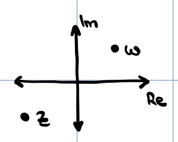
The analysis of differentiable functions $f:\mathbb{C}\to\mathbb{C}$.

Since $\mathbb{R}\subseteq\mathbb{C}$, $\mathbb{C}$-analysis can help solve real problems like $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin(x)}{1+x^2}\,dx=\frac{\pi}{e}$.

$\mathbb{C}$ is a set with a "distance function": it is a metric space (there is a notion of measurement).



distance: $|z-\omega|$ (absolute value in $\mathbb{C}$, modulus).

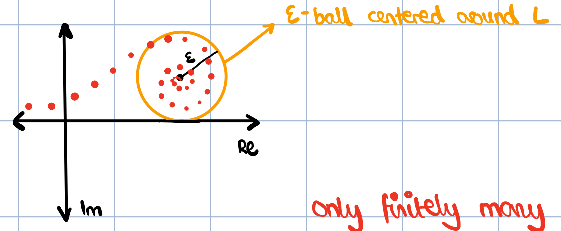
Complex Sequences

A sequence $\{z_n\}_{n\in\mathbb{N}}=\{z_n\}\subseteq\mathbb{C}$, so some map $\varphi:\mathbb{N}\to\mathbb{C}$.

We say $\{z_n\}$ is convergent to $L\in\mathbb{C}$ if the distance between z_n and L gets smaller and smaller, approaching 0.

$\Leftrightarrow$ $\underbrace{\{|z_n-L|\}_{n\in\mathbb{N}}\subseteq\mathbb{R}}$ is convergent to 0.

convergence of a real sequence: $\Leftrightarrow$ $\forall\varepsilon>0,\ \exists N\in\mathbb{N}$ such that $|z_n-L|<\varepsilon,\ \forall n\geq N$.



ε-ball: $B_\varepsilon(L):=\{\omega\in\mathbb{C}:|\omega-L|<\varepsilon\}$

$\parallel$

$B(L,\varepsilon)$ (identical notation)

only finitely many terms of $\{z_n\}$ lie outside any ε-ball.

A function $f:\mathbb{C}\to\mathbb{C}$ is continuous at $z_0\in\mathbb{C}$ if for all sequences $\{z_n\}\subseteq\mathbb{C}$:

$z_n\xrightarrow{n\to\infty}z_0$ implies $f(z_n)\xrightarrow{n\to\infty}f(z_0)$. This is an extension of Sequential Characterization in $\mathbb{R}$.

Remarks:
- Holomorphic vs. analytic functions.

Throughout the course of these notes, we will encounter functions that are said to be holomorphic on $\mathbb{C}$.
(complex differentiable)

A function $f:D\to\mathbb{R}$ is real-analytic on an open set $D\subseteq\mathbb{R}$ if for any $x_0\in D$, if we have the Taylor series

$T_{f,x_0}(x)=\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ converging pointwise to $f(x)$ for x in some δ-neighbourhood of x_0, and $f\in C^\infty(D)$.

The same definition holds for a function being complex-analytic, but for $f:U\to\mathbb{C}$ where $U\subseteq\mathbb{C}$ is an open set,

f is complex-analytic on U $\Leftrightarrow$ f is holomorphic on U.

Remark: Note that any $\mathbb{R}$-analytic function is smooth $(\in C^\infty(\mathbb{R}))$, but not all smooth functions are $\mathbb{R}$-analytic.

Complex Functions

We will see how the concepts from Real Analysis carry over into Complex Analysis. How do they relate?

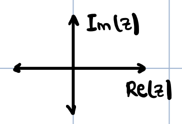
A Real Function: a mapping $f: \mathbb{R} \to \mathbb{R}$ where $\mathbb{R}$ has algebraic properties (a field algebra) and topological property (defining elements being "next to each other")

so we conceptualize $\mathbb{R}$ as a number line, i.e. a one-dimensional continuum (they have the same topological properties).

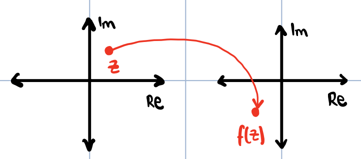
A Complex Function: a mapping $f: \mathbb{C} \to \mathbb{C}$. Fundamentally, $f: \underset{\{a+bi \,:\, a,b \in \mathbb{R}\}}{\mathbb{C}} \longrightarrow \mathbb{C}$ so $\mathbb{R} \subset \mathbb{C}$ is a proper subset.

Moreover, we have an algebraic structure obeying all the axioms of field theory $\Rightarrow$ $\mathbb{C}$ is a field algebra (just like $\mathbb{R}$);

we conceptualize $\mathbb{C}$ as the complex plane :



and for our purposes, $\mathbb{C}$ has a topological structure identical to the plane.



visualizing a mapping $f: \mathbb{C} \to \mathbb{C}$.

example: $f(z) = z+1$ which maps $a+bi \mapsto (a+1)+bi$ (horizontally translating z by 1)

$f(z) = z+i$ which maps $a+bi \mapsto a+(b+1)i$ (vertically translating z by 1)

$f(z) = z^2$; how can we describe this?

The basic idea of polynomials in $\mathbb{R}$ translates to polynomials in $\mathbb{C}$.

Take $z = 2+i$. Then $f(z) = (2+i)^2 = (2+i)(2+i) = (2+i)\cdot 2 + (2+i)i = 4+2i+2i-1 = 3+4i$

Remark: In Euler's form, $f(z) = z^2$ maps $re^{i\theta} \mapsto r^2 e^{i2\theta}$ (modulus-argument form).

The $\mathbb{R}_{\geq 0}$ section of $\mathbb{C}$ can be represented as $re^{i0} = re^0$, so we have $f: re^0 \mapsto r^2 e^{2\cdot 0} = r^2 e^0$ so $f: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

The $\mathbb{R}_{<0}$ section of $\mathbb{C}$ can be represented as $re^{i\pi}$, so we have $f: re^{i\pi} \mapsto r^2 e^{i2\pi} = r^2 e^0$ so $f: \mathbb{R}_{<0} \longrightarrow \mathbb{R}_{\geq 0}$

Difference Between $\mathbb{C}$-analysis and Vector-analysis

We know $f: \mathbb{C} \to \mathbb{C}$ is a complex function, where $\mathbb{C}$ has the topological structure of the 2D-plane. What about multivariable / vector calculus / analysis?

Surely $\mathbb{C}$-functions could have been studied in vector analysis, especially as $f: \mathbb{R}^2 \to \mathbb{R}^2$, since $\mathbb{R}^2$ has the same topological structure as $\mathbb{C}$.

More explicitly, we could identify any $z = x+iy \in \mathbb{C}$ with some $\binom{x}{y} = (x,y) \in \mathbb{R}^2$. What is the point?

Calculus / Analysis is what you get when you mix topology with algebra.

$\mathbb{C}$ and $\mathbb{R}^2$ are topologically homeomorphic, but are not equivalent as algebraic structures (i.e. $\mathbb{C} \not\cong_{\mathbb{R}} \mathbb{R}^2$).

In fact, $\mathbb{R}^2$ is a vector-algebra but $\mathbb{C}$ is a field-algebra, which is more incredible.

Note that $(\mathbb{C}, +) \cong (\mathbb{R}^2, +)$, but not for multiplication.

Essentially, $\mathbb{C} \not\cong \mathbb{R}^2 \Rightarrow \mathbb{C}$-analysis $\not\equiv$ Vector-analysis.

Real and Imaginary Components of Complex Functions

Let $f: \mathbb{C} \to \mathbb{C}$. We can write $f(z) = f(x+iy) = \underbrace{u(x,y)}_{\mathrm{Re}(f)} + \underbrace{v(x,y)}_{\mathrm{Im}(f)} i$ where $u(x,y), v(x,y) \in \mathbb{R}$, $u, v: \mathbb{R}^2 \to \mathbb{R}$
(in many cases)

example: Take $f(z) = z^2 + (3+i)z + 9$, so $f: \mathbb{C} \to \mathbb{C}$ (note that we can have polynomials $f(x) \in \mathbb{C}[x]$).
ring of complex polynomials

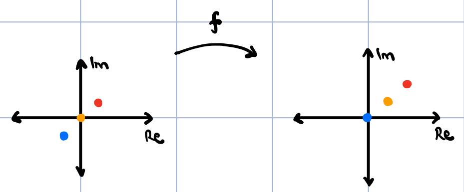
Then $f(z) = f(x+iy) = (x+iy)^2 + (3+i)(x+iy) + 9$

$$= x^2 + 2ixy - y^2 + 3x + 3iy + ix - y + 9$$

$$= \underbrace{(x^2 - y^2 + 3x - y + 9)}_{u(x,y)} + \underbrace{(2xy + 3y + x)}_{v(x,y)} i.$$

Limits of Complex Functions (Part 1, 2)

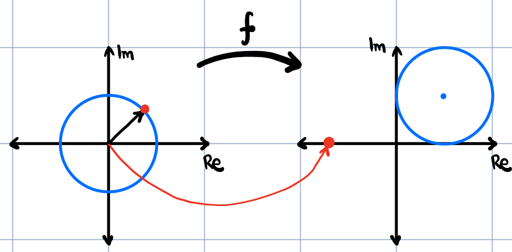
Intuition by example: $f(z) = z + (i+1)$.



so every point moves by a vector ↗.

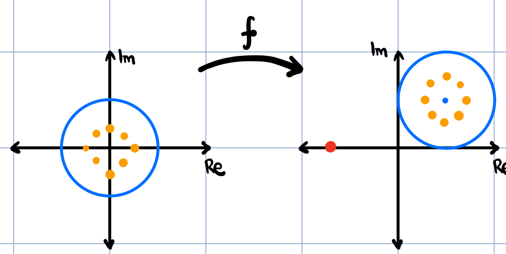
Now take $f(z) = \begin{cases} z + (i+1), & z \in \mathbb{C} \setminus \{0\} \\ -3, & z = 0 \end{cases}$ (note that -3 could have been any $a \in \mathbb{C} \setminus \{i+1\}$)

Let us draw this:



so we have a circle translating its centre to $i+1$.

We have $f(0) = -3$, but we can ask: what is $\lim_{z \to 0} f(z)$? (i.e. what would you expect $f(0)$ to be, from points z surrounding 0?)
↳ what does the value converge to?



so intuitively, we have $\lim_{z \to 0} f(z) = i + 1$ (not necessarily $= f(0)$).

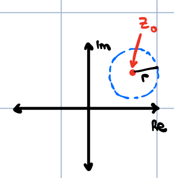
Suppose $f(z) = z/z$. For $z \neq 0$, we have $z \in \mathbb{C}^\times \Rightarrow z^{-1} \in \mathbb{C}^\times$ exists, so $f(z) = 1$, $\forall z \in \mathbb{C} \setminus \{0\}$.

However, $0 \notin \mathbb{C}^\times$ so $z = 0$ cannot exist (undefined), but we have $\lim_{z \to 0} f(z) = 1$ (by consequences of field theory).

Let $f: \mathbb{C} \to \mathbb{C}$. Write $\lim_{z \to z_0} (f(z)) = L$ for some $z_0, L \in \mathbb{C}$. What does this mean?

$$\lim_{z \to 0} (f(z)) = L \iff \underset{\varepsilon \in \mathbb{R}}{\forall \varepsilon > 0}, \ \underset{\delta \in \mathbb{R}}{\exists \delta > 0} \text{ such that } \forall z \in B'(z_0, \delta);\ f(z) \in B(L, \varepsilon).$$

Open Ball $B(z_0, r)$: a subset of $\mathbb{C}$ containing all $z \in \mathbb{C}$ such that z is within a radius r from some $z_0 \in \mathbb{C}$.
open neighbourhood (centre: z_0, radius: r)



i.e. $B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$

Closed Ball $\overline{B}(z_0, r)$: an open ball, inclusive of the boundary of the ball.
closed neighbourhood

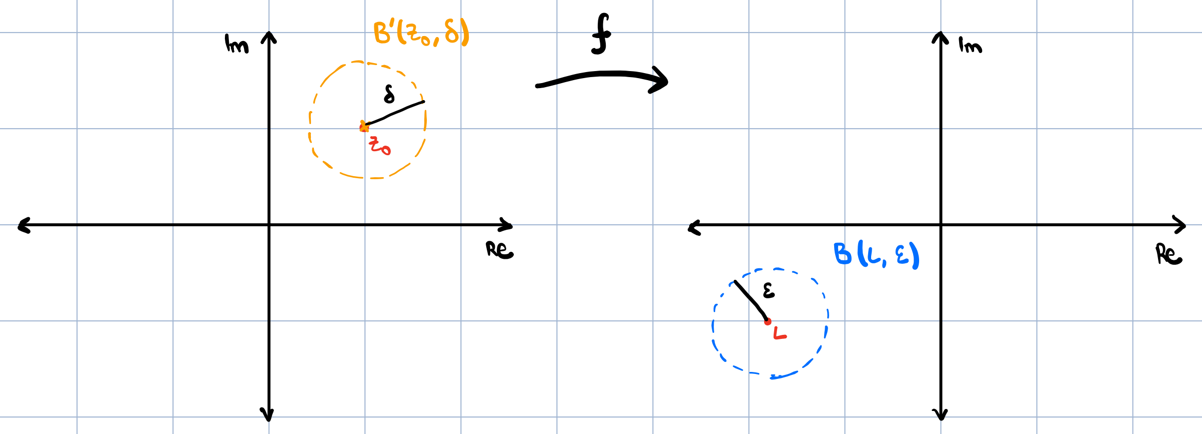
Deleted Open Ball $B'(z_0, r)$: an open ball, exclusive of the centre z_0 itself.
deleted open neighbourhood

i.e. $B'(z_0, r) = B(z_0, r) \setminus \{z_0\}$

Deleted Closed Ball $\bar{B}'(z_0, r)$: a closed ball, exclusive of the centre z_0 itself.

deleted closed neighbourhood

i.e. $\bar{B}'(z_0, r) = \bar{B}(z_0, r) \setminus \{z_0\}$



for any open ball around L of radius ε, there must exist a δ-ball such that for any element in the δ-ball, it maps to some element in the ε-ball.

note: we are not implying surjectivity.

For $f(z) = \begin{cases} z+(i+1), & z \in \mathbb{C}\setminus\{0\}; \\ -3, & z = 0. \end{cases}$ we can show that $\lim_{z\to 0} f(z) = i+1$ by choosing $\delta = \varepsilon$ (or even any $\delta \leq \varepsilon$).

Continuity of Complex Functions

Let $f: \mathbb{C} \longrightarrow \mathbb{C}$. Then f is said to be continuous at $z_0 \in \mathbb{C}$ if $f(z_0) = \lim_{z\to z_0} f(z)$.

If f is continuous in some region $D \subseteq \mathbb{C}$, then we say f is continuous over D. If $D = \mathbb{C}$, then f is continuous over $\mathbb{C}$.

<u>example</u>: $f(z) = z + (2+i)$. Everything is shifted by $\binom{2}{1}$ in $\mathbb{C}$. It is obviously continuous.

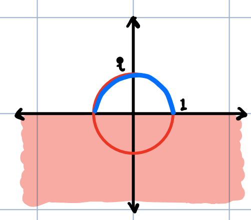
$f(z) = 2z$. This stretches everything by a factor of 2. This is less obviously continuous ($f: re^{i\theta} \longmapsto 2re^{i\theta}$).

$f(z) = \begin{cases} 2z, & z \neq 1+i; \\ -7, & z = 1+i. \end{cases}$ has a discontinuity at $z = 1+i$.

$f(z) = z^2$ maps $re^{i\theta} \longmapsto r^2e^{i2\theta}$; is this continuous?

Consider $\mathbb{C}$ with $\mathrm{Im}(z) \geq 0$ since $f(z)$ is not injective.

Proving $\lim_{z\to z_0} f(z) = f(z_0)$ is "boring" but intuitively, f is continuous.



the half unit circle wraps around to form a full unit circle when $f(z)$ is applied.

everything gets stretched and twisted except 0 and 1.

<u>Proof</u>: Given $\varepsilon > 0$, set $\delta = \min\{1, \frac{\varepsilon}{(1+2|z_0|)}\}$. For all z,

$|z - z_0| < \delta \Rightarrow |z - z_0| < 1$ ($\Rightarrow |z + z_0| < 1 + 2|z_0|$ by the triangle inequality)

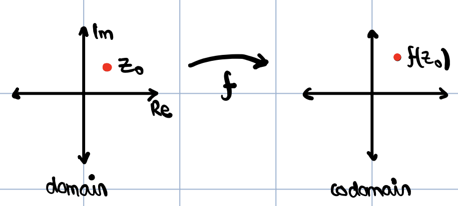
and $|f(z) - f(z_0)| = |z+z_0||z-z_0| < (1+2|z_0|)\dfrac{\varepsilon}{(1+2|z_0|)} = \varepsilon$. $\square$

Complex Differentiation

Truly different from differentiation in vector analysis. In $\mathbb{R}^2$, you often compute partial derivatives, directional derivatives etc., as gradients.

Complex differentiation is much less intuitive, and much more abstract.

Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be continuous on $\mathbb{C}$.



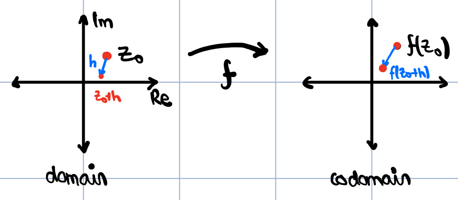
What does differentiation at $z_0 \in \mathbb{C}$ mean?

- Take the quotient function $\dfrac{f(z_0+h) - f(z_0)}{h}$, but what is h here?

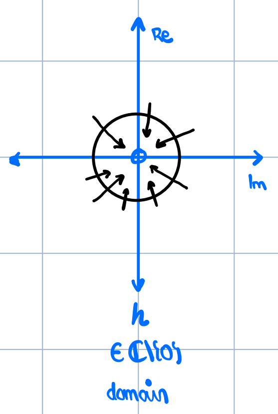
$\mathbb{R}$-analysis: some $h \in \mathbb{R}$;
partial diffs: some small direction increment
directional: some vector increment

h is any "small" complex number.

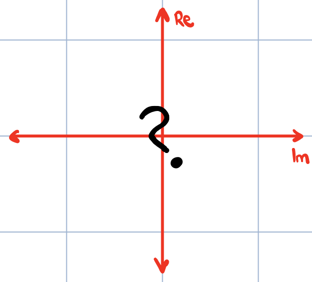
We don't specify the direction of h, so complex derivatives are TRULY 2-dimensional.



{we vary $h \in \mathbb{C}\setminus\{0\}$ in our domain}



$\longrightarrow \dfrac{f(z_0+h)-f(z_0)}{h}$



and we ask what the limit $\underbrace{h \to 0}_{\text{from ALL directions}}$ does to this function.

i.e. Does $\lim\limits_{h\to 0}\left(\dfrac{f(z_0+h)-f(z_0)}{h}\right)$ exist, for $z_0 \in \mathbb{C}$? $(= f'(z_0))$.

Doing so for all $z \in \mathbb{C}$, we get the complex derivative $f'(z)$.

We say $f: \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if:

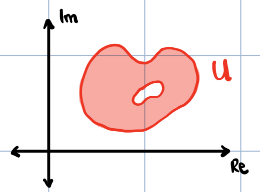
- $f(z)$ is continuous at z_0, i.e. $\lim\limits_{z\to z_0} f(z) = f(z_0)$;
- $\lim\limits_{h\to 0}\left(\dfrac{f(z_0+h)-f(z_0)}{h}\right) = f'(z_0)$ exists.

if f is complex differentiable at all $z_0 \in \mathbb{C}$, we say f is complex differentiable on $\mathbb{C}$.

Differentiability is a local property: we only care about what happens around $z_0 \in \mathbb{C}$ $\Rightarrow$ the domain $\mathrm{dom}(f)$ need not be the entirety of $\mathbb{C}$.

$f: \mathbb{C} \longrightarrow \mathbb{C}$ differentiable at z_0?

domain can be any open set $U \subseteq \mathbb{C}$



} can be as strange as you want, provided U is an open set.

what could this be?

(holds for any metric space)
$\mathbb{C}$ is a metric space

Def: An open set $U \subseteq \mathbb{C}$ if boundary points of U are not a part of the set itself, i.e. every $p \in U$ is surrounded only by points also in U.

Mathematically, $\forall z_0 \in U$, $\exists \varepsilon > 0$ such that $B(z_0, \varepsilon) \subseteq U$. This is the standard topological space $(\mathbb{C}, \mathcal{T})$.

Def: Let $f: U \to \mathbb{C}$ such that $U \subseteq \mathbb{C}$ is an open set. We say f is (complex) differentiable if $\lim\limits_{z\to z_0}\left(\dfrac{f(z)-f(z_0)}{z-z_0}\right)$.

Linear Approximation: $\Leftrightarrow$ there is a function $\Delta_{f,z_0}: U \to \mathbb{C}$ with

For all sequences $(z_n) \subseteq U\setminus\{z_0\}$ with $z_n \xrightarrow{n\to\infty} z_0$, the sequence $\dfrac{f(z_n)-f(z_0)}{z_n - z_0}$ converges.

$$f(z) = f(z_0) + (z-z_0)\cdot \Delta_{f,z_0}(z) \quad \text{for all } z \in U,$$

and Δ_{f,z_0} is continuous at z_0.

$=$

$\underbrace{f(z_0+h)}_{f(z)} = f(z_0) + f'(z)\cdot h$, rewriting $z = z_0 + h \in U$.

then we have $f'(z_0) := \Delta_{f,z_0}(z_0) = \lim\limits_{z\to z_0}\left(\dfrac{f(z)-f(z_0)}{z-z_0}\right)$.

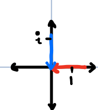
example:

- $f: \mathbb{C} \to \mathbb{C}$, $f(z) = m\cdot z + c$ for $m, c \in \mathbb{C}$.

  We have $f(z) = m\cdot z + c = \underbrace{(m\cdot z_0 + c)}_{f(z_0)} + (z - z_0)\underbrace{m}_{\Delta_{f,z_0}(z)}$ $\Rightarrow f'(z_0) = m$.

- $f: \mathbb{C} \to \mathbb{C}$, $f: z \mapsto \bar{z}$ (complex conjugation). Is this differentiable at $z_0 = 0$?

  We have $f'(0) = \lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z-0} = \lim\limits_{z\to 0}\dfrac{\bar{z}}{z}$ which does not exist:

  Consider approaching 0 from different directions  and we see that the limits are not the same.

  i.e. by Sequential Characterization,

$$z_n = \tfrac{1}{n} \Rightarrow z_n \xrightarrow{n\to\infty} 0: \ \frac{\bar{z}}{z} = \frac{(\frac{1}{n})}{(\frac{1}{n})} = 1 \xrightarrow{n\to\infty} 1 \neq -1 \xleftarrow{n\to\infty} -1 = \frac{(\frac{i}{n})}{(-\frac{i}{n})} = \frac{z_n^*}{\bar{z}_n^*} : 0 \xleftarrow{n\to\infty} z_n^* \Leftarrow \frac{-i}{n} = z_n^*$$

Holomorphic and Entire Functions

Def: Let $f: U \longrightarrow \mathbb{C}$ where $U \subseteq \mathbb{C}$ is an open subset. We say f is holomorphic (on U) if f is complex differentiable at every $z_0 \in U$.

If $U = \mathbb{C}$, i.e. if f is holomorphic on $\mathbb{C}$, then we say f is entire.

Properties:

(a) f is holomorphic $\Rightarrow$ f is continuous

(b) $f, g: U \to \mathbb{C}$ holomorphic $\Rightarrow$ $f+g$, $f \cdot g$ are holomorphic

(c) sum / product / quotient / chain rule(s) for derivatives hold from $\mathbb{R}$-analysis.

example:

- $f(z) \in \mathbb{C}[z]$ are all entire functions, $f(z) = a_m z^m + \ldots + a_1 z^1 + a_0$, $a_0, \ldots, a_m \in \mathbb{C}$.
  complex polynomial

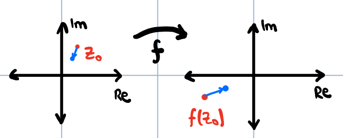
- $f: \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic.
  note: whenever we remove finitely many points from $\mathbb{C}$, we still have an open set.

- $f: \mathbb{C} \setminus S \longrightarrow \mathbb{C}$, $f(z) = \frac{p(z)}{q(z)}$ where $p(z), q(z) \in \mathbb{C}[z]$ and $S = \{\text{zeroes of } q\} = \{\alpha \in \mathbb{C} : q(\alpha) = 0\}$, is holomorphic.

Complex Derivative vs. Jacobian Matrix

{"worthy" comparison of complex derivatives in vector calculus}

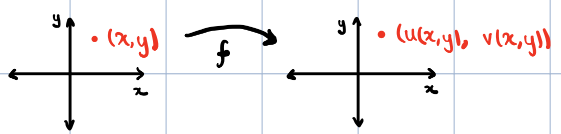
Let $f: \mathbb{C} \to \mathbb{C}$.



We are interested in the change in the codomain between $f(z_0)$ and $f(z_0+h)$.

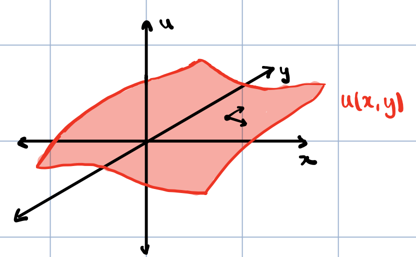
$$(f(z_0+h) - f(z_0)) = f'(z_0)h = \Delta_{f,z_0}(z)h)$$

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$. Since $\mathbb{R}^2$ and $\mathbb{C}$ are topologically "identical", say that this function is analogous to $f: \mathbb{C} \to \mathbb{C}$. Every f has such an equivalence.



The matrix of partial derivatives or the Jacobian matrix represents a linear transformation.

Multivariable Calculus: What is "proper" differentiability? Let's graph $u(x,y)$:
$u: \mathbb{R}^2 \to \mathbb{R}$



If we take the directional derivative with respect to some unit vector $\vec{\alpha}$, $D_{\vec{\alpha}} u(x,y)$, we want $D_{\vec{\alpha}} u(x,y) = \vec{\nabla u} \cdot \vec{\alpha}$.

{Remark: For a recap on vector calculus, see onenote: IB Physics HL: Fields (b0).}

where $\vec{\nabla u} := \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$, so $\vec{\nabla u} \cdot \vec{\alpha} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \frac{\partial u}{\partial x}\alpha_x + \frac{\partial u}{\partial y}\alpha_y = D_{\vec{\alpha}} u(x,y) \Leftrightarrow u$ is properly differentiable.

i.e. we need the directional derivatives in every direction to be related to the partial derivatives as above.

For $f: \mathbb{R}^2 \to \mathbb{R}^2$ to be "properly" differentiable, both $u(x,y)$ and $v(x,y)$ must satisfy the above.

Then we can build the Jacobian matrix which describes how to change vectors in the domain into vectors in the codomain:

$$J = Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$ for $f: \mathbb{R}^2 \to \mathbb{R}^2$. It describes a linear transformation. Note that J is a function as a matrix.

If we want to transform some vector $\begin{pmatrix} h_x \\ h_y \end{pmatrix}$ by J, we have $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} h_x + \frac{\partial u}{\partial v} h_y \\ \frac{\partial v}{\partial x} h_x + \frac{\partial v}{\partial y} h_y \end{pmatrix}$. Why is $J = Df$?

We can take some infinitesimal $h = \begin{pmatrix} h_x \\ h_y \end{pmatrix} \in \mathbb{R}^2$, then we can find the resultant of h in the codomain by the linear transformation.

Vectors are being stretched and rotated relative to one another.

Now, in $\mathbb{C}$-analysis, we have $f(z) = f(z_0) + f'(z)\, h \Rightarrow f(z+h) - f(z) = f'(z)h.$
$\overset{\parallel}{z+h}$

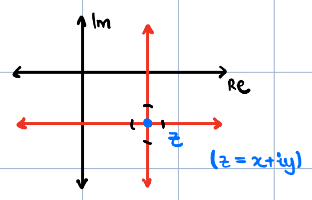
For some $z_0 \in \mathbb{C}$, we have $f'(z_0)$ a constant, so $f(z_0+h) - f(z_0) = f'(z_0)h$, where, since $h \in \mathbb{C}$, we have h being transformed by $f'(z_0) = re^{i\theta}$.

$\Rightarrow$ every holomorphic function has an equivalent $f: \mathbb{R}^2 \to \mathbb{R}^2$ that is vector differentiable. Note that linear transformations are more general, since

 perpendicular vectors can be rotated/stretched independently.

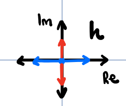
The Cauchy-Riemann Equations : necessary but not sufficient conditions for f to be holomorphic.

Let $f: \mathbb{C} \to \mathbb{C}$. Then $f'(z) = \lim_{h\to 0} \left(\frac{f(z+h) - f(z)}{h} \right) \}$ to exist, the limit must converge to the same value from all directions.



we reduce the problem of the ε-ball into two separate intervals.

Cauchy-Riemann : why don't we equate equations approaching from the $\mathbb{R}$-axis and the imaginary axis?



Consider the decomposition $f(z) = u(x,y) + iv(x,y)$, where $u, v: \mathbb{R}^2 \to \mathbb{R}$.

(1) Considering change along Re-axis:

Then we have $\lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{u(x+h,y) + iv(x+h,y) - (u(x,y) + iv(x,y))}{h}$ where we only consider changes in $\mathrm{Re}(z)$, and $h \in \mathbb{R}$.

$$= \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \left(\frac{u(x+h,y) - u(x,y)}{h} \right) + i \lim_{h\to 0} \left(\frac{v(x+h,y) - v(x,y)}{h} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

which are partial derivatives of "nice" functions in $\mathbb{R}^2$;

(2) Considering changes along Im-axis:

Then we have $\lim_{h\to 0} \left(\frac{u(x, y+h) + iv(x, y+h) - (u(x,y) + iv(x,y))}{ih} \right)$ where we only consider changes in $\mathrm{Im}(z)$, and $h \in \mathbb{R}$.

$x + i(y+h)$

$$= i^{-1} \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \left(\frac{u(x,y+h) - u(x,y)}{h} \right) + \lim_{h\to 0} \left(\frac{v(x,y+h) - v(x,y)}{h} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Cauchy-Riemann Equations

(3) Equating both components: $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

Stronger Cauchy-Riemann : Partial Derivative Continuity at $z = z_0$.

Theorem (Looman-Menchoff) : If $f: \underset{\text{open set}}{U} \to \mathbb{C}$ decomposes as $f(z) = u(x,y) + iv(x,y)$ where $z = x+iy \in \mathbb{C}$ and $u, v: \mathbb{R}^2 \to \mathbb{R}$,

then f is holomorphic on $U \subseteq \mathbb{C} \Leftrightarrow$ (a) f satisfies the Cauchy-Riemann Equations ;

(b) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous in U.

Remark: (0) This is a generalization of Goursat's Theorem. The reduction of Looman-Menchoff to a point is a false assertion.
(see: Goursat, Fréchet differentiability)

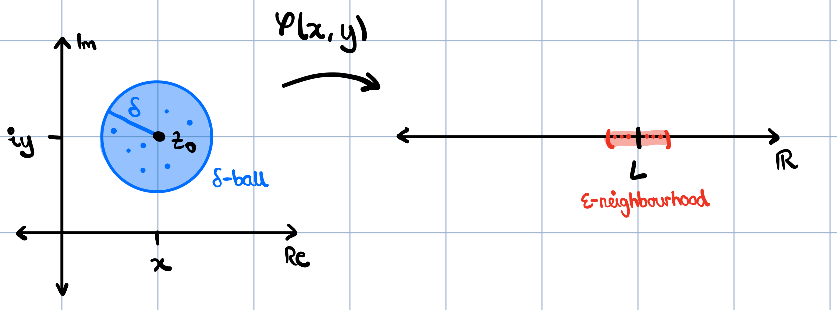
(a) We want the partial derivatives to (at least) exist in some open ball $B(z_0, r)$;

(b) then we can ask about whether they are continuous.

What does it mean for a partial derivative (in this case) to be continuous? Suppose these partial derivatives exist.

Let $\varphi = \frac{\partial o}{\partial o}$ denote arbitrarily one of our derivatives. We have $\varphi: \mathbb{R}^2 \to \mathbb{R}$. Every $z \in \mathbb{C}$ gets mapped to some $r \in \mathbb{R}$.
$(x,y) \longmapsto r \in \mathbb{R}$

Say $\lim_{z\to z_0} \varphi(x,y) = L \in \mathbb{R}$. Then $\forall \varepsilon > 0$, $\exists B'(z_0, \delta)$ such that $|\varphi(x,y) - L| < \varepsilon$.
(all ε-neighbourhoods around L) (δ-ball)



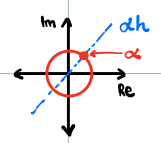
any ε-neighbourhood of L has a corresponding δ-ball $(\delta > 0)$ such that all values of $\varphi(x,y) \in \varepsilon$-neighbourhood $\Rightarrow (x,y) \in \delta$-ball.

Then $\varphi(x,y)$ is continuous at $z_0 \in \mathbb{C}$ if $\varphi(x,y)\big|_{z_0}$ exists, and $\lim_{z \to z_0} \varphi(x,y) = \varphi(x,y)\big|_{z_0}$.

Remember that $\varphi(x,y)$ is of the form $\frac{\partial \circ}{\partial \circ}$.

Proof (Looman-Menchoff): It suffices to show that the continuity of $\varphi(x,y)$ in $U \subseteq \mathbb{C}$ $\Rightarrow$ derivatives in all directions in some neighbourhood are equal.

(1) Take a unit $\alpha \in \mathbb{C}^\times$ such that $\alpha = \alpha_x + i\alpha_y$ where $\alpha_x, \alpha_y \in \mathbb{R}$. We are interested in αh, where $h \in \mathbb{R}$. $(\alpha h = \alpha_x h + i\alpha_y h)$



Construct the complex derivative around $z_0 = x + iy \in \mathbb{C}$ in the direction of α:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \left(\frac{u(x+\alpha_x h,\, y+\alpha_y h) + iv(x+\alpha_x h,\, y+\alpha_y h) - \big(u(x,y) + iv(x,y)\big)}{\alpha_x h + i\alpha_x h} \right)$$

where h varies, and α has arbitrary direction.

We will add and subtract terms:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \left(\frac{u(x+\alpha_x h,\, y+\alpha_y h) - u(x+\alpha_x h,\, y) + u(x+\alpha_x h,\, y) - u(x,y) + i\big[v(x+\alpha_x h,\, y+\alpha_y h) - v(x+\alpha_x h,\, y) + v(x+\alpha_x h,\, y) - v(x,y)\big]}{\alpha_x h + i\alpha_y h} \right)$$

but we cannot quite separate these into partial derivatives yet, since we are dividing by $\alpha h = \alpha_x h + i\alpha_y h$ (what a mess!).

(2) Simplifying the expression (messy!)

