Quantum Computing

Quantum Computing Basics.

Ket $= |0\rangle = \begin{vmatrix}1\\0\end{vmatrix}$ Bra $= \langle 0| = (1 \;\; 0)$

$|1\rangle = \begin{vmatrix}0\\1\end{vmatrix}$ $\langle 1| = (0 \;\; 1)$

Hilbert space :-

Complex ~~conjugate~~ Multidimentional vector space where inner product of any pair of elements is defined as exponential of 2D space to n dimensions and complex cofficient.

$\alpha\hat{i} + \beta\hat{j}$ :: Elements in complex Hilbert space is given by $\begin{vmatrix}\alpha\\\beta\end{vmatrix}$ ← Linear combination of orthogonal basis

complex numbers.

bra $\langle 0| = (1 \;\; 0)$

bra $\langle 1| = (0 \;\; 1)$

bra $\langle 0|$ is conjugat transpose of a Ket $|0\rangle$

Ket $= \begin{vmatrix}\alpha\\\beta\end{vmatrix} = |\alpha^*, \beta^*|$ ← bra

$\alpha = a + ib$ complex conjugate $a^* = a - ib$

Dagger.

Complex Conjugat $A = A^\dagger = \overline{A^T}$

$\alpha\hat{i} + \beta\hat{j}$ Complex No. Orthogonal unit vector.

Ket $|\psi\rangle = \begin{vmatrix}a_1\\a_2\\\vdots\\a_n\end{vmatrix}$ bra $\langle\phi| = [b_1^* \; b_2^* \; \cdots \; b_N^*]$

$\therefore$ bra-ket $= \langle\phi|\psi\rangle = \langle 0|0\rangle = [1 \;\; 0]\begin{bmatrix}1\\0\end{bmatrix} = 1$

This is $\langle 1|0\rangle = [0 \;\; 1]\begin{bmatrix}1\\0\end{bmatrix} = 0$

inner product (vector dot product.)

$\therefore$ bra-ket $= \langle\phi|\psi\rangle = b_1^* a_1 + b_2^* a_2 + \cdots + b_N^* a_N$

Outer product = Ket-bra $= \langle\psi|\phi\rangle = (b_1^* \; b_2^* \cdots b_N^*)\begin{bmatrix}a_1\\a_2\\\vdots\\a_N\end{bmatrix} = \begin{vmatrix}a_1 b_1^* & a_1 b_2^* & \cdots & a_1 b_N^*\\ a_2 b_1^* & \cdots & \cdots & \cdots\\ \vdots & & & \vdots\\ a_N b_1^* & \cdots & \cdots & a_N b_N^*\end{vmatrix}$

$|\psi\rangle \in H$

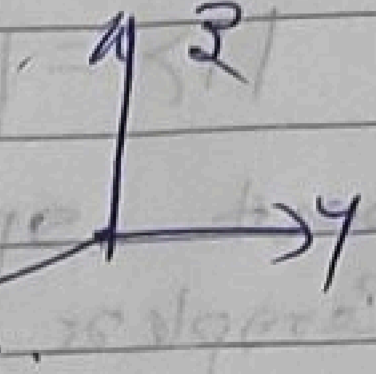
$\langle\psi|\psi\rangle = 1$ – Normalization cond^n.

unit vector with length = 1

if $|\psi\rangle \in H$, any $\lambda|\psi\rangle \in H$

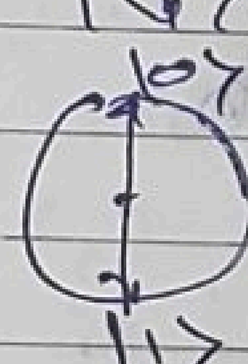
$|\psi\rangle$ & $\lambda|\psi\rangle$ belongs to same state

Example of 2 level quantum system.
plan of polarization is described by two perpendicular state.



Vertical $\updownarrow$ — $|V\rangle$
Horizontal $\rightarrow$ $|H\rangle$
$|P\rangle = \alpha|H\rangle + \beta|V\rangle$ = 2D complex linear vector space
Mathematically
$|H\rangle = |0\rangle$
$|V\rangle = |1\rangle$ | physical reflection of a qubit.

(Ex) Spin of 1/2 particle (e^- & protone) exist in one of the two spin state.
$|\uparrow\rangle - |0\rangle \quad \therefore |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$
$|\downarrow\rangle = |1\rangle \quad\quad \in \mathbb{C}^2$



Qubit state space
$\langle 0|1\rangle = \langle 1|0\rangle = 0$ orthogonal basis
$\langle 0|0\rangle = \langle 0|0\rangle = 1$ normalized state
$\mathbb{C}^2$ = 2D complex linear vector space.
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
$\langle\psi|\psi\rangle = (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$
$= |\alpha|^2 + |\beta|^2 = 1$
Parametrized ψ in terms of 2 angle
$\alpha = \cos\frac{\theta}{2} e^{i\phi_\alpha} \; ; \; \beta = \sin\frac{\theta}{2} e^{i\phi_\beta}$
$|\psi\rangle = \cos\frac{\theta}{2} e^{i\phi_\alpha}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi_\beta}|1\rangle$
(*) $= e^{i\phi_\alpha}\left[\cos\frac{\theta}{2}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi}|1\rangle\right] \quad \{\phi = \phi_\beta - \phi_\alpha$
$\therefore \psi$ can be represented by
$\theta \in [0, \pi] \quad \phi \in (0, 2\pi)$

Single Qubit

Vector in Complex 2D Hilbert Space.
General Qubit State (State vector)

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha \,\&\, \beta \in \mathbb{C}$$

$|0\rangle \,\&\, |1\rangle$ = Orthogonal basis states

Complex No. $\alpha = a + ib$ OR $re^{i\theta}$ = polar coordinates

Modulus $\therefore |\alpha| = \sqrt{a^2+b^2}$ OR r

Qubit Probability:
probability of qubit in standard form
$\{|0\rangle\, |1\rangle\}$

$$\therefore P(|q\rangle = |0\rangle) = |\langle 0|q\rangle|^2$$
$$= |\langle 0|(\alpha|0\rangle + \beta|1\rangle)|^2$$
$$= |\alpha\underbrace{\langle 0|0\rangle}_{1 = \text{Normalized}} + \beta\underbrace{\langle 0|1\rangle}_{0 = \text{orthogonal state}}|^2$$

$$\therefore P(|q\rangle = |0\rangle) = |\alpha|^2$$

$$P(|q\rangle = |1\rangle) = |\langle 1|q\rangle|^2$$
$$= |\langle 1|(\alpha|0\rangle + \beta|1\rangle)|^2$$
$$= |\alpha\underbrace{\langle 1|0\rangle}_{0} + \beta\underbrace{\langle 1|1\rangle}_{1}|^2$$

