Astronomy Club, IITK
Hyperion 2024-25

'Whispers from the Abyss'

Instructions

- You are supposed to type your answers in a `pdf` file and submit the same.
- Your solutions must be properly formatted. Any unconventional notation used must be explicitly stated.
- Attach any relevant code snippets and graphs which were utilized in analysis of the problems.
- You must submit all the solutions in English only.
- An introduction summary of not more than 200 words highlighting the basic workflow needs to be included at the beginning of the final report.
- The solutions must be your own and all references must be duly cited. Any form of plagiarism will not be tolerated under any circumstances and would lead to disqualification.
- Relying solely on AI for final submission could lead to disqualification.
- The deadline of submission is **23:59 hrs of 16th March**. After the deadline, a 5 marks penalty will be imposed for every 15 minutes delay. No submission will be entertained post 1 hour of deadline.
- In case of any discrepancy the decision of the Astronomy Club IITK would be final and binding. Be sure that all the submissions would be evaluated with due diligence and utmost fairness.

All the best!
Cosmos Is Within Us

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Let the quest begin...

Astronomy, a millennia old discipline, has enchanted thinkers since times immemorial and continues to do so till today. Be it ancient sagas immortalized in the stars, philosophical queries of our existence, scientific quests to understand the dark and arcane abyss called the universe, or simply observing the bewitching cosmic dance of the stars and planets through a telescope, astronomy has been a refuge for all. Contributing to this never-ending cosmic reverie, fossicking through the dark abyss and swirling black holes would certainly be intriguing. Presenting to you, your comrade in this celestial journey and our beloved Case Orator- **"Hyperion"** .

A brief introduction to Hyperion: Our case curator possesses a never ending bag of problems and he picks out oddities he wishes to seek answers to when he finds a partner to help him out. Now, he has found you! (Cue Until I found you by Stephen Sanchez ;)) Today, Hyperion has picked out a rather perplexing problem, one of darkness and spirals, one that is inescapable and all consuming, one that is—. Okay, that's enough poetry, to put it simply, Hyperion has decided to venture into the puzzle of black holes and the space-time fabric.

Black holes are some of the most mysterious and powerful objects in the universe. They are regions in space where gravity is so strong that nothing, not even light, can escape. Most black holes are formed when massive stars run out of fuel and collapse under their own gravity. This collapse compresses all the star's mass into a tiny space, creating what is called a singularity—a point of infinite density. The boundary around a black hole, beyond which nothing can escape, is called the event horizon.

Another amusing part of black holes is their Quasi Normal modes. Black holes are almost never isolated. They constantly interact with astronomical matter around. These interacting entities usually comprise dark matter, accretion disk and the like. Although the mass of these entities is much lower than that of black holes, they still manage to perturb the black holes. This perturbation is what gives rise to Quasi Normal modes. They can be detected from the gravitational waves they create. Earlier they were thought to have only theoretical existence, but with the advent of LIGO, proof of their actuality has been found.

Now Hyperion needs your help to navigate through this puzzle. Can you satiate his curiosity through your answers?

1. The Schwarzschild metric describes the spacetime around a **static, spherically symmetric, uncharged** mass. The most general form of a **static, spherically symmetric** metric in four-dimensional spacetime:

$$ds^2 = -A(r)c^2dt^2 + B(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\,,$$

where $A(r)$ and $B(r)$ are functions to be determined. State all assumptions clearly and provide justifications for each:

a. Why we assume spherical symmetry. **(1)**

b. Why we assume time independence. **(1)**

c. Why the vacuum field equations apply. **(1)**

Show that the Schwarzschild solution is:

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\,,$$

(5)

Interpret the meaning of the **Schwarzschild radius** $r_s = \frac{2GM}{c^2}$ and discuss its physical significance (event horizon and escape velocity). **(2)**

2. Given the Schwarzschild metric in four dimensions:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \, ,$$

where

$$f(r) = 1 - \frac{2GM}{r}. \, ,$$

a. **Determine the effective potential** $V(r)$ that describes the propagation of perturbations (scalar, electromagnetic, or gravitational) in the curved spacetime of a Schwarzschild black hole. **(5)**

b. Using this potential, **derive the equation** that governs the perturbations. **(3)**

c. **Discuss how the solutions** to this equation, subject to the boundary conditions of purely ingoing waves at the horizon and purely outgoing waves at infinity, correspond to quasi-normal modes (QNMs). **(2)**

3. Black hole perturbations are largely governed by the following second order differential equation

$$\frac{d^2\psi}{dr_*^2} + \Big[\omega^2 - V(r)\Big]\psi = 0\,,$$

where r is the radial coordinate, $V(r)$ is the QNM potential and ω is the QNM angular velocity[1]. Note that the equation also incorporates the tortoise coordinate r^*, defined as

$$dr_* = \frac{dr}{f(r)}\,,$$

with $f(r)$ being the metric function for a black hole[1]. In this case, the metric function is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{R^2}\,.$$

a. How is this metric function different from the one proposed by Schwarzschild? **(1)**

b. Metric functions are used to calculate the event horizon of black holes. Under what conditions do both yield same/similar results? (**2**)

c. Why do we call r^* a tortoise coordinate? (**1**)

d. Perform a simple change of variables in the above equation to get an equation in terms of r rather than r^*. (Have the answer in terms of $V(r)$ for part (b)) (**6**)

4. Using the equation derived in the previous questions, analyze the behavior of scalar perturbations in the AdS black hole spacetime.

a. Determine the asymptotic behavior of the wavefunction in two critical regions: near the event horizon and at large distances from the black hole. Justify your answers mathematically. (**10**)

b. How does the AdS confining potential influence these behaviors compared to a Schwarzschild black hole? (**5**)

5. Gravitational wave astronomy began in 2015 with the detection of **GW150914** by **LIGO**. After the discovery of this, we were able to test the theories of general relativity, including the existence of quasinormal modes. Your task is to download the time series data of **GW150914** and also any one of the gravitational wave events in the latest data release by LIGO. Load the data in **Python** and prepare frequency domain graphs of each of each of the two events using suitable methods. Clearly describe all the steps for downloading the data and obtaining these graphs, along with the reason for using any specific method. **(20)**

6. Now comes the most interesting application of quasinormal modes. The earlier relations you derived contain the relations of **quasinormal modes** with the **mass of the black hole**. The task is to use the data from the above two GW events to calculate the mass of the black holes from where the gravitational wave originated. Do the required analysis describing all the steps with proper reasoning. Your approach should be physically and mathematically consistent. Don't forget that actual black holes are rotating. **(35)**

`Hyperion would like to remind you that like a gravitational wave, the impact you create can travel across eternity even in silence.` ☺