Consider the term
$$\frac{u(x+\alpha_x h,\, y) - u(x,y)}{\alpha_x h + i\alpha_y h} = \frac{1}{\alpha_x + i\alpha_y}\left(\frac{u(x+\alpha_x h,\, y) - u(x,y)}{h} \right)$$
$$= \frac{\alpha_x}{\alpha_x + i\alpha_y}\left(\frac{u(x+\alpha_x h,\, y) - u(x,y)}{\alpha_x h} \right)$$
which is what we want, since $h \to 0 \Rightarrow \alpha_x h \to 0$. ($\alpha_x \in \mathbb{R}$ is finite)
$$= \frac{\alpha_x}{\alpha_x + i\alpha_y} \cdot \frac{\partial u}{\partial x}\bigg|_{(x,y)}$$
(Fréchet differentiability)

Similarly, we have
$$\frac{v(x+\alpha_x h,\, y+\alpha_y h) - v(x+\alpha_x h,\, y)}{\alpha_x h + i\alpha_y h} = \frac{\alpha_y}{\alpha_x + i\alpha_y} \cdot \frac{\partial v}{\partial y}\bigg|_{(x+\alpha_x h,\, y)}$$

Now taking limits:
$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{\alpha_y}{\alpha_x + i\alpha_y} \cdot \frac{\partial v}{\partial y}\bigg|_{\underset{\displaystyle\downarrow\ (x,y)}{(x+\alpha_x h,\, y)}} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{\alpha_y}{\alpha_x + i\alpha_y} \cdot \frac{\partial v}{\partial y}\bigg|_{(x,y)}$$
by continuity as $h \to 0$.

(which is why we insist on the continuity of $\varphi(x,y)$)

(3) Putting things together:

$$\frac{1}{\alpha_x + i\alpha_y}\left(\alpha_y \underset{\substack{\| \\ -\frac{\partial v}{\partial x}}}{\frac{\partial u}{\partial y}} + \alpha_x \frac{\partial u}{\partial x} + i\left(\alpha_y \underset{\substack{\| \\ \frac{\partial u}{\partial x}}}{\frac{\partial v}{\partial y}} + \alpha_x \frac{\partial v}{\partial x} \right) \right).$$

Now we apply the Cauchy-Riemann Equations:

$$= \frac{1}{\alpha_x + i\alpha_y}\left((\alpha_x + i\alpha_y)\frac{\partial u}{\partial x} + \underset{\substack{\| \\ i(\alpha_x + i\alpha_y)}}{(-\alpha_y + i\alpha_x)}\frac{\partial v}{\partial x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

which is independent of direction. $\square$

Remarks: (For some more rigour):

(a) Each map $f:\mathbb{C}\longrightarrow\mathbb{C}$ induces a map $f_{\mathbb{R}}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ (and vice versa)

example: $f:\mathbb{C}\longrightarrow\mathbb{C}$, $z\longmapsto z^2$, $x+iy\longmapsto(x+iy)^2$ induces $f_{\mathbb{R}}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$, $\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}x^2-y^2\\2xy\end{pmatrix}$ carrying the same information as f.

(b) A map $f_{\mathbb{R}}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is called totally differentiable at $\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathbb{R}^2$ if there is a matrix $J\in\mathbb{R}^{2\times2}$:

$$f_{\mathbb{R}}\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\underbrace{f_{\mathbb{R}}\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)+J\left(\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)}_{\text{linear approximation}}+\underbrace{\phi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)}_{\text{error term}}\quad\text{where}\quad\frac{\phi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)}{\underbrace{\left\|\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|}_{\substack{\text{(Euclidean) Norm}\\=\text{length}}}}\xrightarrow[\substack{\text{length of diff.}\\\text{vector}\to0}]{\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x_0\\y_0\end{pmatrix}}0$$

(c) J is called the Jacobian matrix of $f_{\mathbb{R}}$ at $\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathbb{R}^2$.

(d) We have $f:\mathbb{C}\longrightarrow\mathbb{C}$ (complex) differentiable at $z_0\in\mathbb{C}$ if there is $f'(z_0)\in\mathbb{C}$ and a function $\phi:\mathbb{C}\to\mathbb{C}$ with

$$f(z)=f(z_0)+f'(z_0)\cdot(z-z_0)+\phi(z)\quad\text{where}\quad\frac{\phi(z)}{z-z_0}\xrightarrow{z\to z_0}0.$$

$\uparrow$ multiplication in $\mathbb{C}$

(e) In which cases does a matrix-vector multiplication represent a multiplication of complex numbers?

Say $w,z\in\mathbb{C}$. Then $w\cdot z=(a+ib)(x+iy)=(a\cdot x-b\cdot y)+i(b\cdot x+a\cdot y)$.

$$\equiv\begin{pmatrix}a&-b\\b&a\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax-by\\bx+ay\end{pmatrix}$$

$\Rightarrow$ Theorem ($\mathbb{C}$-differentiability at $z_0\in\mathbb{C}$):

$f:\mathbb{C}\to\mathbb{C}$ is complex differentiable at $z_0=x_0+iy_0\in\mathbb{C}$

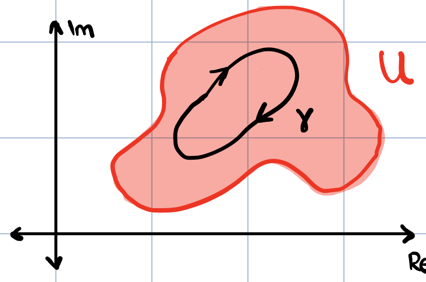
$\Leftrightarrow$ $f_{\mathbb{R}}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is totally differentiable at $\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathbb{R}^2$, and $J\in\mathbb{R}^{2\times2}$ at $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ has the form $\begin{pmatrix}a&-b\\b&a\end{pmatrix}$

$\Leftrightarrow$ For $f_{\mathbb{R}}\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}$ where $u,v:\mathbb{R}^2\to\mathbb{R}$ differentiable at (x_0,y_0) the Cauchy-Riemann Equations are satisfied at point (x_0,y_0).

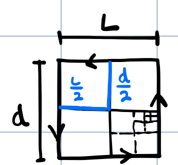
Theorem (Cauchy): Let $\gamma:[a,b]\longrightarrow\mathbb{C}$ be a function mapping an interval $[a,b]$ to $\mathbb{C}$ where $\gamma(a)=\gamma(b)$.

Then the contour integral along the closed loop γ given by $\int_\gamma f(z)\,dz=0$ for $f:\mathbb{C}\to\mathbb{C}$

if f is holomorphic on $\underbrace{\text{an open set}}_{\substack{\text{a simply connected open set:}\\\text{there are no singularities in } U.}}$ $U\subseteq\mathbb{C}$ and γ is a simply closed contour in U.



(0) Start with a rectangle:



R_0 and consider $\int_R f(z)\,dz=\underset{\in\mathbb{C}}{\eta(R_0)}=\underbrace{\eta(R_1^1)+\eta(R_1^2)+\eta(R_1^3)+\eta(R_1^4)}_{\text{contour integrals around each rectangle}}$. We want to show this is 0.

Observe that at least one of $\eta(R_1^1),\dots,\eta(R_1^4)$ must have modulus $\geq\frac{1}{4}|\eta(R_0)|$.

Without loss of generality, say $|\eta(R_1^1)|\geq\frac{1}{4}|\eta(R_0)|$. Then we repeat the process: let $\eta(R_1^1)=\eta(R_2^1)+\ldots+\eta(R_2^4)$.

We have the above; say $|\eta(R_2^1)|\geq\frac{1}{4}|\eta(R_1^1)|\geq\frac{1}{16}|\eta(R_0)|$ by transitivity.

We get a sequence of nested rectangles $R_n\subset R_{n-1}\subset\ldots\subset R_2\subset R_1\subset R_0$. Then $|\eta(R_n)|\geq\frac{1}{4^n}|\eta(R_0)|$.

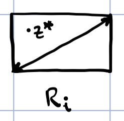
From topology: the intersection of a sequence of countably infinite nested regions is non-empty $\Rightarrow$ they converge to a point.

$(\exists z^* \in \mathbb{C}$ such that $z^* \in R_n \subset \ldots \subset R_0$, i.e. $z^* \in R_i,\ \forall i \in \mathbb{N})$.

By convergence: $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $R_n \subset B(z^*, \varepsilon)$.

(Lemma): It suffices to prove that we can always find some $N \in \mathbb{N}$ such that $R_N \subset B(z^*, \varepsilon)$, where $z^* = \bigcap_{i=0}^{\infty} R_i$ $(\neq \phi$ from topology).

Proof: Take an arbitrary R_i. We need $\varepsilon >$ (largest possible distance in R_i).



(maximum distance (worst case) diagonal)

By Pythagoras and the modulus conditions above, we have

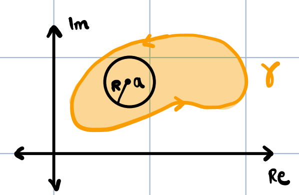
(greatest distance in R_i) $= \frac{1}{2^i}\sqrt{L^2+d^2} < \varepsilon \Rightarrow i = N > \log_2\left(\frac{\sqrt{L^2+d^2}}{\varepsilon}\right)$ true by Archimedean property.

$\triangle$

(0.1) Assume f is holomorphic. Then $\lim_{\substack{h\to 0\\ h\in\mathbb{C}}}\left(\frac{f(z+h)-f(z)}{h}\right) = f'(z)$. Suppose we want to calculate $f'(z^*)$, where $z^* \in \mathbb{C}$.

We have $f'(z^*) = \lim_{\substack{h\to 0\\ h\in\mathbb{C}}}\left(\frac{f(z^*+h)-f(z^*)}{h}\right) = \lim_{z\to z^*}\left(\frac{f(z)-f(z^*)}{z-z^*}\right) \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0$ such that $|z - z^*| < \delta \Rightarrow \left|\frac{f(z)-f(z^*)}{z-z^*}\right| < \varepsilon$.

Theorem (Cauchy's Integral Formula): Let $f: \mathbb{C} \to \mathbb{C}$ holomorphic. Then $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega - a}\,d\omega$ where γ must be positively-oriented and enclose $a \in \mathbb{C}$ (and have winding number $\text{wind}(\gamma) = 1$).



(1) Since γ encloses $a \in \mathbb{C}$, then $\exists R \in \mathbb{R},\ R > 0$, such that $B(a, R)$ is within γ.

Note that $B(a, r)$ is enclosed by γ for any $0 < r \leq R$. By Cauchy's Theorem,

we know $\int_{\gamma(a,R)} f(z)\,dz = \int_{\gamma(a,r)} f(z)\,dz$ where $\gamma(a,r)$ denotes the contour of $B(a,r) \subseteq B(a,R)$. Note that γ is a circle.

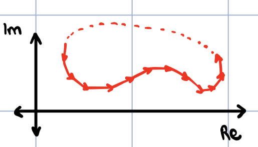
(2) Deformation Theorem $\Rightarrow$ $\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega-a}\,d\omega = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(\omega)}{\omega-a}\,d\omega = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(a)}{\omega-a}\,d\omega - \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(\omega)-f(a)}{\omega-a}\,d\omega.$

$$\frac{f(a)}{2\pi i}\int_{\gamma(a,r)} \frac{1}{\omega-a}\,d\omega = \frac{f(a)}{2\pi i}(2\pi i) = f(a)$$

So we have $\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega-a}\,d\omega = f(a) - \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(\omega)-f(a)}{\omega-a}\,d\omega.$ We want to bound the modulus of the right term.

(3) $\left|\frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(\omega)-f(a)}{\omega-a}\,d\omega\right| <$ Upper Bound

Think about breaking the contour into small incremental complex numbers:



and then we take each increment, transform it somehow, and all all the transformed increments, giving the result.

Q: What is the maximum modulus of the result? We need to consider the straight contour the length of γ:

$=$ (length of contour) $\cdot$ (|maximum complex number that the function maps onto|) (all increments stretched to their maximum moduli and aligned)

$= \underbrace{2\pi r}_{\text{length of } \gamma(a,r) \text{ (circle)}} \cdot (|\max|)$ } how do we figure this out?

We have $\left|\frac{f(\omega)-f(a)}{\omega-a}\right| = \frac{|f(\omega)-f(a)|}{r}$ (since our contour is circular),

and now we use continuity: $\forall \varepsilon > 0,\ \exists \delta > 0$ such that if $\omega \in B(a, \delta)$, then $|f(\omega) - f(a)| < \varepsilon$.

$$\Rightarrow \frac{|f(\omega)-f(a)|}{|\omega - a|} < \frac{\varepsilon}{r}.$$

Now we have $\left| \int_{\gamma(a,r)} \frac{f(w)-f(a)}{w-a}\, dw \right| < 2\pi r \cdot \frac{\varepsilon}{r} = 2\pi\varepsilon, \quad \forall \varepsilon > 0.$ From modulus >0, we have:

$$\Rightarrow \left| \int_{\gamma(a,r)} \frac{f(w)-f(a)}{w-a}\, dw \right| = 0 \Rightarrow \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)-f(a)}{w-a}\, dw = 0$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a}\, dw = f(a) + 0 = f(a). \qquad \square$$

Power Series in Complex Analysis

(example: $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$)

For a sequence of complex numbers $a_0, a_1, \dots,$ the function $f: D \longrightarrow \mathbb{C}$ with $D := \{ z \in \mathbb{C} : \sum_{k=0}^{\infty} a_k (z-z_0)^k \text{ is convergent} \}$

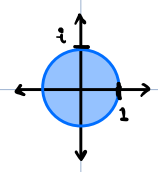
$$z \longmapsto \sum_{k=0}^{\infty} a_k (\underbrace{z-z_0}_{\text{expansion point}})^k$$

is called a power series.

example:

- Geometric series $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z|<1$,

so $|z|<1$ restriction $\Rightarrow D = B(0,1)$ shown graphically:

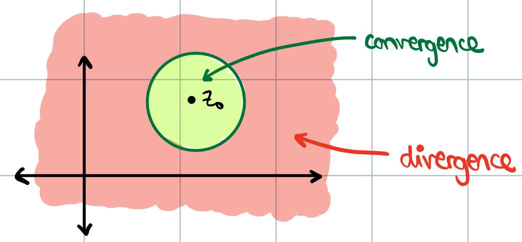


"(Fact):" For a power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, there is a maximal r such that $B(z_0, r) \subseteq D$.

More precisely, $\exists r \in [0,\infty) \cup \{\infty\}$ such that $\begin{cases} B(z_0,r) \subseteq D & \text{for } r \in [0,\infty); \\ D = \mathbb{C} & \text{for } r = \infty. \end{cases}$

If $r \neq \infty$, then $D \neq \mathbb{C} \Rightarrow$ $\underbrace{\text{somewhere on the boundary}}_{\text{varies case-by-case}}$ of $B(z_0, r)$ and everywhere outside, we have divergence.

(for $z \in \mathbb{C} \setminus \overline{B}(z_0, r)$ the power series diverges)



We have then the radius of convergence given by r.

Theorem (Cauchy-Hadamard): For a power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, we have a radius of convergence r for $B(z_0, r)$ given by $\frac{1}{r} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} \in [0,\infty) \cup \{\infty\}$ (note: $\frac{1}{\text{"}0\text{"}} \to \infty$, $\frac{1}{\infty} \to \text{"}0\text{"}$ for our purposes)

If $f: \mathbb{C} \to \mathbb{C}$ is analytic, thus the decomposition into $u, v: \mathbb{R}^2 \to \mathbb{R}$ satisfies Cauchy-Riemann, then u and v also satisfy Laplace's equation $\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and are called harmonic functions.

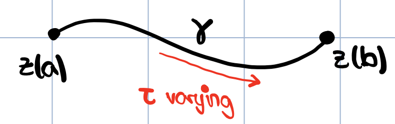
Integration in the Complex Domain: Cauchy's Theorem

An arc ab is defined by a parametric equation $z = z(\tau)$ with $a = z(\alpha)$ and $b = z(\beta)$, in which τ runs through a continuous interval $\alpha \leqq \tau \leqq \beta$,

and $z(\tau) = x(\tau) + iy(\tau)$ is a continuous function of τ on this interval.

We can define γ to be the set of points $\{z(\tau) : \alpha \leq \tau \leq \beta\}$.

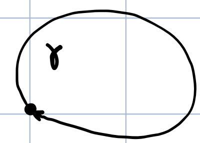
"The orientation of the arc is fixed by its parametrization: the point z describes the arc in the positive sense as τ increases from α to β."



An oriented arc is called a path.

The arc γ is said to be simple if the arc never crosses itself (except possibly at the endpoints). Then $z(t_1) = z(t_2) \Leftrightarrow t_1 = t_2$.
except endpoints

A simple arc that crosses at endpoints $(z(a) = z(b))$, we say γ is a simple closed curve, or a Jordan curve.



note that $(a \leqq t \leqq b)$ and $(\alpha \leqq \tau \leqq \beta)$ are being used interchangeably.

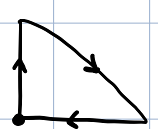
A curve/arc is said to be smooth if:

- $z(\tau)$ has a continuous derivative on $[a, b]$;
- $z'(\tau)$ is non-zero on (a, b);
- $z(\tau)$ is one-to-one on $[a, b)$. } if the other two conditions are met but $z(a) = z(b)$, then γ is a smooth closed curve.

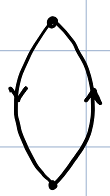
A contour is a piecewise smooth curve: that is, $z(t)$ is continuous but $z'(t)$ is only piecewise continuous.

If $z(t)$ with $z(a) = z(b)$ defines a contour, we say it is a simple closed contour.

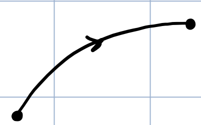
Contours are the sets that complex integration is defined on.



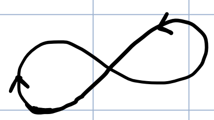
simple closed curve,
simple closed contour
(piecewise continuous derivative)



simple closed contour



simple smooth arc



closed arc/curve,
closed contour

Theorem (Jordan Curve): A simple closed curve divides $\mathbb{C}$ into two sets: the interior, which is bounded;
the exterior, which is unbounded.

i.e. every continuous path connecting points in different regions intersects the curve.

More generally, since $\mathbb{C} \simeq \mathbb{R}^2$ are topologically homeomorphic,

Let C be a Jordan curve in $\mathbb{R}^2$. Then its complement $\mathbb{R}^2 \setminus C$ consists of exactly $\overbrace{\text{two connected components}}^{\text{see topology: connected spaces}}$, as stated above.

Remark: The complement of a Jordan arc in $\mathbb{R}^2$ is connected.

Note: Cauchy-criterion $\Rightarrow$ absolute convergence $\Rightarrow$ convergence.

Exploration of $\exp(z)$ for $z \in \mathbb{C}$. Why can we justify certain properties?

(i) Consider $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ with radius of convergence $R = \infty$.

($\Rightarrow \exp(z)$ is convergent on $\mathbb{C}$).

Remark: Consider $(k!)^{\frac{1}{k}} = (1 \cdot 2 \cdot \ldots \cdot (k-1) \cdot k)^{\frac{1}{k}} \geq \left(\lfloor \frac{k}{2} \rfloor^{\frac{k}{2}}\right)^{\frac{1}{k}} = \lfloor \frac{k}{2} \rfloor^{\frac{1}{2}} \geq \left(\frac{k}{2} - 1\right)^{\frac{1}{2}} \xrightarrow{k \to \infty} \infty$.

We can use this when computing R.

Remark: ratio test success $\Rightarrow$ root test success.

Note the properties $\exp(\omega + z) = \exp(\omega)\exp(z)$. More colloquially, we can write $\exp(z) = e^z$, for $\omega, z \in \mathbb{C}$.

$\exp(z)$ is $\mathbb{C}$ differentiable with $\frac{d}{dz}\exp(z) = \exp(z)$.

(ii) Consider $\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$ and $\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ both with $R = \infty$, so convergent on $\mathbb{C}$.

We must be careful here, but we have an equivalence:

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \underset{\substack{\uparrow \\ \text{uses absolute convergence on } \mathbb{C}}}{``="} \underbrace{\sum_{\ell=0}^{\infty} \frac{(iz)^{2\ell}}{(2\ell)!}}_{\text{even}} + \underbrace{\sum_{\ell=0}^{\infty} \frac{(iz)^{2\ell+1}}{(2\ell+1)!}}_{\text{odd}} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} z^{2\ell} + i \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} z^{2\ell+1}$$

$$= \cos(z) + i\sin(z) \quad \text{for } z \in \mathbb{C}.$$

We can decompose e^z and get a geometric interpretation:

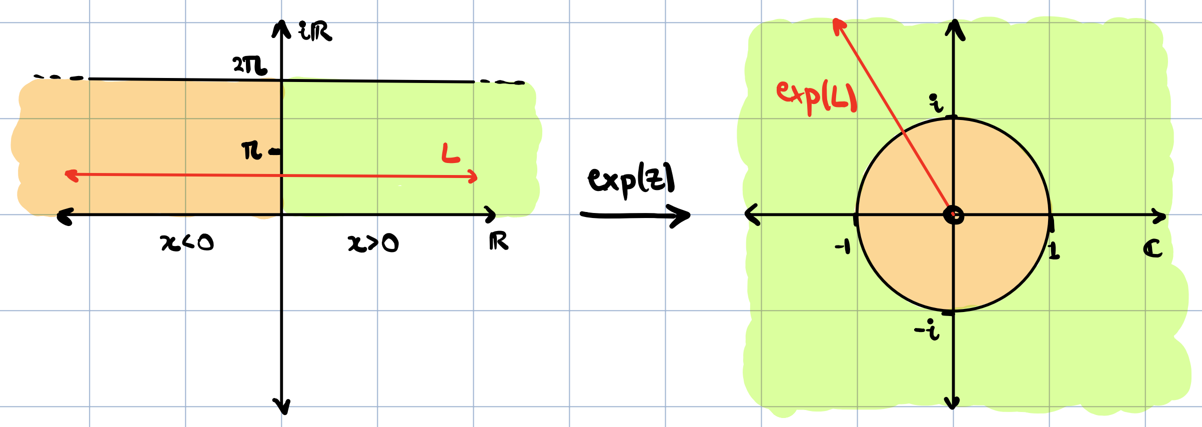
Write $z = x + iy$ for $x, y \in \mathbb{R}$. Then $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$

$= e^x\cos(y) + i(e^x\sin(y))$ $(= u(x,y) + iv(x,y))$

where notice that $e^x\cos(y), e^x\sin(y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$. (Remark: these are harmonic functions).

Note: $\exp(z) = e^z$ for $z \in \mathbb{C}$ is 2π periodic, i.e. $e^z = e^{z + i2\pi}$ (since $\sin(y), \cos(y) : \mathbb{R} \to \mathbb{R}$ are 2π-periodic).

It suffices to choose some "fundamental domain" for e^z, say $y \in [0, 2\pi)$.



(takes rectangular $\to$ polar).

Note: For $z \in \mathbb{C}$, $\exp(z)$ is not injective, so it's not invertible. This makes the theory of $\mathbb{C}$-logarithms funky.

Complex number exponentiation and matrices have a correspondence: generally, complex number $\longleftrightarrow$ Matrix

eg. $e^{x+iy} = e^x(\cos(y) + i\sin(y)) \longleftrightarrow e^{\begin{pmatrix} x & -y \\ y & x \end{pmatrix}} = \begin{pmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{pmatrix}$

A conformal map is a function that locally preserves angles, but not necessarily lengths.

Formally, let U and V be open subsets of $\mathbb{R}^n$ (under standard topology). A function $f: U \to V$ is conformal at a point $u_0 \in U$ if it preserves angles between directed curves through u_0, as well as preserving orientation.

A transformation is conformal $\Leftrightarrow$ the Jacobian at each point is a positive scalar $\cdot$ a rotation matrix (orthogonal with determinant 1).

In $\mathbb{R}^2$, conformal maps $\longleftrightarrow$ locally invertible $\mathbb{C}$-analytic functions.

For $\mathbb{R}^{n>2}$, see Liouville's theorem (later). Conformality generalizes into maps between Riemannian manifolds.

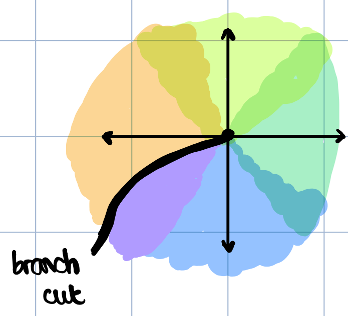
Theorem: Let $U \subseteq \mathbb{C}$ be an open subset. A function $f: U \to \mathbb{C}$ is conformal $\Leftrightarrow$ f is holomorphic and its derivative is non-zero on U.

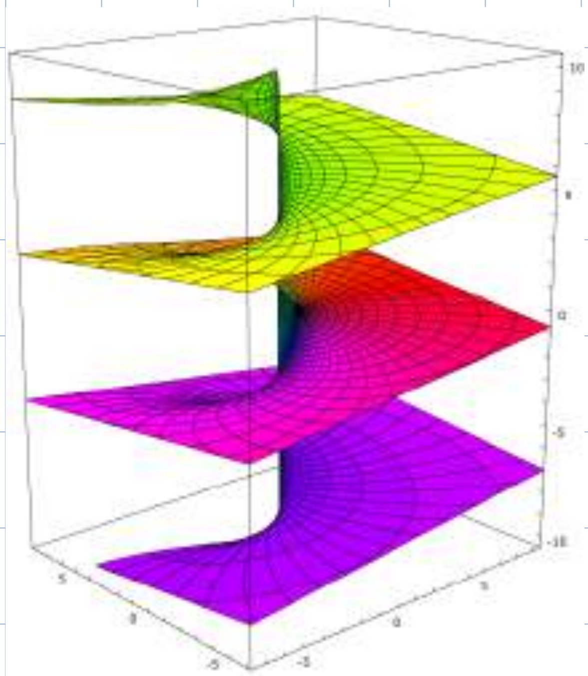
Proof: Outside scope. $\square$

Some functions are not so well-behaved. For instance, $f(z) = z^{\frac{1}{2}}$ has two solutions for each $z \in \mathbb{C}$ but periodicity 4π. To combat this, we introduce a branch cut.

A branch point is a point at which various "sheets" (branches) of a function come together.

A branch cut is a curve in $\mathbb{C}$ such that it is possible to define a single analytic branch of a multi-valued function on $\mathbb{C}$ except that curve.





A standard example is the logarithm.

For $z = re^{i\theta} \in \mathbb{C}$, then

$\log(z) = \log(r) + i\theta$ but there is ambiguity because of e^z being periodic.

A branch cut of the logarithm is a continuous function $L(z)$ giving a logarithm of z, for all z in a connected open set of $\mathbb{C}$.

When "crossing" the branch cut, the logarithm has a jump discontinuity of $2\pi i$.

Remark: The logarithm can be made continuous by gluing together countably many copies (sheets) of $\mathbb{C}$ along the branch cut. See Measure Theory maybe.

See Riemann surfaces for more.

Broadly, branch points fall into three categories:

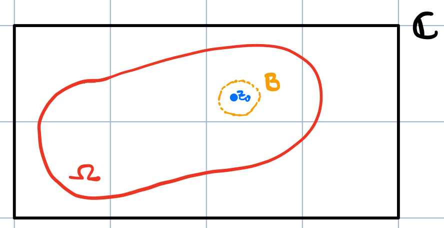
1) Algebraic branch points

2) Transcendental branch points

3) Logarithmic branch points

Algebraic branch points

Let $\Omega \subseteq \mathbb{C}$ be an open connected set and $f: \Omega \to \mathbb{C}$ a holomorphic function. If f is not constant, then the set of critical points $(f'(z)=0)$ has no limit point in Ω. So each critical point z_0 of f lies at the center of an (open) ball $B(z_0, r)$ such that the closure $\bar{B}$ contains no other critical point of f.



Let γ be the boundary of $B(z_0, r)$ taken with its positive orientation.

The winding number of $f(\gamma)$ with respect to a point z_0 is called the ramification index of z_0.

If the ramification index >1, then the corresponding critical value $f(z_0)$ is an algebraic branch point.

Equivalently, z_0 is a ramification point if there exists a holomorphic function ϕ defined in a neighbourhood of z_0 such that $f(z) = \phi(z)(z-z_0)^k + f(z_0)$ for $k \in \mathbb{N}$, $k>1$.

Transcendental and logarithmic branch points

Let g be a global analytic function defined on a deleted open ball $B'(z_0, r)$ around z_0.

Then g has a transcendental branch point if z_0 is an essential singularity of g such that the analytic continuation of a function element once around some simple closed curve γ surrounding z_0 produces a different function element.

Connections to Algebraic Geometry

The notion of branch points can be generalized to mappings between arbitrary algebraic curves.

Let $f: X \to Y$ be a morphism of algebraic curves. By pulling back rational functions on Y to rational functions on X, we have $K(X)$ a field extension of $K(Y)$ with $[K(X): K(Y)] < \infty \Rightarrow f$ is finite.

Assume f is finite. For a point $P \in X$, the ramification index e_P is defined as follows:

Let $Q = f(P)$ and let t be a local uniformizing parameter at P, i.e. t is a regular function defined in a neighbourhood of Q with $t(Q)=0$ whose differential is nonzero.

Then $e_P = v_P(t \circ f)$ where v_P is the valuation in the local ring of regular functions at P. If $e_P > 1$, then f is said to be ramified at P. Then Q is called a branch point.

We return to our notes here.

Paths and line integrals

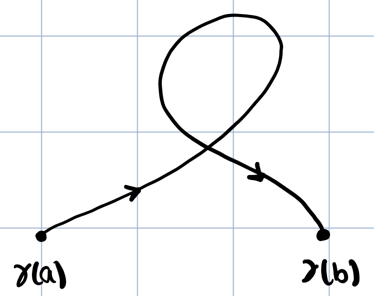
A **piecewise smooth continuous (p.s.c.) path** is a function $\gamma : [a,b] \to \mathbb{C}$ with $[a,b] \subseteq \mathbb{R}$ compact such that

- γ is **piecewise smooth**, i.e. for $a = a_0 < a_1 < \ldots < a_n = b$, the restriction $\gamma|_{(a_{i-1}, a_i)}$ is differentiable and bounded;
- γ is **continuous**.

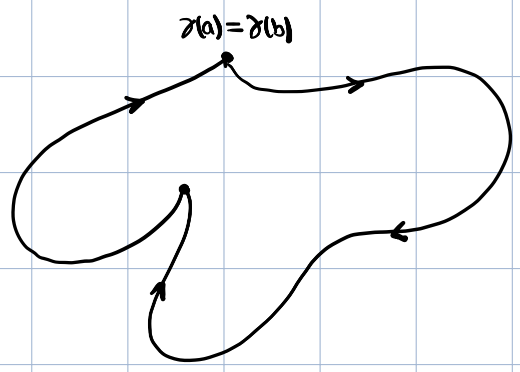
We let γ^* denote the **trace** or **image of** γ.

We say that γ is **closed** if $\gamma(a) = \gamma(b)$.

We say that γ is **simple** if $\gamma(s) = \gamma(t) \Rightarrow s = t$, or $s, t \in \{a, b\}$.



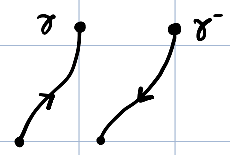
non-simple
smooth



simple
closed
(Jordan curve)

Given $\gamma : [a,b] \to \mathbb{C}$ as above, the **backwards path** $\gamma^- : [a,b] \to \mathbb{C}$ is defined as $\gamma^-(t) = \gamma(a+b-t)$.

Remark: $(\gamma)^* = (\gamma^-)^*$ (they have the same traces).



Two paths $\gamma : [a,b] \to \mathbb{C}$ and $\lambda : [c,d] \to \mathbb{C}$ are **"equivalent"** if there is a function φ such that
(one interval is nicely parametrized in terms of the other)

$\varphi : [a,b] \to [c,d]$ satisfying

- φ continuous;
- φ piecewise smooth;
- $\varphi'(t) > 0$ wherever differentiable.

so that $\lambda = \gamma \circ \varphi$.

Note: From the inverse function theorem on $\mathbb{R}$, $\varphi^{-1} : [c,d] \to [a,b]$ also satisfies the above. Thus we have an equivalence relation.

Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth continuous path (i.e. a contour). Let $f : \gamma^* \to \mathbb{C}$ be continuous.

We define the **line integral of f over γ** by

$$\int_\gamma f = \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

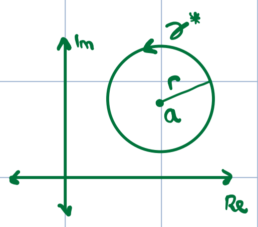
may be undefined at finitely many points } not an issue since integration (on $\mathbb{R}$) ignores single points, i.e. $\int_{[a,b]} = \int_{(a,b)}$ }

Note that each of $\int_a^b \mathrm{Re}(f(\gamma(t))\,\gamma'(t))\,dt$ and $\int_a^b \mathrm{Im}(f(\gamma(t))\,\gamma'(t))\,dt$ (decomposition)

are ordinary Riemann integrals in $\mathbb{R}$ of bounded piecewise continuous functions.

(i) Fix $a \in \mathbb{C}$ and let $r \in \mathbb{R}$, $r > 0$. Let $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$.

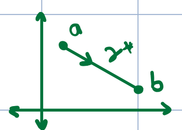
Circles



If $f : \gamma^* \to \mathbb{C}$ is continuous, then $\int_\gamma f = \int_0^{2\pi} f(a + re^{it}) \underbrace{ire^{it}}_{\gamma'(t)} \, dt = ir \int_0^{2\pi} f(a + re^{it}) e^{it} \, dt$.

(ii) Let $a, b \in \mathbb{C}$ distinct. Define $\underset{\subseteq \mathbb{C}}{[a,b]} := \{ \underbrace{(1-t)a + tb}_{\gamma(t)} : t \in [0,1] \subseteq \mathbb{R} \}$. Then note $\gamma'(t) = b - a$.

Line segments



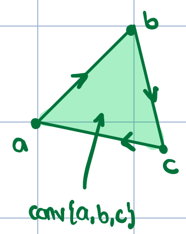
If $f \in \underset{\subseteq \mathbb{C}}{[a,b]} \to \mathbb{C}$ is continuous, then $\int_{[a,b]} f = \int_0^1 f((1-t)a + tb)(b-a) \, dt$.

(iii) Let $a, b, c \in \mathbb{C}$ distinct and $T = \text{conv}\{a, b, c\} = \{ \alpha_a a + \alpha_b b + \alpha_c c : \alpha_{a,b,c} \in [0,1], \ \alpha_a + \alpha_b + \alpha_c = 1 \}$ (convex hull)

Triangles ↓ Polyhedra

with boundary $\partial T = [a,b] \cup [b,c] \cup [c,a] \subseteq \mathbb{C}$.
(bounded in a topological sense)

$=: [a, b, c]$. ← implied piecewise smooth paths on each segment.



Then if $f : \partial T \to \mathbb{C}$ continuous, we have

$$\int_{\partial T} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f.$$

By generalization, we can extend to n-sided polyhedra.

Theorem: Let $\gamma : [a,b] \to \mathbb{C}$ be a contour with $f, g : \gamma^* \to \mathbb{C}$ continuous functions. Then we have:

i) For $\alpha, \beta \in \mathbb{C}$, $\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$.

ii) If $\lambda : [c,d] \to \mathbb{C}$ is "equivalent to" γ, then $\int_\gamma f = \int_\lambda f$.

iii) $\int_{\gamma^-} f = -\int_\gamma f$.

iv) $\left| \int_\gamma f \right| \leq \sup_{z \in \gamma^*} |f(z)| \underbrace{\int_a^b |\gamma'(t)| \, dt}_{\text{length}(\gamma)}$.
compact

Proof (i): Easy exercise.

