EE 208

Control Engineering Lab

Experiment-8: Stability and instability of nonlinear systems.

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OBJECTIVE: -

To examine stability features of a given nonlinear system by observing movement of eigenvalues across the s-plane.

Given: -

We have been provided with differential equations that represent a simplified model of an overhead crane. The equations are:

$$[m_L + m_C].\ddot{x}_1(t) + m_L.l.[\ddot{x}_3(t).\cos x_3(t) - \dot{x}_3^2(t).\sin x_3(t)] = u(t) \quad \text{...equ (1)}$$

$$m_L.[\ddot{x}_1(t).\cos x_3(t) + l.\ \ddot{x}_3(t)] = -m_L g.\sin x_3(t) \quad \text{...equ (2)}$$

Here, constants are:

m_C: Mass of trolley; 10 kg.
m_L: Mass of hook and load; the hook is again 10 kg, but the load can be zero to several hundred kg's,
but constant for a particular crane operation.
l: Rope length; 1m or higher, but constant for a particular crane operation.
g: Acceleration due to gravity, 9.8 ms^{-2}

Variables are:

- Input:
- u: Force in Newtons, applied to the trolley.
- Output:
- y: Position of load in metres, $y(t) = x_1(t) + l.\sin x_3(t)$
- States:
- x_1: Position of trolley in metres.
- x_2: Speed of trolley in m/s.
- x_3: Rope angle in rads.
- x_4: Angular speed of rope in rad/s.

Linearization

To linearize the system, we first need to calculate the steady state the steady state points as point of operations and linearize the system at that point.

The given system has four states and one input.
It could be physically realised that the state x2 is the derivative of state x1 as speed of the trolley is the rate of change of the position.
It could also be seen that state four x4 (angular speed) is a derivative of state x3 which is the rope angle.

Conditions for equilibrium:

Derivative of all states need to be zero, which implies both x2 (derivative of x1) and x4 (derivative of x3) would be zero.

$$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \\ \dot{x}3 \\ \dot{x}4 \end{bmatrix} = \begin{bmatrix} \dot{x}1 \\ ddx1 \\ \dot{x}3 \\ ddx3 \end{bmatrix} = \begin{bmatrix} f1\ ([x, \dot{x}\ , \theta, \dot{\theta}]^T, u(t)) \\ f2\ ([x, \dot{x}\ , \theta, \dot{\theta}]^T, u(t)) \\ f3\ ([x, \dot{x}\ , \theta, \dot{\theta}]^T, u(t)) \\ f4\ ([x, \dot{x}\ , \theta, \dot{\theta}]^T, u(t)) \end{bmatrix}$$

Solving eqn 1 and 2

$$ddx1 = \frac{(l*ml*\sin(x3)*dx3^\wedge 2 + u + g*ml*\cos(x3)*\sin(x3))}{(-\ ml*\cos(x3)^\wedge 2 + mc + ml)}$$

$$ddx3 = \frac{-(\ \ l*ml*\cos(x3)*\sin(x3)*dx3^\wedge 2 + u*\cos(x3) + g*mc*\sin(x3) + g*ml*\sin(x3)\ \ )}{(l*(-\ ml*\cos(x3)^\wedge 2 + mc + ml))}$$

For linearisation, we computed the following Jacobians:

$$A\ (x_0, u_0) = J_X\ (f\ (x, u))\ |_{\ (x0,\ u0)},\ B\ (x_0, u_0) = J_u\ (f\ (x, u))\ |_{\ (x0,\ u0)}$$

Where J is the Jacobian and (x_0, u_0) is the point of linearisation. We computed the Jacobians using the MATLAB script (attached at the end). The state space representation can be given as:

$$\delta\dot{x} = A\ (x_0, u_0)\ \delta_{\ X} + B\ (x_0, u_0)\ \delta_{\ u}$$

Case 1: Equating the value of ddx1 to zero and substituting the value of x3 in ddx3 and obtaining the relation in terms of u.

