X_1, \ldots, X_n$ are systems having classical states set $\Sigma_1, \ldots \Sigma_n$ resp^{ly}.

The classical state set of the n-tuple $(X_1 \ldots X_n)$, viewed as a single compound system, is the cartesian product.

$$\Sigma_1 \times \ldots \times \Sigma_n = \{(a_1, \ldots, a_n) : a_1 \in \Sigma_1, \ldots, a_n \in \Sigma_n\}$$

Example

If $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{0,1\}$, then the classical state set of (X_1, X_2, X_3) is..

(010)

$$\rightarrow \Sigma_1 \times \Sigma_2 \times \Sigma_3 = \{(0,0,0), (0,0,1), (011), (100), (101), (110), (111)\}$$

- Cartesian products of classical state sets are ordered lexicographically (i.e. dictionary ordering): left to right.
- We assume the individual classical state sets are already ordered.
- significance decreases from left to right.

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Example:
The cartesian product $\{1,2,3\} \times \{0,1\}$ is ordered like this:

$(1,0)$ $(1,1)$, $(2,0)$ $(2,1)$, $(3,0)$, $(3,1)$

$\{0,1\} \times \{0,1\} = 00, 01, 10, 11$

Cartesian product

probabilistic states in multiple system

probabilistic states of compound systems associate probabilities with the cartesian product of the classical state sets of individual systems.

Example:
This is a probabilistic state of a pair of bits (X,Y):

$$\Pr((X,Y) = (0,0)) = \frac{1}{2}$$
$$\Pr((X,Y) = (0,1)) = 0$$
$$\Pr((X,Y) = (1,0)) = 0$$
$$\Pr((X,Y) = (1,1)) = \frac{1}{2}$$

vector column.

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

← probability associated with state 00
← " " 01
← " " 10
← " " 11

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Definition :

For a given probabilistic state of (X, Y), we say that X and Y are independent.

$$P_r((X, Y) = (a, b)) = P_r(X = a) P_r(Y = b)$$

for all $a \in \Sigma$ and $b \in \Gamma$.

Suppose that a probabilistic state of (X, Y) is expressed as a vector

$$|\pi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

The systems X and Y are independent if there exist probabilistic vectors.

$$|\phi\rangle = \sum_{a \in \Sigma} q_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b \in \Gamma} r_b |b\rangle \quad \text{.... chk}$$

such that $p_{ab} = q_a r_b$ for all $a \in \Sigma$ and $b \in \Gamma$.

Example

The probabilistic state of a pair of bits (X, Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probability vectors:

$$|\phi\rangle = \frac{3}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

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Example :.
For the probabilistic states

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

of two bits (X, Y) we have that X and Y are not independent.
If they were, we would have numbers q_0, q_1, r_0, r_1 such that

$$q_0 r_0 = \frac{1}{2}$$

$$q_0 r_1 = 0$$

$$q_1 r_0 = 0$$

$$q_1 r_1 = \frac{1}{2}$$

Tensor products of vectors :-

Definition
The tensor product of two vectors.

$$|\phi\rangle = \sum_{a \in \Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b \in \Gamma} \beta_b |b\rangle$$

is the vector

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalently, the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by this condition:

$$\langle ab|\pi\rangle = \langle a|\phi\rangle \langle b|\psi\rangle \quad (\text{for all } a \in \Sigma \text{ and } b \in \Gamma)$$

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Definition.

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle.$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Example.

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle.$$

Alternative notation for tensor products:

$$|\phi\rangle|\psi\rangle = |\phi\rangle \otimes |\psi\rangle$$
$$|\phi\rangle \otimes |\psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

Following our convention for ordering the elements of cartesian product sets, we obtain this specification for the tensor product of two column vectors:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 \\ \vdots \\ \alpha_1 \beta_k \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_2 \beta_k \\ \vdots \\ \alpha_m \beta_1 \\ \vdots \\ \alpha_m \beta_k \end{pmatrix}$$

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Example:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \\ \alpha_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \\ \alpha_3 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \end{pmatrix}$$

multiplying vector by scalar.

observe the following expression for tensor products of standard basis vectors:

$$|a\rangle \otimes |b\rangle = |a\rangle b\rangle = |ab\rangle$$

Alternatively, writing (a, b) as an ordered pair rather than a string, we could write.

$$|a\rangle \otimes |b\rangle = |(a,b)\rangle$$

but it is more common to write

$$|a\rangle \otimes |b\rangle = |a,b\rangle$$

(It is a standard convention in mathematics to eliminate parenthesis when they do not serve to add clarity or remove ambiguity.)