(ii): Let $\varphi : [c,d] \to [a,b]$ give equivalence, so $\lambda = \gamma \circ \varphi$. We let $a = a_0 < a_1 < \ldots < a_n = b$ be points where γ is not differentiable.

Then we have $\int_\lambda f = \int_{\gamma \circ \varphi} f = \int_c^d f(\gamma \circ \varphi(s)) (\gamma \circ \varphi)'(s) \, ds$

(summing over intervals) $= \sum_{j=1}^n \int_{\varphi^{-1}(a_{j-1})}^{\varphi^{-1}(a_j)} f(\gamma \circ \varphi(s)) \underbrace{\gamma'(\varphi(s)) \varphi'(s)}_{\text{chain rule}} \, ds$

(change of variables) $= \sum_{j=1}^n \int_{a_{j-1}}^{a_j} f(\gamma(t)) \gamma'(t) \, dt = \int_\gamma f$. $\square$

Remark: We use $t = \varphi(s)$ so $\sum_{j=1}^{n} \int_{\varphi^{-1}(a_{j-1})}^{\varphi^{-1}(a_j)} f(\gamma \circ \varphi(s))\, \gamma'(\varphi(s))\, \varphi'(s)\, ds$ changes as above.

$\Rightarrow \varphi^{-1}(t) = s$

$\varphi(s) = \varphi(\varphi^{-1}(a_j))$

(iii): Similar to (ii), but the chain rule brings a negative sign.

(iv): $$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$= \lim_{n\to\infty} \left|\sum_{j=1}^{n} f\left(\gamma\left(a+\tfrac{b-a}{n}(j-1)\right)\right) \gamma'\left(a+\tfrac{b-a}{n}(j-1)\right) \tfrac{b-a}{n}\right| \quad \text{(left Riemann sum)}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{\left|f\left(\gamma\left(a+\tfrac{b-a}{n}(j-1)\right)\right)\right|}_{\leq \max_{z\in\gamma^*}|f(z)|} \left|\gamma'\left(a+\tfrac{b-a}{n}(j-1)\right)\right| \tfrac{b-a}{n}$$

(we can take max. $\because$ compactness)

$$\leq \max_{z\in\gamma^*} |f(z)| \underbrace{\int_a^b |\gamma'(t)|\,dt}_{\text{Length}(\gamma)}. \qquad \square$$

$\max_{z\in\gamma^*} \rightsquigarrow \sup_{z\in\gamma^*}$

Theorem (Fundamental Theorem of Calculus for line integrals):

Suppose $U \subseteq \mathbb{C}$ is open and $f: U \to \mathbb{C}$ is continuous and has a primitive $F: U \to \mathbb{C}$ (so $F' = f$ on U).

Then for any piecewise smooth continuous path $\gamma: [a,b] \to \mathbb{C}$ (contour), we have

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

Corollary: If γ is a closed contour, then $\int_\gamma f = 0$.

Proof: The proof of the chain rule shows that, where $\gamma'(t)$ exists,

$(F\circ\gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$. If $a = a_0 < \ldots < a_n = b$ are points of non-differentiability, then

$$\int_\gamma f = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} (F\circ\gamma)'(t)\,dt = \sum_{j=1}^{n} \left[F(\gamma(a_j)) - F(\gamma(a_{j-1}))\right] = F(\gamma(b)) - F(\gamma(a)). \qquad \square$$

Remark: $(F\circ\gamma)(t) = u(t) + i v(t)$, so apply $\mathbb{R}$-analysis Fundamental Theorem of Calculus to each.

"Special case": Let $f(z)=\frac{1}{z}$ on $U=\mathbb{C}\setminus\{0\}$. Let $\gamma(t)=re^{it}$ for $t\in[0,2\pi]$ and $r>0$ fixed.
$\in\mathbb{R}$

Then $\int_\gamma f = \int_0^{2\pi}\frac{ire^{it}}{re^{it}}\,dt = i\int_0^{2\pi}dt = 2\pi i \neq 0$. Huh?

$\Rightarrow f(z)=\frac{1}{z}$ has no primitive on $\mathbb{C}\setminus\{0\}$. Again, this hints that $\ln z$ needs more nuanced care.

Remark: If $n\in\mathbb{Z}\setminus\{-1\}$, and $g(z)=z^n$, and $U=\begin{cases}\mathbb{C}, & n\geq 0;\\ \mathbb{C}\setminus\{0\}, & n<0;\end{cases}$ then $g(z)$ has a primitive $G(z)=\frac{1}{n+1}z^{n+1}+C$.

$\therefore$ Over the closed path γ as before, we have $\int_\gamma g=0$.

Hence, the power series $g(z)=\sum_{k=0}^{\infty}c_k(z-a)^k$ on $B(a,R)$ has a primitive,

$G(z)=\sum_{k=0}^{\infty}\frac{1}{k+1}c_k(z-a)^{k+1}$ so again, $\int_\gamma g=0$.

Theorem (Cauchy-Goursat): Let $U\subseteq\mathbb{C}$ be open, and $f:U\to\mathbb{C}$ be continuous on U, and holomorphic on $U\setminus\{w_0\}$. everywhere on U, except possibly at some point w_0.

Then if $a,b,c\in U$ such that $T:=\operatorname{conv}\{a,b,c\}\subset U$, we have $\int_{[a,b,c]}f=0$.

Remark: The proof will follow. Here are some consequences.

A set $S\subseteq\mathbb{C}$ is called *star-like* if there exists some $\underset{\text{base point}}{z_0\in S}$ such that for any $z\in S$, $[z_0,z]=\{(1-t)z_0+tz:\ t\in[0,1]\}\subseteq S$.

(i) Let $C\subseteq\mathbb{C}$ be convex. Then C is star-like with any base point $z_0\in C$.

(ii) $\mathbb{C}\setminus\underbrace{[-\infty,0]}_{\text{closed ray}}$ is star-like.

Theorem (Existence of Primitives):

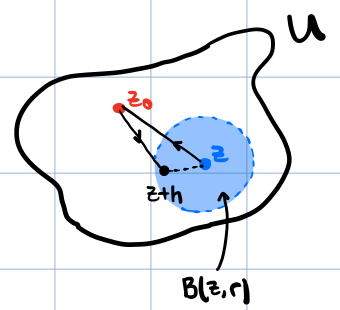
Let $U\subseteq\mathbb{C}$ be open and star-like. Then if $f:U\to\mathbb{C}$ is continuous on U and holomorphic on $U\setminus\{w_0\}$,

then f admits a primitive F on U.

Proof: Let $z_0\in U$ be a base-point showing U is star-like. Define $F:U\to\mathbb{C}$ as $F(z)=\int_{[z_0,z]}f$ where $[z_0,z]\subseteq U$.

Given $z\in U$, we use openness: let $r>0$ such that $B(z,r)\subseteq U$.

If $h\in\mathbb{C}$ with $0<|h|<r$, then



$$F(z+h)-F(z)=\int_{[z_0,z+h]}f-\int_{[z_0,z]}f=\int_{[z_0,z+h]}f+\int_{[z,z_0]}f \quad\leftarrow \text{backwards path}$$

$$=\int_{[z_0,z+h]}f+\int_{[z,z_0]}f+\underbrace{\int_{[z+h,z]}f+\int_{[z,z+h]}f}_{0}$$

Notice $T=\operatorname{conv}\{z_0,z+h,z\}=\bigcup_{w\in[z+h,z]}[z_0,w]\subseteq U$ (by star-like).

Also notice, for fixed $z\in\mathbb{C}$, we have $\int_{[z,z+h]}f(z)\,d\zeta = f(z)h$

$$=\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|$$

$$=\left|\frac{1}{h}\int_{[z,z+h]}[f(\zeta)-f(z)]\,d\zeta\right|$$

$$\leq \frac{1}{\cancel{|h|}} \max_{\zeta\in[z,z+h]} |f(\zeta)-f(z)|\cancel{|h|} \xrightarrow{h\to 0} 0 \quad \text{since } f \text{ continuous.} \qquad \square$$

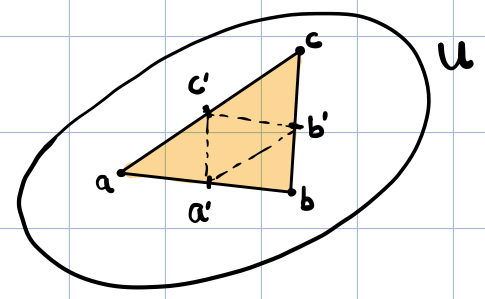
Remark: The idea behind "star-like" is to introduce the idea of a simply connected set.

Set-up (Cauchy-Goursat): We choose triangles since they're easy.

Note that $T = \operatorname{conv}\{a,b,c\}$ is compact (use open cover definition or Heine-Borel in $\mathbb{R}^2$).

We denote the diameter as $\operatorname{diam}(T) := \max\{|a-b|, |b-c|, |a-c|\}$ the furthest two points can be.

length as $\operatorname{length}(\partial T) := \operatorname{length}([a,b,c]) = |b-a| + |c-b| + |c-a|$.



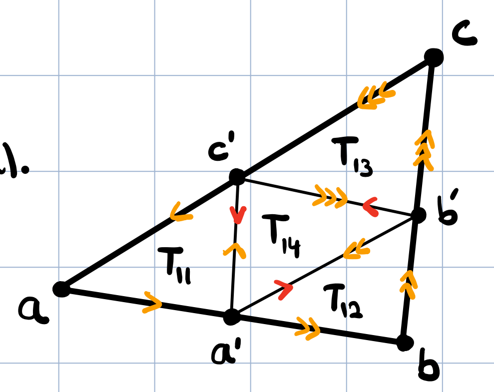
Proof (Cauchy-Goursat): Let a', b', c' be midpoints of $[a,b]$, $[b,c]$, $[c,a]$ respectively.

We split our triangle as shown. Recall the following:

$$\int_{[a,b,c]} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f = \int_{\partial T} f \quad \text{(boundary integral).}$$

$$= \int_{[a,a',c']} f + \int_{[a',b,b']} f + \int_{[b',c,c']} f + \int_{[c',a',b']} f \quad \text{(since things cancel).}$$

$$= \sum_{j=1}^{4} \int_{\partial T_{1j}} f \quad \text{as per our diagram.}$$



So we have $\left|\int_{\partial T} f\right| \leq \sum_{j=1}^{4} \left|\int_{\partial T_{1j}} f\right|$ and hence, for some j, we have $\left|\int_{\partial T} f\right| \leq 4\left|\int_{\partial T_{1j}} f\right|$.

Label this $T_{1j} = T_1$. Notice $\operatorname{diam}(T_1) = \frac{1}{2}\operatorname{diam}(T)$ and $\operatorname{Length}(\partial T_1) = \frac{1}{2}\operatorname{length}(\partial T)$.

We continue by recursion, giving a sequence of such choices:

$T \supset T_1 \supset T_2 \supset \dots$ with $\left|\int_{\partial T_n} f\right| \leq 4\left|\int_{\partial T_{n+1}} f\right|$ and $\operatorname{diam}(T_{n+1}) = \frac{1}{2}\operatorname{diam}(T_n)$

and $\operatorname{length}(\partial T_{n+1}) = \frac{1}{2}\operatorname{length}(\partial T_n)$.

By nested compact set theorem, $\bigcap_{n=1}^{\infty} T_n \neq \emptyset$ and, in fact, $\bigcap_{n=1}^{\infty} T_n = \{z_0\}$ (single point) as $\operatorname{diam}(T_n) \xrightarrow{n\to\infty} 0$.

Now we break into two cases:

1. $z_0 \neq w_0$ so f is differentiable at z_0.

By linear approximation lemma, for $z \in U$,

$$|f(z)-f(z_0)| = |(z-z_0)f'(z_0)| \leq |z-z_0|\,|\underbrace{E(z-z_0)}_{\text{some error term}}| \quad \text{with} \quad \lim_{h\to 0} E(h) = 0.$$

{ We don't care explicitly about this error function except this.

Notice that if we define $g(z) = f(z_0) + (z-z_0)f'(z_0)$ then g has a primitive since it is a polynomial.

So for any n, we have $\int_{\partial T_n} g = 0$ since ∂T_n defines a closed contour.

$$\Rightarrow \left|\int_{\partial T_n} f\right| = \left|\int_{\partial T_n} (f-g)\right| = \left|\int_{\partial T_n} (f(z)-f(z_0)-(z-z_0)f'(z_0))\,dz\right|$$

$$\leq \sup_{z\in\partial T_n} \underbrace{\left|(f(z)-f(z_0)-(z-z_0)f'(z_0)\right|}_{|z-z_0||E(z-z_0)|} \text{length}(\partial T_n)$$

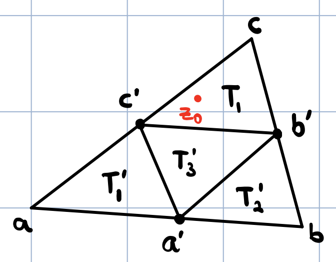
(note: how and why?)

$$\leq \sup_{z\in T_n} \underbrace{|z-z_0|}_{\leq \text{diam}(T_n)}|E(z-z_0)| \frac{1}{2^n} \text{length}(\partial T)$$

$$\leq \sup_{z\in T_n} \frac{1}{4^n}|E(z-z_0)| \text{ diam}(T) \text{ length}(\partial T).$$

Hence $$\left|\int_{\partial T} f\right| \leq 4^n \left|\int_{\partial T_n} f\right| \leq \sup_{z\in T_n} \underbrace{|E(z-z_0)|}_{\leq \frac{1}{2^n}|z-z_0|} \underbrace{\text{diam}(T) \text{ length}(\partial T)}_{\text{fixed finite constants}} \xrightarrow{n\to\infty} 0. \quad \triangle$$

2 $z_0 = w_0$, so f is continuous at w_0 but not necessarily differentiable.



Recall that $w_0 \in T_n$ for all n, since $\{z_0\} = \{w_0\} = \bigcap_{n=1}^{\infty} T_n$.

From above, $$\int_{\partial T} f = \int_{T_1} f + \underbrace{\sum_{j=1}^{3}\int_{T_j'} f}_{=0 \text{ from case 1}} = \int_{T_1} f$$

so $$\left|\int_{\partial T} f\right| = \left|\int_{\partial T_n} f\right| \leq \max_{z\in\partial T_n} |f(z)| \text{ length}(\partial T_n)$$

$$\leq \underbrace{\max_{z\in T} |f(z)|}_{\text{bounded}} \frac{1}{2^n} \text{length}(\partial T) \quad \bullet \text{ since } \partial T_n \subset T_n \subset T$$

$$\xrightarrow[n\to\infty]{} 0. \quad \left(\text{i.e. } \int_{\partial T} f = 0\right). \quad \triangle \ \square$$

(Principal Branch of Logarithm)

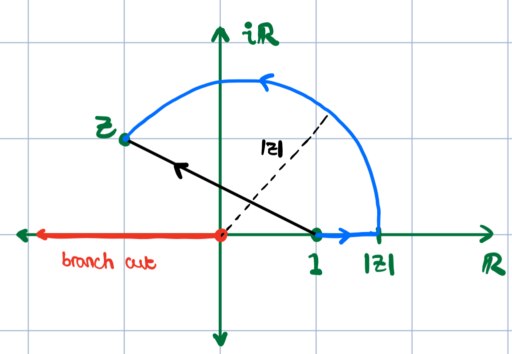
Consider $\mathbb{C} \setminus (-\infty, 0]$ which is star-like with base-point $-1 \in \mathbb{C}$.

notice how this excludes the branch cut

Define $\text{Log}(z) := \int_{[1,z]} \frac{d\zeta}{\zeta}$ and pick $z\in\mathbb{C}$. Then $z = |z|e^{i\theta}$ for $\theta \in (-\pi, \pi)$

Then $$\text{Log}(z) = \int_{\underset{\subseteq\mathbb{C}}{[1,z]}} \frac{d\zeta}{\zeta} = \int_{\underset{\subseteq\mathbb{R}}{[1,|z|]}} \frac{d\zeta}{\zeta} + \int_{\underset{\subseteq\mathbb{C}}{\gamma}} \frac{d\zeta}{\zeta}$$ where $\gamma: [0,\theta] \longrightarrow \mathbb{C}$, $t \longmapsto |z|e^{it}$

$$= \int_1^{|z|} \frac{dt}{t} + \int_0^{\theta} \frac{i|z|e^{it}}{|z|e^{it}}\,dt = \log(|z|) + i\underbrace{\theta}_{\text{argument}}$$



Theorem (Cauchy, for star-like sets):

Let $U \subseteq \mathbb{C}$ be open and star-like, and $f: U \longrightarrow \mathbb{C}$ be continuous on U and holomorphic on $U \setminus \{w_0\}$.

Then for any closed contour $\gamma: [a,b] \longrightarrow U$, we have $\int_\gamma f = 0$.

Remark: First, we explore some consequences and computations.

(I) Computation of some $\mathbb{R}$-integrals:

(i) Let $a \in \mathbb{R}$. Compute $\displaystyle\int_{-\infty}^{\infty} \cos(at) e^{-t^2}\, dt = \lim_{R \to \infty} \underbrace{\int_{-R}^{R} \cos(at) e^{-t^2}\, dt}_{\text{Riemann integrable}}$

We have two cases.

CASE 1: $a = 0$. We will show later that $\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$.

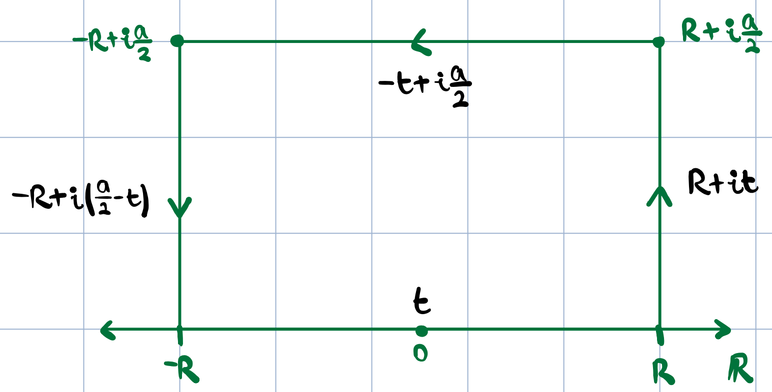
CASE 2: $a \neq 0$. Take $a > 0$ by symmetry. Where do we even begin?

Strategy: Let $f(z) = e^{-z^2}$ holomorphic on $\mathbb{C}$ ("entire") (by chain rule and power series).

Consider the closed contour γ_R shown:

(in $\mathbb{C}$, which is star-like.)

This choice is non-obvious and may be very hard to find arbitrarily.



By Cauchy, we have $\int_{\gamma_R} f = 0$. We decompose this into our desired sum:

$$\Rightarrow \int_{[-R,R]} f + \int_{[R,\, R+i\frac{a}{2}]} f + \int_{[R+i\frac{a}{2},\, -R+i\frac{a}{2}]} f + \int_{[-R+i\frac{a}{2},\, -R]} f = 0. \quad (\star)$$

We work with each independently.

- $\displaystyle\int_{[-R,R]} f = \int_{-R}^{R} e^{-t^2}\, dt \xrightarrow{R \to \infty} \sqrt{\pi}$ from CASE 1.

- $\displaystyle\int_{[R,\, R+i\frac{a}{2}]} f = \int_0^{\frac{a}{2}} e^{-\gamma_R^2} \gamma_R'\, dt = \int_0^{\frac{a}{2}} e^{-(R+it)^2} i\, dt = i \int_0^{\frac{a}{2}} e^{-(R^2+2iRt-t^2)}\, dt.$

  We want to show this approaches 0, so we consider the modulus:

  $$\left| \int_{[R,\, R+i\frac{a}{2}]} f \right| \leq \max_{t \in [0,\frac{a}{2}]} \Big| e^{-(R^2-t^2)} \underbrace{e^{-2iRt}}_{|\cdot| = 1} \Big| \cdot \text{length}([0, \tfrac{a}{2}]) = \max_{t \in [0,\frac{a}{2}]} e^{-R^2+t^2} \cdot \frac{a}{2}$$

  (since maximum occurs at largest $t = \frac{a}{2}$): $\displaystyle = e^{-R^2 + \frac{a^2}{4}} \cdot \frac{a}{2} \xrightarrow{R \to \infty} 0.$

  $$\Rightarrow \int_{[R,\, R+i\frac{a}{2}]} f \xrightarrow{R \to \infty} 0.$$

- $\displaystyle\int_{[-R+i\frac{a}{2},\, -R]} f \xrightarrow{R \to \infty} 0$ by the same computation above, but flipped sign (this does not affect modulus!).

(a miracle) • $\int_{[R+i\frac{a}{2},\, -R+i\frac{a}{2}]} f = \int_{-R}^{R} e^{-\gamma_R^2}\gamma_R'\,dt = \int_{-R}^{R} e^{-(-t+i\frac{a}{2})^2}(-1)\,dt$

$$= -\int_{-R}^{R} e^{-(t^2-2i\frac{a}{2}t-\frac{a^2}{4})}\,dt$$

$$= -\int_{-R}^{R} e^{-t^2}e^{\frac{a^2}{4}}e^{iat}\,dt$$ and now we decompose:

$$= -e^{\frac{a^2}{4}}\int_{-R}^{R} \underbrace{e^{-t^2}}_{\text{even function}}(\cos(at)+\overbrace{i\sin(at)}^{\text{odd function}})\,dt$$ and (even)·(odd) = (odd), and our interval is symmetric;

$\Rightarrow \int_{-R}^{R} e^{-t^2}\, i\sin(at)\,dt = 0.$

$$= -e^{\frac{a^2}{4}}\int_{-R}^{R} e^{-t^2}\cos(at)\,dt$$

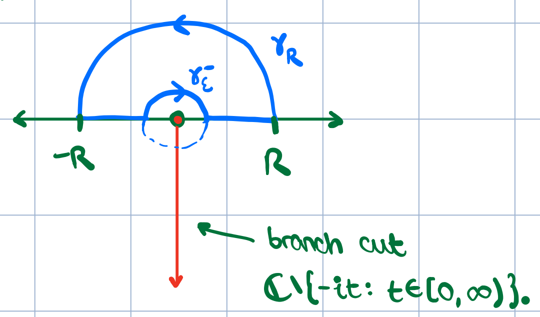
(*) $\Rightarrow \lim_{R\to\infty}\left(-\int_{[R+i\frac{a}{2},\, -R+i\frac{a}{2}]} f\right) = \lim_{R\to\infty}\left(\int_{[-R,R]} f\right) = \sqrt{\pi}$

so we have $e^{\frac{a^2}{4}}\int_{-R}^{R} e^{-t^2}\,dt \xrightarrow{R\to\infty} \sqrt{\pi} \Rightarrow \int_{-R}^{R} e^{-t^2}\cos(at)\,dt \xrightarrow{R\to\infty} e^{-\frac{a^2}{4}}\sqrt{\pi}.$

(ii) Compute $\int_0^\infty \frac{1-\cos(x)}{x^2}\,dx$. This is integrable since $\Leftarrow \left(0\le \frac{1-\cos(x)}{x^2}\le \frac{2}{x^2}\right)$ absolutely integrable.

Strategy: Let $f:\mathbb{C}\setminus\{0\}\longrightarrow\mathbb{C}$ be given by $f(z)=\frac{1-e^{iz}}{z^2}$, (holomorphic) with $\mathrm{Re}(f(x)) = \frac{1-\cos(x)}{x^2}$.

Let $\gamma_{R,\varepsilon}$ be a closed path with $0<\varepsilon<R$, given by a "dented semicircle."



Hence by Cauchy's:

$$\int_{\gamma_{R,\varepsilon}} f = \int_{[\varepsilon,R]} f + \int_{\gamma_R} f + \int_{[-R,-\varepsilon]} f + \int_{\gamma_\varepsilon^-} f = 0.$$

• $\int_{\gamma_R} f = \int_0^\pi f(\gamma_R)\,\gamma_R'\,dt = \int_0^\pi \frac{1-e^{iRe^{it}}}{(Re^{it})^2}\, iRe^{it}\,dt$ since $\gamma_R = Re^{it}$, $t\in[0,\pi]$.

$$= \frac{i}{R}\int_0^\pi \frac{1-e^{iR[\cos(t)+i\sin(t)]}}{e^{it}}\,dt.$$ Now we consider the modulus:

We have $\left|\int_{\gamma_R} f\right| \le \frac{1}{R}\max_{t\in[0,\pi]}\left(1+e^{-R\sin(t)}\right)\cdot\pi = \frac{1}{R}\cdot 2\pi \xrightarrow{R\to\infty} 0.$

- $\displaystyle\int_{[-R,-\varepsilon]} f + \int_{[\varepsilon,R]} f = \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x}\,dx$

$$= \int_{-R}^{-\varepsilon} \frac{1-\cos x - \overbrace{i\sin x}^{\text{odd}}}{\underbrace{x^2}_{\text{even}}}\,dx + \int_{\varepsilon}^{R} \frac{1-\cos x - i\sin x}{x^2}\,dx$$ and $\frac{[\text{odd}]}{[\text{even}]} = [\text{odd}] \to 0.$

so by symmetry,

$$= \int_{-R}^{-\varepsilon} \frac{1-\cos x}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{1-\cos x}{x^2}\,dx = 2\int_{\varepsilon}^{R} \frac{1-\cos x}{x^2}\,dx$$ by evenness.

- $\displaystyle\int_{\gamma_\varepsilon^-} f.$ Recall our linear approximation for $0<|h|$ "small":

$$e^h - 1 = e^h - e^0 = h\exp'(0) + h\overbrace{E(h)}^{\text{error term}}$$

$$= h(1+E(h)) \quad \text{with } \lim_{|h|\to 0} E(h) = 0.$$

Then $$\int_{\gamma_\varepsilon^-} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_\varepsilon} \frac{e^{iz}-1}{z^2}\,dz \quad \text{(reverse path)}$$

$$= \int_0^{\pi} \frac{e^{i\varepsilon e^{it}}-1}{(\varepsilon e^{it})^2}\,\underbrace{i\varepsilon e^{it}}_{\gamma'(t)}\,dt$$

$$= \frac{i}{\varepsilon}\int_0^{\pi} \frac{e^{i\varepsilon e^{it}}-1}{e^{it}}\,dt$$ and by the linear approximation:

$$= \frac{i}{\varepsilon}\int_0^{\pi} \frac{(i\varepsilon e^{it})(1+E(i\varepsilon e^{it}))}{e^{it}}\,dt$$

$$= -\int_0^{\pi}(1+E(i\varepsilon e^{it}))\,dt = -\pi - \int_0^{\pi} E(i\varepsilon e^{it})\,dt$$ and we analyze the modulus:

with $$\left|\int_0^{\pi} E(i\varepsilon e^{it})\,dt\right| \leq \max_{t\in[0,\pi]} |E(i\varepsilon e^{it})| \cdot \overbrace{\pi}^{\text{length}} \xrightarrow{|\varepsilon|\to 0^+} 0.$$

$$= -\pi - \int_0^{\pi} E(i\varepsilon e^{it})\,dt$$ analyze modulus:

where $$\left|\int_0^{\pi} E(i\varepsilon e^{it})\,dt\right| \leq \max_{t\in[0,\pi]} |E(i\varepsilon e^{it})|\,\overbrace{\pi}^{\text{length}}$$ but

as $\varepsilon \to 0^+$, $E(i\varepsilon e^{it}) \to 0$

Thus $$0 = \lim_{R\to\infty}\lim_{\varepsilon\to 0^+}\int_{\gamma_{R\varepsilon}} f = 2\int_0^{\infty}\frac{1-\cos x}{x^2}\,dx - \pi.$$ Note: $\left[\int_{\varepsilon}^{R}\frac{1-\cos x}{x^2}\,dx \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^{\infty}\frac{1-\cos x}{x^2}\,dx\right]$

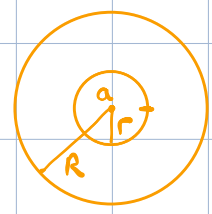
don't worry since we have absolute convergence + bounded.

$$\Rightarrow \int_0^{\infty}\frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}.$$

Cauchy's Integral Formula (on a disc)

Let $R>0$ and $a\in\mathbb{C}$. Let $f: D(a,R)\longrightarrow\mathbb{C}$ be holomorphic. Let $0<r<R$ and $\gamma(t)=a+re^{it}$.

Then for $z\in D(a,r)$, we have $f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta$.



Proof: Fix $z\in D(a,r)$. Treat this as a constant.

Let for $\zeta\in D(a,R)$, $g(\zeta)=\begin{cases}\frac{f(\zeta)-f(z)}{\zeta-z}, & \text{if } \zeta\neq z;\\ f'(z), & \text{if } \zeta=z.\end{cases}$

Notice
- g holomorphic on $D(a,R)\setminus\{z\}$ (quotient rule);
- f holomorphic at $z \Rightarrow g$ is continuous at z $\left(\lim_{\zeta\to z} g(\zeta)=f'(\zeta)\right)$.

Thus by Cauchy's Theorem,

$$0=\underbrace{\int_\gamma}_{\text{closed contour, smooth}} g(\zeta)\,d\zeta=\int_\gamma\frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta$$

$$=\int_\gamma\left[\frac{f(\zeta)}{\zeta-z}-\frac{f(z)}{\zeta-z}\right]d\zeta$$

$$=\int_\gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta-\overset{\text{constant}}{f(z)}\underbrace{\int_\gamma\frac{d\zeta}{\zeta-z}}_{=2\pi i}$$

$$\Rightarrow\int_\gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta=2\pi i\,f(z).$$ We are done.

Corollary (Taylor's Theorem)

Let $R>0$ and $f(z): D(\overset{\in\mathbb{C}}{a},R)\rightarrow\mathbb{C}$ holomorphic. Then $f(z)=\sum_{k=0}^{\infty}c_k(z-a)^k$ for any $z\in D(a,R)$,

with each $c_k=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta=\frac{f^{(k)}(a)}{k!}$.

Proof: Observe that for z,ζ with $|z-a|<r=|\zeta-a|<R$, and $\frac{1}{\zeta-z}=\frac{1}{\zeta-a}\cdot\frac{1}{1-\frac{z-a}{\zeta-a}}$ with $\left|\frac{z-a}{\zeta-a}\right|<1$.

So $$\frac{1}{\zeta-z}=\underbrace{\frac{1}{\zeta-a}\sum_{k=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^k}_{\text{converges uniformly}}$$ and $$f(z)=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$$=\frac{1}{2\pi i}\int_\gamma f(\zeta)\sum_{k=0}^{\infty}\frac{(z-a)^k}{(\zeta-z)^{k+1}}\,d\zeta$$

$$=\frac{1}{2\pi i}\lim_{n\to\infty}\sum_{k=0}^{n}\frac{f(\zeta)}{(\zeta-z)^{k+1}}(z-a)^k\,d\zeta$$

$$=\sum_{k=0}^{\infty}\underbrace{\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta}_{c_k}(z-a)^k$$

$$=\sum_{k=0}^{\infty}c_k(z-a)^k.$$ $\square$

$\mathbb{R}$-analysis is complicated — a short tour.

The open "intervals" $I=(a,b)$ in $\mathbb{R}$ are open connected subsets of $\mathbb{R}$ (under standard topology).

In $\mathbb{C}$, we have three broad generalizations: discs, convex sets and various connected open sets.

Let $n\in\mathbb{N}$. Differentiable $\mathcal{D}^n(I) := \{f: I\to\mathbb{R} \mid f \text{ is } n\text{-times differentiable}\}$

Continuously differentiable $C^n(I) := \{f: I\to\mathbb{R} \mid f^{(n)} \text{ exists and is continuous}\}$

Infinitely differentiable $C^\infty(I) := \{f: I\to\mathbb{R} \mid f^{(n)} \text{ is differentiable } \forall n\in\mathbb{N}\}$

Analytic $\mathcal{A}(I) := \{f: I\to\mathbb{R} \mid \forall c\in I, \exists R>0 \text{ such that } (c-R,c+R)\subseteq I \text{ and}$
$f(x)=\sum_{k=0}^{\infty} c_k(x-c)^k \text{ for } x\in(c-R,c+R).\}$

Notice the proper containment $\mathcal{D}^1(I) \supsetneq C^1(I) \supsetneq \mathcal{D}^2(I) \supsetneq C^2(I) \supsetneq \cdots \supsetneq C^\infty(I) \supsetneq \mathcal{A}(I)$.

For instance, let $I=(-1,1)\subseteq\mathbb{R}$ and $f(x)=\begin{cases} x^2\sin(\frac{1}{x}), & x\neq 0;\\ 0, & x=0.\end{cases}$ Notice $f\in\mathcal{D}^1(I)\setminus C^1(I)$ since f' is not continuous on I.

We also have $C^\infty(I)=\bigcap_{n=1}^{\infty} C^n(I) = \bigcap_{n=1}^{\infty}\mathcal{D}^n(I)$. But we don't have this for $\mathcal{A}(I)$!

Take $g(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0;\\ 0, & x=0.\end{cases}$ Turns out $g\in C^\infty(\mathbb{R})$ but $g\notin\mathcal{A}(\mathbb{R})$. It is not analytic.

$\mathbb{R}$-analysis is so messy. $\mathbb{C}$-analysis is much nicer.

Let $U\subseteq\mathbb{C}$ be open, and define $\mathcal{H}(U) := \{f: U\to\mathbb{C} \mid f \text{ holomorphic on } U\}$.

Recall: If $a\in\mathbb{C}$ and $R>0$, and $f\in\mathcal{H}(D(a,R))$, then for each $z\in D(a,R)$, we have $f(z)=\sum_{k=0}^{\infty} c_k(z-a)^k$
★
with $c_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta$ $\quad (\gamma(t)=a+re^{it},$
$t\in[0,2\pi])$

Lemma: If $U\subseteq\mathbb{C}$ is open, then $\mathcal{D}^1(U)=\mathcal{H}(U)=\mathcal{A}(U)$.

Proof: If $z_0\in U$, find $R>0$ such that $D(z_0,R)\subseteq U$ (possible by topology).

Then, we see that for any $z\in D(z_0,R)$, we have $f(z)=\sum_{k=0}^{\infty}c_k(z-z_0)^k$ by (★).

This this holds for all discs, we are done, and by "definition" we have $f\in\mathcal{A}(U)$. $\square$

Remark: We have not rigorously defined $\mathcal{A}(U)$!

Corollary: Let $U\subseteq\mathbb{C}$ be open, and $f\in\mathcal{H}(U)$. Then $f'\in\mathcal{H}(U)$.

Proof: $f\in\mathcal{H}(U) \underset{\text{by above}}{\Rightarrow} f\in\mathcal{A}(U) \Rightarrow f\in C^\infty(U) \Rightarrow f'\in C^\infty(U) \Rightarrow f'\in C^1(U) \Rightarrow f'\in\mathcal{H}(U)$. $\square$

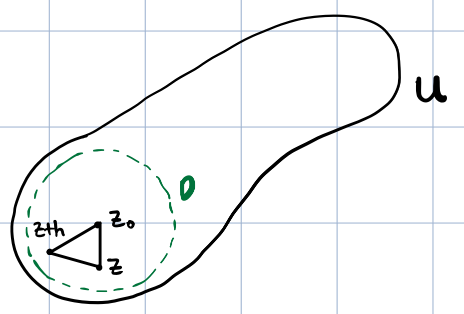
Theorem (Morera's):

Let $U \subseteq \mathbb{C}$ be open, and $f: U \to \mathbb{C}$ be continuous with $\displaystyle\int_{[a,b,c]} f = 0$ whenever $T = \operatorname{conv}\{a,b,c\} \subseteq U$. Then $f \in \mathcal{H}(U)$.

