MTH 308: Numerical Analysis & Scientific Computing I
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur January 31, 2019

Lab 4

Steps you should follow :

1. Start a Matlab session by typing

   ```
   $ matlab &
   ```

   at the command prompt.

2. In the Matlab command window, type

   ```
   >> diary lab4_ID
   ```

   where ID stands for your roll number. For example, if your roll number is 12345, then the command will be `diary lab4_12345`; if your roll number is 123456, then the command will be `diary lab4_123456`. This will create a file named `lab4_ID` in the present working directory. PLEASE DO NOT EDIT THIS FILE.

3. Do your lab assignment – create a separate file to write your scripts/functions, if required; once done, at the Matlab's command prompt, type

   ```
   >> diary off
   ```

4. Attach this file (that is, `lab4_ID`) and any other Matlab code file that you may have created for the labwork in an **email with subject Lab4-ID** and send it to

   mth308.iitk@gmail.com

   before the end of the lab session, that is, by 4:00 pm. Note that **late submissions will not get any credit**.

Consider the one-dimensional equation,

$$-u''(x) = f(x), \quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0. \tag{2}$$

The interval $[0,1]$ can be discretized uniformly by taking the $n+2$ points

$$x_i = ih, \quad i = 0, \dots, n+1,$$

where $h = 1/(n+1)$. Because of the *Dirichlet* boundary conditions (see equation (2)), the values $u(x_0)$ and $u(x_{n+1})$ are known. At every other point, an approximation u_i is sought for the exact solution $u(x_i)$. If the approximation

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

is used in equation (1) at point x_i for $i = 1, \ldots, n$, then the unknowns u_i, u_{i-1}, u_{i+1} satisfy the relation

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

where $f_i = f(x_i)$. Notice that for $i = 1$ and $i = n$, the equation will involve u_0 and u_{n+1} respectively, which are known quantities, both equal to zero in this case. Thus, for $n = 6$, the linear system obtained is of the form

$$Ax = b$$

where

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

For $n = 2^m, m = 3, \ldots, 12$, solve the problem for $f(x) = x^3$ and $f(x) = x\cos x$ to obtain the respective relative error $e_n(u)$ given by

$$e_n(u) = \frac{\max_{1\le i\le n} |u_i - u(x_i)|}{\max_{1\le i\le n} |u(x_i)|}.$$

What do you observe about the ratio $e_n(u)/e_{n/2}(u)$ as n increases?

Some extra practice homework problems (not to be submitted as part of the lab assignment and will not be graded).

I. Let A be an $n \times n$ real symmetric matrix. Show that the following statements are equivalent:

(a) A is positive definite, that is $x^T Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(b) All eigenvalues of A are positive.

(c) The determinants of the leading principal sub-matrices of A are positive.

(d) The pivots of A are positive.

II. Check that the hypotheses and conclusions of the theorem regarding LU decomposition of tridiagonal matrices are true for

$$A_n = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

Find a general formula for $A_n = LU$. Hint: Consider the cases $n = 3, 4, 5$, and then guess the general pattern and verify it.

III. Consider now the one-dimensional version of the convection-diffusion equation with Dirichlet boundary conditions,

$$-au'' + bu' = 0, \quad \text{in } (0,1), \tag{3}$$
$$u(0) = 0, \; u(1) = 1. \tag{4}$$

(a) Verify that the exact solution to the above equation is given by

$$u(x) = \frac{1 - e^{Rx}}{1 - e^{R}}$$

where $R = b/a$.

(b) Consider the approximate solution provided by numerical scheme

$$b\frac{u_{i+1} - u_{i-1}}{h} - a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

on the grid of size $n+2$ where $x_i = ih$, $h = 1/(n+1)$ and u_i approximates $u(x_i)$ for $i = 1, \ldots, n$. Set up the linear system and solve for various values of n, plot the exact and approximate solutions and observe the behavior of the relative error e_n when $h > 2/R$ and when $h \leq 2/R$; also notice the difference in behavior when $b < 0$.

(c) Now, repeat the exercise in part (b) for the approximation

$$b\frac{u_{i+1} - u_i}{h} - a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

and for approximation

$$b\frac{u_i - u_{i-1}}{h} - a\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

and comment on the results.