The tensor product of two vectors is bilinear.

1. Linearity in the first argument:

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$

$$(\alpha|\phi\rangle) \otimes |\psi\rangle = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

2. Linearity in the second argument:

$$|\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$

$$|\phi\rangle \otimes (\alpha|\psi\rangle) = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

eg. $|0\rangle \otimes |0\rangle \otimes |1\rangle$

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Independence and tensor products for three or more systems.

If $|\phi_1\rangle, \cdots, |\phi_n\rangle$ are vectors, then the tensor product

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

is defined by the equation

$$\langle a_1 \cdots a_n|\psi\rangle = \langle a_1|\phi_1\rangle \cdots \langle a_n|\phi_n\rangle$$

Equivalently, the tensor product of three or more vectors can be defined.

$$|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle = (|\phi_1\rangle \otimes \cdots \otimes |\phi_{n-1}\rangle) \otimes |\phi_n\rangle$$

Measurements of probabilistic states

Measurements of compound systems work in the same way as measurements of single systems - provided that all of the systems are measured.

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Example:

Suppose that two bits (X, Y) are in the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

measuring both bits yields the outcome 00 with probability 1/2 and the outcome 11 with probability 1/2.

Question

Suppose two systems (X, Y) are together in some probabilistic state. What happens when we measure X and do nothing to Y?

Answer: ① The probability to observe a particular classical state $a \in \Sigma$ when just X is measured as

$$\Pr(X = a) = \sum_{b \in \Gamma} \Pr((X, Y) = (a, b)).$$

② There may still exist uncertainty about the classical state of Y, depending on the outcome of the measurement:

$$\Pr(Y = b \mid X = a) = \frac{\Pr((X, Y) = (a, b))}{\Pr(X = a)}$$

These formulas can be expressed using the Dirac notation as follows.

Suppose that (X, Y) is in some probabilistic state:

$$\sum_{(a,b) \in \Sigma \times \Gamma} p_{ab}|ab\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab}|a\rangle \otimes |b\rangle =$$

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$$= \sum_{a\in\Sigma} |a\rangle \otimes \left(\sum_{b\in\Gamma} p_{ab} |b\rangle \right)$$

1. The probability that a measurement of X yields an outcome $a \in \Sigma$ is

$$\Pr(X=a) = \sum_{b\in\Gamma} p_{ab}$$

2. Conditioned on the outcome $a \in \Sigma$, the probabilistic state of Y becomes

$$\frac{\sum_{b\in\Gamma} p_{ab}|b\rangle}{\sum_{c\in\Gamma} p_{ac}}$$

Measurements of probabilistic states:

Example:

Suppose (X, Y) is in the probabilistic state

$$\frac{1}{12}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{3}|10\rangle + \frac{1}{3}|11\rangle$$

We write this vector as follows:

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

case 1: the measurement outcome is 0.

$$\Pr(\text{outcome is } 0) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle}{\frac{1}{3}} = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$$

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Case 2: the measurement outcome is 1.

$$\Pr(\text{outcome is } 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle}{\frac{2}{3}} = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

operations on probabilistic states

probabilistic operations on compound systems are represented by stochastic matrices having rows and columns that correspond to the cartesian product of the individual systems classical state sets.

Example :-

A controlled-NOT operation on two bits X and Y:

If X=1, then perform a NOT operation on Y, otherwise do nothing.

X is the control bit that determines wheather or not a NOT operation is applied to the target bit Y.

Action on standard basis	matrix representation
$\lvert 00\rangle \mapsto \lvert 00\rangle$ $\lvert 01\rangle \mapsto \lvert 01\rangle$ $\lvert 10\rangle \mapsto \lvert 11\rangle$ $\lvert 11\rangle \mapsto \lvert 10\rangle$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Example:

Here is a different operation on two bits (X, Y): with probability $1/2$, set Y to be equal to X, otherwise set X to be equal to Y.

The matrix representation of this operation is as follows:

$$\begin{pmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1/2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Question:

Suppose we have two probabilistic operations, each on its own system, described by stochastic matrices:

1. M is an operation on X.
2. N is an operation on Y

If we simultaneously perform the two operations, how do we describe the effect on the compound system (X, Y)?

M, N are matrices.

Tensor product of matrices

Definition:

The tensor product of two matrices

$$M = \sum_{a,b \in \Sigma} \alpha_{ab} |a\rangle\langle b| \quad \text{and} \quad N = \sum_{c,d \in \Gamma} \beta_{cd} |c\rangle\langle d|$$

is the matrix

$$M \otimes N = \sum_{a,b \in \Sigma} \sum_{c,d \in \Gamma} \alpha_{ab} \beta_{cd} |ac\rangle\langle bd|$$

combined operation of X and Y is.