Proof: Fix $z_0 \in U$ and find $R > 0$ such that $D(z_0, R) \subseteq U$. ← recall this is star-like!

Define $F: D(z_0,R) \to \mathbb{C}$ with $F(z) = \displaystyle\int_{[z_0,z]} f(\zeta)\, d\zeta$. Claim: Then $F \in \mathcal{H}(D(z_0,R))$.

Let $z \in D(z_0,R)$ and $h \neq 0$ so that $z+h \in D(z_0,R)$.



Therefore, $\operatorname{conv}\{z_0, z+h, z\} \subset D(z_0,R)$.

By Cauchy-Goursat:

$$\int_{[z_0,\, z+h,\, z]} f = 0 = \underbrace{\int_{[z_0,\, z+h]} f}_{=\,F(z+h)} + \int_{[z+h,\, z]} f + \underbrace{\int_{[z,\, z_0]} f}_{=\,-\int_{[z_0,z]} f \,=\, -F(z)} \quad \text{(by construction)} \implies 0 = F(z+h) + \int_{[z+h,\, z]} f - F(z)$$

we reverse the path in the next step.

So then
$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,\, z+h]} f - \frac{1}{h}\int_{[z,\, z+h]} \overbrace{f(z)}^{\text{constant}}\, d\zeta$$

$$= \frac{1}{h}\left(\int (f(\zeta) + f(z))\, d\zeta\right) \quad \text{and now we consider modulus:}$$

$$\left|\frac{1}{h}\left(\int (f(\zeta) + f(z))\, d\zeta\right)\right| \leq \frac{1}{|h|} \max_{\zeta \in [z,\, z+h]} |f(\zeta) - f(z)| \cdot \underbrace{\text{length}([z, z+h])}_{|h|}$$

$$= \max_{\zeta \in [z,\, z+h]} |f(\zeta) - f(z)| \xrightarrow{h \to 0} 0 \quad \text{by continuity.}$$

Hence $F \in \mathcal{H}(D(z_0,R))$ with $F'(z) = f(z)$.

Hence f admits a primitive on $D(z_0,R)$, so $f = F' \in \mathcal{H}(D(z_0,R))$ (by Corollary)

Since our disc was arbitrary, we can do this for any disc in U. Therefore, $f \in \mathcal{H}(U)$. $\square$

Remark: The idea here is to show that $F' = f$, based on our construction. We do this by Cauchy-Goursat and then showing

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \xrightarrow{h \to 0} 0$$ so $F' = f$. Then since $F \in \mathcal{H}(U) \Rightarrow F \in C^\infty(U) \Rightarrow F' \in \mathcal{H}(U)$ (Corollary) we are done.

Lemma: "Cauchy's Estimates"

Let $U \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(U)$. Let $z_0 \in U$ and $R > 0$ such that $\overset{\text{closed}}{\bar{D}}(z_0, R) \subset U$.

Then for $n \in \mathbb{N}$, we have $|f^{(n)}(z_0)| \leq \dfrac{n!}{R^n} \max\limits_{z \in \underset{\text{circle}}{C}[z_0, R]} |f(z)|$.

Proof: Let $R' > R$ such that $\bar{D}(z_0, R) \subset D(z_0, R') \subseteq U$. (Such an R' exists by topology).

Then by Taylor's theorem, we have $c_n = \dfrac{f^{(n)}(z)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$ with $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$.

$$\text{Thus} \quad |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \max_{\zeta \in \gamma^*} \left| \frac{f(\zeta)}{\underset{\underset{R}{\shortparallel}}{(\zeta - z_0)}^{n+1}} \right| \cdot \overset{2\pi R}{\cancel{\text{length}(\gamma)}} = \frac{n!}{R^n} \max_{\zeta \in \gamma^*} |f(\zeta)|$$

since $\zeta - z_0$ is the radius whenever $\zeta \in \gamma^*$

$$= \frac{n!}{R^n} \max_{\zeta \in C[z_0, R]} |f(\zeta)|. \qquad \square$$

Remark: Cauchy's estimates say things about local information at z_0 using neighbouring properties.

Corollary: Let $U \subseteq \mathbb{C}$ be open and connected. Let $f \in \mathcal{H}(U)$ with $f' = 0$ on U. Then $f|_U$ is constant.

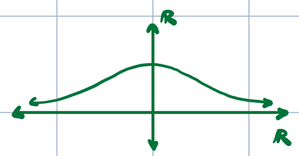
Proof: If $z_0, z_1 \in U$, then we have a contour $\gamma : [0,1] \to U$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

Then by the Fundamental Theorem for line integrals,

$$f(z_1) - f(z_0) = f(\gamma(1)) - f(\gamma(0)) = \int_\gamma f' = \int_\gamma 0 = 0.$$

i.e. $f(z_1) = f(z_0)$, $\forall z_0, z_1 \in U$. $\square$

Theorem (Liouville's): Let $f : \mathbb{C} \to \mathbb{C}$ be entire ($f \in \mathcal{H}(\mathbb{C})$). Say $M = \sup\limits_{z \in \mathbb{C}} |f(z)| < \infty$, so f is bounded on $\mathbb{C}$. Then f is constant.



Remark: Notice how $\mathbb{R}$-analysis is not as nice. Take $e^{-t^2} : \mathbb{R} \to \mathbb{R}$. Clearly, this is bounded on $\mathbb{R}$.

but Liouville's says $e^{-z^2} : \mathbb{C} \to \mathbb{C}$ cannot be bounded on $\mathbb{C}$, since $e^{-z^2} \in \mathcal{H}(\mathbb{C})$.

In fact, it is crazily unbounded.

Proof: For any $z \in \mathbb{C}$ and any $R > 0$, we have $\bar{D}(z, R) \subseteq \mathbb{C}$.

By Cauchy's estimates,

$$|f'(z)| \leq \frac{1!}{R} \max_{\zeta \in C(z,R)} |f(\zeta)| \leq \frac{M}{R}. \quad \text{Thus} \quad |f'(z)| \leq \underbrace{\lim_{R \to \infty} \frac{M}{R}}_{\text{since } \bar{D}(z, \infty) \subseteq \mathbb{C}} = 0 \Rightarrow f'(z) = 0, \ \forall z \in \mathbb{C}.$$

(Corollary) $\Rightarrow f$ is constant.

$\square$

Theorem (Fundamental Theorem of Algebra)

Let $p(z) \in \mathbb{C}[z]$ be non-constant. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$, i.e. z_0 is a root.

Remark: Notice the statement would be false for $\mathbb{R}$. This is the beauty of $\mathbb{C}$.

Proof: If $|p(z_0)| > 0$ for all $z \in \mathbb{C}$, then $f(z) = \frac{1}{p(z)} \in \mathcal{H}(\mathbb{C})$. We construct a contradiction from here.

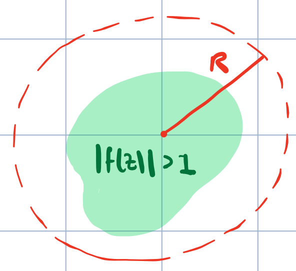
Write $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ with $a_n \neq 0$, and $a_i \in \mathbb{C}$.

For $z \neq 0$, we have $\frac{p(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \xrightarrow{|z| \to \infty} a_n \overset{\neq 0}{.}$

Hence, $f(z) = \frac{z^n}{p(z)} \cdot \frac{1}{z^n} \xrightarrow{|z| \to \infty} a_n^{-1} \cdot 0 = 0.$

we can choose any ε here!

Thus, there is $R > 0$ such that $|z| > R \Rightarrow |f(z)| < 1$. We have some "cut-off".



Notice that f is continuous on the compact disc $\bar{D}(0, R)$, so a finite maximum

$$M = \max_{z \in \bar{D}(0,R)} |f(z)| < \infty \text{ exists. Thus } |f(z)| \leq \max\{M, 1\}, \ \forall z \in \mathbb{C}.$$

Since f is bounded and entire, Liouville's says f is constant. Contradiction. $\lightning$

(we said $p(z)$ non-constant!)

Lemma: Let $p(z) \in \mathbb{C}[z]$ satisfy $\deg(p(z)) = n$, for some $n \in \mathbb{N}$. Then $p(z) = a \prod_{i=1}^{n} (z - c_i)$ where $a \in \mathbb{C}$, and $c_i \in \mathbb{C}$.

roots not necessarily distinct

Proof: Fundamental Theorem of Algebra $\Rightarrow \exists z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Then consider $p(z) = (z - z_0) q_1(z)$ with $\deg(q_1(z)) = n - 1$. Induct on this degree. $\square$

Upcoming: (zeros, poles, residues; curves, winding numbers, homotopy.)
$\Rightarrow$ simply connected sets

Let $U \subseteq \mathbb{C}$ be open. We say $a \in U$ is a cluster point of U if $(B(a,r) \cap U) \setminus \{a\} \neq \emptyset$ for all $r > 0$.

In other words, we may say a is a limit point of U (see Topology).

Theorem (zero sets): Let $U \subseteq \mathbb{C}$ be connected, and $f \in \mathcal{H}(U)$, and $Z_U(f) = \{z \in U : f(z) = 0\}$ (the set of zeros of f in U).

Then either $f \equiv 0$, or $Z_U(f)$ admits no cluster points in U.

Proof: 1 Let $z_0 \in U$ be a cluster point of $Z_U(f)$. (i.e. any z_0-neighbourhood of z_0 contains other elements of $Z_U(f)$.)

We will show that if $R > 0$ such that $D(z_0, R) \subseteq U$, then $D(z_0, R) \subseteq Z_U(f)$.

Suppose for a contradiction that f is not constantly zero on $D(z_0, R)$.

Then, since $f \in \mathcal{H}(U) \Rightarrow f \in \mathcal{A}(U)$, we can write

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \quad \text{for } z \in D(z_0, R) \quad \text{with } c_0 = f(z_0) = 0, \text{ and } c_m \neq 0 \text{ for some } m \geq 1$$

smallest such

since f is not constantly zero.

$$\Rightarrow f(z) = \sum_{k=m}^{\infty} c_k (z - z_0)^k = (z - z_0)^m \underbrace{\sum_{k=m}^{\infty} c_k (z - z_0)^{k-m}}_{g(z)}.$$

Since $g(z)$ is given by a power series, we have

$g \in \mathcal{H}(U) \Rightarrow g$ continuous on $D(z_0, R)$,

and $g(z_0) = c_m \neq 0$.

Choose some sequence $\{z_n\}_{n \in \mathbb{N}}^{\neq z_0}$ with $z_n \xrightarrow{n \to \infty} z_0$. Then by continuity, we have $\lim_{n \to \infty} g(z_n) = g(z_0) \neq 0$.

For large enough $n \in \mathbb{N}$, we get $z_n \in D(z_0, R)$ and $0 = f(z_n) = (z_n - z_0)^m g(z_n) \neq 0$. Contradiction. ↯

Therefore, f is constantly zero on $D(z_0, R)$ for any $R > 0$ such that $D(z_0, R) \subseteq U \Rightarrow f \equiv 0$ on U. □

$(\Rightarrow D(z_0, R) \subseteq Z_U(f))$

2 We show that $Z_U(f)$ admits no cluster points in U. This proof is rather topological.

Let $W = Z_U(f)^{\circ}$ (interior).

Notice that by (1), we have $Z_U(f)$ admits a cluster point in $U \Rightarrow W \neq \emptyset$. Also note that W is open.

Consider $z_0 \in U \setminus W$ (closed), and for every $r > 0$, if $D(z_0, r) \cap W \neq \emptyset$, then z_0 is a cluster point of W in U.

Hence, z_0 is a cluster point of $Z_U(f)$ in U, so $z_0 \in W$ by (1).

If $z_0 \in U \setminus W$, then $\exists r > 0$ such that $D(z_0, r) \subseteq U$. Hence $V = U \setminus W$ is open, so $U = \underbrace{V \cup W}_{\text{both open}}$ and $W \cap V = \emptyset$.

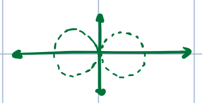
Since $U \subseteq \mathbb{C}$ is connected, exactly one of V, W is empty.

Thus, either $Z_U(f) = U$ or $Z_U(f)$ admits no cluster points in U. □ □

Remark: We can have isolated zeros, but no dense cluster of zeros (unless the function itself is zero).

Non-example: $\mathcal{U} = D(-1,1) \cup D(1,1)$. Define $f(z) := \begin{cases} 0, & z \in D(-1,1); \\ z^2, & z \in D(1,1). \end{cases}$ Then $Z_{\mathcal{U}}(f)$ dense but $f(z) \neq 0$ constantly.

disconnected regions



$\mathcal{U} = \mathbb{C} \setminus \{0\}$ and $g(z) = \sin\left(\frac{1}{z}\right) \in \mathcal{H}(\mathcal{U})$.

Then $Z_{\mathcal{U}}(g)$ contains $\left\{\frac{1}{2\pi n}\right\}_{n \in \mathbb{N}}$ with $\lim_{n\to\infty} \left(\frac{1}{2\pi n}\right) = 0$, so $0 \in \mathbb{C}$ is a cluster point.

(But notice $0 \notin \mathcal{U}$.)
so we are fine!

Corollary: If $\mathcal{U} \subseteq \mathbb{C}$ is connected and open, and $f, g \in \mathcal{H}(\mathcal{U})$, and there is $S \subseteq \mathcal{U}$ such that

- $f|_S = g|_S$ and
- f admits a cluster point in $\mathcal{U}$,

then $f = g$.

Proof: Notice $Z_{\mathcal{U}}(f-g) \supseteq S$ and hence has a cluster point in $\mathcal{U}$. So $f - g = 0$ on $\mathcal{U}$. $\square$

Theorem (Order of Zero): Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected. Let $f \in \mathcal{H}(\mathcal{U})$ be non-constant, and let $z_0 \in Z_{\mathcal{U}}(f)$.

Then there exist $m \in \mathbb{N}$ and $g \in \mathcal{H}(\mathcal{U})$ such that $f(z) = (z-z_0)^m g(z)$ and $g(z_0) \neq 0$ (existence).

In fact, $m = \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$ (uniqueness).

Proof: As usual, let $R > 0$ such that $D(z_0, R) \subseteq \mathcal{U}$. Write $f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ for $z \in D(z_0, R)$ ($f(z) \in \mathcal{H}(\mathcal{U})$).

Since f is non-constant, $\exists m \in \mathbb{N}$ such that $c_m \neq 0$. Let $m = \min\{k \in \mathbb{N} : c_k \neq 0\}$.

Notice $c_0 = f(z_0) = 0$ since $z_0 \in Z_{\mathcal{U}}(f)$.

From Cauchy's Integral Formula (Corollary), we have $c_k = \frac{f^{(k)}(z_0)}{k!}$ so $c_m = \min\{k \in \mathbb{N} : c_k \neq 0\}$

$= \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$.

Construct $g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & \text{for } z \neq z_0; \\ c_m & \text{for } z = z_0. \end{cases}$ Notice that $g \in \mathcal{H}(\mathcal{U} \setminus \{z_0\})$ and g is continuous at z_0,

so $\lim_{z \to z_0} g(z) = \lim_{z\to z_0} \left(\sum_{k=m}^{\infty} c_k (z-z_0)^{k-m}\right) = c_m \neq 0$.

By Cauchy-Goursat: $\int_{[a,b,c]} g = 0$ for $\operatorname{conv}\{a,b,c\} \subseteq \mathcal{U}$, since $g \in \mathcal{H}(\mathcal{U} \setminus \{z_0\})$.

By Morera's: $g \in \mathcal{H}(\mathcal{U})$ since g continuous and $\int_{[a,b,c]} g = 0$ as above. $\square$

Remark: Infinite order zeros cannot exist for any non-constant holomorphic function. We can always factor out zeros and the factorization preserves holomorphy.

Let $U \subseteq \mathbb{C}$ be open, and $z_0 \in U$, and $f \in \mathcal{H}(U \setminus \{z_0\})$.

We call z_0 a singularity of f which can be categorized as follows:

- removable singularity: if $\lim_{z \to z_0} f(z)$ exists.
- pole: if for some $n \in \mathbb{N}_{\geq 1}$, we have $\lim_{z \to z_0} (z-z_0)^n f(z)$ exists.
- essential singularity: if z_0 is neither removable, nor a pole.

Holomorphic extensions of functions.

Let $f \in \mathcal{H}(U \setminus \{z_0\})$ such that z_0 is a removable singularity. Define $\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \neq z_0; \\ \lim_{z \to z_0} f(z) & \text{if } z = z_0. \end{cases}$ This is an obvious candidate as an extension.

Then $\tilde{f} \in \mathcal{H}(U \setminus \{z_0\})$ and continuous on U

Cauchy-Goursat: $\int_{[a,b,c]} \tilde{f} = 0$ whenever $\operatorname{conv}\{a,b,c\} \subseteq U$.

Morera's: $\Rightarrow \tilde{f}$ has a primitive $\Rightarrow \tilde{f} \in \mathcal{H}(U)$.

} Cauchy-Goursat-Morera (CGM) method

Therefore, hereafter, if f has a removable singularity, we will identify f with $\tilde{f}$ and not distinguish them.

Theorem (Order of a pole): Let $U \subseteq \mathbb{C}$ be open, and $z_0 \in U$ be a pole for some $f \in \mathcal{H}(U \setminus \{z_0\})$.

Then $\exists m \in \mathbb{N}$ such that for any $R > 0$ with $D(z_0, R) \subseteq U$,

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_0)^m} + \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + \sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\text{Laurent series with finite principal part}} \quad \text{where } a_k \in \mathbb{C}, \text{ and } \sum_{k=0}^{\infty} a_k (z-z_0)^k \text{ convergent on } D(z_0, R).$$

Remark: If $a_{-m} \neq 0$, then m is the order of the pole.

Proof: Notice that since z_0 is not removable, we have f non-constant on $D'(z_0, R)$. (deleted)

Let $n \in \mathbb{N}$ be minimal such that $\lim_{z \to z_0} (z-z_0)^n f(z)$ exists.

Then if $g(z) := (z-z_0)^n f(z)$, then $g \in \mathcal{H}(D'(z_0, R))$ and by the CGM-method, we can extend g homomorphically to $D(z_0, R)$.

Since $g \in \mathcal{H}(D(z_0, R)) = \mathcal{A}(D(z_0, R))$, we can write $g(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ on $D(z_0, R)$.

Let $l = \min\{k \in \mathbb{N}_{\geq 0} : c_k \neq 0\}$. Notice $l < n$ since z_0 is not removable for f.
why?

Then, on $D(z_0, R)$, we have $f(z) = \frac{g(z)}{(z-z_0)^n} \sum_{k=l}^{\infty} c_k (z-z_0)^{k-n} = \frac{c_l}{(z-z_0)^{n-l}} + \ldots + \frac{c_{n-1}}{(z-z_0)} + \sum_{k=n}^{\infty} c_k (z-z_0)^{k-n}$

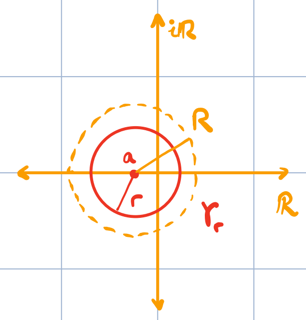
and we are done by renaming coefficients.

$\square$

Given $U \subseteq \mathbb{C}$ open, some $z_0 \in U$ and $f \in \mathcal{H}(U \setminus \{z_0\})$ and z_0 a pole for f,

we define the residue of f at z_0 by $\text{Res}_{z_0}(f) := a_{-1}$ where f is written as in the last theorem:

$$\frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$



Theorem (Residue I: for a circle): Let $a \in \mathbb{C}$ and $f \in \mathcal{H}(D'(a,R))$ $(R>0)$.

Let γ_r define a circle: $\gamma_r(t) = a + re^{it}$, $t \in [0, 2\pi]$, $0<r<R$.

Then $2\pi i \, \text{Res}_a(f) = \int_{\gamma_r} f$.

Proof: Use term-by-term integration. We have

$$\int_{\gamma_r} f = \int_{\gamma_r} \cancel{\frac{a_{-m}}{(z-z_0)^m}} + \dots + \cancel{\frac{a_{-2}}{(z-z_0)^2}} + \frac{a_{-1}}{(z-z_0)} + \cancel{\sum_{k=0}^{\infty} a_k (z-z_0)^k} \, dz$$

$$= \int_{\gamma_r} \frac{a_{-1}}{(z-z_0)} dz = a_{-1} \int_{\gamma_r} \frac{dz}{(z-z_0)} = 2\pi i \, \text{Res}_a(f). \qquad \square$$

Computing Residues:

- simple pole z_0: $\text{Res}_{z_0}(f) = \lim_{z \to z_0} (z-z_0) f(z)$.

Say $f = \frac{h}{g}$ where $z_0 \in Z_D(g)$ (zero of g) and z_0 order 1 for f, with $h, g \in \mathcal{H}(D(z_0, R))$.

Then $\lim_{z \to z_0} (z-z_0) \frac{h(z)}{g(z)} = \lim_{z \to z_0} \frac{h(z)}{\left(\frac{g(z)-g(z_0)}{z-z_0}\right)} = \frac{h(z_0)}{g'(z_0)}$ $\to$ holomorphic $\Rightarrow$ continuity; $\leftarrow$ by definition of derivative

hence $\text{Res}_{z_0}\left(\frac{h}{g}\right) = \frac{h(z_0)}{g'(z_0)}$. Notice $g'(z_0) \neq 0$ so we can stop here.

- higher order poles:

Lemma: If z_0 is a pole of f with order $m>1$, then $\text{Res}_{z_0}(f) = \lim_{z \to z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^{m-1} f(z) \right)$.

Proof: Exercise.

An example of the Residue Theorem: Gaussian Integral (Remark: a bit rough)

Let $f(z) = \dfrac{e^{-z^2}}{1+e^{-2az}}$ where $a = \zeta_8\sqrt{\pi} = \dfrac{1+i}{\sqrt{2}}\sqrt{\pi} = e^{i\frac{\pi}{4}}\sqrt{\pi}$. Notice $a^2 = \zeta_8^2\pi = i\pi$.

0.1 Identity: $f(z) - f(z+a) = e^{-z^2}$

Sketch: Compute manually. We have $f(z) - f(z+a) = \dfrac{e^{-z^2}}{1+e^{-2az}} - \dfrac{e^{-(z+a)^2}}{1+e^{-2a(z+a)}} = \dots = e^{-z^2}$.

This identity is the key to our choice of $f(z)$.

Our goal is $\int_{-\infty}^{\infty} e^{-z^2}\,dz$.

0.2 Singularities of $f(z)$

Notice that the numerator e^{-z^2} is entire, so it has no poles. We consider the denominator $1+e^{-2az}$ instead:

0.2.1 Zeroes of $1+e^{-2az}$

We have $e^{-2az} = -1 \iff -2az = (2k+1)i\pi \iff z = -\frac{a}{2}(2k+1)$ for $k \in \mathbb{Z}$.

We care specifically about the pole $z = \frac{a}{2}$.

0.2.2 $z = \frac{a}{2}$ is an order-1 pole of $f(z)$

By definition, we need $\frac{d}{dz}(1+e^{-2az})\Big|_{\frac{a}{2}} \neq 0$. We have $\frac{d}{dz}(1+e^{-2az}) = -2ae^{-2az}$

so $\frac{d}{dz}(1+e^{-2az})\Big|_{\frac{a}{2}} = -2ae^{-a^2} = -2ae^{-i\pi} \neq 0$.

0.3 Computing Residue ($Res_{\frac{a}{2}}(f)$)

We have shown $z = \frac{a}{2}$ is a simple pole. Hence $Res_{\frac{a}{2}}(f) = \dfrac{e^{-z^2}}{-2ae^{-2az}}\Bigg|_{\frac{a}{2}} = \dfrac{e^{-i\frac{\pi}{4}}}{-2e^{i\pi/4}\sqrt{\pi}(-1)} = \dfrac{1}{2e^{i\pi/2}\sqrt{\pi}}$

$$= \frac{1}{2\sqrt{\pi}i}.$$

1 Setting up a path integral

1.1 Defining $\gamma_{\frac{a}{2}}$:

Let $\gamma_{\frac{a}{2}} = \frac{a}{2} + \frac{|a|}{2}e^{it}$ for $t \in [-\pi, \pi]$. By the Residue Theorem,

$$\int_{\gamma_{\frac{a}{2}}} f = 2\pi i\, Res_{\frac{a}{2}}(f) = \sqrt{\pi}.$$

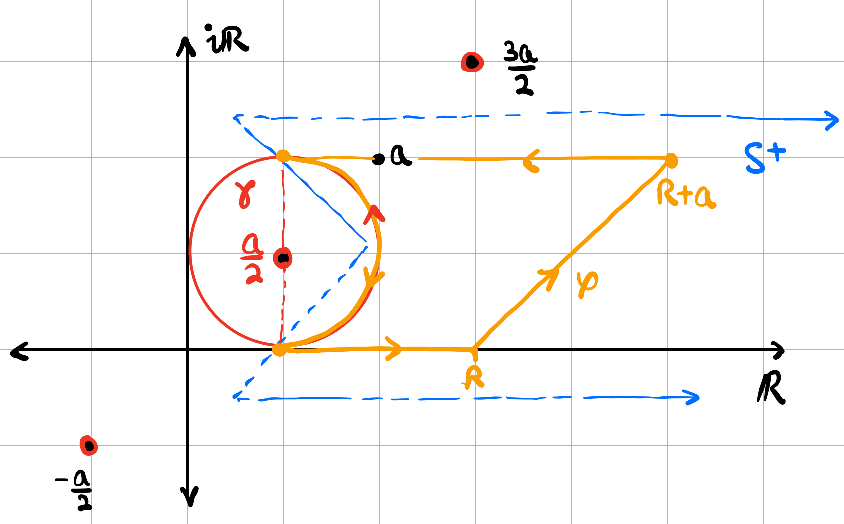
1.2 Creating a parallelogram

Let $L, R \in \mathbb{R}$ such that $L, R > |a|$.

Construct some S^+ such that $S^+ \subseteq \mathbb{C}$ is star-like.

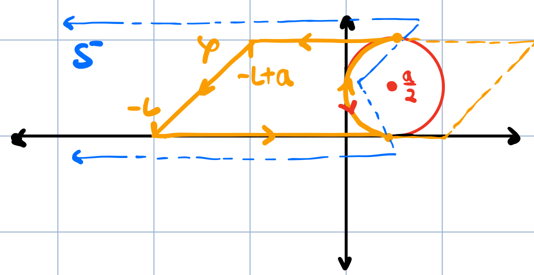
Since S^+ is star-like, contains no poles and φ closed,

$$\int_{\gamma_{\frac{a}{2}}\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}} f = \int_{[Re(\frac{a}{2}),\, R]} f + \int_{[R,\, R+a]} f + \int_{[R+a,\, Re(\frac{a}{2})+i\,Im(a)]} f$$



Remark: Since $\int_\varphi f = 0$ and $f \in \mathcal{H}(S^+)$. Also notice S^+ can have base-point anywhere along its center.

By similar reasoning, we construct the left side:



2 Completing our parallelogram and using the Residue Theorem

We have

$$\int_{[-L,R]} f + \int_{[R,R+a]} f + \int_{[R+a,\,-L+a]} f + \int_{[-L+a,\,-L]} f = \int_{\gamma_{\frac{a}{2}}} f = 2\pi i \operatorname{Res}_{\frac{a}{2}}(f) = \sqrt{\pi}.$$

Now we compute. Parametrize $[R, R+a]$ with $\{R+ta : t \in [0,1]\}$.

2.1 $|f(R+ta)|$ is bounded.

We have $$|f(R+ta)| \underset{\text{by some computation}}{\leq} \frac{e^{-R^2 - tR\sqrt{\pi}}}{1-e^{R/\sqrt{\pi}}} \leq \frac{e^{-R^2}}{1-e^{R/\sqrt{\pi}}}$$ (by choosing $t=0$).

2.2 $\left|\int_{[R,R+a]} f\right|$ and $\left|\int_{[-L+a,-L]} f\right| \xrightarrow[R\to\infty]{L\to\infty} 0.$

By our parametrization, $$\int_{[R,R+a]} f = \int_0^1 f(R+ta)\,dt \leq \max_{t\in[0,1]} |f(R+ta)| \cdot \text{length}([R, R+a])$$

$$\leq \frac{e^{-R^2}}{1-e^{-R/\sqrt{\pi}}}|a| \xrightarrow{R\to\infty} 0.$$

and we can repeat the same process for $\left|\int_{[-L+a,-L]} f\right| \xrightarrow{L\to\infty} 0.$

2.3 Compute the remaining integrals.

We have $$\int_{[-L,R]} f + \int_{[R+a,\,-L+a]} f = \int_{[-L,R]} f - \int_{[-L+a,\,R+a]} f = 2\pi i \operatorname{Res}_{\frac{a}{2}}(f) = \sqrt{\pi}$$
($[-L+a, R+a]$ ← backwards path)

$$= \int_{-L}^{R} f(t)\,dt - \int_{-L}^{R} f(t+a)\,dt$$ but these are Riemann integrals, so

$$= \int_{-L}^{R} (f(t) - f(t+a))\,dt$$

$$= \int_{-L}^{R} e^{-t^2}\,dt \xrightarrow[L\to\infty]{R\to\infty} \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$ Done!

by our identity (0.1)

Theorem (Residue II: Multiple poles in a circle):

Let $a \in \mathbb{C}$ and $R > 0$ such that $z_1, \dots, z_n \in D(a,R)$ are distinct poles for $f \in \mathcal{H}(D(a,R) \setminus \{z_1, \dots, z_n\})$.

Then for r such that $\max\{|z_1 - a|, \dots, |z_n - a|\} < r < R$,

we have $\displaystyle\int_{\gamma_r} f = \sum_{k=1}^{\infty} \operatorname{Res}_{z_k}(f)$.

closed circle

Corollary: We can replace γ_r above with

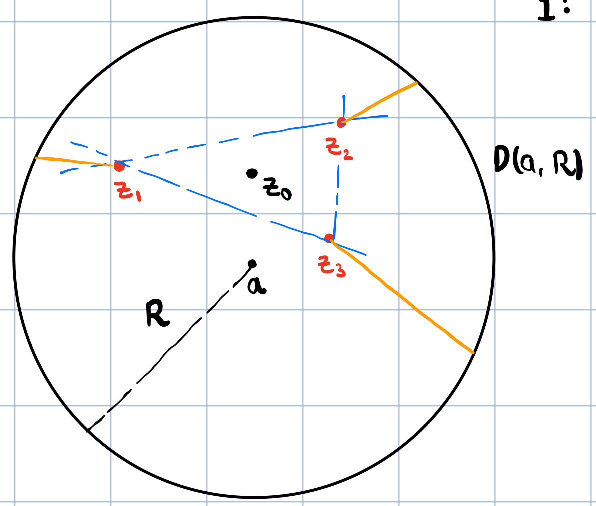
- semi-circles;
- convex quadrilaterals.

provided "orientation" is preserved.

This gives us motivation to study winding curves.

Proof (Pictures) of Theorem: We will show the statement for $n = 3$. The proof can be generalized beyond that.

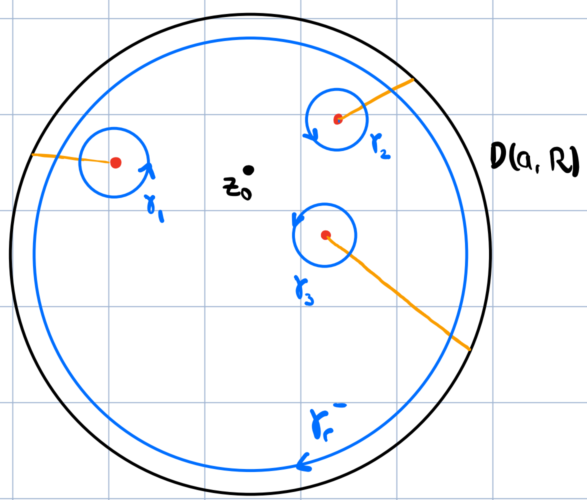
1: Draw lines between all the poles. Then $\exists z_0 \in D(a,R)$ such that z_0 does not lie



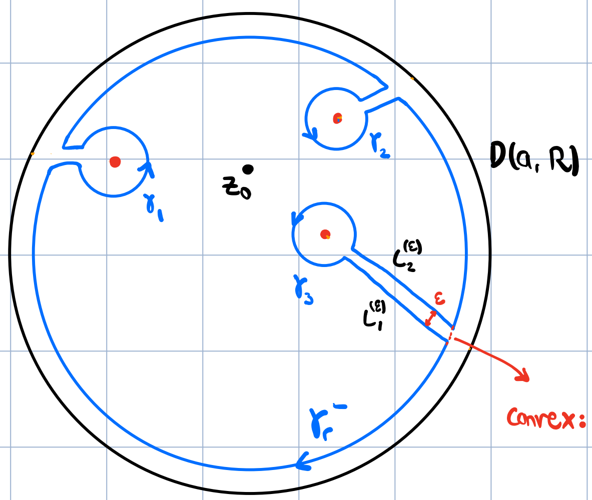
2: Construct a star-like set by drawing rays casted by z_0.

Take $\underset{\text{star-like}}{\mathcal{U}} = D(a,R) \setminus \{\text{rays}\}$.

We want to be able to use Theorem (Residue I), so let's analyze:



connect paths
"multiple keyhole"



Convex: $\displaystyle\int_{L_1^{(\varepsilon)}} f + \int_{L_2^{(\varepsilon)}} f \xrightarrow{\varepsilon \to 0^+} 0$ by local primitives on convex sets.

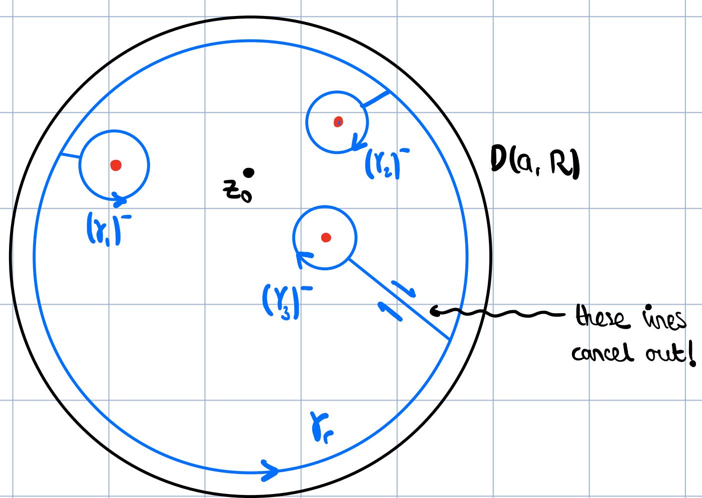
Let Γ_ε connect our circular arc segments with straight lines $L_i^{(\varepsilon)}$, where $\varepsilon \sim$ width.

Then $\rightleftarrows\!\odot$ $(\gamma_i^{(\varepsilon)})^-$ can be these "keyhole" segments.

4: In the above diagram, take $\varepsilon \to 0^+$.

Let $\Gamma = $ "$\displaystyle\lim_{\varepsilon \to 0^+} \Gamma_\varepsilon$."

