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Important Questions

Module-I (Matrix)

(Problems on Inverse)

1. Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix by using elementary transformations. **(2011)** Ans: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Employing elementary transformations, find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **(2018)** Ans: $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

3. Find the inverse of the matrix M by applying elementary transformations $M = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ Ans: $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

4. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 1 & 1 \end{bmatrix}$ without first evaluating the product.

5. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, determine two non-singular matrix P and Q such that $PAQ = I$. Hence find A^{-1}. **(2014, 2019)**

Ans: $A^{-1} = QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

5. find the inverse of $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$,. **(2021)** Ans: $A^{-1} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(Problems on Rank)

1. Use elementary transformations to reduce the matrix A to triangular form $A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$. Hence find the rank of A.

Ans: $A \approx \begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 1 & -6 & -3\\ 0 & 0 & 33 & 22\\ 0 & 0 & 0 & 0\end{bmatrix}$. **Rank of A = 3 because number of non zero rows are 3** **(2010)**

2. Reduce the matrix A to its normal when A = $\begin{bmatrix}1 & 2 & -1 & 4\\ 2 & 4 & 3 & 4\\ 1 & 2 & 3 & 4\\ -1 & -2 & 6 & -7\end{bmatrix}$. Hence find the rank of A. **(2019)** Ans: $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

Rank of A = 3

3. Prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the rank of the matrix $\begin{bmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{bmatrix}$ is less than three.

4. Find the rank of the following matrices by reducing into normal form (or canonical form)

(i) $\begin{bmatrix}1 & -1 & 2 & -3\\ 4 & 1 & 0 & 2\\ 0 & 3 & 0 & 4\\ 0 & 1 & 0 & 2\end{bmatrix}$ **(2012)** (ii) $\begin{bmatrix}3 & 2 & -1\\ 4 & 2 & 6\\ 7 & 4 & 5\end{bmatrix}$ **(2016)** (iii) $\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 11 & -5 & 3\\ 2 & -5 & 3 & 1\\ 4 & 1 & 1 & 5\end{bmatrix}$ **(2015)** (iv) $\begin{bmatrix}0 & 1 & 2 & -1\\ 1 & 2 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$ **(2018)**

Ans: (i) 4 (ii) 2 (iii) 2 (iv) 4

5. Find the value of P for which the matrix $A=\begin{bmatrix}3 & P & P\\ P & 3 & P\\ P & P & 3\end{bmatrix}$ is of rank 1. **(2012)**

6. Find all values of μ for which rank of the matrix $A=\begin{bmatrix}\mu & -1 & 0 & 0\\ 0 & \mu & -1 & 0\\ 0 & 0 & \mu & -1\\ -6 & 11 & -6 & 1\end{bmatrix}$ is of rank 3

7. Use elementary transformations to find the rank of matrix $A=\begin{bmatrix}-2 & -1 & 3 & -1\\ 1 & 2 & -3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}$. **(2018)** Ans:. **Rank of A = 3**

8. Find two non-singular matrix P and Q such that PAQ is in the normal form of the matrix and hence find the rank of matrix

$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ Ans: Rank of $A \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

8. Find two non-singular matrix P and Q such that PAQ is in the normal form of the matrix A where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ **(2021)**

Ans: $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(Problems on System of Linear equations)

1. Solve the system of linear equations , show that the equations $x+y+z=3$, $x+2y+3z=4$, $x+4y+9z=6$.
Ans: $x=2$, $y=1$ and $z=0$

2. Using matrix method, show that the equations $3x+3y+2z=1$, $x+2y=4$ $10y+3z=-2$ $2x-3y-z=5$ are consistent and hence obtain the solutions for *x*, *y* and *z*. **(2010)** **Ans:** $x=2$, $y=1$ **and** $z=-4$

3. Investigate for consistency of the following equations and if possible, find the solutions: $4x-2y+6z=8$, $x+y-3z=-1$ and $15x-3y+9z=21$. **Ans:** $x=1$, $y=3k-2$ **and** $z=k$

4. Test the consistency of the following equations and if possible, find the solutions: $10y+3z=0$, $3x+3y+z=1$ and $2x-3y-z=5$. **(AKTU-2018)** **Ans:** inconsistent, no solution