Example:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

This is stochastic matrix.

doing nothing is represented by Identity matrix.

Example:

Resetting a bit X to the 0 state and doing nothing to a bit Y yields this operation on (X,Y):

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Quantum states:

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems classical state sets.

Example:

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle \otimes |0\rangle^{Y} + \frac{i}{2}|0\rangle \otimes |1\rangle^{Y} - \frac{1}{2}|1\rangle \otimes |0\rangle^{Y} - \frac{i}{2}|1\rangle \otimes |1\rangle^{Y}$$

$$\frac{3}{5}|0\rangle \otimes |0\rangle^{Y} - \frac{4}{5}|1\rangle \otimes |1\rangle^{Y}$$

$$|0\rangle \otimes |1\rangle$$

$\boxed{OR}$

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

- Tensor products of quantum state vectors are also quantum state vectors.

Let $|\phi\rangle$ be a quantum state vector of a system X and let $|\psi\rangle$ be a quantum state vector of a system Y. The tensor product

$$|\phi\rangle \otimes |\psi\rangle$$

is then a quantum state vector of the system (X, Y). States of this form are called <u>product states</u>. They represent independence between the system X and Y.

Note: [one system X, Y in differet locations.
X declares ϕ, Y declare initialise ψ state
Comb (X, Y) forming single ψ. Even in two diff. labs.]

Entangled state:

Evaluate

$$|\phi\rangle \otimes |\psi\rangle = \underbrace{\frac{1}{\sqrt{2}}|00\rangle}_{\phi} + \underbrace{\frac{1}{\sqrt{2}}|11\rangle}_{\psi}$$

$$\langle 0|\phi\rangle\langle 1|\psi\rangle = \langle 01|\phi \otimes |\psi\rangle = 0$$

Substitute ϕ and ψ value

$$= \langle 01|\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\langle 01|00\rangle + \frac{1}{\sqrt{2}}\langle 01|11\rangle$$

$$= \frac{1}{\sqrt{2}}\underbrace{\langle 0|0\rangle}_{1} \times \underbrace{\langle 1|0\rangle}_{0} + \underbrace{\langle 0|1\rangle}_{0} \times \underbrace{\langle 1|1\rangle}_{1}$$

$$= 0$$

multiplying $\langle 0|\phi\rangle = 0$ or $\langle 1|\phi\rangle = 0$.
which contradicts the fact that

$$\langle 0|\phi\rangle\langle 0|\psi\rangle = \langle 00|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle 1|\phi\rangle\langle 1|\psi\rangle = \langle 11|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

are both nonzero.

$|00\rangle \longrightarrow |0\rangle \otimes |0\rangle$ not follows

Combinely two qubit represent state.

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no single qubit can't represent state.

- All quantum computation done in entangled state.

product state of two qubit

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq (a_0|0\rangle + b_0|1\rangle \otimes (a_1|0\rangle + b_1|1\rangle \otimes \cdots \otimes a_{n-1}|0\rangle + b_{n-1}|1\rangle)$$

$$|a_k|^2 + |b_k|^2 = 1.$$

2n Complex numbers
n pairs of Complex no.

$$(c_1 c_2 \cdots c_{2n}) + (c_1' + c_2' + \cdots + c_n')$$
$$= c_1 + c_1', c_2 + c_2' + \cdots c_n + c_n')$$

we will get sequence of 2n Complex no.

Bell state :-

Show that scalar product is zero for 1st two states.

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\phi^+\rangle|\phi^-\rangle = \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(\underbrace{|00\rangle|00\rangle}_{1} - \underbrace{|00\rangle|11\rangle}_{0} + \underbrace{|11\rangle|00\rangle}_{0} - \underbrace{|11\rangle|11\rangle}_{1})$$

$$= \frac{1}{\sqrt{2}}(1-1) = 0$$

$|\phi^+\rangle|\phi^-\rangle = 0$ Hence proof.

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Measurements of quantum states:

$$\langle a_1 a_2 a_n \rangle = |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_m^n\rangle$$

$\therefore$ m^n entries will be in basis state.

Measurements of compound systems work in the same way measurements of single systems provided that all of the system are measured.

