Lab 5

Steps you should follow :

1. Start a Matlab session by typing

   ```
   $ matlab &
   ```

   at the command prompt.

2. In the Matlab command window, type

   ```
   >> diary lab5_ID
   ```

   where ID stands for your roll number. For example, if your roll number is 12345, then the command will be `diary lab5_12345`; if your roll number is 123456, then the command will be `diary lab5_123456`. This will create a file named `lab5_ID` in the present working directory. PLEASE DO NOT EDIT THIS FILE.

3. Do your lab assignment – create a separate file to write your scripts/functions, if required; once done, at the Matlab's command prompt, type

   ```
   >> diary off
   ```

4. Attach this file (that is, `lab5_ID`) and any other Matlab code file that you may have created for the labwork in an **email with subject Lab5-ID** and send it to

   mth308.iitk@gmail.com

   before the end of the lab session, that is, by 4:30 pm. Note that **late submissions will not get any credit**.

For a sequence (x_k) of vectors in $\mathbb{R}^n$ given by $x_k = M^k x_0$ where M is a real $n \times n$ matrix, the *convergence factor* is the limit

$$\rho = \lim_{k \to \infty} \left(\frac{\|x_k\|}{\|x_0\|} \right)^{1/k}.$$

The *convergence rate* τ is the (natural) logarithm of the inverse of the convergence factor, that is

$$\tau = -\log \rho.$$

The above definition depends on the initial vector x_0, so it may be termed *specific convergence factor*. A *general* convergence factor can also be defined by

$$\phi = \lim_{k\to\infty} \left(\max_{x_0 \in \mathbb{R}^n} \frac{\|x_k\|}{\|x_0\|} \right)^{1/k}.$$

This factor satisfies

$$\phi = \lim_{k\to\infty} \left(\max_{x_0 \in \mathbb{R}^n} \frac{\|M^k x_0\|}{\|x_0\|} \right)^{1/k} = \lim_{k\to\infty} \|M^k\|^{1/k} = \rho(M),$$

where $\rho(M)$ denotes the spectral radius of M. Note that the last equality holds for any matrix norm and can be shown using Jordan canonical form (here, we will assume this result without getting into the proof).

Now, consider the $n \times n$ matrix

$$A_n = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

and the column vector $b_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ of n ones.

Note that the exact solution x_n^{EX} can be obtained according to

$$(x_n^{EX})_j = \begin{cases} n/2, & j = 1, \\ n-1, & j = 2, \\ 2x_{j-1} - x_{j-2} - 1, & j = 3, \dots, n-1, \\ (x_{j-1} + 1)/2, & j = n. \end{cases}$$

For $n = 2^m, m = 3, \dots, 8$, solve $A_n x = b_n$

1. using Gauss-Seidel iterations to obtain the sequence of approximations $x_{n,k}^{GS}$: to start the iterations, choose $x_{n,0}^{GS} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ and continue iterating until $\|x_{n,k}^{GS} - x_n^{EX}\|_\infty / \|x_n^{EX}\|_\infty < 10^{-10}$, and

2. using Gauss-Jacobi iterations to obtain the sequence of approximations $x_{n,k}^{GJ}$: to start the iterations, choose $x_{n,0}^{GJ} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ and continue iterating until $\|x_{nk}^{GJ} - x_n^{EX}\|_\infty / \|x_n^{EX}\|_\infty < 10^{-10}$.

What are the general and specific convergence factors for Gauss-Seidel and Gauss-Jacobi iterations? For computing eigenvalues, you may use Matlab's `eig` function (type `help eig` for more information). What do you observe when you change 10^{-10} to 10^{-11} in your code?

An extra practice homework problem (not to be submitted as part of the lab assignment and will not be graded).

Consider the $n \times n$ matrix

$$A_n(\alpha) = \begin{bmatrix} \alpha & -1 & & & & & \\ -1 & \alpha & -1 & & & & \\ & -1 & \alpha & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & \alpha & -1 & \\ & & & & -1 & \alpha & -1 \\ & & & & & -1 & \alpha \end{bmatrix}$$

where α is a real parameter. Verify that the eigenvalues of $A_n(\alpha)$ are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \quad j = 1, \ldots, n,$$

where

$$\theta = \frac{\pi}{n+1},$$

and that an eigenvector associated with λ_j is

$$u_j = \begin{bmatrix} \sin(j\theta) & \sin(2j\theta) & \cdots & \sin(nj\theta) \end{bmatrix}^T.$$

1. Under what condition on α does this matrix become positive definite.

2. Will Gauss-Jacobi and Gauss-Seidel iterations converge when $\alpha = 2$? What will their respective convergence factor be?