5. Investigate, for what values of λ and μ do the system of equations $x+y+z=6$, $x+2y+3z=10$ and $x+2y+\lambda z=\mu$ have (i) no solution (ii) unique solution (iii) infinite solution? **(AKTU-2016, 2018)**

Ans: $C = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & \lambda-3 & \vdots & \mu-10 \end{bmatrix}$ (i) $\lambda=3$, $\mu\neq10$ (ii) $\lambda\neq3$, μ may have any value (iii) $\lambda=3$, $\mu=10$

6. Determine the values of λ and μ such that the system $2x-5y+2z=8$, $2x+4y+6z=5$ and $x+2y+\lambda z=\mu$ has (i) no solution (ii) a unique solution (iii) infinite number of solutions.

Ans: $C = \begin{bmatrix} 2 & -5 & 2 & \vdots & 8 \\ 0 & 9 & 4 & \vdots & -3 \\ 0 & 0 & \lambda-3 & \vdots & \mu-5/2 \end{bmatrix}$ (i) $\lambda=3$, $\mu\neq5/2$ (ii) $\lambda\neq3$, μ may have any value (iii) $\lambda=3$, $\mu=5/2$

7. Determine the values of *a* and *b* for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$ has (i) a unique solution (ii) no solution (iii) infinitely many solutions.

Ans: $C = \begin{bmatrix} 1 & -3 & 1-a & \vdots & b+1 \\ 0 & 7 & 4+5a & \vdots & -5b-2 \\ 0 & 00 & -6-2a & \vdots & 3b-1 \end{bmatrix}$ (i) $a\neq-3$, *b* may have any value (ii) $a=-3$, $b\neq1/3$ (iii) $a=-3$, $b=1/3$

8. Show that the equations $-2x+y+z=a$, $x-2y+z=b$ and $x+y-2z=c$ have no solution unless $a+b+c=0$. In which case they have infinitely many solutions? Find these solutions when a = 1, b = 1, c= -2

Ans: $C=\begin{bmatrix}1 & 1 & -2 & \vdots & c\\ 0 & -3 & 3 & \vdots & b-c\\ 0 & 0 & 0 & \vdots & a+b+c\end{bmatrix}$ (i) if $a+b+c\neq 0$, $rank$ of $C=3\neq rank\ of\ A$ then system is inconsistent, have no solution (ii) if $a+b+c=0$, $rank$ of $C=2=rank\ of\ A<Number\ of\ unknown$ then system is consistent, have infinite solution $x=k-1$, $y=k-1$ and $z=k$

9. Find the value of λ such that the following equations have unique solution: $\lambda x+2y-2z-1=0$, $4x+2\lambda y-z-2=0$ and $6x+6y+\lambda z-3=0$ and use matrix method to solve these equations when $\lambda=2$. **(MTU-2018)**

Ans: $\lambda\neq 2$, when $\lambda=2$, $x=1/2+(k_1-k_2)$, $y=k_2$ and $z=k_1$ where k_1 and k_2 are arbitrary

10. For what values of k, the equations $x+y+z=1$, $2x+y+4z=k$ and $4x+y+10z=k^2$ have a solution and solve them completely in each case.

Ans: $C=\begin{bmatrix}1 & 1 & 1 & \vdots & 1\\ 0 & -1 & 2 & \vdots & k-2\\ 0 & 0 & 0 & \vdots & k^2-3k+2\end{bmatrix}$ **when** $k=1$, $x=\frac{3}{2}(1-k)$, $y=k$ and $z=\frac{1}{2}(k-1)$, **when** $k=2$, $x=1-3\lambda$, $y=2\lambda$ and $z=\lambda$, where λ is arbitrary

11. Apply the matrix method to solve the system of equations $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$.

(UPTU-2014) Ans: $x=-1$, $y=4$ and $z=4$

12. Test the consistency and hence solve the following set of equations: $10y+x3z=0$, $3x+3y+z=1$, $2x-3y-z=5$, $x+2y=4$.