Equating ddx1=0:
Since dx3=0
We have:

$$u(t) + m_l\, g \sin(x_3) \cos(x_3) = 0 \qquad \text{…. equ (2)}$$

which implies:

$$\sin^{-1}\left[\frac{-2u(t)}{mlg}\right] = x_3 \qquad \text{….equ (3)}$$

On equating ddx3=0:
On putting x3 derived from equation 1 in expression of ddx3

We have:

$u(t)\ [\sqrt{(mlg)^2 - 4u^2} - (ml + mc)2g] = 0$ …. equ (4)

which implies either:

$u(t) = 0$ or

$u = \pm \frac{g}{2}\sqrt{(4mc^2 - 3ml^2 + 8ml\ mc)}$ …. equ (5)

Since, mc = 10 kg (<< ml) and ml is of the order of few hundred kg's.
Therefore, the term under the sq. root (in eqn (4)) always comes out to be negative and non-zero value of u becomes imaginary.

Possible point, $Z_{ss} = [x1 \quad x2 \quad x3 \quad x4 \quad u] = [x1 \quad 0 \quad 0 \quad 0 \quad 0]$

Case 2: Equating the value of ddx3 to zero first and substituting the value of x3 in ddx1 and obtaining the relation in terms of u.

$x_3 = \tan^{-1}[\frac{-u(t)}{g(ml+mc)}]$

Substituting in ddx1:

$u(t) + m_l\, g \sin(x_3) \cos(x_3) = 0$

We get:

$u(t) = m_l\, g\, u(t)\, g\, (m_l + m_c) / (g^2 (m_l + m_c)^2 + u^2)$ …. equ (6)

Implies: $u(t)=0$

And two imaginary values of u(t)

$u^2(t)= g^2 (m_l + m_c) (-m_c)$ …. equ (7)

Therefor we get the same steady state point as that in Case I.

Linearizing the system corresponding to $Z_{ss} = [x1 \quad x2 \quad x3 \quad x4 \quad u] = [x1 \quad 0 \quad 0 \quad 0 \quad 0]$

$$A_{new} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & (g*ml)/mc & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(g*ml^2 + g*mc*ml)/(1*mc*ml) & 0 \end{bmatrix}$$

$$B_{new} = \begin{bmatrix} 0 \\ 1/mc \\ 0 \\ -1/(l*mc) \end{bmatrix}$$

$$C_{new} = \begin{bmatrix} 1 & 0 & l & 0 \end{bmatrix}$$

$$Z_{ss} = \begin{bmatrix} x1 & x2 & x3 & x4 & u \end{bmatrix} = \begin{bmatrix} x1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Analysis:

Eigen Values:

As the poles of the linearised system are the eigenvalues themselves, the stability of the system depends on the eigen values of the matrix A.

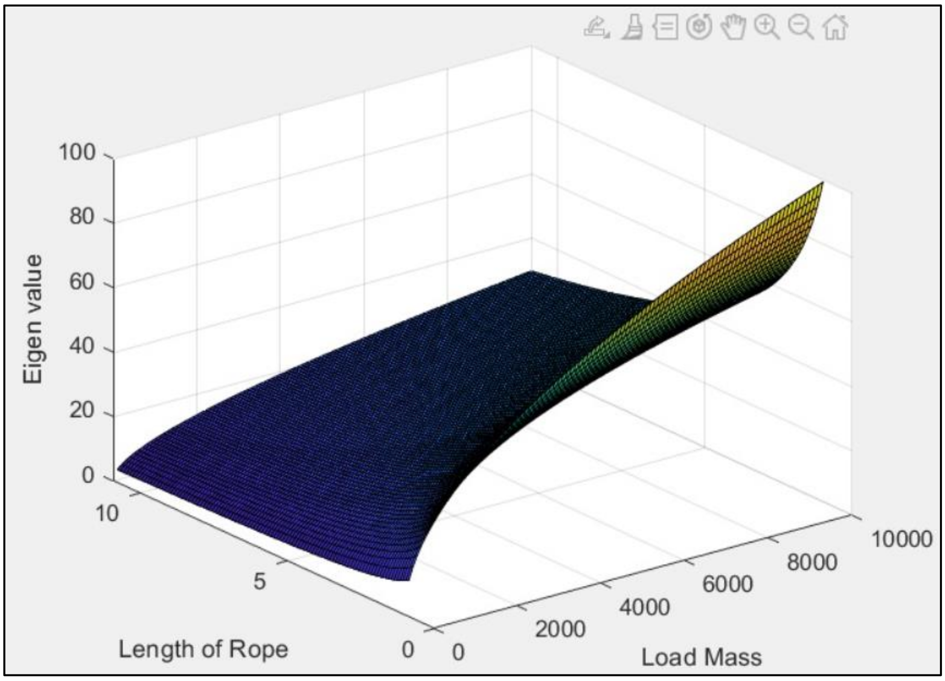
- The system is stable if all of the poles have negative real parts.
- The system is unstable if either of the poles has a positive real part.
- The system is marginally stable if all poles are purely imaginary.
- The system may be unstable or marginally stable, if any of the poles is at the origin.

