MTH 308: Numerical Analysis & Scientific Computing I
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur April 4, 2019

Lab 11

Steps you should follow :

1. Start a Matlab session by typing

   ```
   $ matlab &
   ```

   at the command prompt.

2. In the Matlab command window, type

   ```
   >> diary lab11_ID
   ```

   where ID stands for your roll number. For example, if your roll number is 12345, then the command will be `diary lab11_12345`; if your roll number is 123456, then the command will be `diary lab11_123456`. This will create a file named `lab11_ID` in the present working directory. PLEASE DO NOT EDIT THIS FILE.

3. Do your lab assignment – create a separate file to write your scripts/functions, if required; once done, at the Matlab's command prompt, type

   ```
   >> diary off
   ```

4. Attach this file (that is, `lab11_ID`) and any other Matlab code file that you may have created for the labwork in an **email with subject Lab11-ID** and send it to

   mth308.iitk@gmail.com

   before the end of the lab session, that is, by 4:30 pm. Note that **late submissions will not get any credit**. In the case that your diary file is too big to be sent as an email attachment, upload it on you Google Drive (the cloud storage space associated with your Google/Gmail account) and share the link with mth308.iitk@gmail.com.

Repeat the following exercise for functions $f : [-1, 1] \to \mathbb{R}$ given by

i. $f(x) = \sin(x)$

ii. $f(x) = \sin(|x|)$

iii. $f(x) = \sin(x\sqrt{|x|})$

iv. $f(x) = 1/(1 + 25x^2)$

1. Write a program to compute an approximation to f in $\mathcal{P}_n$, denoted by f_n^ℓ, that interpolates f at the $(n+1)$ equidistant points in $[-1, 1]$, that is, $x_j = -1 + 2j/n, j = 0, 1, \ldots, n$. Complete the following table:

n	$e_n^{\ell,\infty} = \|f - f_n^\ell\|_\infty$	$\frac{\log\left(e_{n-1}^{\ell,\infty}/e_n^{\ell,\infty}\right)}{\log(n/(n-1))}$
2		—
3		
4		
5		
6		
7		
8		
9		
10		

Comment on the results.

2. Write a program to compute an approximation to f in $\mathcal{P}_n$, denoted by f_n^c, that interpolates f at the $(n+1)$ Chebyshev points in $[-1, 1]$, that is, $x_j = \cos((2j+1)\pi/(2n+2)), j = 0, 1, \ldots, n$. Complete the following table:

n	$e_n^{c,\infty} = \|f - f_n^c\|_\infty$	$\frac{\log\left(e_{n-1}^{c,\infty}/e_n^{c,\infty}\right)}{\log(n/(n-1))}$
2		—
3		
4		
5		
6		
7		
8		
9		
10		

Comment on the results.

To compute the L^∞ errors for the tables, use

$$\|f - p_n\|_\infty = \|f - p_n\|_{L^\infty([-1,1])} \approx \max_{0 \le j \le 1000} |f(x_j^e) - p_n(x_j^e)|$$

where $x_j^e = -1 + 2j/1000$.
You might find it easier to implement the code if you write a function for obtaining the Lagrange basis functions for a given set of interpolation points which can then be used to construct respective Lagrange interpolating polynomials. For example, one implementation might look like:

```
function [ ljx ] = lagrangeBasis(xj,j,x)

    % xj - an array containing the interpolation points
    % j  - the jth basis function, j from 1 to length(xj)
    % x  - an array of evaluation points

    n = length(xj);
    [neval,meval] = size(x);
    ljx = ones(neval,meval);

    for k = 1:n
        if k ~= j
             ljx = ljx .* (x-xj(k)) ./ (xj(j)-xj(k));
        end
    end

end
```

An extra practice homework problem (not to be submitted as part of the lab assignment and will not be graded).

I. For each $n = 1, \ldots, 10$, plot the two approximations (corresponding to the two interpolation grids) obtained in the lab and the exact function on the same plot. Now repeat the exercise for $n = 11, \ldots, 30$ and comment on the results.

II. Repeat the lab exercise for Bernstein approximation as follows – by rescaling the Bernstein polynomial of degree n for approximation of functions in $C([0, 1])$, derive an expression for approximation of continuous functions on $[a, b]$. Use it to approximate f, by say f_n^b on $[-1, 1]$. Write a program to carry out the computations. Complete the following table:

n	$e_n^{b,\infty} = \|f - f_n^b\|_\infty$	$\frac{\log\left(e_{n-1}^{b,\infty}/e_n^{b,\infty}\right)}{\log(n/(n-1))}$
2		—
3		
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Comment on the results.

III. Let f be a real-valued function on an interval J and $x_0, \dots, x_n$ be $n+1$ distinct points in J. Suppose $p_k \in \mathcal{P}_k$ denotes the Lagrange interpolating polynomial for f at the first $k+1$ points $x_0, \dots, x_k$.

(a) Prove that $p_n(x) - p_{n-1}(x) = c(x - x_0) \cdots (x - x_{n-1})$ for some constant c. Let us call this constant the n-th divided difference of f at the x_i and denote it by $f[x_0, \dots, x_n]$.

(b) Prove that $f[x_{\sigma(0)}, \dots, x_{\sigma(n)}] = f[x_0, \dots, x_n]$ for any σ that permutes the set $\{0, 1, \dots, n\}$. In other words, show that permuting the interpolation points does not change the divided difference value.

(c) Prove the recursion relation

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$$

with $f[x] := f(x)$.

(d) Show that

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n](x - x_0) \cdots (x - x_{n-1}).$$

Prove that the error is given as

$$f(x) - p_n(x) = f[x_0, \dots, x_n, x](x - x_0) \cdots (x - x_n).$$

(e) If $f \in C^n(J)$, show that there exists a point ξ in the interior of J such that

$$f[x_0, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi).$$