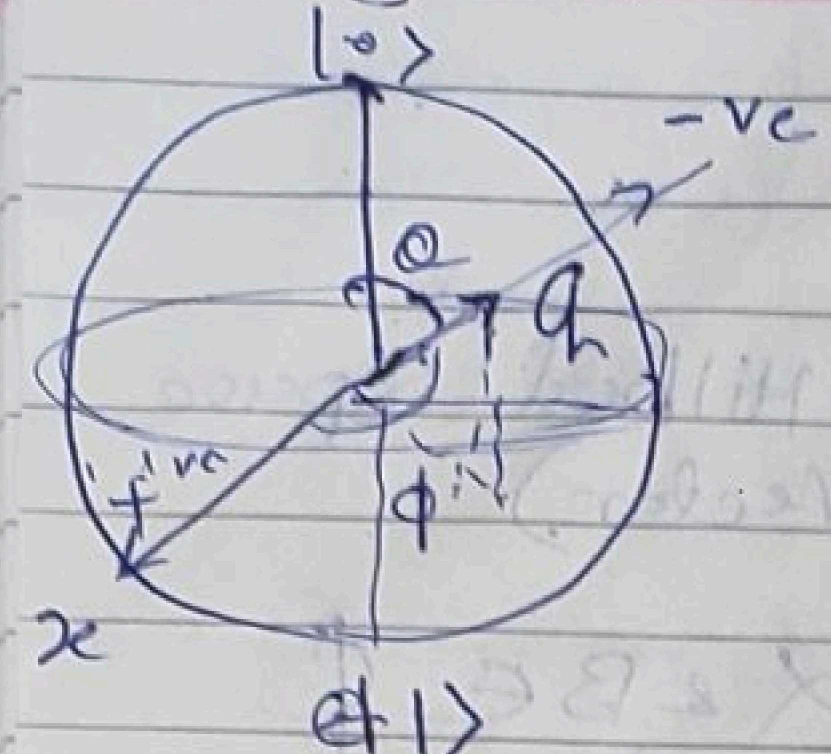
$$P(|q\rangle = |1\rangle) = \beta^2$$

$\therefore |\alpha|^2 + |\beta|^2 = 1 \qquad \therefore \langle q|q\rangle = 1$ Normalized

Probability summing '1'

α & β are probability amplitude

Qubid Represent^n in polar for.



$|q\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha = r_1 e^{i\theta_1}$

$\beta = r_2 e^{i\theta_2}$

As q is normalized

$\boxed{r_1^2 + r_2^2 = 1}$

	0	30	45	60	90
Cos θ	1	$\frac{\sqrt{3}}{2}$	$1/\sqrt{2}$		0
Sin θ	0	$\frac{1}{2}$		$\frac{\sqrt{3}}{2}$	1

	0	30	45	60	90 $\pi/2$
Cos					
Sin	0	$1/2$	$1/\sqrt{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$1/\sqrt{2}$	$1/2$	0

$r_1 = \cos\frac{\theta}{2}$ $\quad r_2 = \sin\frac{\theta}{2}$

Represent^n of q to θ towards north pole.

if $\theta = 0$

$\cos\theta = 0$

$\therefore r_1 = 0$ & q becomes 0

$\alpha = 0$.

π $\quad \sin\frac{\pi}{2} = 0$

$\therefore r_2 = 0$ due to $\beta = 0$

Complex part

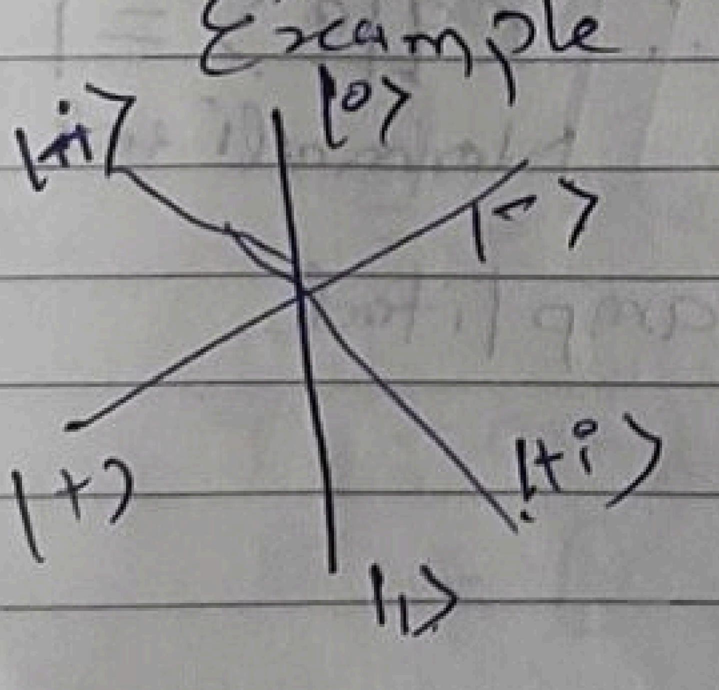
$\theta_2 = \theta_1 + \phi$ relative phase.

$\therefore |q\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\theta}\sin\frac{\theta}{2}|1\rangle$

q makes θ angle with z axis & take project^n so angle with x axis φ

Qubit relative phase (Rotation around z axis has no analogous in classic computing like superposition.

Example of Single Qubits :



$|0\rangle = \begin{vmatrix}1\\0\end{vmatrix}$ $\quad |1\rangle = \begin{vmatrix}0\\1\end{vmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ 1/\sqrt{2}\end{pmatrix}$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$

$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \begin{vmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{vmatrix}$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = \begin{vmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{vmatrix}.$$

Measuring state $|-\rangle$ state

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

on the basis $\{0, 1\}$

$$P(|-\rangle \,|\, |0\rangle) = |\langle 0|-\rangle|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

α.

$$P(|-\rangle \,|\, |1\rangle) = |\langle 1|-\rangle|^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

β

$$P = \frac{1}{2} + \frac{1}{2} = 1$$

OR

$$\langle 0|-\rangle = |0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(\underbrace{\langle 0|0\rangle}_{1} - \frac{1}{\sqrt{2}}\underbrace{\langle 0|1\rangle}_{0})$$

$$= \frac{1}{\sqrt{2}} = \alpha$$

$$\langle 1|-\rangle = |1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(\underbrace{\langle 1|0\rangle}_{0} - \frac{1}{\sqrt{2}}\langle 1|1\rangle)$$

$$= \frac{1}{\sqrt{2}} - 1/\sqrt{2} = \beta$$

$$= \alpha^2 + \beta^2 = 1$$

$\Rightarrow$ Measuring state $|+\rangle$ state on the basis $\{0, 1\}$

$$\langle 0|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\underbrace{\langle 0|0\rangle}_{1} + \frac{1}{\sqrt{2}}\underbrace{\langle 0|1\rangle}_{0}$$

$$= \frac{1}{\sqrt{2}} = \alpha$$

$$\langle 1|+\rangle = \frac{1}{\sqrt{2}}\langle 1|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}(\underbrace{\langle 1|0\rangle}_{0} + \underbrace{\langle 1|1\rangle}_{1}) = \frac{1}{\sqrt{2}} = \beta$$