$$\text{Eigen Values of Matrix A} = \begin{bmatrix} 0 \\ 0 \\ -\left(\left(\frac{-g*(mc+ml)}{l*mc}\right)^{\frac{1}{2}}\right) \\ \left(\frac{(-g*(mc+ml))}{l*mc}\right)^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \lambda 1 \\ \lambda 2 \\ \lambda 3 \\ \lambda 4 \end{bmatrix}$$

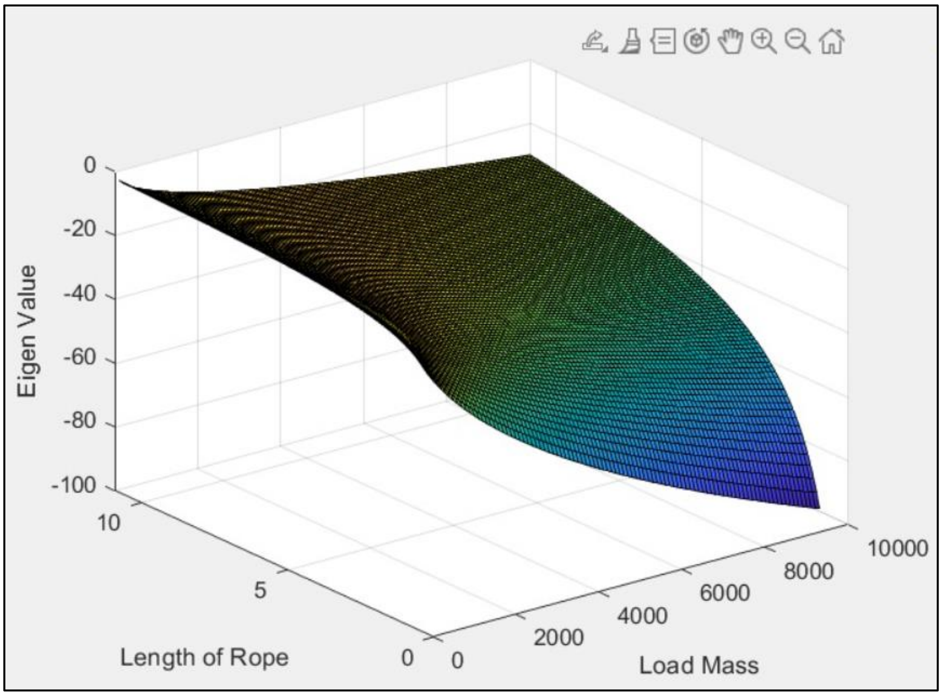
As we can see, all the poles have the real part equal to zero (two poles are at the origin and other two are on the imaginary axis), the linearised system is always marginally stable or unstable (if there are no zeros at the origin).

