MTH 308: Numerical Analysis & Scientific Computing I
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur

February 28, 2019

Lab 7

Steps you should follow :

1. Start a Matlab session by typing

   ```
   $ matlab &
   ```

   at the command prompt.

2. In the Matlab command window, type

   ```
   >> diary lab7_ID
   ```

   where ID stands for your roll number. For example, if your roll number is 12345, then the command will be `diary lab7_12345`; if your roll number is 123456, then the command will be `diary lab7_123456`. This will create a file named `lab7_ID` in the present working directory. PLEASE DO NOT EDIT THIS FILE.

3. Do your lab assignment – create a separate file to write your scripts/functions, if required; once done, at the Matlab's command prompt, type

   ```
   >> diary off
   ```

4. Attach this file (that is, `lab7_ID`) and any other Matlab code file that you may have created for the labwork in an **email with subject Lab7-ID** and send it to

   mth308.iitk@gmail.com

   before the end of the lab session, that is, by 4:30 pm. Note that **late submissions will not get any credit**. In the case that your diary file is too big to be sent as an email attachment, upload it on you Google Drive (the cloud storage space associated with your Google/Gmail account) and share the link with mth308.iitk@gmail.com.

Use the Newton's method to compute an eigenvalue λ and corresponding eigenvector x of an $n \times n$ matrix A. Note that if we define $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$f(x, \lambda) = \begin{bmatrix} Ax - \lambda x \\ x^T x - 1 \end{bmatrix},$$

then $f(x, \lambda) = 0$ precisely when λ is an eigenvalue and x is a corresponding normalized eigenvector. Start by verify that the Jacobian matrix $f'(x, \lambda)$ is given by

$$f'(x, \lambda) = \begin{bmatrix} A - \lambda I & -x \\ 2x^T & 0 \end{bmatrix}.$$

A reasonable starting guess is to take x_0 to be an arbitrary normalized nonzero vector ($x_0^T x_0 = 1$) and take $\lambda_0 = x_0^T A x_0$. Test your program on the $n \times n$ matrix

$$A_n(\alpha) = \begin{bmatrix} \alpha & -1 & & & & & \\ -1 & \alpha & -1 & & & & \\ & -1 & \alpha & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & \alpha & -1 & \\ & & & & -1 & \alpha & -1 \\ & & & & & -1 & \alpha \end{bmatrix}$$

with a real parameter α, where the eigenvalues are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \quad j = 1, \ldots, n,$$

$$\theta = \frac{\pi}{n+1},$$

and a corresponding eigenvector

$$x_j = \begin{bmatrix} \sin(j\theta) & \sin(2j\theta) & \cdots & \sin(nj\theta) \end{bmatrix}^T.$$

In particular, test your program on matrices $A_n(2)$ (that is, $\alpha = 2$) for $n = 2, 4, 8, 16, 32, 64$ and 128. Note that the sequence obtained through Newton iterations will converge to one of the eigen-pairs that may not necessarily correspond to the dominant eigenvalue.

An extra practice homework problem (not to be submitted as part of the lab assignment and will not be graded).

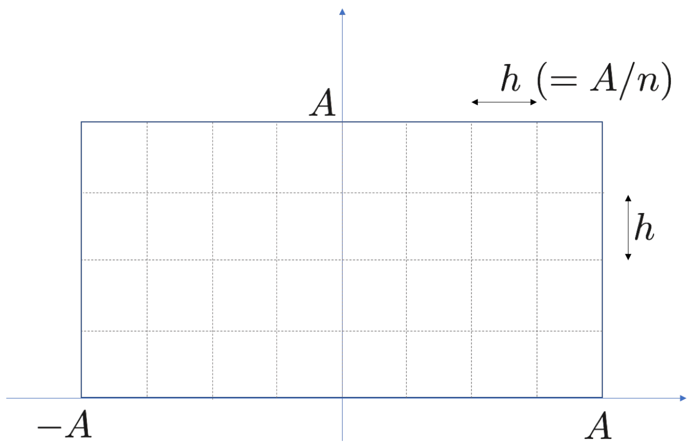
I. Suppose x^* satisfies $f(x^*) = 0$. The basin of attraction of x^* is the set of x so that if $x_0 = x$ then $x_k \to x^*$ as $k \to \infty$. If $f'(x^*)$ is non-singular, the basin of attraction of x^* under Newton's method includes at least a neighborhood of x^*, because Newton's method is locally convergent. Now consider the two dimensional problem of finding roots of $f(z) = z^2 - 1$, where $z = x + iy$. Written out in its real components, $f(x, y) = (x^2 - y^2 - 1, 2xy)$. The basin of attraction of the solution $z^* = 1((x^*, y^*) = (1, 0))$ includes a neighborhood of $z = 1$ but surprisingly many many other points in the complex plane. This *Mandelbrot set* is one of the most beautiful examples of a two dimensional fractal.

(a) Show that the Newton iteration is $z_{m+1} = z_m - (z_m^2 - 1)/(2z_m)$.

(b) Set $e_m = z_m - 1$ and show that $e_{m+1} = \frac{1}{2}e_m^2/(1 + e_m)$.

(c) Use this to show that if $|e_m| < 1/2$, then $|e_{m+1}| < \frac{1}{2}|e_m|$. Argue that this implies that the basin of attraction of $z^* = 1$ includes at least a disk of radius 1/2 about z^*. In addition, z_0 is in the basin of attraction of $z^* = 1$ if $|z_m - 1| < 1/2$ for some m.

(d) Use the previous part to make a picture of the Mandelbrot set. Consider three parameters A, n, N and divide the rectangle $|x| < A$, $0 < y < A$ into a regular grid of small cells of size $h \times h$ where $h = A/n$. Start Newton's method from the center of each cell. Color the cell if $|z_m - 1| < 1/2$ for some $m < N$. See how the picture depends on the parameters A, n and N.



II. Study the convergence of Newton's method applied to solving the equation $f(x) = x^2 = 0$. The root $x^* = 0$ is degenerate in that $f'(x^*) = 0$. The Newton iterates are x_m satisfying $x_{m+1} = x_m - f(x_m)/f'(x_m)$. Show that the local convergence in this case is linear, which means that there is an $\alpha < 1$ with $|x_{m+1} - x^*| \leq \alpha|x_m - x^*|$.

III. The function $f(x) = x/\sqrt{1 + x^2}$ has a unique root: $f(x) = 0$ only for $x = 0$.

(a) Show that the Newton' method succeeds if and only if $|x_0| < 1$.

(b) Draw graphs to illustrate the first few iterates when $x_0 = 0.5$ and $x_0 = 1.5$. Note that the Newton's method for this problem has local cubic convergence which is even faster than the more typical local quadratic convergence

(c) Use the Taylor expansion to second order to derive the approximation

$$f(x') \approx C(\overline{x})f(\overline{x})^2 = \frac{1}{2}\frac{f''(\overline{x})}{f'(\overline{x})^2}f(\overline{x})^2$$

where $x' = \overline{x} - f(\overline{x})/f'(\overline{x})$ and then use the approximation to explain the cubic convergence for Newton's method in this case.