If $|\psi\rangle$ a quantum state of a system $(X_1 \cdots X_n)$ and every one of the systems is measured, then each n-tuple

$$(a_1 \cdots a_n) \in \Sigma_1 \times \cdots \times \Sigma_n$$

(or string $a_1 \cdots a_n$) is obtained with probability

$$|\langle a_1 \cdots a_n|\psi\rangle|^2$$

Question:
Suppose two systems $(X\,Y)$ are together in some quantum state. What happens when we measure X and do nothing to Y?

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_{ab} |ab\rangle$$

If just X is measured, the probability for each outcome $a \in \Sigma$ to appear must therefore be equal to

$$P_r(\text{outcome is } a) = \sum_{b\in\Gamma} |\langle ab|\psi\rangle|^2 = \sum_{b\in\Gamma} |\alpha_{ab}|^2$$

Example:
suppose that (X,Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and X is measured.

Soln: We begin by writing

$$|\psi\rangle = \underset{\text{commn}}{\underset{=}{|0\rangle}} \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + \underset{\text{commn out}}{\underset{=}{|1\rangle}} \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

suppose that (X,Y) is in the state

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

and X is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\|\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right\|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

in which case the state of (X,Y) becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle}{\sqrt{\frac{3}{4}}} = |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)$$

The probability for the measurement to result in the outcome 1 is

$$\left\|\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right\|^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

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in which case the state of (X, Y) becomes

$$|1\rangle \otimes \frac{\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{1}{4}}} = |1\rangle \otimes \left(\frac{i}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Same method is used when Y is measured instead of X.

Y is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{5}{8}}} = \left(\sqrt{\frac{4}{5}}|0\rangle + \frac{i}{5}|1\rangle\right) \otimes |0\rangle$$

The probability for the measurement to result in the outcome 1 is

$$\left\| \frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{3}{8}}} = \left(\sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle\right) \otimes |1\rangle$$

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Unitary operations on multiple system:

Quantum operations on compound systems are represented by unitary matrices whose rows and columns correspond to the Cartesian product of the classical state sets of the individual systems.

$\langle \psi_1 | \psi_1 \rangle = 1$
$\langle \psi_2 | \psi_1 \rangle = 0.$

$$U \cdot U^{\dagger} = I$$

scalar product of itself = 1. (Normalization property)

Unitary operations preserves normalized state or preserving scalar product.

$U|x\rangle$ & $U|y\rangle$ then scalar product of these should be equal to $\langle x|y\rangle$.

$$\langle x | U^{\dagger} U | y \rangle = \langle x | y \rangle$$

$$U^{\dagger} U = I$$ ...... resolution Identity

U is a matrix and v is another matrix.

$$U \otimes V (|\phi\rangle \otimes |\psi\rangle) = U|\phi\rangle \otimes V|\psi\rangle)$$

$$\therefore [U^{\dagger}] = [U^{*}]^{T} \qquad (\ ) \otimes (\ )$$

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example:

Suppose that X and Y are systems that share the same classical state set Σ. The swap operation on the pair (X, Y) exchange the contents of the two systems:

$SWAP|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$

It can be expressed using the Dirac notation as follows:

$$SWAP = \sum_{a,b \in \Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

for instance, when X and Y are qubits, we find that

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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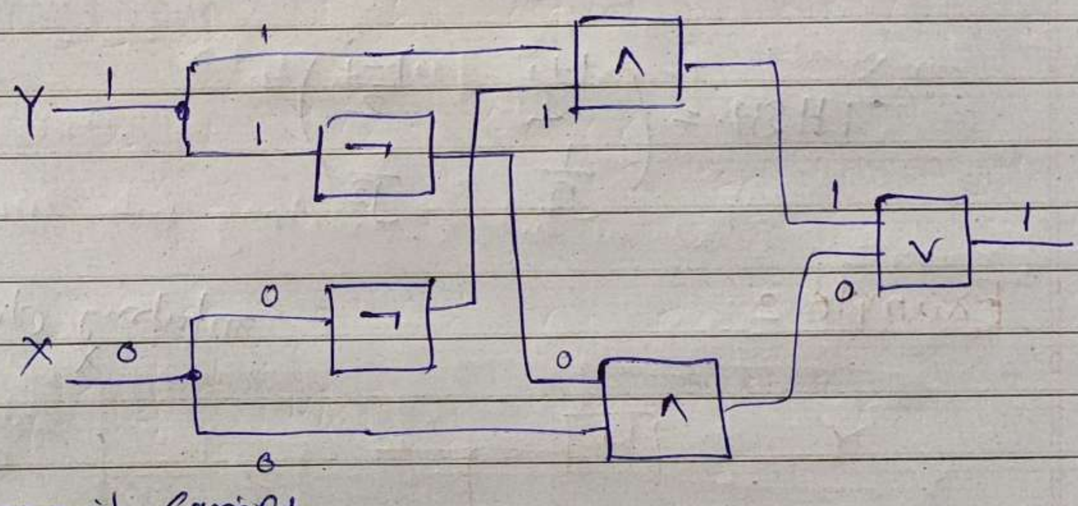
Quantum circuits.