$\therefore$ It is observed that probability for $|+\rangle$ & $|-\rangle$ are identical for standard basis

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

Measuring State

$|\psi\rangle = \frac{1+2i}{\sqrt{5}}|0\rangle - \frac{2}{\sqrt{5}}|1\rangle$

On standard basis

$P(|\psi\rangle \text{ on } |0\rangle) = |\langle 0|\psi\rangle|^2 = \left(\frac{1+2i}{\sqrt{5}}\right)^2 = \frac{1+4}{5} = \frac{5}{5}$

$P(|\psi\rangle \text{ on } |1\rangle) = |\langle 1|\psi\rangle|^2 = \left(\frac{-2}{\sqrt{5}}\right)^2 = \frac{4}{5}$

$P(\langle\psi|0\rangle) + P(\langle\psi|1\rangle) = \cancel{\frac{4}{5}} + \frac{5}{5} = 1$

→ Consider state $|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{2}}|\psi\rangle$

Calculate probability $= \cancel{0}1$

Measuring state $|0\rangle$ & $|1\rangle$

$P(|\psi\rangle \text{ on } |0\rangle) = |\langle 0|\psi\rangle|^2 = \left|\left(\frac{2}{\sqrt{5}}\right)\right|^2 = \frac{4}{5}$

$P(|\psi\rangle \text{ on } |1\rangle) = |\langle 1|\psi\rangle|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5}$

$\frac{4}{5} + \frac{1}{5} = \cancel{0}1$

for Measuring Quantum state we have to extract classical informa^n from the Quantum state with specific probability.

This process is governed by Born rule which state that probability of measuring a particular classical state is the absolute value squared of entry in the quantum state vector coordinating to that state.

This superposition is a fundamental aspect of Q. computing allowing qubits to represent & process information in way that classical bits cannot.

* Find the matrix representation of
(1) $|0\rangle\langle 0|$ (2) $|0\rangle\langle 1|$ (3) $\langle 1|0\rangle$ (4) $\langle 1|1\rangle$
in $\{|0\rangle, |1\rangle\}$ basis.

(1) $|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}$ (2) $|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$ (3) $\langle 1|0\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

Physical quantities

Linear transformation operator → Hermitian operator, $H = (H^*)^T = H^\dagger$ dagger
Linear transformation operator → Unitary operator

H is self adjoint Matrix.
$H^\dagger$ = complex conjugate transpose

(1) Hermitian Matrix have real eigen value
$\cancel{H|x}$ $H|V\rangle = \lambda|V\rangle$ λ eigen value, $|V\rangle$ eigen state of H

(2) H. Matrices are diagonalized
$\begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}_{2\times 2}$ admit a diagonal form in orthogonal basis.

Examples of Hermitian operations for 2 level System.

Matrix Representation

outer product

$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1.1 & 1.0\\0.1 & 0.0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

$|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1.0 & 1.1\\0.0 & 0.1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$

$\langle 1|0\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0.1 + 1.0 = 0$

$\langle 1|1\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0.0 + 1.1 = 1$

inner product

do $|1\rangle\langle 0|$
$|1\rangle\langle 1|$

Spin along y axis

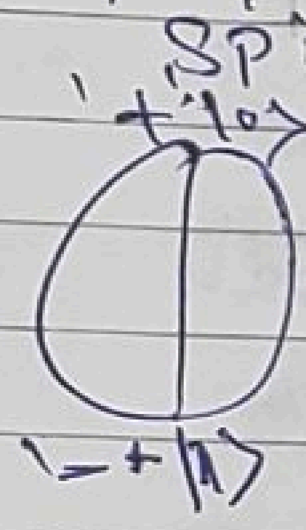
$$S_y = \frac{\hbar}{2}\left(|+i\rangle\langle +i| - |i\rangle\langle -i|\right)$$

Examples of Hermitian operation for 2 level system

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (S_z^{*})^{T} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_z$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (S_x)^{\dagger} = S_x$$

Physical qualities



Spin $-\frac{1}{2}$ system: $|0\rangle \equiv |\pm \tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = |\mp \tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_z|0\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-\hbar}{2}|0\rangle$$

$$S_z|1\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-\hbar}{2}|1\rangle$$

i.e. S_z must be associated with spin along z direc^n.

$+\hbar/2$ $|0\rangle$ $\quad -\hbar/2$ $|1\rangle$

$$\Rightarrow S_x|0\rangle = \frac{\hbar}{2}|1\rangle$$
$$S_x|1\rangle = \frac{\hbar}{2}|0\rangle$$
Not eigen value equa^n

$$S_x(|+\rangle) = S_x\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}\right)(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}\right)\left(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{1}\right) = \frac{\hbar}{2}|+\rangle$$

$$S_x|-\rangle = \frac{-\hbar}{2}|-\rangle$$

S_x is a observable correspond to component along x-direct^n

$\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)$ $\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)$

$\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\begin{pmatrix}0&0\\0&1\end{pmatrix}$

Outer Product

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$\begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$= \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)$$

Unitary Operations.
which satisfied,

$$U^\dagger U = I = U U^\dagger$$

identity M.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \Rightarrow H|0\rangle = |+\rangle = \begin{vmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{vmatrix}$$

$$H|1\rangle = |-\rangle = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$\Rightarrow H^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = H = H \text{ is Hermiton. M.}$$

$$HH^\dagger = H^2 = \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_{2\times 2}$$

$$H^\dagger H = H^2 = I$$

$|\psi(t_0)\rangle$ = state of a system at time 't_0'

- state at time 't' is obtained by the action of the arbitary operator on the initial state

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$$

Hamition operator

H operator associated with energy of a system

Quantum state description with complex no. entries
Quantum information is represented by density Matrix. So it is essential to study

Q. Information used to model the effect of noise on one quantum computation or state of one piece of an entangled pair.
D. Matrix is for Quantum information theory & Q. cryptography.

Classical information is relevant to an introduction of quantum information.
bit has classical state 0 & 1.
6 die has states 0, 1, 2, 3, 4, 5, 6.
fan has 4 state 'ON', 'OFF', 'high', 'low'.

∴ X is a bit $= \Sigma = \{0, 1\}$.
die $= \Sigma = \{1, 2, 3 \ldots 6\}$. same with fan.

if X is a bit & '0' with probability $3/4$ then state '1' with prob. $1/4$

$$Pr(X=0) = 3/4 \quad \& \quad Pr(X=1) = 1/4$$

Column vector satisfied two properties
1. All entries non zero.
2. The sum of entries is equal to 1.

Classical operation
It is deterministics where each classical state $a \in \Sigma$ is transformed into $f(a)$ for some function 'f' of the form.

$$f : \Sigma \to \Sigma$$

e.g. $\Sigma = \{0, 1\}$ there are '4' function.
f_1, f_2, f_3 & f_4 which are represented by

a	$f_1(a)$	a	$f_2(a)$	a	$f_3(a)$	a	$f_4(a)$
0	0	0	0	0	1	0	1
1	0	1	1	1	0	1	1

for a = 1, the '0' i/p flipped to '1' & '1' to '0'

f^n 1st & 4th are constant

$f_1(a) = 0$ & $f_4(a) = 1$

$f_2(a) = a$ for $a \in \Sigma$ = f_2 is identity f^n

$f_3(a)$ is fun^n $f_3(0) = 1$, $f_3(1) = 0$ = NOT fun

The action of deterministic opera^n on probabilistic state can be represented by Matrix - vector multiplication.