By Cauchy's theorem on $\mathcal{U}$, we have $\displaystyle\int_{\Gamma_\varepsilon} f = 0$,



$$\Rightarrow \int_{\gamma_r} f + \sum_{j=1}^{3} \int_{\gamma_j^-} f = \int_{\Gamma_\varepsilon} f = 0$$

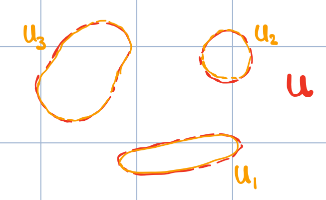
backwards

$$\Rightarrow \int_{\gamma_r} f = \sum_{j=1}^{3} \int_{\gamma_j} f = \sum_{j=1}^{3} 2\pi i \operatorname{Res}_{z_j}(f)$$ by Theorem (Residue I). We are done. $\square$

Star-like regions are restricting us. We want to generalize to arbitrary curves. **GOAL:** Winding numbers.

Topological Notes:

Let $\mathcal{U} \subseteq \mathbb{C}$. Then:

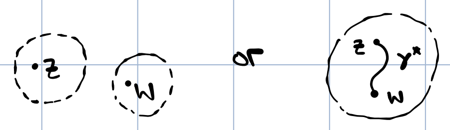


(1) $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ such that each $\mathcal{U}_n$ is a connected open set with $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$ for $m \neq n$.

Call each $\mathcal{U}_n$ a connected component of $\mathcal{U}$.

Idea If $z \in \mathcal{U}$, then the path component $\mathcal{U}_z = \{w \in \mathcal{U} : \exists\ \underset{\text{contour (piecewise smooth continuous)}}{\gamma} : [0,1] \to \mathcal{U}$ with $\gamma(0) = z,\ \gamma(1) = w\}$.

Remark: If $z, w \in \mathcal{U}$, then either $\mathcal{U}_z \cap \mathcal{U}_w = \emptyset$ or $\mathcal{U}_z = \mathcal{U}_w$. ← just by reverse paths!



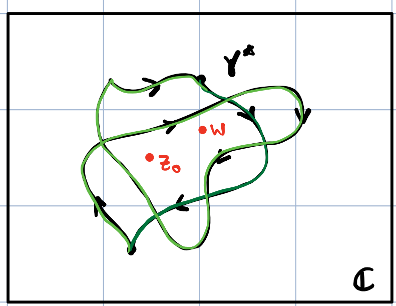
The Gaussian rationals $\mathbb{Q} + i\mathbb{Q} = \mathbb{Q}(i)$ are dense in $\mathbb{C}$ and are countable $(|\mathbb{Q}(i)| = \aleph_0)$.

Thus $\mathcal{U} = \bigcup_{z \in \mathcal{U}} \mathcal{U}_z = \bigcup_{k=1}^{\infty} \mathcal{U}_{q_k}$ where $q_k \in \mathbb{Q}(i)$. Exercise: Show that we can select $\leq \aleph_0$ disjoint components.

(2) If $K \subset \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ admits a unique component which is unbounded.

Note $K \subset D(0,R) \subset \bar{D}(0,R)$. Let V be the component of $\mathcal{U}$ containing $\partial \bar{D}(0,R)$. This is unique (exercise).

Theorem (Winding Numbers): Let γ be a closed contour in $\mathbb{C}$. Then for any $z_0 \in \overset{\text{open}}{\mathbb{C} \setminus \gamma^*}$,



(1) the quantity $W_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$ is always in $\mathbb{Z}$.

(2) if w and z belong to the same component of $\mathbb{C} \setminus \gamma^*$, then $W_\gamma(z) = W_\gamma(w)$.

(3) if z is in the unbounded component, then $W_\gamma(z) = 0$.

Proof: (1) Parametrize $\gamma : [0,1] \to \mathbb{C}$, and let $0 = a_0 < a_1 < \ldots < a_n = 1$ be points where $\gamma'(a_j)$ does not exist.

Then for $z_0 \in \mathbb{C} \setminus \gamma^*$, we have $\int_\gamma \frac{dz}{z - z_0} = \int_0^1 \frac{\gamma'(t)}{\gamma(t) - z_0}\, dt$ which is a Riemann integral.

Define, for $t \in [0,1]$, $\psi(t) := \exp\left(\int_0^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right)$ so notice $\psi(0) = 0$ and $\psi'(t) = \psi(t) \frac{\gamma'(t)}{\gamma(t) - z_0}$ by chain rule and the Fundamental Theorem of Calculus, except at $a_0, \ldots, a_n$.

$$\varphi(t) := \frac{\psi(t)}{\gamma(t) - z_0} \quad \text{so} \quad \varphi'(t) = \frac{\psi'(t)}{\gamma(t) - z_0} - \psi(t) \frac{\gamma'(t)}{(\gamma(t) - z_0)^2} = 0.$$

Notice $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$ are real-valued functions, so by Mean Value Theorem, are constant on each $[a_{j-1}, a_j]$.

Thus φ is locally constant on $[0,1]$ and continuous $\Rightarrow \varphi$ is constant.

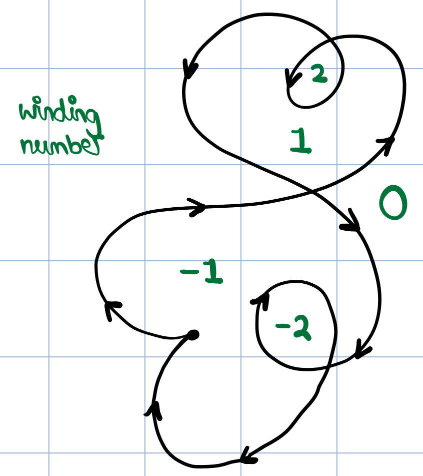
Thus $$\frac{1}{\gamma(0) - z_0} = \frac{\psi(0)}{\gamma(0) - z_0} = \varphi(0) = \varphi(1)$$

$$= \frac{\exp\left(\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right)}{\gamma(1) - z_0}$$ but γ is closed, so $\gamma(0) = \gamma(1)$. Multiply both sides by denominator:

$$\Rightarrow 1 = \exp\left(\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds\right) = \exp\left(\int_\gamma \frac{dz}{z - z_0}\right)$$ and we take Log:

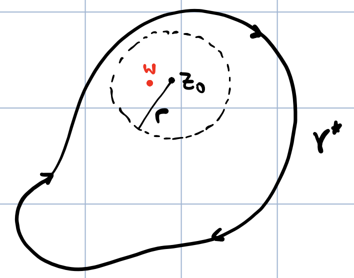
$$\Rightarrow \int_\gamma \frac{dz}{z - z_0} \in 2\pi i \mathbb{Z} \Rightarrow W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}$$ as required. $\square$ (1)

Proof (2): We will show that $W_\gamma : \mathbb{C}\setminus\gamma^* \to \mathbb{Z}$ is continuous component-wise.



Fix $z_0 \in \mathbb{C}\setminus\gamma^*$ which is open since γ^* is compact.

$\Rightarrow$ Choose $r>0$ such that $D(z_0, r) \subset \mathbb{C}\setminus\gamma^*$.



Let $w \in D(z_0, \frac{r}{2})$.

Then $|W_\gamma(z_0) - W_\gamma(w)|$ by definition:

$$= \left| \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} - \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} \right|$$

$$= \left| \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{z-z_0} - \frac{1}{z-w}\right] dz \right|$$ and we combine fractions and simplify:

$$= \frac{1}{2\pi}\left| \int_\gamma \frac{z_0-w}{(z-z_0)(z-w)}\, dz \right|.$$ Recall that since $z \in \gamma^*$ we have $|z-z_0| > r$ and $|z-w| > \frac{r}{2}$ by choice.

$$\leq \frac{1}{2\pi} \max_{z\in\gamma^*} \left(\frac{|z_0-w|}{|z-z_0||z-w|} \right) \cdot \text{length}(\gamma) \leq \frac{1}{2\pi}\frac{|z_0-w|}{(r\cdot\frac{r}{2})} \cdot \text{length}(\gamma) = \frac{1}{\pi r^2}\cdot \text{length}(\gamma)\, |z_0-w|$$

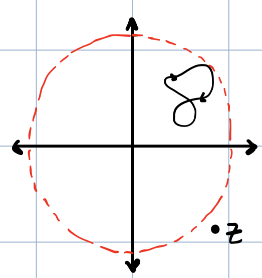
but this shows Lipschitz continuity ($\Rightarrow$ continuity).

We've shown that $W_\gamma(z) \in \mathbb{Z}$. Let us pullback the function:

Let $m \in \mathbb{Z}$. We have $U_m = W_\gamma^{-1}(D(m, \frac{1}{2})) = W_\gamma^{-1}(\{m\})$ since W_γ is $\mathbb{Z}$-valued.

Recall $\mathbb{C}\setminus\gamma^* = \bigsqcup_{m\in\mathbb{Z}} U_m$ with each U_m open and $n\neq m \Rightarrow U_m \cap U_n = \emptyset$ (each disjoint).

Thus, if V is a component of $\mathbb{C}\setminus\gamma^*$, we see $V = U_m$ for some m, since we violate connectedness of V otherwise. $\square$ (2)



Proof (3): Let W denote the unique unbounded component of $\mathbb{C}\setminus\gamma^*$.

If $|z| > \max_{\zeta\in\gamma^*}|\zeta|$ then $|W_\gamma(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}\right|$

$$\leq \frac{1}{2\pi}\max_{\zeta\in\gamma^*}\frac{1}{|\zeta-z|}\cdot\text{length}(\gamma) \leq \frac{1}{2\pi}\max_{\zeta\in\gamma^*}\left|\frac{1}{|\zeta|-z}\right|\cdot\text{length}(\gamma)$$

$\xrightarrow{|z|\to\infty}$ 0. since γ^* compact $\Rightarrow |\zeta|$ bounded.

But $W_\gamma(z)$ constant on W, so $W_\gamma(z)$ must be 0 by the above. $\square$ $\square$

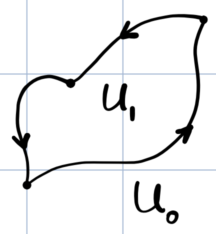
Let $m \in \mathbb{Z}$ be fixed and nonzero. Consider $\gamma_{m,R}(t) = a + Re^{imt}$ for $R>0$, $a\in\mathbb{C}$ and $t\in[0, 2\pi]$.

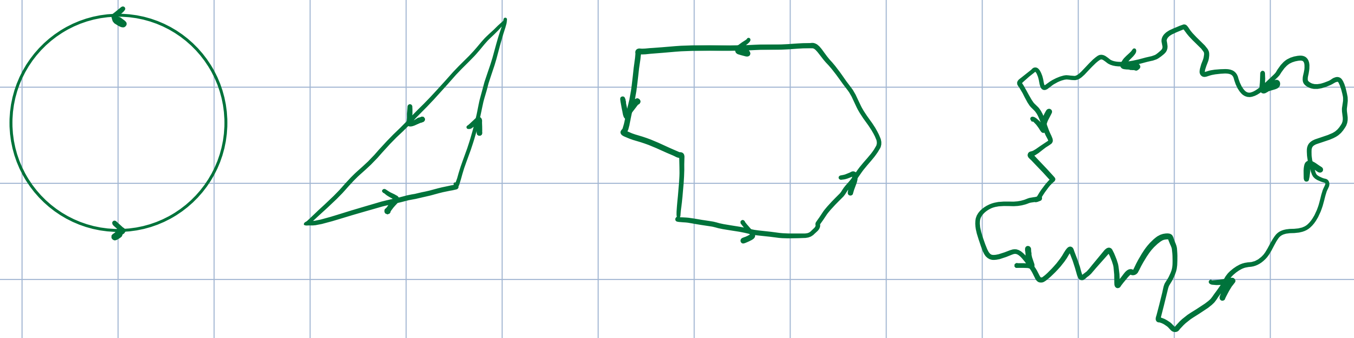
Then $W_{\gamma_{m,R}}(z) = m \iff z \in D(a, R)$.

A "toy path" is a simple closed contour $\gamma : [0,1] \to \mathbb{C}$ (arbitrary compact set) so that $\mathbb{C} \setminus \gamma^* = U_1 \cup U_0$ where,

(counter-clockwise oriented, evident exterior/interior)

for $k \in \{0,1\}$, we have $U_k = \{z \in \mathbb{C} \setminus \gamma^* : W_\gamma(z) = k\}$, and U_k are connected.





examples of "toy paths"

Theorem (Jordan Curve): Let γ be a Jordan curve. Then $\mathbb{C} \setminus \gamma^*$ has two components:

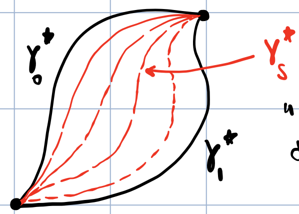
$\mathbb{C} \setminus \gamma^* = \mathcal{I} \cup \mathcal{O}$ with $\mathcal{I}, \mathcal{O}$ connected, and $\mathcal{I} \cap \mathcal{O} = \emptyset$, and $\overline{\mathcal{I}}$ compact.

Proof: Beyond scope. See C&O courses. $\square$

Let $U \subseteq \mathbb{C}$ be open, and $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ be contours.

A contour homotopy in U is a function $H : [0,1] \times [0,1] \longrightarrow U$ with
a) $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$
OR
b) γ_0, γ_1 are closed paths
such that

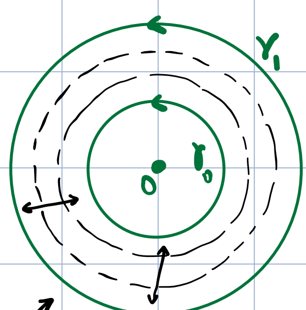
- H is continuous;
- $H(0,t) = \gamma_0(t)$;
- If $s \in [0,1]$, then $H(s,t) = \gamma_s(t)$ defines a contour. ← optional for many authors



If a homotopy exists inside of U, we say γ_0 is homotopic to γ_1 in U.

(i) Let $U = \mathbb{C} \setminus \{0\}$. Let $0 < r < R$. Define $\gamma_0(t) = re^{i2\pi t}$ for $t \in [0,1]$.

$$\gamma_1(t) = Re^{i2\pi t}$$



shrink/grow deformation (concentric circles γ_s)

Define $H : [0,1] \times [0,1] \longrightarrow \mathbb{C} \setminus \{0\}$ to be the convex combination

$$H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$$

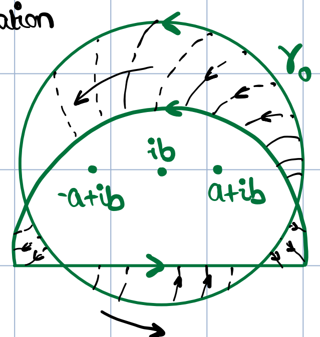
$$= \big((1-s)r + sR\big)e^{i2\pi t}.$$ Then H is a homotopy.

(ii) Let $U = \mathbb{C} \setminus \{-a+ib,\ a+ib\}$ where $a, b \in \mathbb{R}_{>0}$.

Define $\gamma_0(t) = ib + Re^{i2\pi t}$ where $R > \sqrt{a^2+b^2} > a$.

$\gamma_1(t)$: concatenation of $[-R, R]$ and $t \mapsto Re^{it}$, $t \in [0, \pi]$. $\Bigg\}$ $\gamma_1(t) = \begin{cases} (1-2t)(-R) + 2tR, & t \in [0, \frac{1}{2}); \\ Re^{i2\pi(2t-1)}, & t \in [\frac{1}{2}, 1]. \end{cases}$

deformation

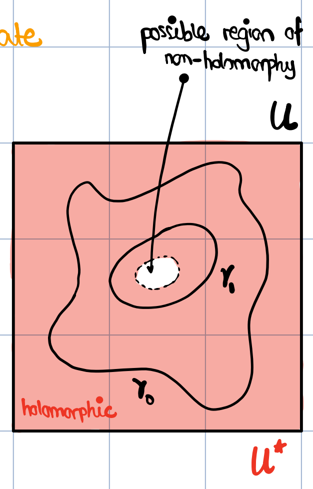


Let $H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$. Then H defines a homotopy.

Usually, it's a lot harder to show that $H([0,1]^2) \subset \mathbb{C} \setminus \{-a+ib,\ a+ib\}$ remains contained.

Theorem (Deformation Theorem):

Let $U \subseteq \mathbb{C}$ be open. Let $\gamma_0, \gamma_1 : [0,1] \to U$ be contours that are contour homotopic in U with intermediate paths closed (i.e. $\gamma_s(0) = \gamma_s(1)$, $\forall s \in [0,1]$). Then for $f \in \mathcal{H}(U^*)$, we have $\int_{\gamma_0} f = \int_{\gamma_1} f$, where U^* is such that γ_0, γ_1 can be contour homotopic.



Remark: The elaborate proof seems unnecessary, but takes care of cases where γ_0, γ_1 are not homotopic to a point.

Proof: Here's the setup. Let $H : [0,1] \times [0,1] \to U^*$ be a homotopy linking γ_0 to γ_1.

Recall:
- H is continuous on $[0,1]^2$;
- $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$;
- Each $\gamma_s(t) = H(s,t)$ defines a contour.

0 Consequences of the continuity of H.

0.1 Let $K = H([0,1] \times [0,1])$. Since the pre-image is compact, we have $K \subset U$ compact by the continuity of H.

Hence $\text{dist}(K, \mathbb{C} \setminus U^*) = \inf\{|z-w| : z \in K, w \in \mathbb{C} \setminus U\} > 0$.

Therefore $\exists r \in \mathbb{R}$ such that $0 < r < \text{dist}(K, \mathbb{C} \setminus U)$.

0.2 A continuous function on a compact set is uniformly continuous.

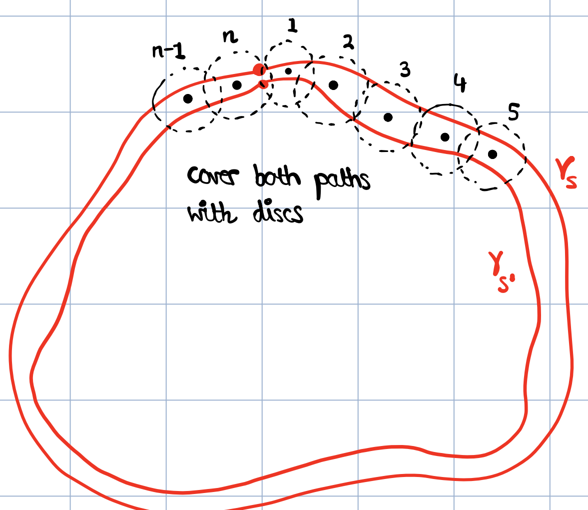
Then $\exists \delta > 0$ such that $\|(s,t) - (s',t')\| < \delta$ in $[0,1] \times [0,1]$.

So we can choose r such that $|H(s,t) - H(s',t')| < \frac{r}{2}$ in U.

Notice if $|s-s'| < \delta$ in $[0,1]$, then for every $t \in [0,1]$, we have

$$|\gamma_s(t) - \gamma_{s'}(t)| = |H(s,t) - H(s',t)| < \frac{r}{2}$$ i.e. $\gamma_s, \gamma_{s'}$ are uniformly less than $\frac{r}{2}$ apart.

1 If $|s-s'| < \delta$ in $[0,1]$, then $\int_{\gamma_s} f = \int_{\gamma_{s'}} f$.



1.1 Let $C = \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there is } 0 = t_0 < t_1 < \ldots < t_n = 1 \\ \text{and open discs } D_1, \ldots, D_n \text{ such that} \\ \gamma_\sigma(t_{j-1}),\ \gamma_\sigma(t_j) \in D_j, \quad \sigma = s, s' \end{array} \right\}$

Let $t^* = \sup C$. Then $t^* = 1$.

If $t^* \neq 1$, then $t < 1$, and we find $t_1, \ldots, t_n$ such that $t_n \leq t^* < t_n + \frac{\delta}{2}$.

Let t_{n+1} be chosen as $t_{n+1} = \min\{t_n + \frac{\delta}{2}, 1\}$.

Then add a disc $D_{n+1} = D(\gamma_s(t_{n+1}), r) \subset U$ since $\gamma_s(t_{n+1}) \in K$.

Since $0 \leq t_{n+1} - t_n < \delta$, we have, for $\sigma = s, s'$,

$$|\gamma_\sigma(t_n) - \gamma_\sigma(t_{n+1})| = |H(\sigma, t_n) - H(\sigma, t_{n+1})| < \frac{r}{2}$$ by uniform continuity;

and $\quad |\gamma_{s'}(t_{n+1}) - \gamma_s(t_{n+1})| = |H(s', t_{n+1}) - H(s, t_{n+1})| < \frac{\varepsilon}{2}$ from earlier;

$\Rightarrow \quad |\gamma_s(t_{n+1}) - \gamma_{s'}(t)| < r$

Hence for $\sigma = s, s'$ we have $\gamma_\sigma(t_n), \gamma_\sigma(t_{n+1}) \in D_{n+1}$. Essentially, by the definition of C, we have $t^* = 1$ and $C = [0,1]$. We've covered our interval.

Conclusion: There is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ and open discs $D_1, \ldots, D_n \subset \mathcal{U}$ such that

$$\gamma_\sigma(t_{j-1}), \gamma_\sigma(t_j) \in D_j \quad \text{for } \sigma = s, s' \text{ and } j = 1, 2, \ldots, n.$$

1.2 Applying theorems.

Notice that each disc D_j is convex, so D_j is star-like. By Theorem (Existence of Primitives), f admits a primitive $F \in \mathcal{H}(D_j)$.

Since $D_j \cap D_{j+1} \neq \emptyset$ for each j, we can adjust by constants so $F_j = F_{j+1}$ on $D_j \cap D_{j+1}$.

Notice that γ_s closed $\Rightarrow D_n \cap D_1 \neq \emptyset$ so $F_n = F_1 + C$ on $D_n \cap D_1$.

$$(\text{FTC in } \mathbb{C}) \Rightarrow \int_{\gamma_s} f = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma_s(t))\, \gamma_s'(t)\, dt = \sum_{j=1}^{n} [F_j(\gamma_s(t_j)) - F_j(\gamma_s(t_{j-1}))] \quad \text{(telescoping series);}$$
$$= F_n(\gamma_s(1)) - F_1(\gamma_s(0)) = C.$$

Similarly $\int_{\gamma_{s'}} f = C$, so then $\int_{\gamma_{s'}} f = \int_{\gamma_s} f$. Remember γ_σ are intermediary paths, so we are not fully done.

1.3 Completion. Let $0 = s_0 < s_1 < \ldots < s_m = 1$ such that $s_j - s_{j-1} < \delta$.

Then by the above, $\int_{\gamma_{s_0}} f = \int_{\gamma_{s_1}} f = \ldots = \int_{\gamma_{s_m}} f$

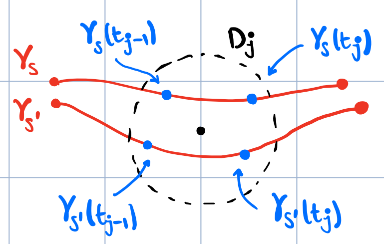
$$\begin{array}{ccc} \int_{\gamma_{s_0}} f & = \ldots = & \int_{\gamma_{s_m}} f \\ \parallel & & \parallel \\ \int_{\gamma_0} f & & \int_{\gamma_1} f \end{array} \quad \text{so} \quad \int_{\gamma_0} f = \int_{\gamma_1} f. \qquad \square$$

Remark: Here's an overview of our strategy.

- 0: We note consequences of uniform continuity. Namely, if (s,t) and (s',t') are close, then

$$\begin{array}{ccc} H(s,t) & \text{and} & H(s',t') \text{ must be close.} \\ \parallel & & \parallel \\ \gamma_s(t) & & \gamma_{s'}(t') \end{array}$$

- 1: 1.1: We show that we can cover any two intermediary paths $\gamma_\sigma(t)$ with finitely many open discs.



$(\gamma_\sigma(t_{j-1}), \gamma_\sigma(t_j) \in D_j)$.

1.2: We let s and s' be close. Then we cover the paths as above, and apply theorems to show $\int_{\gamma_s} f = \int_{\gamma_{s'}} f$.

i.e. the integral over two "close" intermediary paths is the same.

What is $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$? Consider $f(z) = \frac{e^{iz}}{z}$. Notice $f \in \mathcal{H}(\mathbb{C}\setminus\{0\})$ so f admits a primitive on $\mathbb{C}\setminus\{0\}$.

Also notice that $z=0$ is a simple order-1 pole, since $\lim_{z\to 0}(z-0)^1 f(z)$ exists.

(computing residues) $\Rightarrow \operatorname{Res}_{z=0}(f) = \lim_{z\to 0}\left(\frac{e^{iz}}{(z)'}\right) = \lim_{z\to 0} e^{iz} = 1.$ Define $\gamma_\varepsilon = \varepsilon e^{it}$, for $t \in [0,\pi]$.
$\gamma_R = Re^{it}$,

Define Γ to be the concatenation of γ_R, γ_ε^-, $[\varepsilon, R]$, $[-R,-\varepsilon]$ where $R>\varepsilon>0$.

Notice $f \in \mathcal{H}(\Gamma^*)$ and Γ encloses no singularities, so by Cauchy's Theorem, we have $\int_\Gamma f = 0$.

We have $\int_\Gamma f = \int_{\gamma_\varepsilon^-} f + \int_{\gamma_R} f + \int_{-R}^{-\varepsilon} f + \int_{\varepsilon}^{R} f = 0.$

Notice $\int_{\gamma_R} \frac{e^{iz}}{z}\,dz = \int_0^\pi \frac{e^{i(Re^{it})}}{Re^{it}}\, iRe^{it}\,dt$ and $\left|\int_0^\pi \frac{e^{i(Re^{it})}}{Re^{it}}\,dt\right| \leq \frac{1}{R}\max_{t\in[0,\pi]}\left|\frac{e^{i(Re^{it})}}{Re^{it}}\right| \cdot \pi$

$\leq \frac{\pi}{R} \xrightarrow{R\to\infty} 0.$

We also have $\int_{\gamma_\varepsilon^-} f \longrightarrow \lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon^-} f = \pi i$, so $\int_{\gamma_\varepsilon} f = -\pi i$.

Then $\int_{\gamma_R} f + \int_{\gamma_\varepsilon^-} f + \int_{-R}^{-\varepsilon} f + \int_\varepsilon^R f = 0 + \pi i + \int_{-R}^{-\varepsilon} f + \int_\varepsilon^R f = 0.$

$\Rightarrow \int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,dz + \int_\varepsilon^R \frac{e^{iz}}{z}\,dz = -\pi i$ but when we take the limit:

$\downarrow \varepsilon \to 0^+$

$\int_{-\infty}^{0} \frac{e^{iz}}{z}\,dz + \int_0^\infty \frac{e^{iz}}{z}\,dz = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \int_{-\infty}^{\infty} \frac{\cos(z)}{z}\,dz + i\int_{-\infty}^{\infty} \frac{\sin(z)}{z}\,dz$ so $\int_{-\infty}^{\infty} \frac{\sin(z)}{z}\,dz = \pi.$

Corollary: Let $\mathcal{U} \subseteq \mathbb{C}$ be open, and $\gamma_0, \gamma_1 : [0,1] \to \mathcal{U}$ be contours sharing endpoints such that γ_0 and γ_1 are contour homotopic (in $\mathcal{U}$).

Then for any $f \in \mathcal{H}(\mathcal{U})$, we have $\int_{\gamma_0} f = \int_{\gamma_1} f$.

Proof: Let $H : [0,1]\times[0,1] \to \mathcal{U}$ be an endpoint-preserving homotopy with $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$.

Let $\tilde{H} : [0,1]\times[0,1] \to \mathcal{U}$ be given by $\tilde{H}(s,\cdot) = \gamma_s \cdot \gamma_1^-$ a concatenation, where $\gamma_s(t) = H(s,t)$.

Then $\tilde{H}$ is a closed path homotopy linking γ_0 and γ_1.

$\Rightarrow 0 = \int_{\gamma_1} f - \int_{\gamma_1} f = \int_{\gamma_1} f + \int_{\gamma_1^-} f = \int_{\underset{\text{closed}}{\gamma_1\cdot\gamma_1^-}} f \underset{\text{theorem}}{=} \int_{\underset{\text{closed}}{\gamma_0\cdot\gamma_1^-}} f = \int_{\gamma_0} f - \int_{\gamma_1} f.$ $\square$

Let $\mathcal{U}$ be open and $\gamma : [0,1] \to \mathcal{U}$ be a closed contour. We say γ is null-homotopic in $\mathcal{U}$ if there is a constant closed contour $\gamma_0(t) = z_0$ for $z_0 \in \mathcal{U}$ (the "null curve") such that γ and γ_0 are homotopic.

Our goal is to generalize beyond star-like regions.

Proposition: Let $U \subseteq \mathbb{C}$ be open and star-like. Then any closed contour $\gamma:[0,1] \to U$ is null-homotopic in U.

Proof: Let z_0 be a base-point such that U is star-like.

Define $H(s,t) = (1-s)z_0 + s\gamma(t)$ the convex combination of paths, so notice $H(s,t) \in [z_0, \gamma(t)] \underset{\text{by star-like definition}}{\subseteq} U$.

Then H is a contour homotopy, so γ is null-homotopic. $\square$

In topology, a region where all closed paths are null-homotopic is called a simply connected region.

Connectedness in general gets slightly messy; see Topology for more.

Remark: Some of the definitions (e.g. homotopy) we use are not as general as Topology.
Rather, they obey topology on Euclidean $\mathbb{R}^n$ and the like.

Let $U \subseteq \mathbb{C}$ be open. A contour cycle Γ in U is a finite collection of closed contours in U.

Write $\Gamma = \gamma_1 \dotplus \gamma_2 \dotplus \dots \dotplus \gamma_n$ for each closed contour γ_k, and

$\Gamma^* = \gamma_1^* \cup \gamma_2^* \cup \dots \cup \gamma_n^*$ for each trace. Notice $\bigcup_{k=1}^{n} \gamma_k^*$ is a finite union of compact sets, so Γ^* is compact.

If $f:\Gamma^* \to \mathbb{C}$ is continuous, then

$$\int_\Gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f = \sum_{k=1}^{n} \int_0^1 f(\gamma_k(t))\gamma_k'(t)\,dt$$ is a finite sum of Riemann integrals.

reparametrize γ_k over $[0,1] \subset \mathbb{R}$.

and if $z \in \mathbb{C}\setminus\Gamma^*$, then $W_\Gamma(z) = \sum_{k=1}^{\infty} W_{\gamma_k}(z)$.

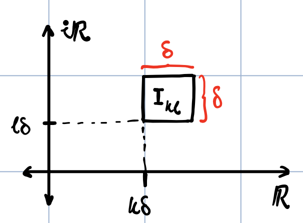
Cauchy's Integral Formula (on a compact set):

Let $U \subseteq \mathbb{C}$ be open. Let $K \subset U$ be compact. Then there is a cycle Γ in $U\setminus K$ for which for any $f \in \mathcal{H}(U)$,

we have $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$ for $z \in K$. In particular, $W_\Gamma(z) = 1$ for each $z \in K$.

Proof: 0 Setting up: Topology

Fix $\delta > 0$ such that $\sqrt{2}\delta < \operatorname{dist}(K, \mathbb{C}\setminus U)$. Notice by compactness, we have $\operatorname{dist}(K, \mathbb{C}\setminus U) > 0$
(since K and $\mathbb{C}\setminus U$ are disjoint).



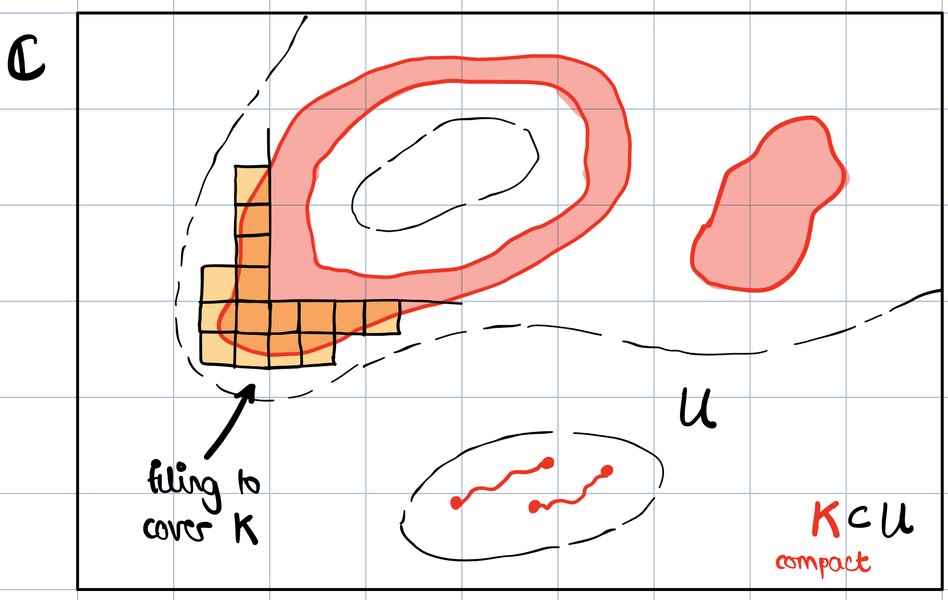
For $k, l \in \mathbb{Z}$, let $I_{kl} = \{z \in \mathbb{C} : \operatorname{Re}(z) \in [k\delta, (k+1)\delta],\ \operatorname{Im}(z) \in [l\delta, (l+1)\delta]\}$.

(Notice this is a tiling of squares each with diagonal $\sqrt{2}\delta$, i.e.)

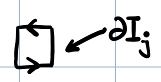
Thus $\operatorname{diam}(I_{kl}) = \sup\{|w-z| : w, z \in I_{kl}\} = \sqrt{2}\delta < \operatorname{dist}(K, \mathbb{C}\setminus U)$.

Let $\mathcal{I} = \{I_1, \dots, I_m\} = \{I_{kl} : (k,l) \in \mathbb{Z}\times\mathbb{Z},\ K \cap I_{kl} \neq \emptyset\}$ where m finite as K compact $\overset{\text{homeomorphic } \mathbb{C} \cong \mathbb{R}^2}{\Longrightarrow}$ K bounded.

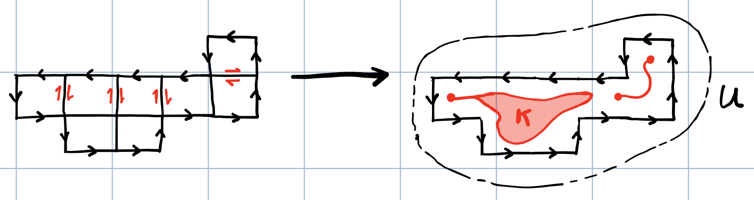
Notice $I_j \subset U$, $\forall I_j \in \mathcal{I}$, since $\operatorname{diam}(I_k) < \operatorname{dist}(K, \mathbb{C}\setminus U)$.



Parametrize each boundary ∂I_j $(I_j \in \mathcal{J})$ such that we have positively oriented contours on each.



The key idea is to invoke Cauchy's Theorem on each square, notice edges that "cancel", and proceed from there.



1 (Cauchy's Integral Formula for squares):

If $\mathcal{H}(\mathcal{U})$, then $f(z) = \frac{1}{2\pi i} \int_{\partial I_j} \frac{f(\zeta)}{\zeta - z} d\zeta$ for $z \in I_j^{\circ}$ (interior of the square).