Movement of Eigen Values

- Eigen values which are zero doesn't depend on the system parameters
- For non-zero eigen values, variation can be seen w.r.t to model parameters as follows



Variation of $\lambda 4$

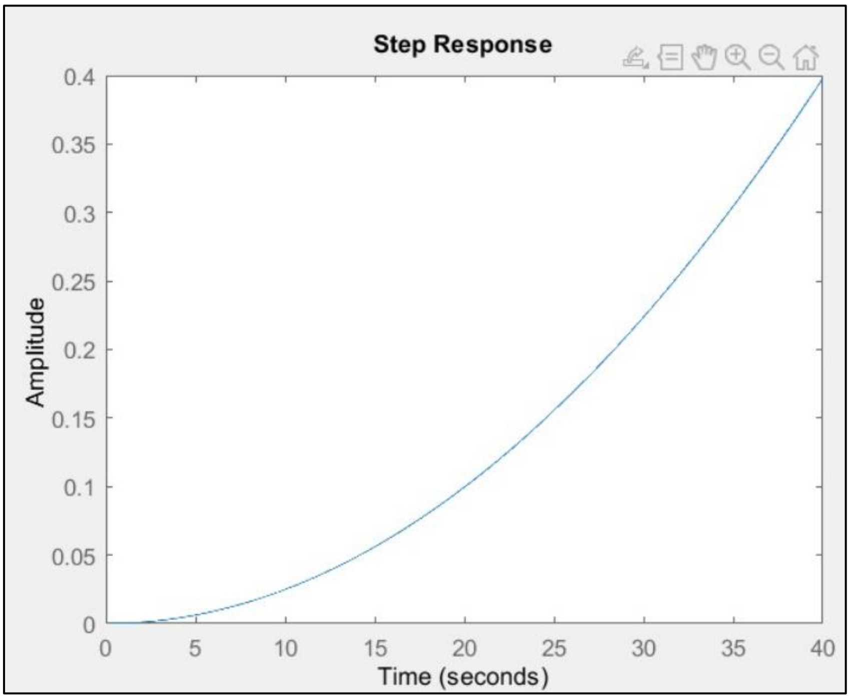


Variation of $\lambda 3$

- The nature of the eigen values remains same i.e., $\lambda 3$ and $\lambda 4$ remain on the imaginary axis after variation in m_L and L (since these parameters can attain only positive values for a physical system)

<u>Step response for the linearized system:</u>

To calculate the step response, first we have to take specific set of values for model parameters.



Eigen Values:

0

0

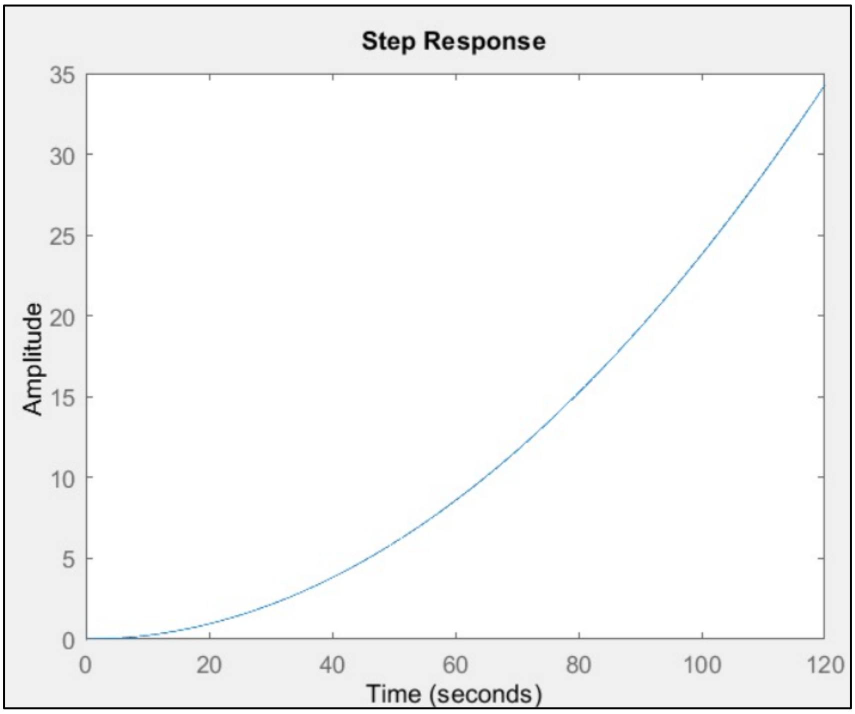
-44.40

44.40

```
t_f =

  0.08 s^2 + 0.1962
  -----------------
   s^4 + 394.4 s^2
```

m_L = 2000 kg, mc = 10 kg, l=1



Eigen Values:

0

0

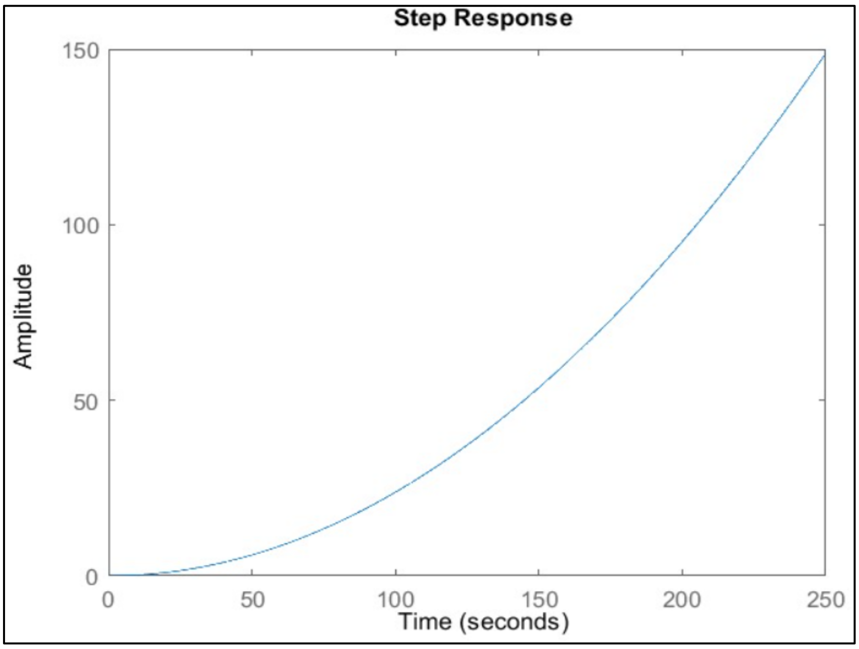
-14.35

14.35

```
t_f =

  0.08 s^2 + 0.1962
  -----------------
   s^4 + 394.4 s^2
```

m_L = 200 kg, m_C = 10 kg, l=1



Eigen Values:

