Lab 9

Steps you should follow :

1. Start a Matlab session by typing

   ```
   $ matlab &
   ```

   at the command prompt.

2. In the Matlab command window, type

   ```
   >> diary lab9_ID
   ```

   where ID stands for your roll number. For example, if your roll number is 12345, then the command will be `diary lab9_12345`; if your roll number is 123456, then the command will be `diary lab9_123456`. This will create a file named `lab9_ID` in the present working directory. PLEASE DO NOT EDIT THIS FILE.

3. Do your lab assignment – create a separate file to write your scripts/functions, if required; once done, at the Matlab's command prompt, type

   ```
   >> diary off
   ```

4. Attach this file (that is, `lab9_ID`) and any other Matlab code file that you may have created for the labwork in an **email with subject Lab9-ID** and send it to

   mth308.iitk@gmail.com

   before the end of the lab session, that is, by 4:30 pm. Note that **late submissions will not get any credit**. In the case that your diary file is too big to be sent as an email attachment, upload it on you Google Drive (the cloud storage space associated with your Google/Gmail account) and share the link with mth308.iitk@gmail.com.

Consider the sequence of polynomials $(p_n(x))_{n\geq 0}$ where p_0 is a constant polynomial, p_1 satisfies

$$p_1(x) = \frac{k_1}{k_0}(x - \beta_0)p_0(x),$$

and for $n \geq 2$, they are given by the recurrence

$$p_n(x) = \frac{k_n}{k_{n-1}}(x - \beta_{n-1})p_{n-1}(x) - \frac{k_n k_{n-2}}{k_{n-1}^2} p_{n-2}(x) \tag{1}$$

where $(\beta_n)_{n\geq 0}$ is a real sequence and $(k_n)_{n\geq 0}$ is a given sequence of positive real numbers. It is easy to see that, after some rearrangements, the recurrence (1) takes the form

$$xp_{n-1}(x) = \frac{k_{n-1}}{k_n} p_n(x) + \frac{k_{n-2}}{k_{n-1}} p_{n-2}(x) + \beta_{n-1} p_{n-1}(x)$$

thus yielding the following equation

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ p_{N-1}(x) \end{pmatrix} = \begin{pmatrix} \beta_0 & k_0/k_1 & 0 & 0 & \cdots & 0 \\ k_0/k_1 & \beta_1 & k_1/k_2 & 0 & \cdots & 0 \\ 0 & k_1/k_2 & \beta_2 & k_2/k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{N-1} \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ p_{N-1}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ (k_{N-1}/k_N) p_N(x) \end{pmatrix}$$

for $N \geq 1$. Clearly, if $x = x_i$ is a root of $p_N(x)$, then x_i is an eigenvalue of the matrix

$$J_N = \begin{pmatrix} \beta_0 & k_0/k_1 & 0 & 0 & \cdots & 0 \\ k_0/k_1 & \beta_1 & k_1/k_2 & 0 & \cdots & 0 \\ 0 & k_1/k_2 & \beta_2 & k_2/k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_{N-1}. \end{pmatrix}$$

Thus, we see that, the eigenvalues $x_1, x_2, \ldots, x_N$ of J_N are the roots of $p_N(x)$. The eigenvector corresponding to the eigenvalue x_i is $(p_0(x_i), p_1(x_i), p_2(x_i), \ldots, p_{N-1}(x_i))^T$.

To find the roots of the sequence of functions $(P_n(x))_{n\geq 0}$ defined recursively by

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$, utilize the above observation for

$$p_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x).$$

Implement this in Matlab to find all roots of $P_4(x)$, $P_8(x)$, $P_{16}(x)$ and $P_{32}(x)$. If needed, you can use Matlab's `eig` function in your implementation.

Remark: This example demonstrates an important application of matrix eigenvalue problems – as we will see later in this course, the roots of $P_n(x)$ play an important role in accurate approximations of integrals.

An extra practice homework problem (not to be submitted as part of the lab assignment and will not be graded).

Consider an $n \times n$ Hermitian matrix A with eigenvalue-eigenvector pairs (λ_i, u_i) for $i = 1, \ldots, n$. Assume that (λ_i, u_i) are known for $i = 1, \ldots, s$, for some $s < n$ and $u_i, i = 1, \ldots, s$ are mutually orthogonal. To compute λ_{s+1}, consider

$$x_0 = z - \sum_{i=0}^{s} \frac{u_i^* z}{u_i^* u_i} u_i$$

for an arbitrary $z \in \mathbb{C}^n \setminus \operatorname{span}\{u_1, \ldots, u_s\}$ and set up the iteration

$$x_{m+1} = Ax_m, \quad m = 0, 1, 2, \ldots \ .$$

1. Show that, for each $m = 0, 1, 2, \ldots$, x_m is orthogonal to u_i, $i = 1, \ldots, s$, that is, $u_i^* x_m = 0$.
2. If $\lambda_{s+1} > \lambda_i, i = s+2, \ldots, n$, then show that $\sigma_m = x_m^* A x_m / x_m^* x_m \to \lambda_{s+1}$ as $m \to \infty$.