Circuits are models of computation:
- Wires carry information
- Gates represent operations

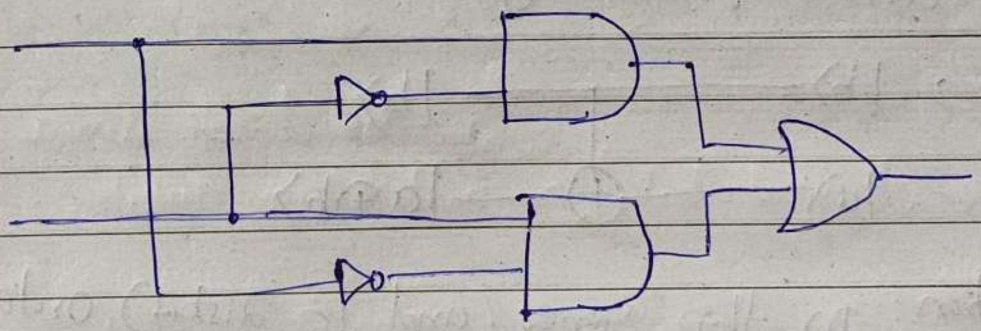
Circuits are always acyclic - information flows from left to right.

Example : Boolean Circuits.
Wires store binary values, gates represent Boolean logic operations such as AND ($\wedge$), OR ($\vee$), NOT ($\neg$) and FANOUT ($\cdot$).



Above circuit carries sequence of operⁿ ended with final value 1.

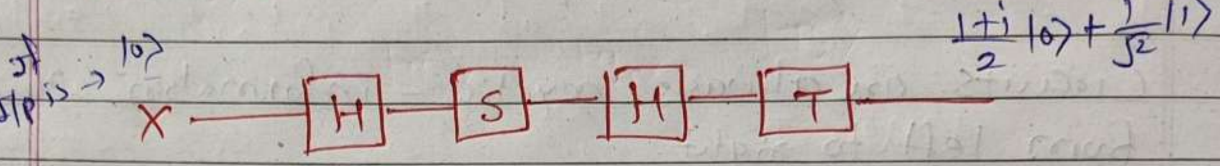
Example: Boolean Circuits.



Information flowing through circuit from left to right.

In the quantum circuit model, the wires represent qubits and the gates represent both unitary operations and measurements.

Example 1:

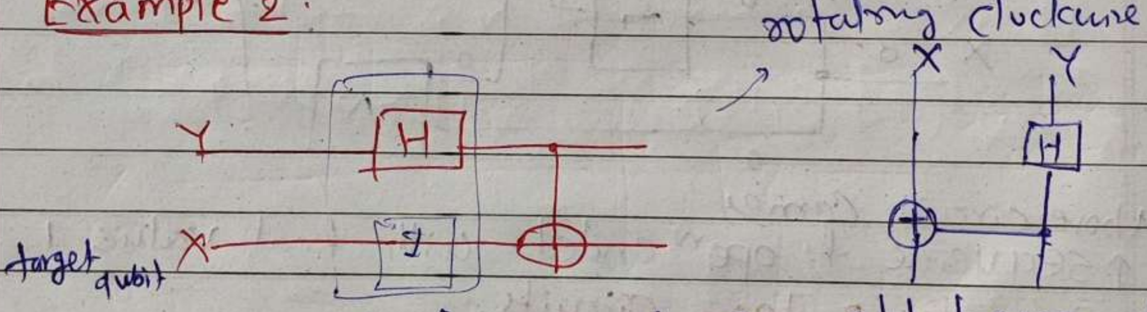


sequence of operation from left to right.