Exercise. Prove using homotopies, or see proof for above.

2 $\frac{1}{2\pi i} \int_{\partial I_j} \frac{f(\zeta)}{\zeta - z} dz = 0$ for $z \in \mathcal{U} \setminus I_j$ (outside the square).

2.1 Let $\varepsilon = \mathrm{dist}(I_j, \mathbb{C} \setminus \mathcal{U}) > 0$. Then we have the following:

- $C_j := I_j + D(z, \varepsilon) = \{ w, w' : \begin{smallmatrix} w \in I_j, \\ w' \in D(z,\varepsilon) \end{smallmatrix} \}$ is open, convex and contained by $\mathcal{U} \setminus \{z\}$.
- $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holomorphic in C_j (so $g(\zeta) \in \mathcal{H}(C_j)$).

So Cauchy's Theorem for star-like applies (since C_j convex $\Rightarrow$ star-like).

3 Let $\Delta = \partial I_1 \dot{+} \ldots \dot{+} \partial I_m$ be a cycle. Then if $z \in \bigcup_{j=1}^{m} I_j^{\circ}$, by (1) and (2) we have

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{f(\zeta)}{\zeta - z} d\zeta$$ by construction.

Let's construct Γ.

3.1 Algorithm for path construction from grid

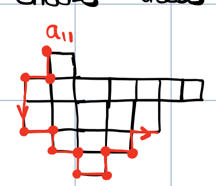
Let $F = \bigcup_{j=1}^{m} I_j$ (compact). Let a_{11} denote the $\underbrace{\text{top-leftmost}}$ corner of ∂F.

exists since compactness $\Rightarrow$ boundedness

we are essentially collecting nodes of our graph connected to node a_{11}.

Let $a_{12}, a_{13}, \ldots, a_{1p_1}$ denote the other corners of ∂F that are "met" by starting at a_{11}.

We'll choose these by traveling counterclockwise from a_{11}, i.e. $[a_{1k}, a_{1k+1}] = \partial F \cap \partial I_k \neq \emptyset$ such that we are oriented correctly.



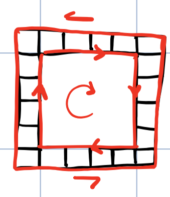
When a corner case is met ( ) then always turn right and continue.

Let $\gamma_1 = [a_{11}, a_{12}] \dot{+} [a_{12}, a_{13}] \dot{+} \cdots \dot{+} [a_{1p_1-1}, a_{1p_1}] \dot{+} [a_{1p_1}, a_{11}]$.

Notice this is finite since our graph is finite.

Either we are done, or we repeat by defining a_{21} and following the same process on $\underbrace{\partial F \setminus \gamma_1^*}_{\text{what remains}}$.

Notice we may be forced into clockwise paths on the inside :



but it shouldn't matter too much.

Continue to form closed polygonal curves until we are done (when we've covered ∂F).

Let $\Gamma = \gamma_1 \dot{+} \gamma_2 \dot{+} \ldots \dot{+} \gamma_n$ for these constructed curves, so Γ is a contour cycle.

3.2 $\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_\Delta \frac{f(\zeta)}{\zeta - z} d\zeta$ when $z \in \bigcup_{j=1}^m I_j^\circ$ (union of interiors).

This is true since we have that segments of $\partial I_j \subseteq \Delta$ have parts that cancel by reverse path.

Therefore, we are left with Γ, so the integrals are equal. Γ only "accepts" exterior segments.

4 Conclusion of Proof. How do we say things about $K \subset U$?

We have $f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ for $z \in \bigcup_{j=1}^m I_j^\circ$ by (3) and (3.4).

However :

- $\overline{\bigcup_{j=1}^m I_j^\circ} = F = \bigcup_{j=1}^m I_j \supset K$; ← closure of union

- both $z \mapsto f(z)$ and $z \mapsto \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ are continuous on $U \setminus \Gamma^*$. (for the second function, recall the proof of the winding number)

Thus for $z \in F^\circ$ (where $K \subset F^\circ$ by construction), we can find a sequence $(z_n)_{n=1}^\infty \subset \bigcup_{j=1}^m I_j^\circ$ with $z = \lim_{n\to\infty} z_n$.

$$\Rightarrow f(z) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z_n} d\zeta$$

↓ commute by continuity of integral

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Notice that for $z \in F^\circ \supset K$, we have

$$W_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - z} d\zeta = 1.$$

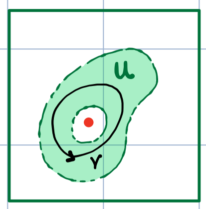
Corollary: If $U \subseteq \mathbb{C}$ is open, and $K \subset U$ ($\neq \emptyset$) is compact, then for $z \in K$, $\exists \gamma$ closed contour in U such that $W_\gamma(z) \neq 0$.

Proof: Construct Γ as described in the above theorem. Recall $z \in K \Rightarrow W_\Gamma(z) = \sum_{k=1}^m W_{\gamma_k}(z) = 1$,

thus $W_{\gamma_k}(z) \neq 0$ for some $k \in \{1, \ldots, m\}$. $\square$

Let $U \subseteq \mathbb{C}$ be open and γ in U be a closed contour. We call γ special in U if $W_\gamma(z) = 0$, for all $z \in \mathbb{C} \setminus U$.

Here's a nonexample:



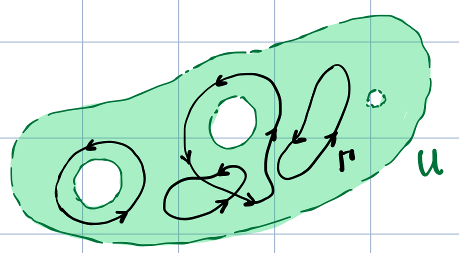
Notice that for • we have $\bullet \in \mathbb{C} \setminus U$ but $W_\gamma(\bullet) \neq 0$.

Remark: If U is star-like (eg. convex), then any closed contour in U is null-homotopic $\Rightarrow$ γ is special for U.

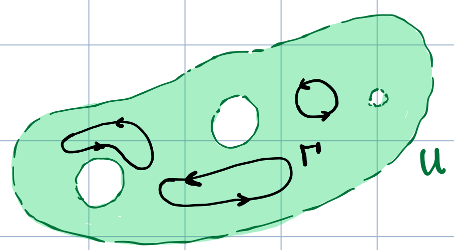
Therefore we are further generalizing away from star-like sets.

Similarly, let Γ be a cycle in U. We call Γ special in U if $W_\Gamma(z) = 0$, for all $z \in \mathbb{C} \setminus U$.

Notice that "specialness" is more flexible for cycles.



here, the cycle Γ is NOT special for U.



the cycle Γ here is special for U.

Here are some resulting properties:

Let Γ be a cycle that is special for U, where $U \subseteq \mathbb{C}$ is open. Let $U_\Gamma = \overset{\text{pullback}}{W_\Gamma^{-1}}(\{0\})$ which is open in $\mathbb{C}$.

(since $W_\Gamma : \mathbb{C} \setminus \Gamma^* \to \mathbb{Z}$ is continuous)

Notice U_Γ contains the unbounded connected component.

Thus $K_\Gamma := \mathbb{C} \setminus U_\Gamma$ is closed and bounded $\Rightarrow$ K_Γ compact by Heine-Borel in $\mathbb{R}^2$ (since $\mathbb{C} \underset{T}{\cong} \mathbb{R}^2$ homeomorphic).

Therefore we have $\underbrace{\mathbb{C} \setminus U \subset U_\Gamma}_{\text{(specialness)}} \Rightarrow K_\Gamma \subset U$. This is very useful.

"support" of Γ
≈"interior"

Theorem (Fubini's): Let $\gamma, \delta : [0,1] \to \mathbb{C}$ be contours and $g : \gamma^* \times \delta^* \to \mathbb{C}$ be continuous.

Then $\int_\delta \int_\gamma g(z, \zeta)\, dz\, d\zeta = \int_\gamma \int_\delta g(z,\zeta)\, d\zeta$. In particular, integration commutes.

Proof: We won't prove this, but here's the main idea. Mild abuse of notation.

$$\text{Consider } \int_\delta \int_\gamma g(z, \zeta)\, dz\, d\zeta = \int_0^1 \left(\int_0^1 g(\gamma(t), \delta(s))\, \gamma'(t)\, dt \right) \delta'(s)\, ds$$

$$= \int_0^1 \left(\int_0^1 g(\gamma(t), \delta(s))\, \gamma'(t)\, \delta'(s)\, dt \right) ds$$

$$\overset{\text{key step to prove}}{=} \int_0^1 \left(\int_0^1 g(\gamma(t), \delta(s))\, \gamma'(t)\, \delta'(s)\, ds \right) dt$$

$$= \int_0^1 \left(\int_0^1 g(\gamma(t), \delta(s))\, \delta'(s)\, ds \right) \gamma'(t)\, dt. \qquad \square$$

Remark: Should be covered in MATH 247 for $\mathbb{R}^n$.

Theorem (Cauchy, for "special" cycles):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open and Δ be a cycle in $\mathcal{U}$ such that it is "special" for $\mathcal{U}$. Then if $f \in \mathcal{H}(\mathcal{U})$, then $\int_\Delta f = 0$.

Proof: Let K_Δ be the supporting set ("interior") for Δ (i.e. $K_\Delta = \mathbb{C} \setminus \mathcal{U}_\Delta$ where $\mathcal{U}_\Delta = W_\Delta^{-1}(\{0\})$).

Since Δ is special for $\mathcal{U}$, we have that $K_\Delta \subset \mathcal{U}$.

By Cauchy's Integral Formula (on compact sets), there exists a cycle Γ in $\mathcal{U} \setminus K_\Delta$ such that

for $f \in \mathcal{H}(\mathcal{U})$ and $z \in K_\Delta$, $f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$. Recall $\Delta^* \subset K_\Delta$ by construction.

Then $\int_\Delta f \overset{(\text{since } \Delta^* \subset K)}{=} \int_\Delta \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta\, dz$. Let Γ be comprised of closed contours $\Gamma = \gamma_1 \dot{+} \dots \dot{+} \gamma_n$ and $\Delta = \delta_1 \dot{+} \dots \dot{+} \delta_m$

$$= \frac{1}{2\pi i} \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{\delta_j} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta\, dz \overset{\text{Fubini's}}{=} \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{j=1}^{m} \int_{\gamma_k} \int_{\delta_j} \frac{\overset{\text{constant in } z}{f(\zeta)}}{\zeta - z} dz\, d\zeta$$

$$= \int_\Gamma f(\zeta) \underbrace{\left(\frac{1}{2\pi i} \int_\Delta \frac{1}{\zeta - z} dz \right)}_{W_\Delta(\zeta) = 0} d\zeta = 0$$ where $\zeta \in \Gamma^* \subseteq \mathcal{U} \setminus K_\Delta = \mathcal{U} \cap \mathcal{U}_\Delta$.

$\square$

Theorem (Cauchy Integral Formula, "special" cycles):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open, and let Γ be a contour cycle in $\mathcal{U}$ which is "special" for $\mathcal{U}$.

Then for $f \in \mathcal{H}(\mathcal{U})$ and $z \in \mathcal{U} \setminus \Gamma^*$, we have $\underbrace{W_\Gamma(z)}_{\text{accounts for both the zero and the nonzero cases.}} f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$.

Remark: There are very few assumptions on $\mathcal{U}$; they are transported onto Γ instead.

Proof: Fix $z \in \mathcal{U} \setminus \Gamma^*$. Define $g : \mathcal{U} \to \mathbb{C}$ by $g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{for } \zeta \neq z; \\ f'(\zeta) & \text{for } \zeta = z. \end{cases}$

Then $g \in \mathcal{H}(\mathcal{U} \setminus \{z\})$ and continuous at z $\overset{CGM}{\Longrightarrow}$ we extend our function, so $g \in \mathcal{H}(\mathcal{U})$. ← recall that CGM relies on local star-like properties.

By Theorem (Cauchy, for "special" cycles), we have

$$0 = \int_\Gamma g = \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \overset{\text{constant}}{f(z)} \underbrace{\int_\Gamma \frac{1}{\zeta - z} d\zeta}_{2\pi i\, W_\Gamma(z)}$$

$$\Rightarrow 2\pi i\, W_\Gamma(z) f(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \quad \square$$

Let $\mathcal{U} \subseteq \mathbb{C}$ be open. Let $\{z_1, \dots, z_n\}$ be a set of isolated (non-cluster) points in $\mathcal{U}$.

We call a function $f : \mathcal{U} \to \mathbb{C}$ meromorphic if $f \in \mathcal{H}(\mathcal{U} \setminus \{z_1, \dots, z_n\})$ and each $z_j \in \{z_1, \dots, z_n\}$ is a pole of f.

Remark: Meromorphic functions can have at most countably many poles.

Theorem (Residue Theorem, "special" cycles):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open, and let $P \subset \mathcal{U}$ be a set with no cluster points in $\mathcal{U}$. Let $f \in \mathcal{H}(\underbrace{\mathcal{U} \setminus P}_{\text{open}})$ be meromorphic with poles P. Then if Γ is a contour cycle in $\mathcal{U} \setminus P$ and is special for $\mathcal{U}$, then

$\int_\Gamma f = 2\pi i \sum_{z \in P} W_\Gamma(z) \operatorname{Res}_z(f)$, and this sum is finite.

Proof: Recall $\mathcal{U}_\Gamma = W_\Gamma^{-1}(\{0\})$ is open and unbounded,

$K_\Gamma = \mathbb{C} \setminus \mathcal{U}_\Gamma$ is compact, and $\Gamma^* \subseteq K_\Gamma \overset{\text{by specialness}}{\subset} \mathcal{U}$.

Then by assumption on P, we have $P \cap K_\Gamma$ finite (otherwise, cluster point in $P \cap K_\Gamma \Rightarrow$ cluster point in $\mathcal{U}$, but P has no cluster points in $\mathcal{U}$).

Denote $P \cap K_\Gamma = \{z_1, \dots, z_n\}$ a set of distinct points. We have finitely many poles to deal with.

For $j = 1, \dots, n$, if $r_j > 0$, then $D'(z_j, r_j) \subset \mathcal{U} \setminus P$, so by Theorem (Order of Poles), we have

$$f(z) = \underbrace{\frac{a_{j-m_j}}{(z-z_j)^{m_j}} + \dots + \frac{a_{j-1}}{(z-z_j)}}_{\text{principal part } p_j(z) \text{ at } z_j} + g_j(z), \quad \text{where } z \in D'(z_j, r_j) \text{ and } g_j \in \mathcal{H}(D'(z_j, r_j)) \text{ is a power series.}$$

Notice $p_j \in \mathcal{H}(\mathcal{U} \setminus \{z_j\})$.

Define $h(z) = f(z) - \sum_{j=1}^n p_j(z)$ for $z \in \mathcal{U} \setminus \{z_1, \dots, z_n\}$. Then notice $\lim_{z \to z_j} h(z)$ exists and equals $g_j(z_j) - \sum_{\substack{k=1 \\ k \neq j}}^n p_k(z_j)$.

Thus, h extends (by $\underset{n \text{ times}}{\text{CGM}}$) to a holomorphic function on $\mathcal{U}$.

By Theorem (Cauchy, for "special" cycles), we have $0 = \int_\Gamma h = \int_\Gamma f - \sum_{j=1}^n \int_\Gamma p_j$, so we want to understand $\int_\Gamma p_j$.

Let $j = 1, \dots, n$. Consider $\int_\Gamma p_j$. We have $\int_\Gamma p_j = \sum_{k=1}^{m_j} \int_\Gamma \frac{a_{j-k}}{(z-z_j)^k} dz$. If $k \geq 2$, then the function admits a primitive, so all those $\int_\Gamma (\dots) dz = 0$.

$$\overset{\substack{(k=1 \text{ term}) \\ \text{survives}}}{=} \int_\Gamma \frac{a_{j-1}}{(z-z_j)} dz = \underbrace{a_{j-1}}_{\operatorname{Res}_{z_j}(f)} \cdot \underbrace{\int_\Gamma \frac{1}{z-z_j} dz}_{2\pi i W_\Gamma(z_j)}$$

$$\Rightarrow \int_\Gamma f = 2\pi i \sum_{j=1}^n W_\Gamma(z_j) \operatorname{Res}_{z_j}(f). \quad \square$$

Remark: In practice, we will usually use Jordan curves, and not special cycles.

Let $0 \leq r < R \leq \infty$ and $a \in \mathbb{C}$. An open annulus is given by $A(a, r, R) := \{z \in \mathbb{C} : r < |z-a| < R\}$

$= D(a, R) \setminus \bar{D}(a, r)$.

If $r = 0$ and $R = \infty$, then $A(a, 0, R) = D'(a, R) = \mathbb{C} \setminus \{0\}$.

The annulus is the natural home of the Laurent series.

A Laurent series is a function $f: A(a,r,R) \longrightarrow \mathbb{C}$ where $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$ which converges on $A(a,r,R)$.

$$= \underbrace{\sum_{k=1}^{\infty} \frac{c_{-k}}{(z-a)^k}}_{\text{principal part}} + \underbrace{\sum_{k=0}^{\infty} c_k (z-a)^k}_{\text{residual part}}$$

(we won't be using this terminology much)

Note that $R \leq \{\text{radius of convergence of } \sum_{k=0}^{\infty} c_k(z-a)^k \text{ (the residual part)}\}$,

and

$\frac{1}{r} \leq \{\text{radius of convergence of } \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-a)^k} \text{ (the principal part)}\}$,

So the series converges uniformly on compact subsets of the annulus.

Proposition: Let $U \subseteq \mathbb{C}$ be open. If $f \in \mathcal{H}(U)$, then f admits a Laurent series on U.

Theorem (Laurent Series):

Let $a \in \mathbb{C}$ and $0 \leq r < R \leq \infty$. If $f \in \mathcal{H}(A(a,r,R))$ then for $z \in A(a,r,R)$, we have that f admits a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$ where if $r < r' < |z-a| < R' < R$, and $\gamma_{r'} := a + r'e^{it}$ for $t \in [0, 2\pi]$,

$\gamma_{R'} := a + R'e^{it}$

then
$$c_k = \begin{cases} \frac{1}{2\pi i} \int_{\gamma_{r'}} f(\zeta)(\zeta-a)^{|k|-1} \, d\zeta & \text{if } k < 0; \\ \frac{1}{2\pi i} \int_{\gamma_{R'}} \frac{f(\zeta)}{(\zeta-a)^{k+1}} \, d\zeta & \text{if } k \geq 0. \end{cases}$$

Proof: Incredibly involved computationally. May include later.

We generalize further by characterizing connectedness.

Let $U \subseteq \mathbb{C}$ be open. A hole for U is a non-empty set $K \subseteq \mathbb{C} \setminus U$ such that:

- K is compact;
- $U \cup K$ is open (i.e. $\partial K \subseteq \partial U$).

- $\{0\} = K$ is a hole for $\mathbb{C} \setminus \{0\}$.
- If $0 < r < R < \infty$ and $a \in \mathbb{C}$, then $\bar{D}(a, r)$ is a hole for $A(a, r, R)$.

Lemma (Holes and Contours): Let $U \subseteq \mathbb{C}$ be open. Then the following are equivalent:

(i) U admits a hole K;

(ii) There is $z \in \mathbb{C} \setminus U$ and a closed contour γ in U such that $W_\gamma(z) \neq 0$ (i.e. $\underset{\text{closed contour}}{\exists \gamma}$ that is NOT special for U).

Proof: Exercise.

Theorem (Simple Connectivity): Let $U \subseteq \mathbb{C}$ be open. Then the following are equivalent:

Homotopy (0) Any closed contour γ in U is null-homotopic.

Specialty (i) Any closed contour γ in U is special for U.

Holes (ii) U admits no hole.

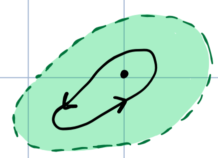
Cauchy's (iii) For any closed contour γ in U, and any $f \in \mathcal{H}(U)$, we have $\int_\gamma f = 0$.

Primitives (iv) Any $f \in \mathcal{H}(U)$ admits a primitive on U.

Logarithm (v) If $f \in \mathcal{H}(U)$ and $0 \notin f(U)$, then there exists $g \in \mathcal{H}(U)$ such that $f = \exp(g)$.

Remark: Showing any of $(i\text{-}v) \Rightarrow (0)$ requires the Riemann Mapping Theorem.

Proof: $(0) \Rightarrow (i)$ Let γ be a closed contour in U, so γ is null-homotopic.



Notice that $g(\zeta) = \frac{1}{\zeta - z} \in \mathcal{H}(U)$ if $z \in \mathbb{C} \setminus U$. Hence we have

$$W_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \underset{\text{Deformation Theorem}}{=} \frac{1}{2\pi i} \int_{[z_0]} \frac{d\zeta}{\zeta - z} = 0$$

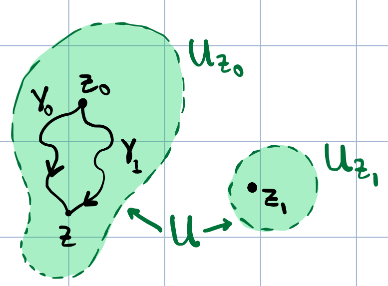
by the Fundamental Theorem of Calculus (line integrals). $\triangle$

$(i) \Rightarrow (ii)$ Contrapositive of above lemma. $\triangle$

$(ii) \Rightarrow (iii)$ Theorem (Cauchy, for "special" cycles). $\triangle$

$(iii) \Rightarrow (iv)$ Notice if $z_0 \in U$ and U_{z_0} is the connected component of z_0, then for $z \in U_{z_0}$ and any contours $\gamma_0, \gamma_1 : [0,1] \to U_{z_0} \subseteq U$ such that $\gamma_0(0) = z_0 = \gamma_1(0)$ and $\gamma_0(1) = z = \gamma_1(1)$, we have $\gamma_0 \cdot \gamma_1^-$ is a closed contour, so for $f \in \mathcal{H}(U)$,

$$\int_{\gamma_0} f - \int_{\gamma_1} f = \int_{\gamma_0} f + \int_{\gamma_1^-} f = \int_{\gamma_0 \cdot \gamma_1^-} f \underset{(iii)}{=} 0.$$



Thus we define $F: U_{z_0} \to \mathbb{C}$ by $F(z) = \int_\gamma f$ where γ is any contour connecting z_0 and z.

By (proof of) Morera's Theorem, $F'(z) = f(z)$.

We similarly define F on each component of U (i.e. choose representatives $z_1, z_2, \dots$ in different components and build F).

at most countably many since $\mathbb{Q}[i] \subset \mathbb{C}$ dense and $|\mathbb{Q}[i]| = \aleph_0$ countable

$\triangle$

(iv) $\Rightarrow$ (v) Since $0 \notin f(U)$, then $\frac{1}{f} \in \mathcal{H}(U)$ by the chain rule. Therefore $f' \in \mathcal{H}(U)$.

Therefore $\frac{f'}{f} \in \mathcal{H}(U)$ and by (iv), there is a primitive F for $\frac{f'}{f}$ on U.

Let $z_1, z_2, \dots$ be representatives of the components of U. For each z_j, let $c_j \in \mathbb{C}$ be such that

why can we do this? $\exp(c_j) = f(z_j) \exp(-F(z_j))$ so $\exp(F(z) + c_j) = f(z)$, for $z \in U_{z_j}$.

Let $g: U \to \mathbb{C}$ be given by $g(z_j) = F(z_j) + c_j$ for $z \in U_{z_j}$.

Define $h = \frac{f}{\exp(g)} = f \exp(-g)$. Then $h' = f' \exp(-g) - \left(\frac{f'}{f}\right) f \exp(-g) = 0$, so h constant on each component.

By some computation, $h = 1$, so $\exp(g) = f$. $\triangle$

(v) $\Rightarrow$ (i) Fix $z_0 \in \mathbb{C} \setminus U$, so $f(z) = z - z_0$ satisfies $0 \notin f(U)$. Hence by above proof,

$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0}$ admits a primitive g on U.

Now if γ is any closed contour in U, we have $W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = 0$ by FTC (line integrals).

since beginning and endpoint meet at the same primitive function

$\triangle$ $\square$

<u>Remarks</u>: If we have (v), then for any $f \in \mathcal{H}(U)$ with $0 \notin f(U)$, with g in (v) we have

$h = \exp(\frac{1}{2} g)$ satisfying $h^2 = f$ implying the existence of square roots.

Logarithm and roots are intimately connected to one another.

Suppose U is connected. The existence of square roots is the fundamental tool to the Riemann Mapping Theorem.

Theorem (Riemann Mapping): Let $U \subsetneq \mathbb{C}$ be open and simply connected. Then there exists a biholomorphic function (a bijective holomorphic function with holomorphic inverse) $f : U \to \mathbb{D}$ where $\mathbb{D} = D(0,1)$.

Riemann mapping

Remark: • f being biholomorphic $\Rightarrow$ f conformal map. Therefore f preserves local shapes up to rotation and scaling.

Poincaré: The Riemann mapping f is unique up to rotation and recentering. If $z_0 \in U$ and ϕ is an arbitrary angle, then $\exists!\, f$ such that $f(z_0) = 0$ and $f'(z_0) = \phi$. Consequence of the Schwarz Lemma.

The Holomorphic Automorphism Group

If $U \subseteq \mathbb{C}$ is open and connected, then let $\mathrm{Aut}(U)$ denote the collection of biholomorphic functions

$\mathrm{Aut}(U) = \{ f : U \to U \mid f \text{ biholomorphic} \}$. Then $(\mathrm{Aut}(U), \circ)$ forms a group.

Explore this more!

Theorem (Argument Principle): Let $U \subseteq \mathbb{C}$ be open, and $P \subset U$ be a set with no cluster points in U and let $f \in \mathcal{H}(U \setminus P)$. Let γ be a Jordan curve in $U \setminus (P \cup Z(f))$ where $Z(f) = \{ z \in \mathbb{C} : f(z) = 0 \}$, such that γ is special for U. Let $U_1 = W_\gamma^{-1}(\{1\}) \subset U$ by specialty.

Then $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = NZ_\gamma(f) - NP_\gamma(f)$ where $NZ_\gamma(f)$ = number of zeros of f counting multiplicity in U_1,

$NP_\gamma(f)$ = number of poles of f counting order in U_1.

Remark: f is meromorphic on U.

Proof: By Cauchy's Residue Theorem, $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \sum_{j=1}^{n} \mathrm{Res}_{z_j}\left(\frac{f'}{f}\right)$ where $\{z_1, \dots, z_n\} = \underbrace{P \cap Z(f) \cap U_1}_{\text{poles for } \frac{f'}{f}}$.

Remark: $\log(gh) = \log(g) + \log(h)$ ← possible nonsense

$\log(gh) = \dfrac{(gh)'}{gh}$ ← not nonsense

So we have the Logarithmic derivative $\dfrac{(gh)'}{gh} = \dfrac{gh' + g'h}{gh} = \dfrac{g'}{g} + \dfrac{h'}{h}$.

When we consider zeros, we have $f(z_j) = 0$, so $\exists m_j \in \mathbb{N}$: $f(z) = (z - z_j)^{m_j} g(z)$ where $g \in \mathcal{H}(U)$ and $g(z_j) = 0$.

$$\Rightarrow \frac{f'}{f} = \frac{m_j (z - z_j)^{m_j - 1}}{(z - z_j)^{m_j}} + \frac{g'(z)}{g(z)}$$

$\Rightarrow \mathrm{Res}_{z_j}\left(\frac{f'}{f}\right) = -m_j$ so we add all of these to get our result. $\square$

Theorem (Rouché): Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected, let $f \in \mathcal{H}(\mathcal{U})$ be non-constant. If $g \in \mathcal{H}(\mathcal{U})$ and γ is a Jordan curve in $\mathcal{U}$ such that $|f(z)-g(z)| < |f(z)|$ for $z \in \gamma^*$, then $NZ_\gamma(f) = NZ_\gamma(g)$ where $NZ_\gamma() = \#$ of zeros with multiplicity in $\mathcal{U}_1 = W_\gamma^{-1}(\{1\})$.

Remark: Often used in analytic number theory. Notice f, g have no poles in $\mathcal{U}$.

Proof: Note that for $z \in \gamma^*$, we have $|f(z)| > 0$ so γ is in $\mathcal{U} \setminus Z(f)$.

Thus $f \circ \gamma$ is a curve in $\mathbb{C} \setminus \mathcal{U}$.

By the Argument Principle:

$$NZ_\gamma(f) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t))} = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{d\zeta}{\zeta}.$$

Notice if $z \in \gamma^* \cap Z(g)$ then we would have $|f(z)-g(z)| = |f(z)-0| = |f(z)|$, violating assumptions.

Therefore, as above, $g \circ \gamma$ is a closed contour in $\mathbb{C} \setminus \{0\}$.

Hence, $$NZ_\gamma(g) = \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} = \frac{1}{2\pi i}\int_{g\circ\gamma} \frac{d\zeta}{\zeta}.$$

It suffices to show that $f\circ\gamma$ and $g\circ\gamma$ are contour homotopic as closed curves in $\mathbb{C}\setminus\{0\}$.

Define a homotopy defined by the intermediate paths

$\gamma_s(t) = (1-s) f\circ\gamma(t) + s\, g\circ\gamma(t)$ where $s \in [0,1]$ so we have a convex combination. Notice $\gamma_s(t)$ is a contour.

It suffices to show that $\gamma_s^* \subset \mathbb{C}\setminus\{0\}$. For $t \in [0,1]$, we have

$$|\gamma_s(t)| = |f(\gamma(t)) + s[g(\gamma(t)) - f(\gamma(t))]| \geq |f(\gamma(t))| - s\underbrace{|g(\gamma(t)) - f(\gamma(t))|}_{<|g(\gamma(t))-f(\gamma(t))|} > |f(\gamma(t))| - s|f(\gamma(t))| \geq 0.$$

so $|\gamma_s(t)| > 0 \Rightarrow 0 \notin \gamma_s^*$ so $\gamma_s^* \subset \mathbb{C}\setminus\{0\}$ defines a closed contour $\Rightarrow f\circ\gamma$ and $g\circ\gamma$ homotopic.

By homotopy equivalence, we have

$$NZ_\gamma(g) = \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} = \frac{1}{2\pi i}\int_{g\circ\gamma}\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = NZ_\gamma(f). \qquad \square$$

Theorem (Fundamental Theorem of Algebra): Let $p(z) \in \mathbb{C}[z]$ with $\deg(p(z)) = n$. Then $|NZ(p)| = n$ (i.e. $p(z)$ has n zeros counting multiplicity in $\mathbb{C}$).

"Proof": (Sketch) The idea is to choose a large enough $R > 0$ such that $R^n > \sum_{k=0}^{n} |a_k| R^k = |a_0| + |a_1|R^1 + \ldots + |a_n|R^n$.

Let $\gamma(t) = Re^{it}$ for $t \in [0, 2\pi]$.

Then check that $|p(z) - z^n| < |z^n|$ for $z \in C(0,R) = \gamma^*$. Therefore $NZ(p) = NZ(z \mapsto z^n)$ so $p(z)$ has at least n zeros. Then check that $p(z)$ has no more than n zeros. $\square$

Theorem (Open Mapping in $\mathbb{C}$-analysis):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected, and $f \in \mathcal{H}(\mathcal{U})$ be non-constant. Then $f(\mathcal{U}) \subseteq \mathbb{C}$ is open.

Remarks: If $V \subseteq \mathcal{U}$ is open, then $f|_V \in \mathcal{H}(V)$ so $f(V)$ is also open, i.e. f is an open map.

Open Mapping (functional analysis) / Banach-Schauder Theorem:

Let X, Y be Banach spaces. Let $\varphi : X \to Y$ be a surjective continuous linear transformation. Then φ is an open map.

Open Mapping (topological groups):

Let G, H be locally compact Hausdorff groups. Let $\varphi : G \to H$ be a surjective continuous homomorphism.

Then φ is an open map if G is σ-compact.

Proof: Let $w_0 \in f(\mathcal{U})$. Since $\mathbb{C}$ is Euclidean under the standard topology, open sets are characterized by the existence of open neighbourhoods. We will find $\varepsilon > 0$ such that $D(w_0, \varepsilon) \subset f(\mathcal{U})$, to show openness.

$w_0 \in f(\mathcal{U}) \Rightarrow w_0 = f(z_0)$ for some $z_0 \in \mathcal{U}$. Let $g(z) = f(z) - w_0 \in \mathcal{H}(\mathcal{U})$ so $g(z_0) = 0$.

By Theorem (Zero Sets), either $g \equiv 0$ or $Z_{\mathcal{U}}(g)$ admits no cluster points in $\mathcal{U}$, but g non-constant

so $Z_{\mathcal{U}}(g)$ admits no cluster points in $\mathcal{U} \Rightarrow \exists r > 0$ such that $g(z) \neq 0$, $\forall z \in D(z_0, r)$.

Thus if $\delta \in (0, r)$, then let $m = \min\limits_{z \in C(z_0, \delta)} |f(z) - w_0| > 0$. Find $\varepsilon \in (0, m)$.

Now we have to show that $D(w_0, \varepsilon) \subseteq f(\mathcal{U})$.

Fix $w \in D(w_0, \varepsilon)$ and define $h(z) = f(z) - w$. Then for fixed δ and $z \in C(z_0, \delta)$, we have

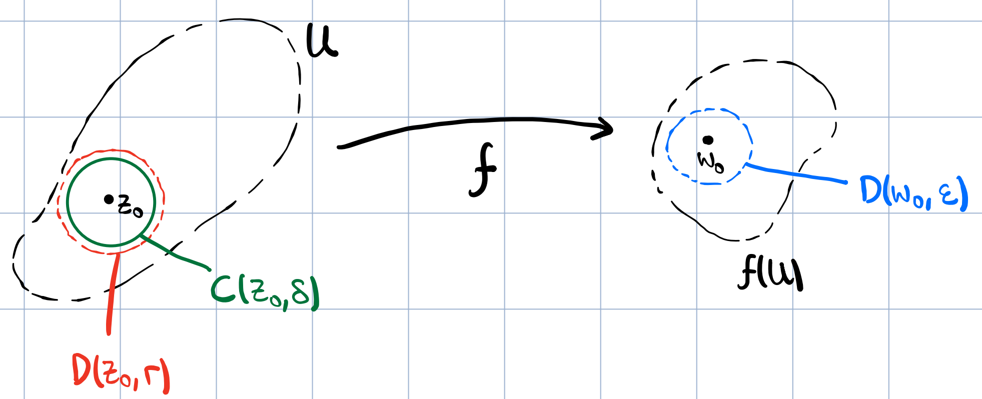
$$|h(z) - g(z)| = |w - w_0| < \varepsilon < m \underset{\text{by definition}}{\leq} |g(z_0)|.$$

By Rouché's Theorem, h has the same number of zeros in $D(z_0, \delta)$ as g, namely one.

$\Rightarrow \exists z \in D(z_0, \delta) \subseteq D(z_0, r) \subseteq \mathcal{U}$ such that $0 = h(z) = f(z) - w$, i.e. $w = f(z)$.