M represent a given fun^n.

$f: \Sigma \rightarrow \Sigma$ which satisfies

$M|a\rangle = |f(a)\rangle$ for every $a \in \Sigma$ such as Matrix always exist & is unique.

M_1, M_2, M_3, M_4 corresponds to fun^n $f_1 f_2 f_3 f_4$

$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

Matrices with exactly '1' in one column n rest of others are zero

$\Sigma = \{0, 1\}$ Bra $\langle 0| = (1\ 0)$

$\langle 1| = (0\ 1)$

$n = 2 \therefore 2^n = 2^2 = 4$ Combination.

$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} (0\ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{2\times 2}$

from above fun^n with '1 bit i/p of a'

a	$f_1(a)$	a	$f_2(a)$	a	$f_3(a)$	a	$f_4(a)$
0	0	0	0	0	1	0	1
1	0	1	1	1	0	1	1

$M_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $M_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $M_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $M_4 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$M_1 M_2 M_3 M_4$ corresponds to fun^n $f_1 f_2 f_3 f_4$

$f_1|0\rangle = 1|0\rangle + 0|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{matrix} 1 \\ 0 \end{matrix}$

$f_1|1\rangle = 1|0\rangle + 0|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{matrix} 0 \\ 0 \end{matrix} = \begin{matrix} 1 \\ 0 \end{matrix}$

$$\underset{M_1}{M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}} \underset{f_1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ same.

$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f_2(|0\rangle) = 1|0\rangle + 0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f_2(|1\rangle) = 1|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Identity

$M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $f_3(|0\rangle) = 1|1\rangle$

$f_3|1\rangle = 0\,1|0\rangle$

$f_3(a)$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

complement

$f_4(a)$

$M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ = Same

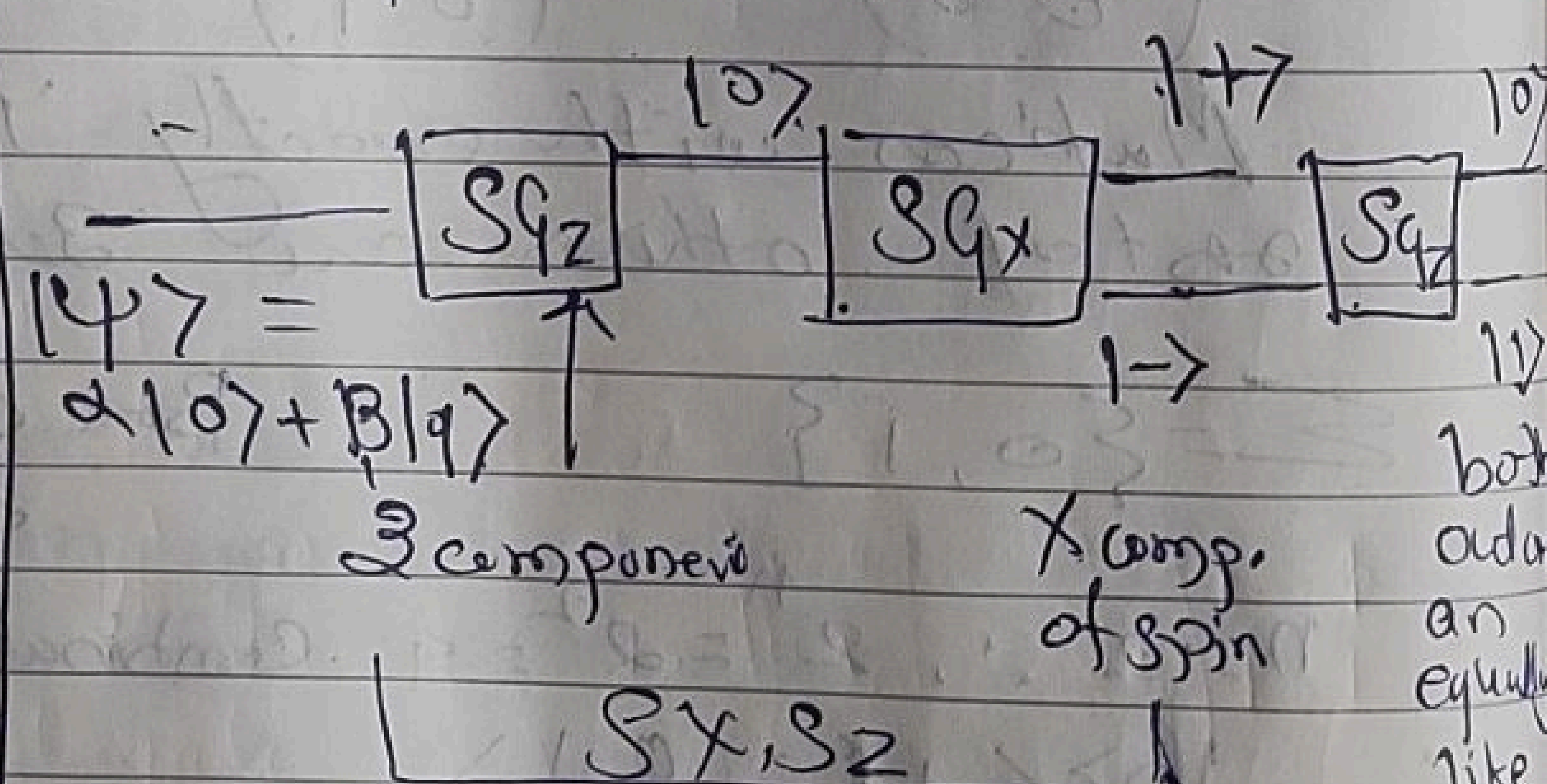
Uncertainty Principle.

$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$

$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$



2 different component of spin which are not compatible, both state are definite

In state S_z, eigen state $|+\rangle$ & $|-\rangle$ outcomes are equally like.

2. State S_x, $|0\rangle$ & $|1\rangle$, both are equally likes

It is not possible to have same state which has a definite value for both S_x & S_y S_z.

Uncertainty Principle:-
S_x & S_z are incompatible with each other but it is not possible to have a state which has a definite value for both S_x & S_y.