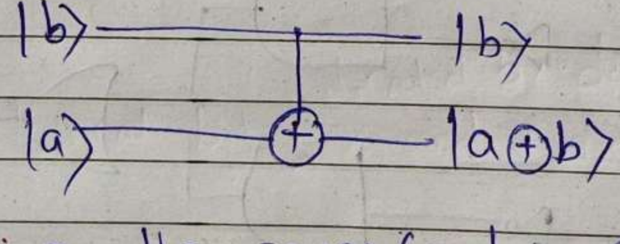
$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

$$THSH = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Example 2:



action of these gate on std. basis states



Convention: In this series (and in Qiskit), ordering qubits from bottom-to-top is equivalent to ordering them left-to-right.

$$I \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

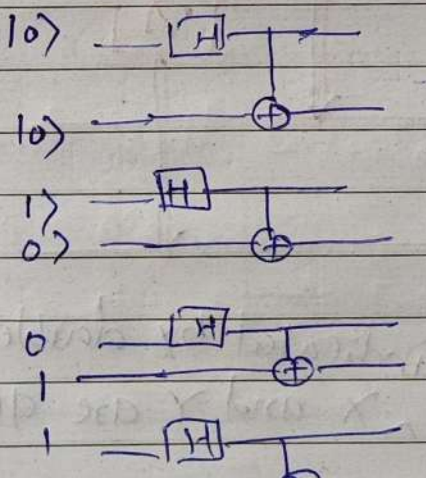
controlled not operation is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 & 0 \end{pmatrix}$$

If I/p



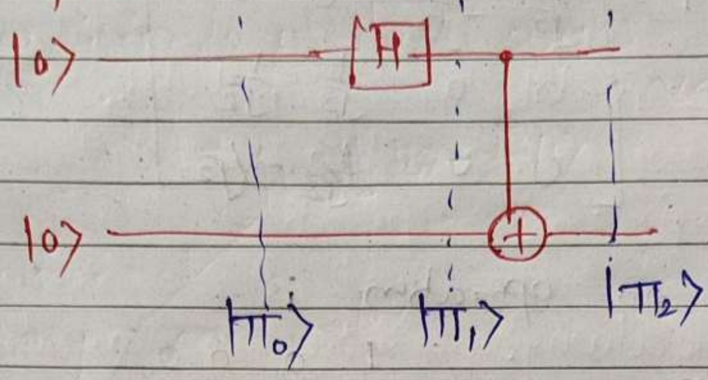
o/p $|\phi^+\rangle$

$\rightarrow |\psi^+\rangle$

$\rightarrow |\phi^-\rangle$

$\rightarrow -|\psi^-\rangle$

Example:

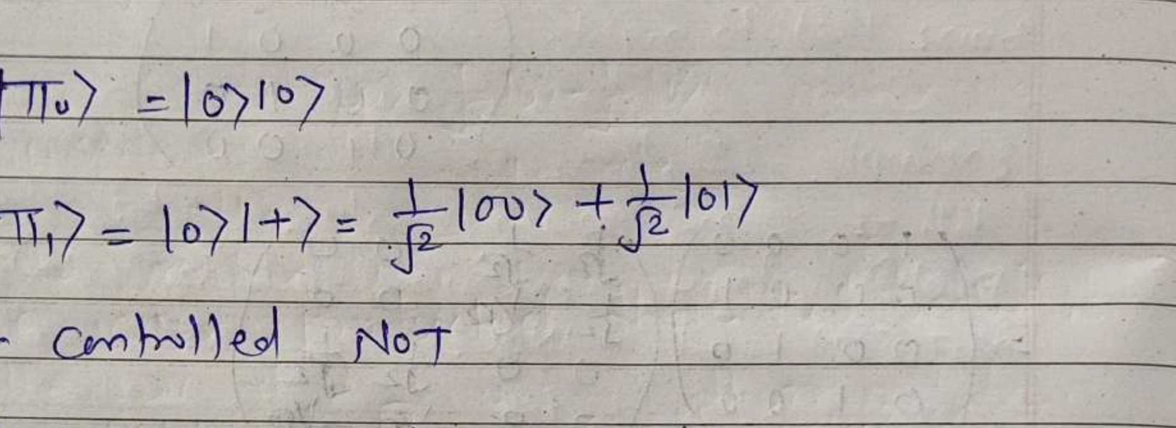


$$H \begin{cases} |\pi_0\rangle = |0\rangle|0\rangle \\ |\pi_1\rangle = |0\rangle|+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle \end{cases}$$