0

0

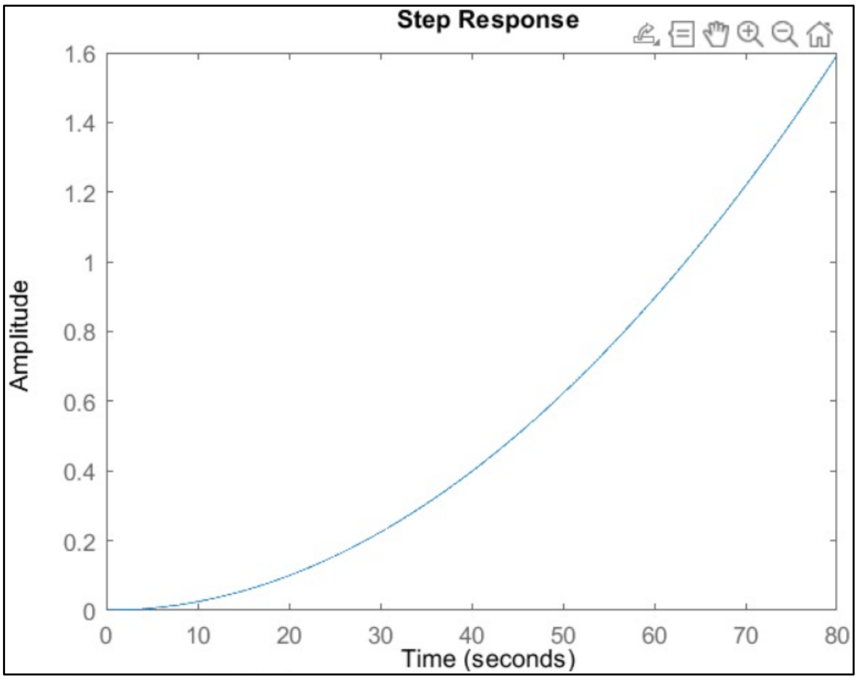
-6.41

6.41

```
t_f =

  0.08 s^2 + 0.1962
  -----------------
   s^4 + 394.4 s^2
```

m_L = 200 kg, m_C = 10 kg, l=5



Eigen Values:

0

0

-19.85

19.85

```
t_f =

  0.08 s^2 + 0.1962
  -----------------
   s^4 + 394.4 s^2
```

m_L = 2000 kg, m_C = 10 kg, l=5

- The given system response is non-oscillatory and undamped, for all combination of model parameters (as the eigen value remain on the imaginary axis making the system critically damped)
- The exponential increase in the step response confirms the instability of the system.
- As the value of m_L increases, rise time of the response also increases.
- As the value of L increases, rise time of the response decreases.

Stability Analysis

The above system comes out to be globally unstable (or marginally stable) due to presence of two zero and two purely imaginary poles. Once the input force (u(t)) is supplied the state and output dynamics of the crane increases exponentially without settling to any steady state value. Only one steady state point is deduced for the above crane system (trivial point) and that too is unstable therefore it can be concluded that the system is globally unstable at that point.

Conclusion:

In the following lab,

- The system is linearized after calculating the steady state point.
- Stability analysis is performed after analysing the eigen values.
- The movement of eigen values is constrained only to the imaginary axis and their movement is governed only by the system parameters (ml, mc, l, g).

MATLAB Script:

```
clear;
clc;

close all;
s=tf('s')

syms x1 dx1 ddx1 x3 dx3 ddx3 ; %state variables and input
syms u;
syms mc ml l g; %Model variables
X = [x1 dx1 x3 dx3];
f2 = (l*ml*sin(x3)*dx3^2 + u + g*ml*cos(x3)*sin(x3))/(- ml*cos(x3)^2 + mc + ml);
f4 = -(l*ml*cos(x3)*sin(x3)*dx3^2 + u*cos(x3) + g*mc*sin(x3) + g*ml*sin(x3))/(l*(- ml*cos(x3)^2 + mc + ml));
F = [dx1 f2 dx3 f4]
F_X = jacobian(F, X);
F_u = jacobian(F, u);
```

Script 1: Linearising the Non-Linear System

```
syms xo
%x1=xo, dx1=0,x3=0,dx3=0,u=0;

A = subs(F_X, {x1,dx1, x3,dx3, u},[xo 0 0 0 0]);
B = subs(F_u, {x1,dx1, x3,dx3, u},[xo 0 0 0 0]);
C=[1 0 1 0];
% A(2,3)=-1*A(2,3);
% A(4,3)=-1*A(4,3);
%sys_mimo = ss(A,B,C,0);
%tf(sys_mimo);
[E,v] = eig(A)
A_n = subs( A, [mc ml l g] , [10 2000 5 9.81] );
B_n = subs( B, [mc ml l g] , [10 2000 5 9.81] );
C_n = subs( C, [mc ml l g] , [10 2000 5 9.81] );
A_m=[0 1 0 0; 0 0 1962 0; 0 0 0 1; 0 0 -197181/500 0];
B_m=[0; 0.1; 0; -1/50];
C_m=[1 0 1 0];

sys_mimo = ss(A_m,B_m,C_m,0);
t_f=tf(sys_mimo)
step(t_f)
```

Script 2: Analysing the Linearised System

```
l=[1:0.1:10.9];
m=[100:100:10000];
[x,y] = meshgrid(m,l);
z =(sqrt(-0.981.*(10+x)./y));
surf(x,y,z, "FaceColor","interp")
```

Script 3: Plotting the variation of the Eigen Values

THANKYOU!