Quantum Gates

Quantum gates nothing but unitory operators, need basis states.
Three types:- Single, 2 qubit & 3 qubit

Single Qubit :- Basis $= \{|0\rangle, |1\rangle\}$

$2^0 = 0 \quad 2^0 = 1$

~~write Basis~~ as a column — Basis $= \left\{ \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \end{vmatrix} \right\}$

get Identity Matrix

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_2$

- 2 Qubit Basis State :-

tensor product

$\left\{ \begin{matrix} |0\rangle \otimes |0\rangle & |0\rangle \otimes |1\rangle & |1\rangle \otimes |0\rangle & |1\rangle \otimes |1\rangle \\ |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ 2^1 \times 0 + 2^0 \times 0 & 2^1 \times 0 + 2^0 \times 1 & ② & 2^1 \times 1 + 2^0 \times 1 \\ = 0 & = ① & & = 2+1 = ③ \end{matrix} \right\}$

In Digital	A	B
	0	0
	0	1
	1	0
	1	1

$|0\rangle \otimes |0\rangle = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$

$\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$

∴ Basis states as column

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_4 = D[1111] = I_4$

⇒ 3 Qubit states Basis $= \{ \overset{0}{|000\rangle}\ \overset{1}{|001\rangle}\ \overset{2}{|010\rangle}\ \overset{3}{|011\rangle}\ \overset{4}{|100\rangle}\ \underset{5}{|101\rangle},\ \underset{6}{|110\rangle}\ \underset{7}{|111\rangle} \}$

$|0\rangle$ to $|2^3 - 1\rangle$

Column Mat

$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & & & & & & & 1 \end{vmatrix}_8 = D[11111111] = I_8)$

Single Quantum Gates → Represented by Unitary operator.

dagger.

$UU^{\dagger} = I \Rightarrow U^{-1} = U^{\dagger}$ | ~~dte~~ $\det U = 1$ | columns satisfies orthogonal condn.

Self product = 1

two different prod = 0 = orthogonal

Any single Qubit Quantum Gate is given by

$U = e^{i\alpha} R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}(\vec{\sigma}\cdot\hat{n})}$ = Rotational operator about arbitary dimension

$U = e^{i\alpha}\left[I\cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}\right]$

let $\alpha = \frac{\pi}{2}$ & $\theta = \pi$

$U = \underbrace{e^{i\pi/2}}_{i}\left[I\underbrace{\cos\frac{\pi}{2}}_{0} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\pi}{2}\right) = (\vec{\sigma}\cdot\hat{n})$

$U = \vec{\sigma}\cdot\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$

'n' with different values of n

$n = (1,0,0)$ $n = (0,1,0)$

$n = (0,0,1)$

$n = (1,0,0)$: $U = \sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ $n = (0,1,0)$: $U = \sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$ $n = (0,0,1)$: $U = \sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$-\boxed{X}-$ $-\boxed{Y}-$ $-\boxed{Z}-$

σ_x Pauli 'X' σ_y Pauli Y σ_z Pauli Z

$U = \vec{\sigma}\cdot\hat{n}$

with $\hat{n} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ ∴ $U = \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z$

$= \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = H$

Hadarmad gate

$H = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ ∴ $H^2 = I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

Pauli Gates:

(1) Pauli X

σ_x	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	1
$\langle 1\|$	1	0

outer product form.

1 → for 1 = $\langle 0|$ & $|1\rangle$ make
$\langle 0| \Downarrow |0\rangle$, $|1\rangle \Downarrow \langle 1|$

$\langle 1|$ make $|1\rangle$, $|0\rangle \Downarrow \langle 0|$

$$\boxed{X = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

outer product form of pauli X

$$X|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle$$

$$= \underbrace{\langle 1| * |0\rangle}_{0} \quad \underbrace{\langle 0| * |0\rangle}_{1}$$

$$X|0\rangle = |1\rangle$$

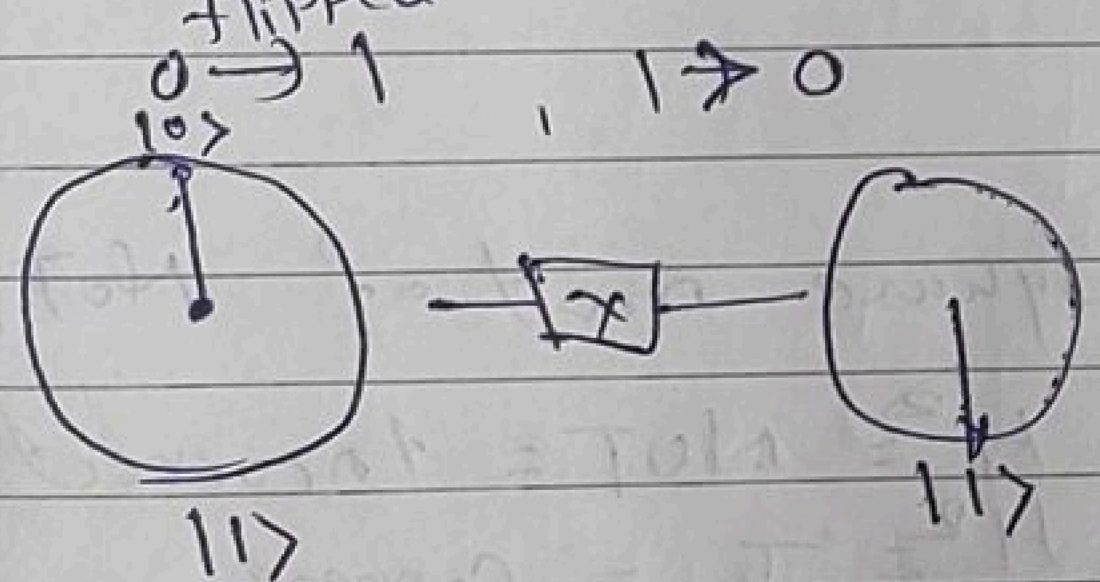
it is given by $|0\rangle$ —[X]— $|1\rangle$

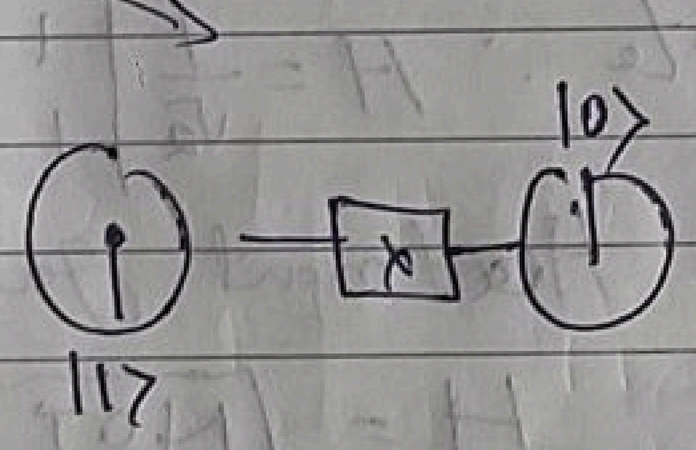
Similarly

$$X|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle$$

$$X|1\rangle = |0\rangle \quad \therefore \quad |1\rangle —[X]— |0\rangle$$

flipped
0 → 1 , 1 → 0





Quantum NOT gate = Pauli X

Pauli Y - σ

σ_y	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	$-i$
$\langle 1\|$	i	0

$$Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$Y|0\rangle = (-i|0\rangle\underbrace{\langle 1|}_{0} + i|1\rangle\underbrace{\langle 0|}_{1})|0\rangle$$

$$Y|0\rangle = i|1\rangle \quad , \quad Y|1\rangle = -i|0\rangle$$

$|0\rangle$ —[Y]— $i|1\rangle$ $\qquad$ $|1\rangle$ —[Y]— $-i|0\rangle$

pauli Z

σ_z	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	1	0
$\langle 1\|$	0	-1

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$Z|0\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle$$

$$\boxed{Z|0\rangle = |0\rangle}$$

$$Z|1\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle = -|1\rangle$$

$$Z|1\rangle = -|1\rangle = \text{phase changed}$$

$|0\rangle - \boxed{Z} - |0\rangle \qquad |1\rangle - \boxed{Z} - -|1\rangle$ — phase change