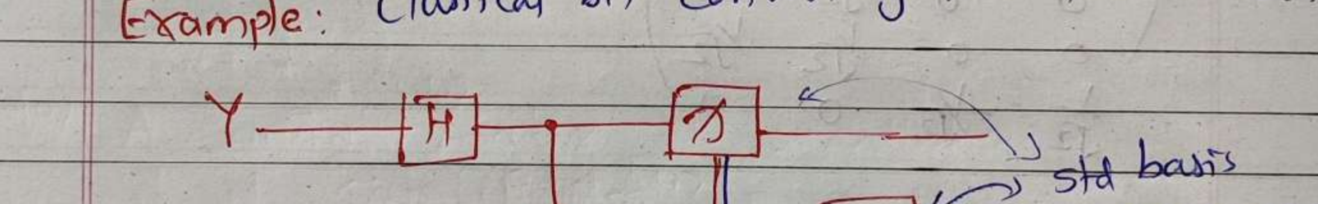
then controlled NOT

$$|\pi_2\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\phi^+\rangle$$

Example: classical bit ~~converting into~~ quantum ckt



A, B are classical bits indicated by double lines. X and Y are qubits.



Notations.

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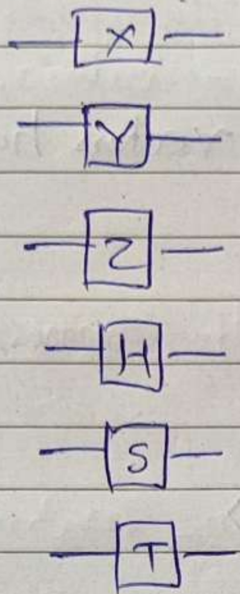
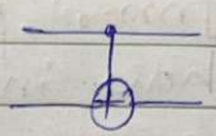
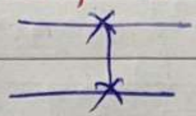
single qubit gates:-



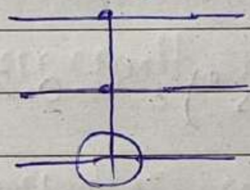
CONTROLLED-NOT



Swap Gate

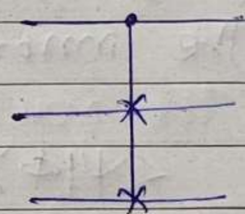


Toffoli gate



Controlled not gate

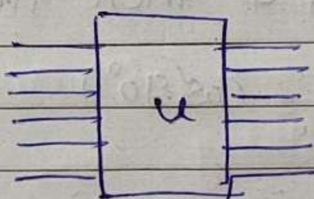
Fredkin Gate



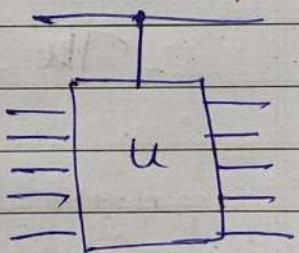
Controlled swap gate

It is also sometimes Convenient to view arbitrary unitary operations as gates.

Unitary operation



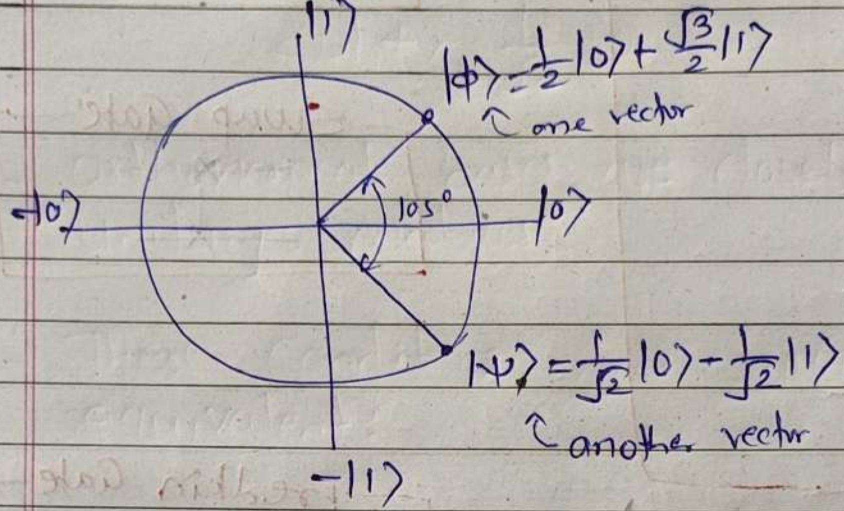
Controlled unitary operation



Inner products

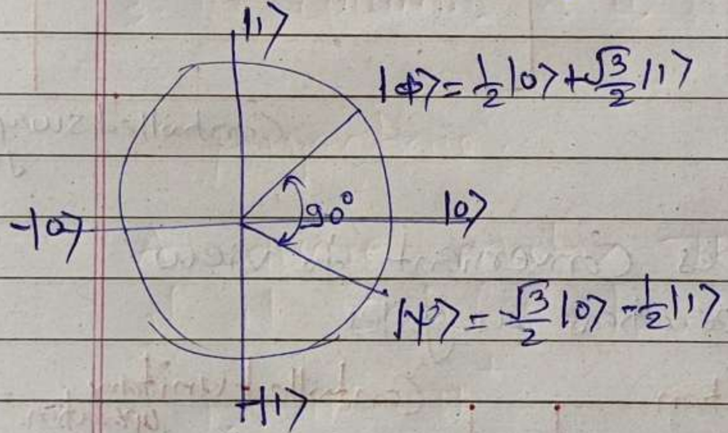
how inner product works

vectors having real no. entry.