But since $w \in D(w_0, \varepsilon)$ was arbitrary, we see this holds for any $w \in D(w_0, \varepsilon)$, so

$D(w_0, \varepsilon) \subseteq f(D(z_0, \delta)) \subseteq f(\mathcal{U})$. $\square$



Theorem (Maximum Modulus Principle):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(\mathcal{U})$ be non-constant. If $V \overset{\neq \emptyset}{\subseteq} \mathcal{U}$ open such that $\bar{V} \subseteq \mathcal{U}$ with $\bar{V}$ compact,

then for $z_0 \in V$, we have $|f(z_0)| < \max_{z \in \partial V} |f(z)|$, i.e. the maximum modulus value occurs on the boundary.

Proof: Assign $M = \max_{z \in \bar{V}} |f(z)| > 0$ since $V \neq \emptyset$ and f non-constant. Notice by compactness, M is well-defined.

If there was $z_0 \in V$ such that $f(z) = M$, then we find $\varepsilon > 0$ such that $D(f(z_0), \varepsilon) \subseteq f(V)$.

But $\left| f(z_0) + \frac{f(z_0)}{|f(z_0)|} \frac{\varepsilon}{2} \right| = |f(z_0)| \cdot \left| 1 + \underbrace{\frac{\varepsilon}{2|f(z_0)|}}_{>0} \right|$

$$\Rightarrow M = |f(z_0)| < |f(z_0)| \cdot \left| 1 + \frac{\varepsilon}{2|f(z_0)|} \right| \quad \text{and} \quad \left| f(z_0) + \frac{f(z_0)}{|f(z_0)|} \frac{\varepsilon}{2} - f(z_0) \right| < \varepsilon$$

so for some $z \in V$, we have $f(z) = f(z_0) + \frac{f(z_0)}{|f(z_0)|} \frac{\varepsilon}{2}$ so $|f(z)| > M$. $\lightning$ $\square$

Theorem (Injectivity): Let $\mathcal{U} \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(\mathcal{U})$ be injective on $\mathcal{U}$. Then the following hold:

1) $f'(\mathcal{U}) \neq 0$ for all $z \in \mathcal{U}$ (i.e. $f'(z)$ is non-vanishing on $\mathcal{U}$);

2) $f^{-1} \in \mathcal{H}(f(\mathcal{U}))$ where $f(\mathcal{U})$ is open since f is non-constant.

Proof (1): Suppose for a contradiction there is $z_0 \in \mathcal{U}$ with $f'(z_0) = 0$ (so f' is vanishing on $\mathcal{U}$).

Set-up: Find $R > 0$ such that $\bar{D}(z_0, R) \subset \mathcal{U}$ and $\underline{f'(z_0) \neq 0}$ on $D'(z_0, R)$.

can be done since f non-constant
$\Rightarrow f' \neq 0$ identically and we know
$f'(z_0)$ is an isolated zero (no cluster points).

By Corollary to Theorem (Cauchy's Integral Formula), for $z \in D(z_0, R)$, we have

$$f(z) = f(z_0) + \underbrace{\cancel{c_1}}_{0 \text{ since } f'(z_0)=0}(z - z_0) + \sum_{k=2}^{\infty} c_k (z - z_0)^k = f(z_0) + \sum_{k=n}^{\infty} c_n (z - z_0)^k$$

where $n \geq 2$ such that c_n is the first nonzero term.

We can rewrite: $f(z) = f(z_0) + c_n (z - z_0)^n + G(z)$ where $G(z) = (z - z_0)^{n+1} g(z)$ with $g \in \mathcal{H}(D(z_0, R))$ and $g(z_0) \neq 0$.

Now find $r \in (0, R)$ so that $|c_n| > rM$ where $M = \max_{z \in \bar{D}(z_0, R)} |g(z)|$ $\left(= \max_{z \in C(z_0, R)} |g(z)| \right)$ by Maximum Modulus Principle.

Notice that $|c_n| r^n > r^{n+1} M$. Then let $w \in \mathbb{C} \setminus \{0\}$ be small enough so that $|c_n| r^n - |w| > r^{n+1} M$.

Define $H(z) = f(z) - f(z_0) - w$. Notice that $H(z) = c_n (z - z_0)^n - w + G(z)$.

Now we proceed.

where M was defined

For $z \in C(z_0, r) \subset \bar{D}(z_0, R)$, we have

$$|F(z) - H(z)| = |F(z) - (f(z) - f(z_0) - w)| = |G(z)| \text{ by definition;}$$

$$= |z - z_0|^{n+1} |g(z)|.$$

But since $g \in C(z_0, r)$, we have $|z-z_0|^{n+1}|g(z)| \leq r^{n+1}M < |c_n|r^n - |w| \leq |c_n(z-z_0)^n - w| = |F(z)|$

$\Rightarrow$ $|G(z)| < |F(z)|$ on $C(z_0, r)$. Now we can use Rouché's Theorem:

Notice that F has at least $n \geq 2$ zeros (with multiplicity) on $D(z_0, r)$, since

$$0 = c_n(z-z_0)^n - w \Rightarrow \frac{w}{c_n} = (z-z_0)^n \quad \text{where} \quad \left|\frac{w}{c_n}\right| < r^2 \quad \text{(exercise: check this).}$$

By Rouché's, H has at least two zeros in $D(z_0, r)$. ← why does H have this property?

We have $H(z_0) = f(z_0) - f(z_0) - w = -w \neq 0$ and $H'(z) = f'(z) \neq 0$ on $D(z_0, r) \subset D(z_0, R)$

$\Rightarrow$ the zeros of H are distinct, since each has order one (by Theorem (Order of Zero)).

Since H is just a linear translation, the equation $f(z) = f(z_0) + w$ has at least two solutions,

contradicting injectivity. $\lightning$ $\square$

(2): By Theorem (Open Mapping), if $V \subseteq \mathcal{U}$ is open, then $f(V)$ is open.

We have $V = f^{-1}(f(V)) = \{w \in \mathbb{C} : f^{-1}(w) \in f(\mathcal{U})\}$ so we see that f^{-1} is continuous (pullback of open set stays open).

If $w, w_0 \in f(\mathcal{U})$ so $w = f(z)$ and $w_0 = f(z_0)$ for $z, z_0 \in \mathcal{U}$, and $z - z_0 = f^{-1}(w) - f^{-1}(w_0)$,

then $\dfrac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \dfrac{z - z_0}{f(z) - f(z_0)}$ so by continuity, as $w \to w_0$, we have $f(z) \to f(z_0)$ so

$$\lim_{w \to w_0}\left(\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0}\right) = \lim_{z \to z_0}\left(\frac{z - z_0}{f(z) - f(z_0)}\right) = \frac{1}{f'(z_0)} \quad \text{so} \quad f^{-1\prime}(z) = \frac{1}{f'(z)} \Rightarrow f^{-1} \in \mathcal{H}(f(\mathcal{U})). \quad \square$$

If $\mathcal{U} \subseteq \mathbb{C}$ is open and connected, a branch of logarithm is a function $L : \mathcal{U} \to \mathbb{C}$ such that $\exp(L(z)) = z$ for $z \in \mathcal{U}$.

By the above Theorem (Injectivity), we would see $L \in \mathcal{H}(\mathcal{U})$. By simple connectedness characterization, we know that such an L always exists if $\mathcal{U}$ is simply connected (eg. star-like).

Problem: Uniqueness. If L is a branch of logarithm, so is $L + i2\pi k$ for $k \in \mathbb{Z}$.

Point-set Topology: what topology do these sheets of logarithm have?

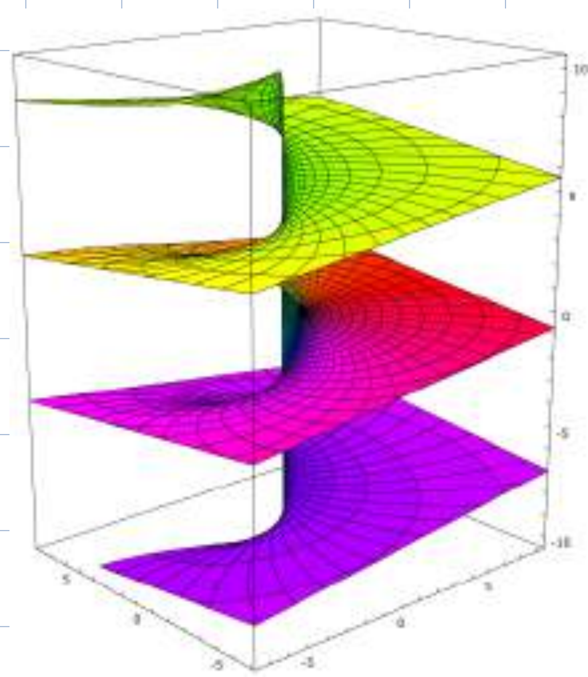
sending each strip S to a new sheet $\Rightarrow$ $\exp(z)$ defines a homeomorphism since exp, Log would be bijective and continuous.

Riemann Surfaces and Log

The various branches of log cannot be glued to give a continuous function $\log: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ because two branches may give different values at points where both are defined.

Consider two copies: $\mathrm{Log}(z)$ on $\mathbb{C}-\mathbb{R}_{\leq 0}$ so $\theta \in (-\pi, \pi)$. Notice they agree on the upper half plane but
$\mathrm{Log}(z)$ on $\mathbb{C}-\mathbb{R}_{\geq 0}$ so $\theta \in (0, 2\pi)$. not on the lower half plane.

Thus we can glue along the upper half plane, but we will have two copies of the lower half.



We can continue by gluing more branches, which gives us a connected surface.

This gives a Riemann surface R associated with $\log z$.

A point on $\mathbb{R}$ can be thought of as a pair $(z, \theta) \in \mathbb{C}\times\mathbb{R}$ so the surface

R can be embedded in $\mathbb{C}\times\mathbb{R} \approx \mathbb{R}^3$.

This way, the branches glue to give a well-defined function $\log_R : R \to \mathbb{C}$.
$(z,\theta) \mapsto \ln|z| + i\theta$

This is the analytic continuation of Log.

There is a projection map $R \to \mathbb{C}\setminus\{0\}$ that "flattens" the Riemann surface, sending $(z,\theta) \mapsto z$. The projection map $R \to \mathbb{C}\setminus\{0\}$ realizes R as a covering space of $\mathbb{C}\setminus\{0\}$. In fact, it is a Galois covering with deck transformation group isomorphic to $\mathbb{Z}$, generated by the homeomorphism $(z,\theta) \mapsto (z, \theta+2\pi)$.

As a complex manifold, R is biholomorphic with $\mathbb{C}$ via $\log_R$, since the inverse $z \mapsto (e^z, \mathrm{Im}(z))$ is holomorphic.

Thus, R is simply connected, so R is the universal cover of $\mathbb{C}\setminus\{0\}$.

A manifold is a topological space that locally resembles Euclidean space. More precisely, an n-dimensional manifold is a topological space such that each point has a neighbourhood homeomorphic to some open subset of n-dimensional Euclidean space.

Circles, lines and curves are 1-manifolds. Surfaces are 2-manifolds.

A Riemann surface X is a complex manifold of complex dimension one. This means that X is a connected Hausdorff space endowed with an atlas of charts to $\mathbb{D} \subset \mathbb{C}$. For every $x \in X$, there is a neighbourhood of x homeomorphic to $\mathbb{D}$, and the transition maps between two overlapping charts are required to be holomorphic.

This goes into the theory of algebraic curves and Riemannian geometry.

We will begin exploring conformal maps. The goal is to better understand the above, and more.

Automorphisms of $\mathbb{D}$

Corollary: If $f \in \mathcal{H}(\mathbb{D})$ then either $|f(z)| < \sup_{z \in \mathbb{D}} |f(z)|$ or f is constant on $\mathbb{D}$.

Proof: By Theorem (Open Mapping),

$f(\mathbb{D}) \subseteq \mathbb{C}$ is open when f is non-constant. Hence, by Maximum Modulus Principle, the supremum is never achieved. $\square$

A rotation of $\mathbb{D}$ is any map of the form $R: \mathbb{D} \to \mathbb{D}$, $z \mapsto e^{i\theta} z$ where $\theta \in \mathbb{R}$. Notice there is a 2π-redundancy.

Lemma (Schwarz's): Let $f: \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then:

(i) $|f(z)| \leq |z|$ for $z \in \mathbb{D}$;

(ii) $|f'(0)| \leq 1$;

(iii) If equality holds in (i) for some $z_0 \in \mathbb{D}\setminus\{0\}$, or equality holds in (ii), then f is a rotation.

Proof (i): Since $f(0) = 0$, we have the Maclaurin series $f(z) = \sum_{k=1}^{\infty} c_k z^k = z g(z)$ where $g(z) = \sum_{k=1}^{\infty} c_k z^{k-1} \in \mathcal{H}(\mathbb{D})$.

If $|z| = r \in (0,1)$ (so $z \in \mathbb{D}\setminus\{0\}$) then $|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{r} \leq \frac{1}{r}$ since $f(\mathbb{D}) \subseteq \mathbb{D}$ (so $|f(z)| \leq 1$, $\forall x \in \mathbb{D}$).

By Maximum Modulus, $|g(z)| \leq \frac{1}{r}$ where $z \in \bar{D}(0,r)$, for any $r \in (0,1)$. So if $z \in \mathbb{D}\setminus\{0\}$,

$\left|\frac{f(z)}{z}\right| \leq g(z) \leq 1$ since the maximum occurs on $\bar{\mathbb{D}} = \bar{D}(0,1)$ where $r = 1$.

$\Rightarrow |f(z)| \leq |z|$. $\triangle$

(ii) Notice $f'(0) = \lim_{z \to 0} \frac{f(z) - \cancel{f(0)}^{0}}{z} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z)$ where $g(z)$ is defined as above. But $g \in \mathcal{H}(\mathbb{D})$

$\Rightarrow \lim_{z \to 0} g(z) = g(0)$ by continuity.

Thus $|f'(0)| \leq |g(0)| \leq 1$ by (i). $\triangle$

(iii) Suppose $|f(z)| = |z_0|$ for some $z_0 \in \mathbb{D}\setminus\{0\}$. Then $g(z_0) = \frac{f(z_0)}{z_0}$ so g attains a supremum within $\mathbb{D}$.

By Maximum Modulus on $\mathbb{D}$, we have $g(z) = c \in \mathbb{C}$ and $|g(z)| = |c| = 1$ so $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Thus $\frac{f(z)}{z} = e^{i\theta} \Rightarrow f(z) = ze^{i\theta}$ so f defines a rotation of $\mathbb{D}$.

Alternatively, if $|f'(0)| = 1$, then by (ii), g attains a supremum in $\mathbb{D} \Rightarrow f$ is a rotation. $\triangle\square$

Theorem (Automorphisms of $\mathbb{D}$):

(i) Let $w \in \mathbb{D}$. Then the function $\psi_w(z) = \frac{w-z}{1-\bar{w}z}$ for $z \in \mathbb{D}$ satisfies:

- $\psi_w(\mathbb{D}) = \mathbb{D}$;
- $\psi_w^{-1} = \psi_w$;
- $\psi_w(0) = w$.

(ii) If $f(z)$ is biholomorphic, then $f(z) = e^{i\theta}\psi_w(z)$ for some $w \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

Proof: (i) Just do algebra. Since $|w|<1$, we have $\psi_w(z) \in \mathcal{H}(D(0, \frac{1}{|w|}))$ and $\bar{\mathbb{D}} \subset D(0, \frac{1}{|w|})$.

If $z \in \partial\mathbb{D}$, so $z\bar{z} = 1$, then $\psi_w(z) = \frac{w-z}{1-\bar{w}z} = z\frac{w\bar{z}-1}{1-\bar{w}z} = z\frac{\overline{(\bar{w}z-1)}}{-(\bar{w}z-1)}$ so $|\psi_w(z)| = 1$.

Since ψ_w non-constant, by Maximum Modulus,

$\psi_w(\mathbb{D}) \subseteq \mathbb{D}$. Now if $z \in \mathbb{D}$, then $(\psi_w \circ \psi_w)(z) = \frac{w - \frac{w-z}{1-\bar{w}z}}{1-\bar{w}\frac{w-z}{1-\bar{w}z}} = z$ by some algebra.

Therefore $\mathbb{D} \subseteq \psi_w(\mathbb{D})$ so $\psi_w(\mathbb{D}) = \mathbb{D}$,

and $\psi_w \circ \psi_w = id_{\mathbb{D}} \Rightarrow \psi_w = \psi_w^{-1}$, and trivially $\psi_w(0) = w$. $\triangle$

(ii) Since f is biholomorphic, we have $f^{-1}: \mathbb{D} \to \mathbb{D}$. Let $w = f^{-1}(0)$ so $f(w) = 0$. Let $g = f \circ \psi_w$ so

$g(0) = f(\psi_w(0)) = f(w) = 0$. Notice g^{-1} exists since g is a composition of invertible functions.

We have $g: \mathbb{D} \to \mathbb{D}$ and $g^{-1}: \mathbb{D} \to \mathbb{D}$ and $g^{-1}(0) = 0$

(Schwarz's) $\Longrightarrow$ $|g(z)| \leq |z|$ and $|g^{-1}(z')| \leq |z'|$ for $z, z' \in \mathbb{D}$. Letting $g(z) = z'$ (possible by bijectivity),

we have $|z'| \leq |g(z)| \Rightarrow |g(z')| = |z'|$ so g is a rotation.

Thus $f = g \circ \psi_w^{-1} = g \circ \psi_w = e^{i\theta}\psi_w$. $\square$

Lemma: Let $z_0 \in \mathbb{C}$, and $r>0$, and $f \in \mathcal{H}(D'(z_0, r))$ be bounded on $D'(z_0, r)$. Then z_0 is a removable singularity.

Proof: Let $M = \sup_{z \in D'(z_0, r)} |f(z)| < \infty$. Then for $z \in D'(z_0, r)$, we have $0 \leq |(z-z_0)f(z)| = |z-z_0||f(z)| \leq |z-z_0|M$.

Thus $\lim_{z \to z_0} (z-z_0)f(z) = 0$, so either $\lim_{z \to z_0} f(z)$ exists and we are done,

or z_0 is a simple pole with $\mathrm{Res}_{z_0}(f) = 0$, so we have the above case. $\square$

Theorem (Casorati-Weierstrass): Let $z_0 \in \mathbb{C}$, and $R>0$, and $f \in \mathcal{H}(D'(z_0, R))$. If z_0 is an essential singularity, then for any $r \in (0, R]$, we have the image $f(D'(z_0, r)) \subseteq \mathbb{C}$ is dense.

Remark: Theorem (Picard): In fact, we have $f(D'(z_0, r)) \supseteq \mathbb{C} \setminus \{w\}$ for some $w \in \mathbb{C}$. We won't prove this.

Proof: Suppose for some $r \in (0, R]$ that $f(D'(z_0, r))$ is not dense in $\mathbb{C}$, for a contradiction.

$\Rightarrow \exists w \in \mathbb{C}$ and some $\delta > 0$ such that $D(w, \delta) \cap f(D'(z_0, r)) = \emptyset$ (this characterization is equivalent to density).

so $|f(z) - w| \geq \delta > 0$ for $z \in D'(z_0, r)$.

Let $g(z) = \frac{1}{f(z) - w}$ so $|g(z)| = \frac{1}{|f(z)-w|} \leq \frac{1}{\delta}$ for $z \in D'(z_0, r)$. Hence by the above Lemma, we have

z_0 a removable singularity for g.

Extend g holomorphically to $D(z_0, r)$.

If $g(z_0) = \lim_{z \to z_0} g(z) \neq 0$ then $f(z) = \frac{1}{g(z)} + w$ for $z \in D(z_0, r')$ where $r' \in (0, r]$ and has a removable singularity, contradicting assumptions. ↯

If $g(z_0) = 0$, by Theorem (Order of Zero), we have $\frac{1}{f(z) - w} = g(z) = (z - z_0)^m h(z)$ where $m \in \mathbb{N}$ and $h(z_0) \neq 0$ and $h \in \mathcal{H}(D(z_0, r))$.

Thus $(z - z_0)^m f(z) = \frac{1}{h(z)} - (z - z_0)^m w \xrightarrow{z \to z_0} \frac{1}{h(z_0)}$ so the limit exists, contradiction. ↯ $\square$

Theorem (Automorphisms of $\mathbb{C}$): Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective.

Then $f(z) = az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$. In other words, $\mathrm{Aut}(\mathbb{C}) = \{ f(z) \in \mathbb{C}[z] : \deg(f) = 1 \}$.

Proof: Since f is non-constant, by Theorem (Open Mapping), we have $R > 0$ such that $f(D(0, R))$ is open.

Since f is injective, we have $f(D(0,R)) \cap f(\mathbb{C} \setminus \bar{D}(0,R)) = \emptyset$ since $D(0,R) \cap \mathbb{C} \setminus \bar{D}(0,R) = \emptyset$.

$\Rightarrow V = f(\mathbb{C} \setminus \bar{D}(0, R))$ is not dense, since $f(D(0,R))$ is open and nonempty with the above property.

Also V is unbounded since otherwise, this would imply f being bounded, which would violate Liouville's.

Let $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $g(z) = f(\frac{1}{z})$, so $g(D'(0, \frac{1}{R})) = f(\mathbb{C} \setminus \bar{D}(0, R)) = V$ is not dense.

Therefore $0 \in \mathbb{C}$ must be a pole of g (since essential singularity $\Rightarrow V$ dense). Thus

by Casorati-Weierstrass, $\exists n \in \mathbb{N} : \lim_{z \to 0} z^n g(z)$ exists. Hence $\lim_{|z| \to \infty} \frac{f(z)}{z^n} = \lim_{|z| \to \infty} \frac{1}{z^n} g(\frac{1}{z}) = \lim_{w \to 0} w^n g(w)$ exists.

by a Liouville-style argument, we have $f \in \mathbb{C}[z]$ a polynomial (exercise).

by injectivity, f has only one zero, say $f(z_0) = 0$.

by Theorem (Injectivity), we have $f'(z_0) \neq 0$ so z_0 is a unique zero of multiplicity 1. Thus $\deg(f) = 1$.

$\Rightarrow f(z) \in \mathbb{C}[z]$ and $\deg(f) = 1$. $\square$

Let X and Y be topological spaces and let $f : X \to Y$ be continuous and injective. We say f is a **topological embedding** if f yields a homeomorphism $X \to f(X)$ (where $f(X)$ carries the subspace topology inherited from Y).

There exists an embedding $\mathbb{C} \hookrightarrow \mathbb{R}^3$ since $\mathbb{C} \cong \mathbb{R}^2$ homeomorphic and $\mathbb{R}^2 \subset \mathbb{R}^3$ is a subspace topology of $\mathbb{R}^3$).

We define the Riemann Sphere $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ ← "mystery" point, with the rules

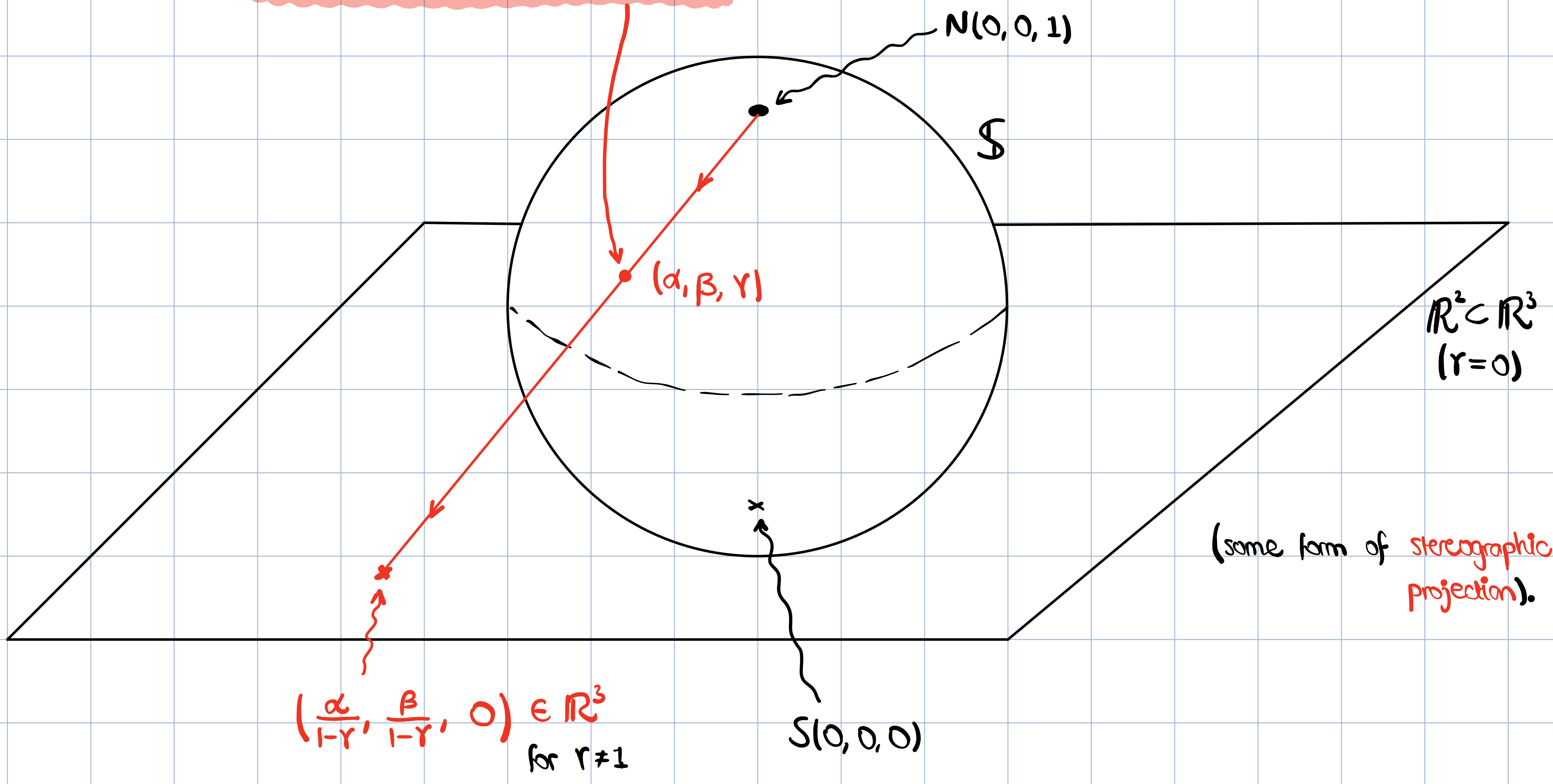
- $z + \infty = \infty$ for $z \in \mathbb{C}$;
- $z \cdot \infty = \infty$ for $z \in \mathbb{C} \setminus \{0\}$;
- $\frac{z}{0} = \infty$ for $z \in \mathbb{C} \setminus \{0\}$;
- $\frac{z}{\infty} = 0$ for $z \in \mathbb{C} \setminus \{0\}$.

Let $\mathbb{C} \hookrightarrow \mathbb{R}^3$, $z \mapsto (\mathrm{Re}(z), \mathrm{Im}(z), 0)$ define an embedding and define the sphere

$$\mathbb{S} := \{(\alpha, \beta, \gamma) : \underbrace{\alpha^2 + \beta^2 + (\gamma - \tfrac{1}{2})^2 = (\tfrac{1}{2})^2}_{(\Leftrightarrow \alpha^2 + \beta^2 = \gamma(1-\gamma))}\}.$$

Consider the convex combination for $\alpha, \beta, \gamma \in \mathbb{S}$

given by $(1-\gamma)\left(\frac{\alpha}{1-\gamma}, \frac{\beta}{1-\gamma}, 0\right) + \gamma(0, 0, 1) \in \mathbb{S}$ (notice as γ varies, this forms $\mathbb{S}$ (almost)).



(some form of stereographic projection).

Now define a map $P : \mathbb{S} \to \mathbb{C}_\infty$ by $P(\alpha, \beta, \gamma) = \begin{cases} \frac{\alpha}{1-\gamma} + i\frac{\beta}{1-\gamma} & \text{when } \gamma \neq 1; \\ \infty & \text{when } \gamma = 1. \end{cases} = \frac{\alpha}{1-\gamma} + i\frac{\beta}{1-\gamma}$ by arithmetic on $\mathbb{C}_\infty$.

We show P^{-1} exists by computation.

Let $z \in \mathbb{C}$. We have $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$ and we solve for $t \in [0, 1]$: $(1-t)(x, y, 0) + t(0, 0, 1)$.

To be in $\mathbb{S}$, we require $((1-t)x)^2 + ((1-t)y)^2 = (1-t)t \Leftrightarrow (1-t)^2(x^2+y^2) = t(1-t)$

$$\text{“}\Leftrightarrow\text{”} \quad (1-t)(x^2+y^2) = t$$

$$\Leftrightarrow \begin{cases} t = 1: & \text{we are at } N(0,0,1); \\ t < 1: & x^2 + y^2 = (x^2+y^2+1)t \end{cases}$$

$$\Rightarrow t = \frac{x^2+y^2}{1+x^2+y^2} = \frac{|z|^2}{1+|z|^2}$$

and notice that $1 - \frac{|z|^2}{1+|z|^2} = \frac{1}{1+|z|^2}$. Thus $P^{-1}(z) = \begin{cases} (0, 0, 1) & \text{if } z = \infty; \\ \left(\frac{\mathrm{Re}(z)}{1+|z|^2}, \frac{\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2}{1+|z|^2}\right) & \text{if } z \in \mathbb{C}. \end{cases}$

We can then assign a topology to $\mathbb{C}_\infty$ such that P is a homeomorphism. Notice then $\mathbb{C}_\infty \cong \mathbb{S}$ are homeomorphic and $\mathbb{S}$ compact topological space $\Rightarrow \mathbb{C}_\infty$ compact.

Möbius Maps

Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected. Define $\mathrm{Aut}(\mathcal{U}) := \{f : \mathcal{U} \to \mathcal{U} \mid f \text{ biholomorphic}\}$.

Recall $GL_2(F) := \{A \in M_{2\times 2}(F) \mid \det(A) \neq 0\}$ is the general linear group of invertible matrices. Consider $GL_2(\mathbb{C})$.

In Linear Algebra, Theorem (Classical Adjoint) gives us a way to compute $A^{-1} \in GL_2(\mathbb{C})$, namely

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C}) \implies A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(Cramer's Rule).}$$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C})$ and $z \in \mathbb{C}_\infty$, define the Möbius map for A by $\mu_A(z) := \frac{az+b}{cz+d}$ with arithmetic on $\mathbb{C}_\infty$.

(i.e. $\frac{a}{c}$ when $\begin{smallmatrix} z=\infty \\ c\neq 0\end{smallmatrix}$; ∞ when $\begin{smallmatrix} z=\infty \\ c=0 \\ z\neq -d/c \end{smallmatrix}$)

For instance:

- $\mu_{\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]}(z) = z+b$ (translation);
- $\mu_{\left[\begin{smallmatrix} w & 0 \\ 0 & 1 \end{smallmatrix}\right]}(z) = wz$ so if $w \in \partial\mathbb{D}$, this is rotation; if $w \in \mathbb{R}$, this is scaling; etc.
- $\mu_{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}(z) = \frac{1}{z} \in \mathbb{C}_\infty$ but $\mu_{\left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]}(z) = \frac{i}{iz} = \frac{1}{z} \in \mathbb{C}_\infty$ so $\mu_{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]} = \mu_{\left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]}$.

Therefore the map $\varphi : GL_2(\mathbb{C}) \longrightarrow \{\mu_A : A \in GL_2(\mathbb{C})\}$, $A \longmapsto \mu_A$ is not injective. We have redundancies.

We collect some facts.

- $GL_2(\mathbb{C})$ forms a group. The special linear group $SL_2(\mathbb{C}) := \{A \in GL_2(\mathbb{C}) : \det(A) = 1\} \subseteq GL_2(\mathbb{C})$ forms a subgroup;
- For $A, B \in M_{n\times n}(F)$, we have $\det(AB) = \det(A)\det(B)$.

Thus $\mu_A \circ \mu_B = \mu_{AB}$ (exercise).

- $\mu_A^{-1} = \mu_{A^{-1}}$ so μ_A has an inverse, and we know how to find it.
- For $\underset{\text{nonzero}}{\alpha \in \mathbb{C}}$, and $I \in GL_2(\mathbb{C})$, we have $\alpha I = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ so $\mu_{\alpha I} = \mathrm{id}_{\mathbb{C}_\infty}$.

$\implies$ if $B \in GL_2(\mathbb{C})$ then $\mu_{\alpha B} = \mu_{(\alpha I)B} = \mu_{\alpha I} \circ \mu_B = \mathrm{id}_{\mathbb{C}_\infty} \circ \mu_B = \mu_B$.

Therefore, we often parametrize Möbius maps over $SL_2(\mathbb{C})$, but even this gives redundancy since $\underset{\text{NOT injective}}{\varphi} : SL_2(\mathbb{C}) \longrightarrow \{\mu_A\}$, $A \longmapsto \mu_A$ sends $\varphi(I) = \mathrm{id}_{\mathbb{C}_\infty} = \varphi(-I)$.

projective special linear group

quotient group

Hence we define $PSL_2(\mathbb{C}) := SL_2(\mathbb{C}) / \{-I, I\}$ which fully parametrizes Möbius maps (i.e. $PSL_2(\mathbb{C}) \overset{\text{bij}}{\longleftrightarrow} \{\mu_A\}$).

Notes: (1) $\mathrm{Aut}(\mathbb{C}) = \left\{\mu_A : A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix};\ \begin{smallmatrix} a, b \in \mathbb{C} \\ a \neq 0 \end{smallmatrix}\right\}$.

(2) $\mathrm{Aut}(\mathbb{D})$ as Möbius maps.

Let $G_{\mathbb{D}} := \left\{A = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C};\ \det(A) = 1\right\}$. Notice $G_{\mathbb{D}} \subseteq SL_2(\mathbb{C})$ forms a subgroup.

$\implies \alpha\bar{\alpha} - \beta\bar{\beta} = 1$

$\implies |\alpha|^2 - |\beta|^2 = 1$

Let $z \in \mathbb{C}$ be generic. Let $A = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in G_{\mathbb{D}}$ so $|\alpha| > |\beta| \geq 0$.

Then $\mu_A(z) = \dfrac{\alpha z+\beta}{\bar\beta z+\bar\alpha} = \dfrac{\bar\alpha}{\alpha}\dfrac{z+\frac{\beta}{\alpha}}{\left(\frac{\bar\beta}{\bar\alpha}\right)z+1}$. Let $w = \dfrac{-\beta}{\alpha}$ (notice $\alpha\neq 0$).

$= -\dfrac{\bar\alpha}{\alpha}\dfrac{w-z}{1-\bar w z} = e^{i\theta}\psi_w(z)$ for some $\theta\in\mathbb{R}$ such that $e^{i\theta} = -\dfrac{\bar\alpha}{\alpha}$

(since note that $\left|\frac{\bar\alpha}{\alpha}\right| = 1$).

Thus $\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta}\psi_w : \begin{smallmatrix}\theta\in\mathbb{R};\\ w\in\mathbb{D}.\end{smallmatrix}\} = \{\mu_A : A\in G_{\mathbb{D}}\}$.

Soon we will show that $G_{\mathbb{D}} \underset{grp}{\cong} SL_2(\mathbb{R})$ so $\mathrm{Aut}(\mathbb{D}) \underset{grp}{\cong} SL_2(\mathbb{R})$.