④ **Hadmard Gate**

H	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\langle 1\|$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$$H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \times |0\rangle \left[|0\rangle\langle 0| + |0\rangle \cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$ gives ⇒ superposition

$$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

∴ $|0\rangle - \boxed{H} - 1/\sqrt{2}[|0\rangle + |1\rangle]$ | $|1\rangle - \boxed{H} - 1/\sqrt{2}(|0\rangle - |1\rangle)$

∴ Hadmard gate gives linear combination of superposition state.

Note: $H = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

Hadmard gate is square root of NOT gate.

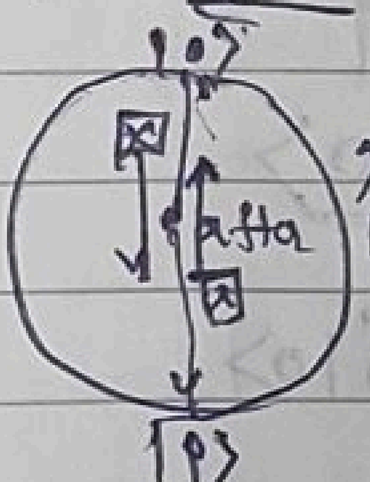
$H = \sqrt{NOT}$ but $H^2 = NOT$ = Incorrect

$H^2 = I$ = Correct.

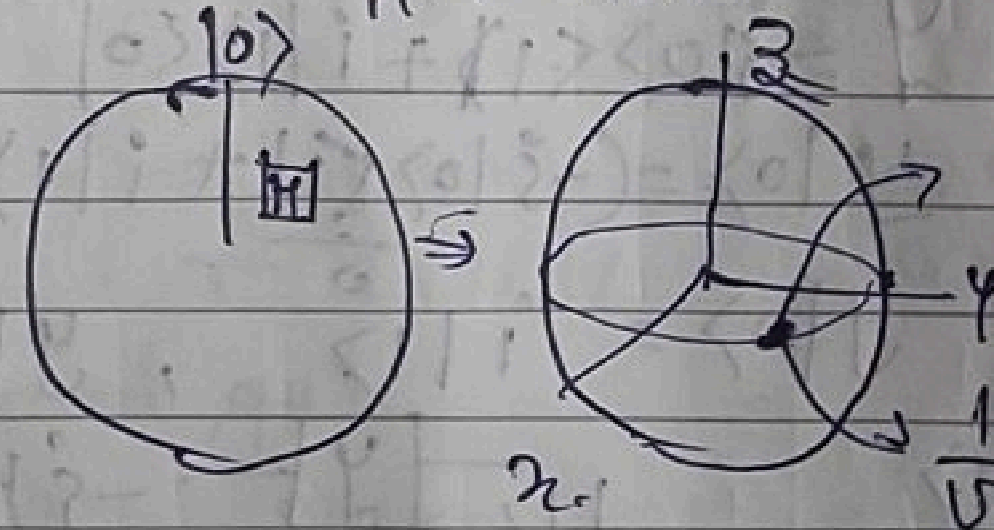
$H = \sqrt{NOT}$ why?

NOT

$|0\rangle - \boxed{X} - |1\rangle$



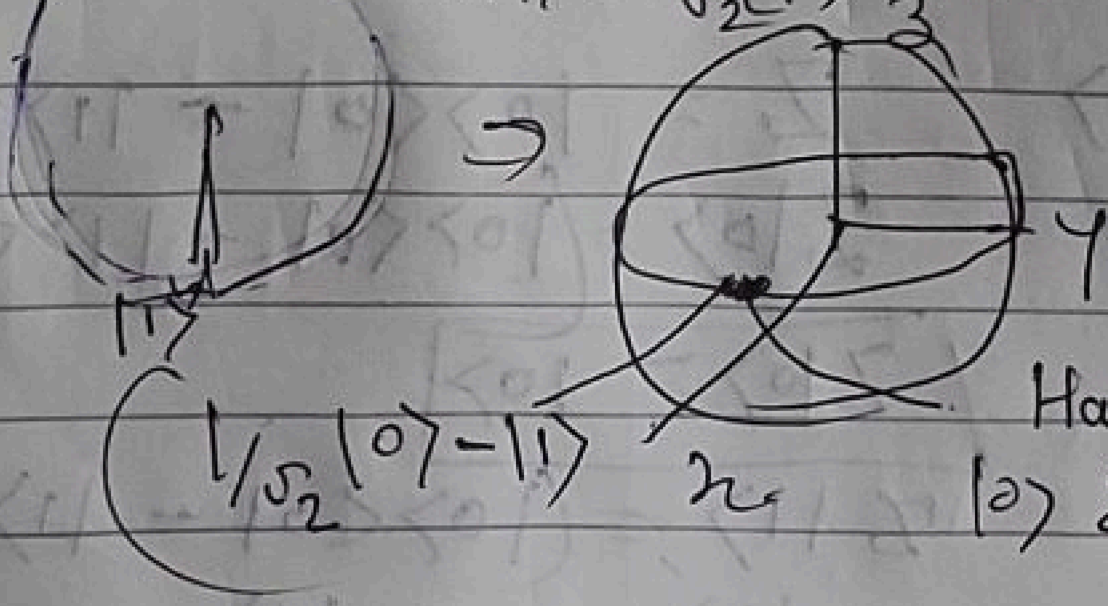
$H|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$



State is in half way bet^n

$\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$

$H|1\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$



So $H = \sqrt{NOT}$

$1/\sqrt{2}\,|0\rangle - |1\rangle$

Half way b/w $|0\rangle$ & $|1\rangle$

$$H = \frac{1}{\sqrt{2}}[X + Z] = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$H|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ $\qquad H|1\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$

$$H|x\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^x|x\rangle\right]$$

Operating H Gate

$H|0\rangle = \frac{1}{\sqrt{2}}$ $\quad H = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

tensor product

$|0\rangle - \boxed{H} -$
$|0\rangle - \boxed{H} -$ $\Big\} \psi_{o/p} = \Rightarrow \psi_f = H|0\rangle \otimes H|0\rangle$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$|0\rangle - \boxed{H} -$
$|0\rangle - \boxed{H} -$ $\Big\} \frac{1}{(\sqrt{2})^2}\sum_{0=0}^{2^{2}-1}|j\rangle$

$$\psi_f = \frac{1}{\sqrt{2}^2}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$= \frac{1}{(\sqrt{2})^2}\left\{\underset{0}{|00\rangle} + \underset{1}{|01\rangle} + \underset{2}{|10\rangle} + \underset{3}{|11\rangle}\right]$$

linear combination of all the numbers lying betn 0 & 3

$\vdots$

$|0\rangle - \boxed{H} -$
$|0\rangle - \boxed{H} -$
$\vdots$
$|0\rangle - \boxed{H} -$ $\Big\} \frac{1}{(\sqrt{2})^n}\sum_{j=0}^{2^{n-1}}|j\rangle$

$\Rightarrow$ Random no. lies in betn '0' to 2^{n-1}

In Classical Comp

eg. If you have to search phone of 'XYZ' from 10'000 entries, you have to search 5000 entries at least.