Ans: $x_1=1$, $x_2=0$ $x_3=1$ **(2018)**

13. Find the values of k for which the system of equations $(3k-8)x+3y+3z=0$, $3x+(3k-8)y+3z=0$ and $3x+3y+(3k-8)z=0$ has a non trivial solution. Ans: when $k=2/3$, $k=11/3$, $k=11/3$. **(AKTU-2022).**

(Problems on Linearly Dependent and independent vectors)

1. Show that the vectors $X_1 = (1, 2, 3)$, $X_2 = (2, -2, 0)$ form a linearly independent set.

2. Show that the vectors $x_1 = (1,2,4)$, $x_2 = (2,-1,3)$, $x_3 = (0, 1, 2)$ and $x_4 = (-3,7,2)$ are linearly dependent and find the relation between them. Ans: $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$ **(UKTU-2012, 2022)**

3. Find whether or not the set of vectors $[1,-2]$, $[2,1]$, $[3,2]$ is linearly dependent or independent. Ans: L. D.

4. Prove that the four vectors $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (0,0,1)$ and $x_4 = (1,1,1)$ in $V_3(c)$ form a linearly dependent set but any three of them are linearly independent.

5. Show that the vectors $x_1 = (2, 3,1,-1)$, $x_2 = (2,3,1,-2)$, $x_3 = (4,6,2,1)$ are linearly dependent. Express one of the vectors as a linear combination of the others. **Ans:** $5x_1 - 3x_2 - x_3 = 0$

6. Show that the column vectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ are linearly independent. Ans: $5x_1 - 3x_2 - x_3 = 0$ **(AKTU-2020)**

7. If the vectors $(0,1,a)$, $(1,a,1)$ and $(a,1,0)$ is linearly dependent, then find the value of *a*. Ans: $0, \pm\sqrt{2}$

Eigen Values and Eigen Vectors

1.Find the eigen values and eigen vectors of the following matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Ans: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow \lambda = 1,1,4$ **(2018)**

2. Find the eigen values and eigen vectors of the following matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ Ans: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow \lambda = 1,3,3$ **(2021)**

3. Find the eigen values and eigen vectors of the following matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ Ans: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow \lambda = 1,1,7$ **(2022)**

35. Find the eigen values and eigen vectors of the following matrices:

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ Ans: (i) 0, 3, 15, $k_1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $k_2\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $k_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ (ii) 3, 2, 5, $\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}$, $k_2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $k_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Cayley-Hamilton theorem

1. Verify Cayley-Hamilton theorem for the following matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence compute A^{-1}.

Also evaluate $A^6 - 6A^5 + 9A^4 - 2A^3 + 12A^2 + 23A - 9I$. **(AKTU-2016, 2018)**

Ans: $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$ and $A^6 - 6A^5 + 9A^4 - 2 + 23A - 9I = 5A - I = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$

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2. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, compute A^{-1}. Also find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. Ans: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$, $A^2 + A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ **(AKTU-2022)**

3. Verify Cayley-Hamilton theorem for the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and hence, compute A^{-1}. Also Also evaluate

$A^8 - 11A^7 - 4A^6 + 3A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$. **(AKTU-2019)**

Ans: $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$, $A^2 + A + I = \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{bmatrix}$

4. Given matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ find adjA by using the Cayley-Hamilton theorem . **(2022)**

Ans: $\lambda^3 - 3\lambda^2 + 5\lambda + 3 = 0$, $A^{-1} = -\frac{1}{3}\left(A^2 - 3A + 5I\right) = -\frac{1}{3}\begin{bmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{bmatrix}$ $\left[\because A^{-1} = \frac{adjA}{|A|} \ So \ adjA = A^{-1}|A|\right]$

$$adjA = A^{-1}|A| = \begin{bmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{bmatrix}$$

5. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}.

Ans:, $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -8 \\ -1 & 0 & 4 \end{bmatrix}$ **(AKTU-2022)**

6. Express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A where matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ **(AKTU-2017)**

Ans:, $138A - 403I$

7. Evaluate the expression $A + 5I + 2A^{-1}$ as a linear polynomial in A where matrix $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ **(AKTU-2017)**

Ans:, $138A - 403I$