The inner product of these two vector is

$$\langle\psi|\phi\rangle = \frac{1-\sqrt{3}}{2\sqrt{2}} = \cos(105°) = -0.2588$$



The inner product of these two vectors is

$$\langle\psi|\phi\rangle = 0 = \cos(90°)$$

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Orthogonality and orthonormality:

- Two vectors $|\psi\rangle$ and $|\phi\rangle$ are orthogonal if their inner product is zero:

$$\langle\psi|\phi\rangle = 0$$

- An orthogonal set $\{|\psi_1\rangle, \ldots |\psi_m\rangle\}$ is one where all pairs are orthogonal:

$$\langle\psi_j|\psi_k\rangle = 0 \quad (\text{for all } j \neq k)$$

- An orthonormal set $\{|\psi_1\rangle \ldots |\psi_m\rangle$ is an orthogonal set of unit vectors:

$$\langle\psi_j|\psi_k\rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad (\text{for all } j \neq k)$$

- An orthonormal basis $\{|\psi_1\rangle, \ldots |\psi_m\rangle\}$ is an orthonormal set that forms a basis (of a given space).

Example

for any classical state set Σ, the set of all standard basis vectors:

$$\{|a\rangle : a \in \Sigma\}$$

is an orthonormal basis.

Example

The set $\{|+\rangle, |-\rangle\}$ is an orthonormal basis for the 2-dimensional space corresponding to a single qubit.

Example

The Bell basis $\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle$ is an orthonormal basis for the 4-dimensional space corresponding to two qubits.

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Example:
The set $\{|0\rangle, |+\rangle\}$ is not an orthogonal set because

$$\langle 0|+\rangle = \frac{1}{\sqrt{2}} \neq 0$$

- orthonormal bases are closely connected with unitary matrices.

- These conditions on a square matrix U are equivalent:

1. The matrix U is unitary (i.e. $U^{\dagger}U = 1 = U \cdot U^{\dagger}$).

2. The rows of U form an orthonormal basis.

3. The columns of U form an orthonormal basis.

Projections:

A square matrix Π is called a projector if it satisfies two properties:

1. $\Pi = \Pi^{\dagger}$ ..... Hermitian
2. $\Pi^2 = \Pi$ .....

projecting image on screen, If u apply projection second time it do nothing.

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Example:
If $\{|\psi_1\rangle, \cdots |\psi_m\rangle\}$ is an orthonormal set, then this is a projection:

$$\Pi = \sum_{k=1}^{m} |\psi_k\rangle\langle\psi_k|$$

$$\Pi^{\dagger} = \left(\sum_{k=1}^{m} |\psi_k\rangle\langle\psi_k|\right)^{\dagger} = \sum_{k=1}^{m} \left(|\psi_k\rangle\langle\psi_k|\right)^{\dagger} = \sum_{k=1}^{m} |\psi_k\rangle\langle\psi_k| = \Pi$$

$$\Pi^2 = \sum_{j=1}^{m}\sum_{k=1}^{m} |\psi_j\rangle\langle\psi_j|\psi_k\rangle\langle\psi_k| = \sum_{k=1}^{m} |\psi_k\rangle\langle\psi_k| = \Pi$$

projective measurements

A collection of projections $\{\Pi_1, \cdots \Pi_m\}$ that satisfies
$$\Pi_1 + \cdots + \Pi_m = 1.$$
describes a projective measurement.

When such a measurement is performed on a system in the state $|\psi\rangle$, two things happen:

1. The outcome $k \in \{1, \cdots ; m\}$ of the measurement is chosen randomly:

$$Pr(\text{outcome is } k) = \|\Pi_k|\psi\rangle\|^2 = \langle\psi|\Pi_k|\psi\rangle$$
Π_k projection; ψ is unit vector

2. The state of the system becomes
$$\frac{\Pi_k|\psi\rangle}{\|\Pi_k|\psi\rangle\|}$$