You may 1st get it in first attemp or may be in last attempt so at least half of entries might search.

$\frac{1000}{1+last}$
2

But in Q. Compu, only $\sqrt{10000}$ searches you have to performed for finding number

IST Gate. $R_{\hat{n}}(\phi) = e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}}$

$\hat{n} = \hat{x}$

About x-axis

$R_x(\phi) = e^{-i\frac{\phi}{2}\sigma_x}$

$= I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_x$

$= \begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} 0 & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & 0 \end{bmatrix}$

$R_x(\phi) = \begin{bmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & 0 \end{bmatrix}$

Rotation operator about X axis

About Y axis

$R_y(\phi) = e^{-i\frac{\phi}{2}\sigma_y}$

$= I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_y$

$= \begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} 0 & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & 0 \end{bmatrix}$

$R_y\phi = \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$

$R_y(\phi)$

About Z axis

$R_z(\phi) = e^{-i\frac{\phi}{2}\sigma_z}$

$= I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_z$

$= \begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} i\sin\frac{\phi}{2} & 0 \\ 0 & -i\sin\frac{\phi}{2} \end{bmatrix}$

$R_z\phi = \begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}$

$R_z(\phi)$

$R_z(\phi) = \begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix} \quad \begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix} \quad \cancel{\begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}}$

Add a global phase multiply with $e^{i\phi/2}$

$R_z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \quad R_z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$ — operation remains same but there is change of global phase.

let $\phi = 2\pi$ — $\boxed{I}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

—$\boxed{I}$—

ψ_i —$\boxed{I}$— ψ_i

let $\phi = \frac{\pi}{2}$ — $\boxed{S}$ —

$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

—$\boxed{S}$—

$S^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix}$

let $\phi = \pi/4$ —$\boxed{T}$— $= T = \sqrt{S}$

$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{(i\pi/2)^{1/2}} \end{bmatrix}$

—$\boxed{T}$—

$T^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$

$T = \sqrt{S}$

$e^{-i\frac{\pi}{2}}$ — $e^{(-i\pi/2)^{1/2}}$

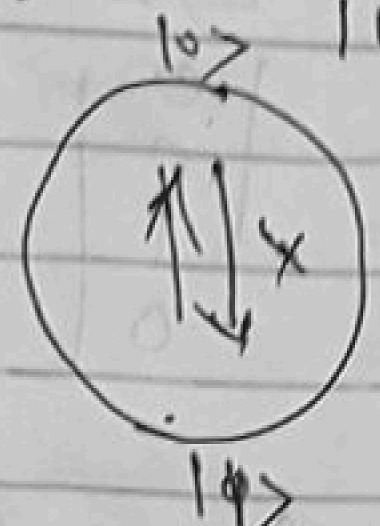
$S \longrightarrow T$

so $T = \sqrt{S}$

NOT Gate

Quantum NOT gate $\equiv$ Pauli X gate

$|0\rangle - \boxed{X} - |1\rangle \qquad |1\rangle - \boxed{X} - |0\rangle$



$\Rightarrow$ Pauli X-gate is giving opposite points on bloch sphere.

Let us consider arbitary state

case I $\quad \psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

$X = (|0\rangle\langle 1| + |1\rangle\langle 0|)$ Pauli X operaⁿ on arbitary state

$X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$

$X|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle = (1)$

(Pauli X is not acting as universal NOT gate)

case (2)

$\psi_{opp} = \begin{cases} \theta = \pi - \theta \\ \phi = \pi + \phi \end{cases} \quad \psi_{opp} = \cos\frac{\pi-\theta}{2}|0\rangle + e^{i(\pi+\phi)}\sin\frac{\pi-\theta}{2}|1\rangle$

$\therefore \psi_{opp} = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle = (2)$

$(1) \neq (2)$

So Pauli X gate is ~~a~~ universal NOT gate

$|0\rangle - \boxed{X} - |1\rangle$

$|1\rangle - \boxed{X} - |0\rangle$

$\alpha|0\rangle + \beta|1\rangle - \boxed{X} - \alpha|1\rangle + \beta|0\rangle$

Not opposite on Bloch sphere

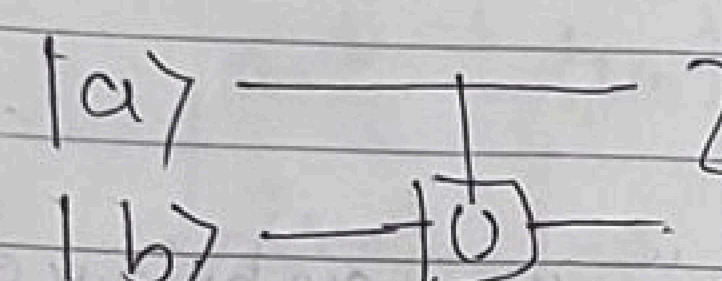
2 Qubit Quantum Gates:

Basis State $\{ |00\rangle \; |01\rangle \; |10\rangle \; |11\rangle \} = \{|0\rangle, \ldots\}$ $\{0,1,2,3\}$

$\quad\quad\quad\quad \hookrightarrow 0 \quad \hookrightarrow 1 \quad \hookrightarrow 2 \quad \hookrightarrow 3$

Eigen state $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = I = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

2 Qubit Quantum Gate = Controlled Unitary Operation $\boxed{CU}$



$\boxed{CU}$. 2^n Representation

$CU = \begin{vmatrix} I_2 & O_2 \\ O_2 & U_2 \end{vmatrix}$ all unitary operator concern

$U = X \begin{vmatrix}0&1\\1&0\end{vmatrix}$ $\quad U = Z = \begin{vmatrix}1&0\\0&-1\end{vmatrix}$ $\quad U = S$

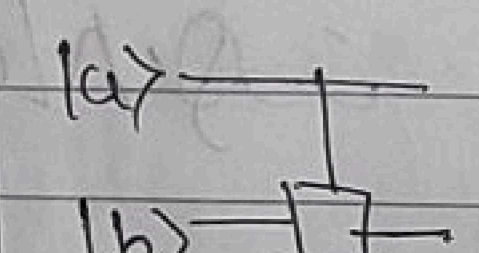
replace X
the controlled X
$\boxed{CNOT}$

$U = S = \begin{vmatrix}1&0\\0&i\end{vmatrix}$

[Contr phase] $\quad \boxed{CZ}$

Replace X — CNOT
controlled Z = CZ
controlled phase [Cphase]

$I = \begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$



$U = X$ CNOT $\quad U = Z$ [CZ] $\quad U = S$ [Cphase]

① CNOT.

$\begin{vmatrix} I_2 & O_2 \\ O_2 & X \end{vmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \Longleftrightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I$

4×4 $\quad$ 0 1 3 2
$\Downarrow$

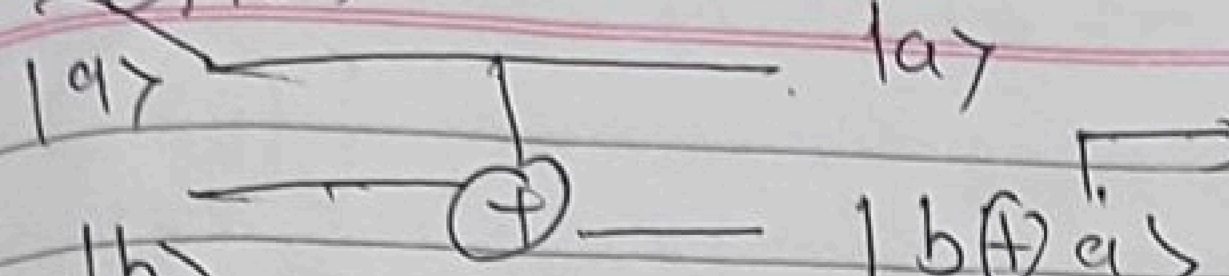
In CNOT
3rd & 4th
Column are interchanged.
$\Downarrow$
$2 \rightleftarrows 3$
$|10\rangle \rightleftarrows |11\rangle$

$2 \rightleftarrows 3$
$|10\rangle \rightleftarrows |11\rangle$
$|10\rangle \xrightarrow{CNOT} |11\rangle$
$|11\rangle \xrightarrow{CNOT} |10\rangle$
$\left.\begin{array}{l}|00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle\end{array}\right\}$ No Change

CNOT
* if first Qbit is $|1\rangle$ then Second Qubit is flipped
$|11\rangle \rightarrow |10\rangle$
$|10\rangle \rightarrow |11\rangle$

controlled bit

CNOT



Modulo 2 addⁿ

0+0
0+1
1+0
1+0

Target bit

$\|a\rangle$	$\|b\rangle$	$\|a\rangle$	$\|b\oplus a\rangle$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

(2) $U = Z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$

$$CU = \begin{vmatrix} I_2 & O_2 \\ O_2 & Z \end{vmatrix} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \end{array}$$

$\Downarrow$ I

$3 \rightarrow -3$

$|11\rangle \xrightarrow[CZ]{} -|11\rangle$

$U = S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Cphase =

$$\text{Cphase} = \begin{vmatrix} I_2 & O_2 \\ O_2 & S \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{vmatrix}$$

$3 \rightarrow i$

$\text{Cphase} = |11\rangle \rightarrow i|11\rangle$

3 Qubit Quantum Gate

$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$

0 1 2 3 4 5 6 7

$\boxed{I_8}$

$D[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

1 bit

$U = e^{i\gamma} R_n(\theta)$

2 bit

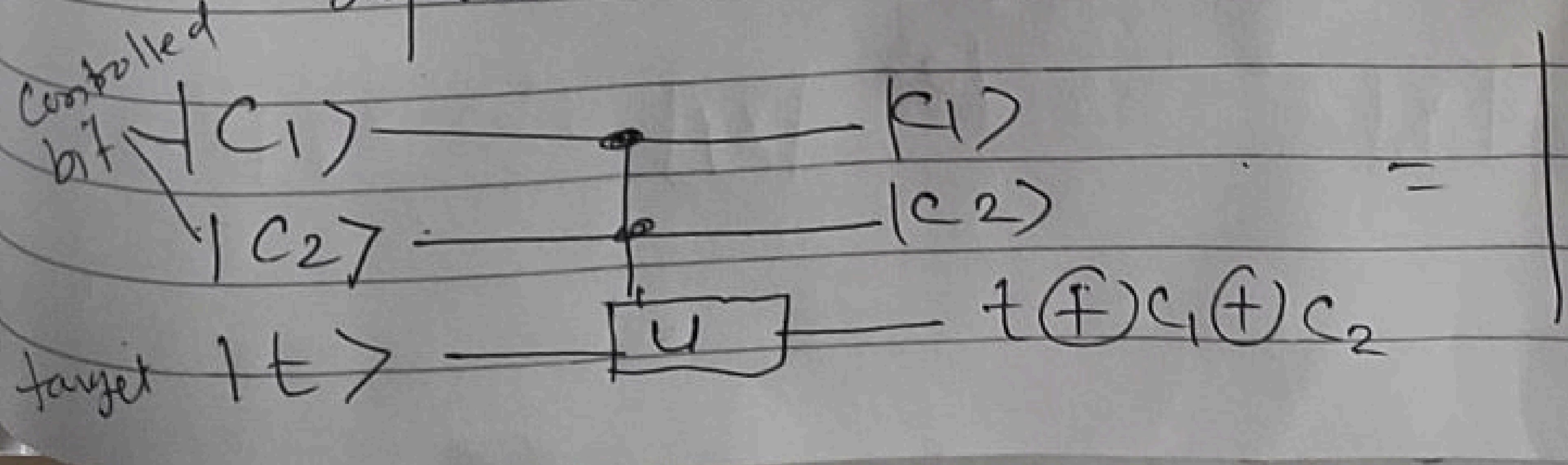
$CU = \begin{bmatrix} I_2 & O_2 \\ O & U \end{bmatrix}$

3 qubit

$$CCU = C^2U = \begin{vmatrix} I_4 & O_4 \\ O_4 & CU \end{vmatrix}.$$

3 Qubit is given by Controlled Controlled Unitary

represented = C^2U



$$\begin{vmatrix} I_4 & O_4 \\ O_4 & U_4 \end{vmatrix}$$

|CNOT| SWAP

TOFFOLI Gate FREDKIN Gate

$$U_4 = CNOT = \begin{array}{c} \\ I_4 \\ \\ \\ \\ \\ O_4 \\ \\ \end{array} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 7 & 6 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \begin{array}{c} \\ O_4 \\ \\ \\ \\ \\ U_4 \\ \\ \end{array}$$

change

TOFFOLI

$6 \rightleftarrows 7$

$6 \rightleftarrows 7$

$|111\rangle \rightleftarrows |111\rangle$

2 bit 11 then flipped 3rd bit. 2bits 11

if first 2bits are '11' then 3rd bit is flipped

SWAP Gate