Theorem (Automorphisms of $\mathbb{C}_\infty$):

If $f:\mathbb{C}_\infty\to\mathbb{C}_\infty$ satisfies:
- f continuous on $\mathbb{C}_\infty$;
- f bijective with $z_0 = f^{-1}(\infty)$ unique;
- $f|_{\mathbb{C}\setminus\{z_0\}} \in \mathcal{H}(\mathbb{C}\setminus\{z_0\})$;

then $f = \mu_A$ for some $A\in GL_2(\mathbb{C})$.

Proof: 1 $f(\infty)=\infty$. Then $f|_{\mathbb{C}_\infty\setminus\{\infty\}} = f|_{\mathbb{C}} \in \mathcal{H}(\mathbb{C})$ and injective $\Rightarrow f\in\mathrm{Aut}(\mathbb{C})$ so $f = \mu_{\left[\begin{smallmatrix}a & b\\ 0 & 1\end{smallmatrix}\right]}$.

2 $f(\infty)\neq\infty$. Let $z_0 = f^{-1}(\infty)$ and $B = \begin{bmatrix} z_0 & b\\ 1 & -z_0\end{bmatrix}$ where $b\in\mathbb{C}$ is such that $\det(B)\neq 0$.

Then $\mu_B(z) = \dfrac{z_0 z+b}{z-z_0}$ taking $z_0\mapsto\infty$, $\infty\mapsto z_0$.

Let $g := f\circ\mu_B$ so
- g bijective
- $g(\infty)=\infty$
- $g\in\mathcal{H}(\mathbb{C}\setminus\{z_0\})$
- g continuous at z_0 (since f continuous at ∞)

$\Rightarrow z_0\in\mathbb{C}$ is a removable singularity for g.

$\overset{CGM}{\Longrightarrow}$ extend g holomorphically to $\mathbb{C}$

$\overset{(1)}{\Rightarrow} g\in\mathrm{Aut}(\mathbb{C})$ so $g = \mu_{\left[\begin{smallmatrix}\alpha & \beta\\ 0 & 1\end{smallmatrix}\right]} = \mu_A$.

so $\mu_A = f\circ\mu_B \Rightarrow \mu_A\circ\mu_B^{-1} = f = \mu_{AB^{-1}}$. $\square$

Remark: If we stare close enough, we implicitly defined some notion of differentiability at $\infty\in\mathbb{C}_\infty$.

Let $\mathcal{U}, V \subseteq \mathbb{C}$ be open. A conformal map is a map $f: \mathcal{U} \to V$ such that $f \in \mathcal{H}(\mathcal{U})$ and f is bijective. Thus

$$\left(\substack{\text{Inversion}\\ \text{theorem}}\right) \implies f^{-1}: V \to \mathcal{U} \text{ such that } f^{-1} \in \mathcal{H}(V) \text{ and } f^{-1} \text{ bijective.}$$

These are essentially biholomorphic maps between two open subsets of $\mathbb{C}$.

We define the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Notice $\mathbb{H} \subseteq \mathbb{C}$ is open. Recall $\mathbb{D} \subseteq \mathbb{C}$ is open.

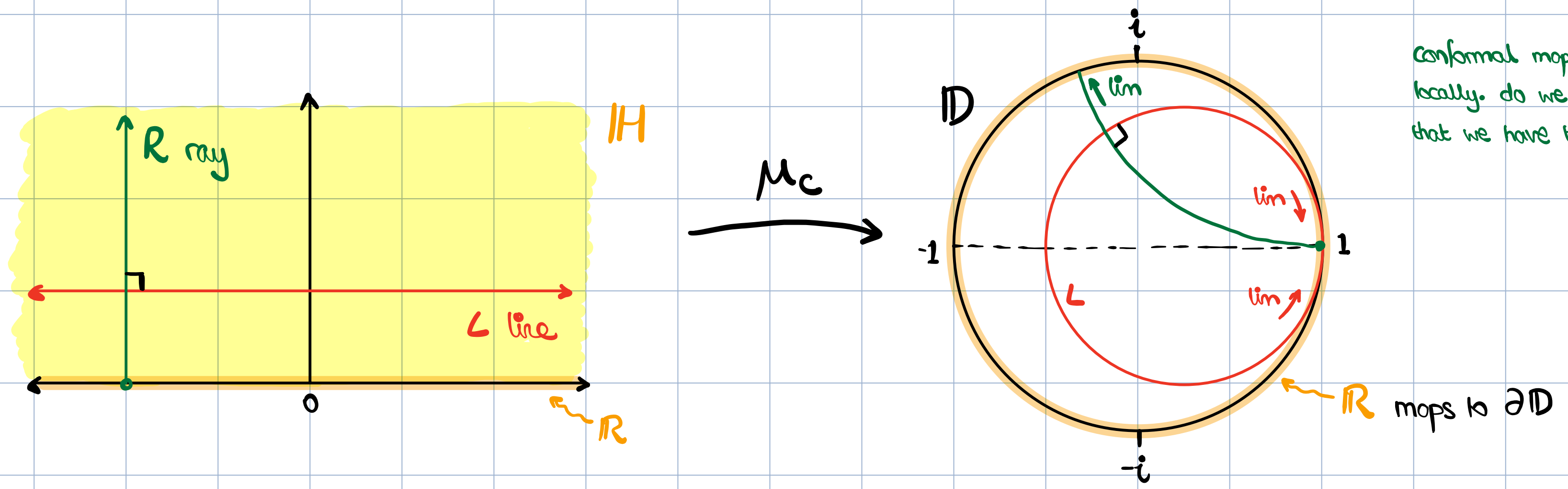
Cayley Map

Notice for $z \in \mathbb{C}$, we have $z \in \mathbb{H} \iff \underbrace{|z-i|}_{\text{distance from } i} < \underbrace{|z+i|}_{\text{distance from } -i}$

$$\iff \left|\frac{z-i}{z+i}\right| < 1 \iff \left|\frac{z-i}{z+i}\right| \in \mathbb{D}.$$ Then we can define a map $\mathbb{H} \to \mathbb{D}$.

We can define a Möbius map μ_C with $C := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \implies C^{-1} = \frac{1}{2i}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. Then μ_C is the Cayley map.

Notice $\mu_C(i) = 0$ and for $x \in \mathbb{R}$, we get $|\mu_C(x)| = 1 \implies \mu_C(x) \in \partial\mathbb{D}$, and $\lim_{|z| \to \infty} \mu_C(z) = 1$.



conformal maps are angle-preserving locally. do we know enough to show that we have this local property?

Theorem (Automorphisms of $\mathbb{H}$): If $f: \mathbb{H} \to \mathbb{H}$ is a conformal map, then $f = \mu_A$ where $A \in SL_2(\mathbb{R})$.

Proof: Let $C = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ the Cayley matrix, so $\mu_C(\mathbb{H}) = \mathbb{D}$ and $\mu_{C^{-1}}(\mathbb{D}) = \mathbb{H}$.

Then notice $\mathbb{D} \xrightarrow{\mu_C^{-1}} \mathbb{H} \xrightarrow{f} \mathbb{H} \xrightarrow{\mu_C} \mathbb{D}$ so $\mu_C \circ f \circ \mu_{C^{-1}} \in \mathrm{Aut}(\mathbb{D})$ so this map is conformal.

But we know what $\mathrm{Aut}(\mathbb{D})$ looks like! Namely, we have $\mu_C \circ f \circ \mu_{C^{-1}} = \mu_A$ where $A = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in SL_2(\mathbb{C})$.

We can then write

$$f = \mu_{C^{-1}} \circ \mu_A \circ \mu_C = \mu_{C^{-1}AC} \quad \text{where } C^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

It suffices to show $C^{-1}AC \in SL_2(\mathbb{R})$. By computation.

We get $C^{-1}AC = \begin{bmatrix} \mathrm{Re}\alpha + \mathrm{Re}\beta & -\mathrm{Im}\beta + \mathrm{Im}\alpha \\ -\mathrm{Im}\alpha + \mathrm{Im}\beta & \mathrm{Re}\alpha - \mathrm{Re}\beta \end{bmatrix} \in GL_2(\mathbb{R})$. But $\det(C^{-1}AC) = \cancel{\det(C^{-1})}^{1} \cancel{\det(A)}^{1} \cancel{\det(C)}^{1} = 1.$ ($\in SL_2(\mathbb{R})$)

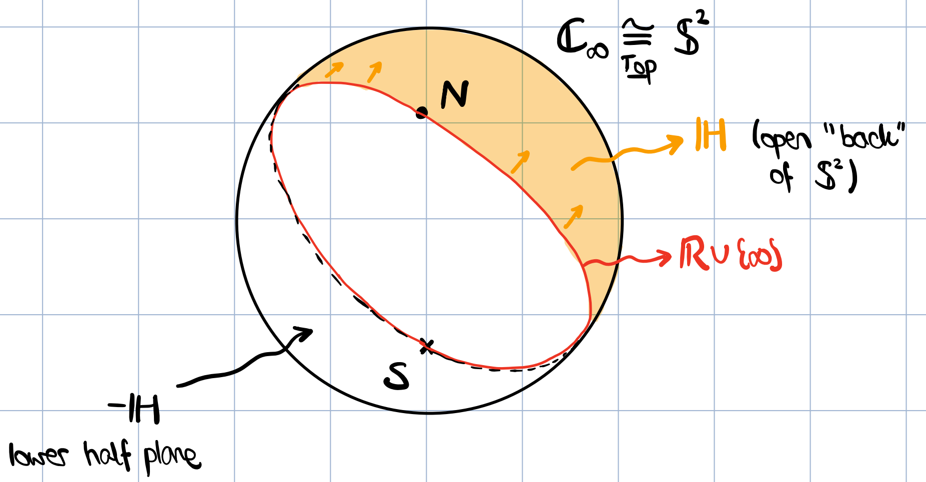
$$\implies C^{-1}AC \in SL_2(\mathbb{R}).$$

Claim: We get all of $SL_2(\mathbb{R})$ this way. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ we solve the equations for a, b, c, d in terms of $\mathrm{Re}\alpha, \mathrm{Re}\beta, \mathrm{Im}\alpha, \mathrm{Im}\beta$. $\square$

Remark: For $A \in SL_2(\mathbb{R})$ we have $\mu_A(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$. We turn $\mathbb{R} \cup \{\infty\}$ into $\partial\mathbb{D}$ and in some sense this preserves the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ of the upper half plane.

$$\implies \mathbb{R} \cup \{\infty\} \underset{\text{Top}}{\cong} S^1 \text{ the circle.}$$

Since μ_A preserves $\mathbb{R} \cup \{\infty\}$, we can visualize $\mathbb{H}$ and $\mathbb{R}$ on the Riemann sphere $\mathbb{C}_\infty$:



Recall the stereographic projection of $\mathbb{S}^2$ onto $\mathbb{C}_\infty$ and this makes complete sense.

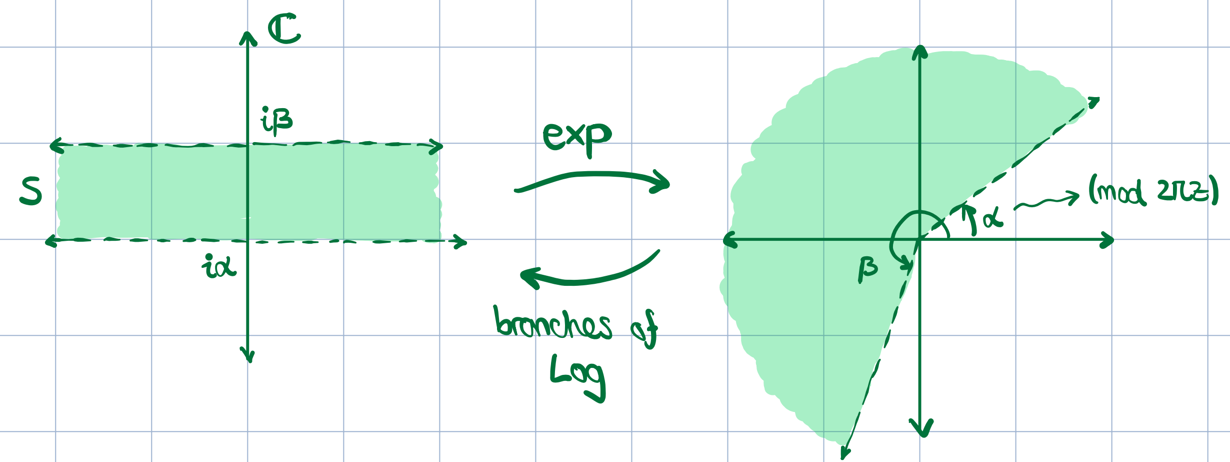
Some Important Conformal Maps

(i) Affine maps, defined on $\mathbb{C}$ (the basic elements that "generate" $\mathrm{Aut}(\mathbb{C})$).

- rotations: $z \mapsto e^{i\theta} z$ for $\theta \in \mathbb{R}$;
- scaling: $z \mapsto rz$ for $r > 0$;
- translations: $z \mapsto z + b$ for $b \in \mathbb{C}$.

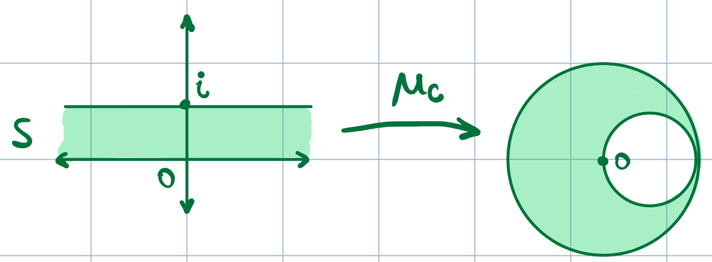
(ii) Exponents and Logarithms

Define a strip $S := \{z \in \mathbb{C} : \alpha < \mathrm{Im}(z) < \beta\}$ where $\underset{\in \mathbb{R}}{\alpha < \beta}$ and $\beta - \alpha \leq 2\pi$.



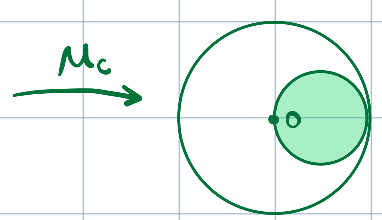
(iii) The Cayley Map (as seen above)

Consider a horizontal strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1\}$. Notice $\mu_c(i) = 0$. Then we have



and the thinner S gets, the more towards the outside of $\mathbb{D}$ it would map to.

$\Rightarrow \mathbb{H} + i$

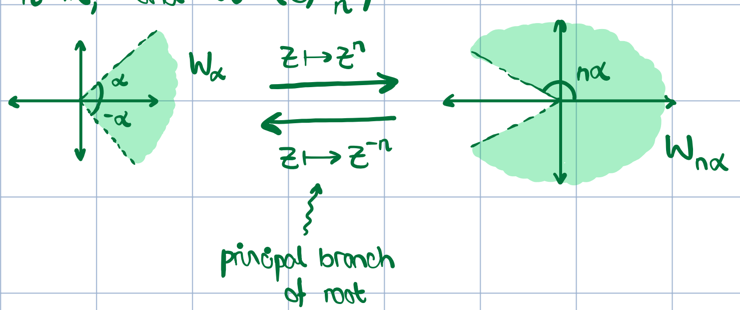


Also notice that the first quadrant maps to $\mathbb{D} \cap \mathbb{H}$.

(iv) Powers and Principal Roots

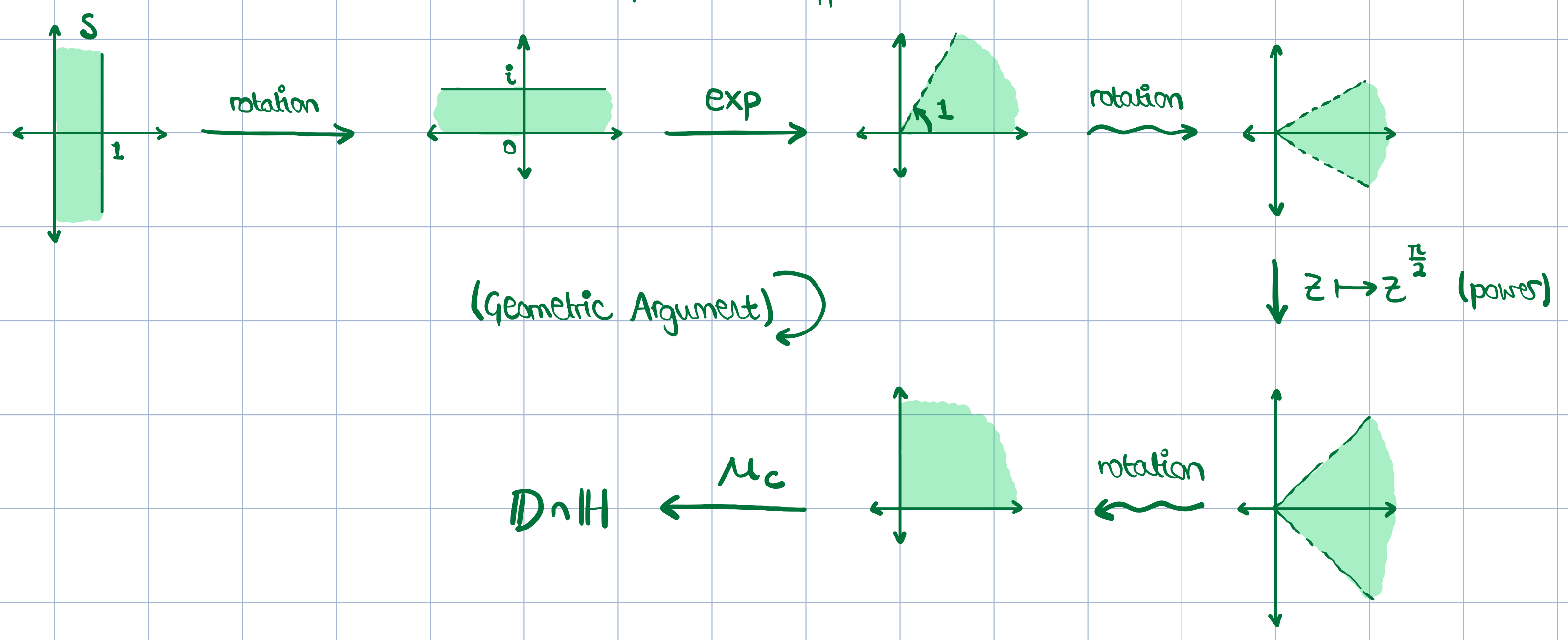
Let $n \in \mathbb{R}$, and $\alpha \in (0, \frac{\pi}{n})$ — this restriction allows our maps to be conformal.

Define the wedge $W_\alpha := \{z \in \mathbb{C} : |\mathrm{Arg}(z)| < \alpha\}$ — principal argument in $(-\pi, \pi]$.



and things stretch since $|r| \mapsto |r|^n$ and vice-versa.
and contract

Exercise: Find a conformal map between $S = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\mathbb{D} \cap \mathbb{H}$.
vertical strip — upper half disc



Further Exploration   We notice that conformal maps are locally angle-preserving. Thus, locally, they take the form of a scaled rotation.

Q: How can we prove this property?

Let $\mathcal{U} \subseteq \mathbb{C} \underset{\text{Top}}{\cong} \mathbb{R}^2$ be open. Let $C^2(\mathcal{U}) := \{u : \mathcal{U} \to \mathbb{R} \mid \text{all second order partial derivatives are continuous.}\}$.

Theorem (Mixed Partials) (really, this is an iterated limits (commuting property) theorem)

$$\Longrightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

homogeneous Laplace's equation ($\Delta u = 0$)

Define the harmonic functions $\text{Har}(\mathcal{U}) := \{u \in C^2(\mathcal{U}) : \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\}$.

Notice: $\text{Har}(\mathcal{U})$ is an $\mathbb{R}$-linear space.

By the Cauchy-Riemann equations, decomposing $f : \mathcal{U} \to \mathbb{C}$ as $\text{Re}(f)$ and $\text{Im}(f)$, we have

$$f \in \mathcal{H}(\mathcal{U}) \Rightarrow \text{Re}(f), \text{Im}(f) \in C^\infty(\mathcal{U}).$$

Proposition: Let $\mathcal{U} \subseteq \mathbb{C} \underset{\text{Top}}{\cong} \mathbb{R}^2$ be open and connected, and let $u \in C^2(\mathcal{U})$ be real-valued ($u : \mathbb{C} \to \mathbb{R}$).

(i) If there is $f \in \mathcal{H}(\mathcal{U})$ such that $u = \text{Re}(f)$, then $u \in \text{Har}(\mathcal{U})$;

(ii) If $u \in \text{Har}(\mathcal{U})$, then $g = \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y} \in \mathcal{H}(\mathcal{U})$;

(iii) If $\mathcal{U}$ is simply connected and $u \in \text{Har}(\mathcal{U})$, then there exists $f \in \mathcal{H}(\mathcal{U})$ such that $u = \text{Re}(f)$.

Proof: (i) Plug and chug. Let $f \in \mathcal{H}(\mathcal{U})$ such that $u = \text{Re}(f)$ and $v = \text{Im}(f)$.

Then $u, v \in C^\infty(\mathcal{U}) \subseteq C^2(\mathcal{U})$ since $f \in C^\infty(\mathcal{U})$.

We calculate the Laplacian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial x}\right)$ by Cauchy-Riemann;

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \quad \text{by Theorem (Mixed Partials).}$$

Thus $\nabla^2 u = 0$ and $u \in C^2(\mathcal{U}) \Rightarrow u \in \text{Har}(\mathcal{U})$. Δ

(ii) Write $\tilde{u} = \frac{\partial u}{\partial x}$ and $\tilde{v} = \frac{\partial u}{\partial y}$, so $g = \tilde{u} + i\tilde{v}$ where $\tilde{u}, \tilde{v} \in C^1(\mathcal{U})$.

We check Cauchy-Riemann for g:

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial^2 u}{\partial x^2} \underset{\nabla^2 u = 0}{=} \frac{-\partial^2 u}{\partial y^2} = \frac{\partial \tilde{v}}{\partial y} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \underset{\text{Mixed Partials}}{=} \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial \tilde{v}}{\partial x}. \quad \Delta$$

(iii) By simple connectivity, we have g (as defined above) admits a primitive $\tilde{f}$ on $\mathcal{U}$.

Consider $\frac{\partial(\text{Re}(\tilde{f}))}{\partial x} \underset{\text{limits commute}}{=} \text{Re}\left(\frac{\partial \tilde{f}}{\partial x}\right) = \text{Re}(g) = \frac{\partial u}{\partial x}$ and similarly

$$\frac{\partial(\text{Re}(\tilde{f}))}{\partial y} \overset{\text{limits commute}}{=} \text{Re}\left(\frac{\partial \tilde{f}}{\partial y}\right) = -i\text{Re}(i\tilde{f}') = \frac{\partial u}{\partial y}.$$

Write $w = Re(\tilde{f}) - u$ so $\frac{\partial w}{\partial x} = 0$ and $\frac{\partial w}{\partial y} = 0$ so w is a constant function, say $C \in \mathbb{C}$.

Then let $f = \tilde{f} - C$ and so $Re(f) = u$. $\triangle$ $\square$

If $u \in Re(f)$ where $f \in \mathcal{H}(\mathcal{U})$ for $\mathcal{U} \subseteq \mathbb{C}$ open, then we call $v = Im(f)$ the harmonic conjugate of u.

Let $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ be given by $u(x+iy) = \log(x^2+y^2)$. Restrict the domain to $\mathbb{C} \setminus (-\infty, 0]$. ← simply connected

not simply connected ↗

Notice $u(x+iy) = \log(x^2+y^2) = 2\log|x+iy| = 2Re(\mathrm{Log}\, z)$ ← principal branch of logarithm

so $u = Re(f) = Re(2\mathrm{Log}(z))$ so $f \in \mathcal{H}(\mathbb{C} \setminus (-\infty, 0])$

$\Rightarrow u \in Har(\mathbb{C} \setminus (-\infty, 0])$.

What is the harmonic conjugate of u?

Exercise: Show that

$$v(x+iy) = \begin{cases} 2\operatorname{arccot}\left(\frac{x}{y}\right) & \text{for } y>0; \\ 2\arctan\left(\frac{y}{x}\right) & \text{for } x>0; \\ 2\operatorname{arccot}\left(\frac{x}{y}\right) - \pi & \text{for } y<0. \end{cases}$$

Corollary: Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected. Let $h \in \mathcal{H}(\mathcal{U})$ and $u \in Har(h(\mathcal{U}))$ (open since h continuous). Then $u \circ h \in Har(\mathcal{U})$.

Proof: Fix $z_0 \in \mathcal{U}$ and take $R>0$ such that $D(z_0, R) \subseteq h(\mathcal{U})$. Then $\exists f \in \mathcal{H}(D(z_0,R))$ so $u = Re(f)$.

← convex

Then $u \circ h = Re(f \circ h) \in Har(\mathcal{U})$ since $f \circ h \in \mathcal{H}(\mathcal{U})$. $\square$

PROPOSITION (Mean Value Property): Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected, and $u \in Har(\mathcal{U})$. If $z_0 \in \mathcal{U}$, and $R>0$ such that $\bar{D}(z_0, R) \subset \mathcal{U}$, then:

(i) (Strong form) $$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta;$$

(ii) (Weak form) $$u(z_0) = \frac{1}{\pi R^2} \iint_{\bar{D}(z_0,R)} u(x+iy)\, d(x,y).$$

Proof: (i) Let $0 < \delta < \operatorname{dist}(\bar{D}(z_0,R), \mathbb{C} \setminus \mathcal{U})$. Then $\bar{D}(z_0,R) \subset D(z_0, R+\delta) \subset \mathcal{U}$.

← convex $\Rightarrow$ simply connected

Hence there is $f \in \mathcal{H}(D(z_0, R+\delta))$ so $Re(f) = u$.

We use the Cauchy Integral Formula.

$$f(z_0) = \frac{1}{2\pi i}\int_{C(z_0,R)} \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{(z_0+Re^{i\theta}) - z_0}\, iRe^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta$$ and then we take the real part of both sides.

$$\Rightarrow \operatorname{Re}(f(z_0)) = u(z_0) = \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\big(f(z_0+Re^{i\theta})\big)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+Re^{i\theta})\,d\theta. \quad \triangle$$

(ii) We have

$$\iint_{\bar{D}(z_0,R)} u(x+iy)\,d(x,y) \overset{\text{polar}}{=} \int_0^{2\pi}\int_0^R u(z_0+Re^{i\theta})\,\underbrace{r}_{\text{Jacobian}}\,dr\,d\theta$$

$$\overset{\text{Fubini's}}{=} \int_0^R\int_0^{2\pi} u(z_0+Re^{i\theta})\,d\theta\,r\,dr \overset{(i)}{=} \int_0^R \underbrace{2\pi u(z_0)}_{\text{constant}}\,r\,dr$$

$$= \pi R^2 u(z_0). \quad \triangle \ \square$$

Theorem (Liouville's): Let $u \in Har(\mathbb{C})$ be bounded. Then u is a constant function.

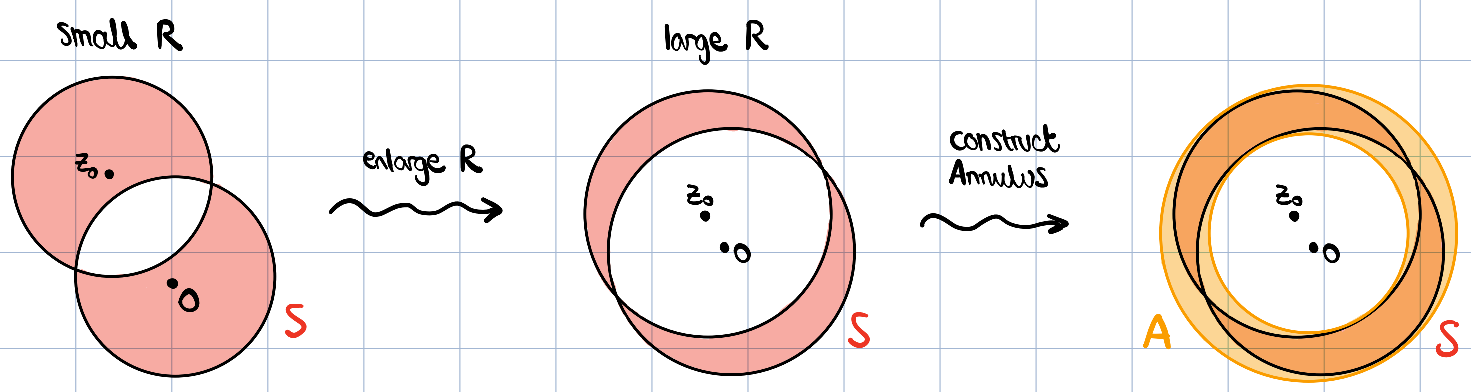
Remark: We give this alternate proof using harmonicity.

Proof: Fix $z_0 \in \mathbb{C}$ and any $R>0$ sufficiently large. By the Mean Value Property, we have

$$u(z_0) - u(0) = \frac{1}{\pi R^2}\iint_{\bar{D}(z_0,R)} u(x+iy)\,d(x,y) - \frac{1}{\pi R^2}\iint_{\bar{D}(0,R)} u(x+iy)\,d(x,y)$$

$$= \frac{1}{\pi R^2}\iint_S u(x+iy)\,d(x,y) \quad \text{where } S = \big(\bar{D}(z_0,R)\setminus \bar{D}(0,R)\big) \cup \big(\bar{D}(0,R)\setminus\bar{D}(z_0,R)\big)$$

$$= \bar{D}(0,R) \,\triangle\, \bar{D}(z_0,R) \quad \text{the symmetric difference.}$$



We construct an Annulus A such that $S \subseteq A$. Notice as R gets large, the annulus gets thinner.

Thus $\operatorname{area}(A) \to \pi(R+|z_0|)^2 - \pi R^2$.

Let $M = \sup_{z\in\mathbb{C}} |u(z)| < \infty$ by assumption, so $|u(z_0)-u(0)| \leq \frac{1}{\pi R^2}\iint_S |u(x+iy)|\,d(x,y)$

$$\leq \frac{1}{\pi R^2}\iint_A M\,d(x,y) = \frac{\operatorname{area}(A)}{\pi R^2} \xrightarrow{R\to\infty} 0.$$

$\Rightarrow |u(z_0)-u(0)| = 0$ so u is a constant function. $\square$

Remark: Some details are excluded in favour of the picture proof.

Theorem (Maximum Principal for Harmonic Functions):

Let $\mathcal{U} \subseteq \mathbb{C}$ be open and connected. Let $u \in Har(\mathcal{U})$ be non-constant. Then u achieves its maximum on $\partial \mathcal{U}$.

Proof: Technical. Refer a textbook.

We know that harmonic functions admit a Mean Value Property ($u \in Har(\mathcal{U}) \Rightarrow u$ admits Mean Value Property).

Does this hold both ways?

Theorem (Dirichlet Problem on a disc):

continuous

Let $\beta : \partial \mathbb{D} \longrightarrow \mathbb{R}$ be continuous. Then there exists $u \in C^1(\overline{\mathbb{D}})$ on $\overline{\mathbb{D}}$ such that $u|_{\partial \mathbb{D}} = \beta$ and $u|_{\mathbb{D}} \in Har(\mathbb{D})$.

Remark: This is a PDE-type result since we are implicitly solving the Laplacian.

Proof: 0 Constructing a plausible formula.

If such a u existed, then for $r \in (0,1)$, we would have the Mean Value Property, so

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\,dt \xrightarrow[\text{needs justification}]{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \beta(e^{it})\,dt.$$

Accepting this, for $z \in \mathbb{D}$, let $\Psi_z(w) = \frac{w-z}{1-\bar{z}w} \in Aut(\mathbb{D})$ and recall this is self-inverting.

Thus we should be able to apply our formula to $u \circ \Psi_z$, so

$$u(z) = u \circ \Psi_z(0) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\Psi_z(e^{it}))\,dt$$

Inspect that $\Psi_z(e^{it}) \in \partial \mathbb{D}$

The equation $\frac{1}{2\pi}\int_0^{2\pi} \beta(\Psi_z(e^{it}))\,dt$ is a Poisson kernel for β. See Fourier analysis and Measure.

For $z \in \mathbb{D}$, define $u(z) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\Psi_z(e^{it}))\,dt$.

1 $u \in Har(\mathbb{D})$ is harmonic.

We define a change of variables. Let $z \in \mathbb{D}$ be fixed. Write $e^{i\theta} = \Psi_z(e^{it}) \Rightarrow \Psi_z(e^{i\theta}) = e^{it}$ (inverting).
(we are defining θ as a function of t)

Applying $\frac{d}{dt}$ to both sides:

$$\Psi_z'(e^{i\theta})\, ie^{i\theta}\frac{d\theta}{dt} = ie^{it}. \quad \text{Notice } \Psi_z'(w) = \frac{(\bar{z}w-1)-(w-z)}{(\bar{z}w-1)^2} = \frac{|z^2-1|}{(\bar{z}w-1)^2}.$$

Since $e^{it} = \Psi_z(e^{i\theta})$ we have

$$\Psi_z'(e^{i\theta})\,\cancel{i}e^{i\theta}\frac{d\theta}{dt} = \cancel{i}\underset{\Psi_z(e^{i\theta})}{\underset{\shortparallel}{e^{it}}} \Rightarrow \Psi_z'(e^{i\theta})\,\overline{\Psi_z(e^{it})}\,e^{i\theta}\frac{d\theta}{dt} = 1.$$

at some point, we somehow change θ to t. Inspect the arithmetic here.

... we should eventually get $\frac{|z|^2-1}{|z-e^{i\theta}|}\frac{d\theta}{dt} = 1.$

Notice that

$$\frac{z+e^{i\theta}}{z-e^{i\theta}} = \frac{(z+e^{i\theta})(\bar{z}+e^{-i\theta})}{|z-e^{i\theta}|^2} = \frac{z\bar{z}+ze^{-i\theta}+\bar{z}e^{i\theta}+1}{|z-e^{i\theta}|^2} = \frac{|z|^2-1+i\,\mathrm{Im}(\bar{z}e^{i\theta})}{|z-e^{i\theta}|^2}.$$

Therefore $\dfrac{|z|^2-1}{|z-e^{i\theta}|^2} = Re\left(\dfrac{|z|^2-1+i\,Im(\bar{z}e^{i\theta})}{|z-e^{i\theta}|^2}\right) = Re\left(\dfrac{z+e^{i\theta}}{z-e^{i\theta}}\right).$

Thus, if $z\in\mathbb{D}$, we have $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\beta(\Psi_z(e^{it}))\,dt$

$$= \frac{1}{2\pi}\int_0^{2\pi}\beta(e^{i\theta})\frac{|z|^2-1}{|z-e^{i\theta}|^2}\frac{d\theta}{dt}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\beta(e^{i\theta})\,Re\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)d\theta$$

$$= Re\Bigg(\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\beta(e^{i\theta})\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)d\theta}_{f(z)}\Bigg) =: Re(f(z)).$$

To show $u\in Har(\mathbb{D})$ we can show $f\in\mathcal{H}(\mathbb{D})$.

1.1 $f(z)\in\mathcal{H}(\mathbb{D})$ is holomorphic.

Let T be a triangle in $\mathbb{D}$. We consider $\displaystyle\int_{\partial T} f = \frac{1}{2\pi}\int_{\partial T}\int_0^{2\pi}\beta(e^{i\theta})\left(\frac{z+e^{i\theta}}{z-e^{i\theta}}\right)d\theta\,dz$

$$\overset{\text{Fubini's}}{=} \frac{1}{2\pi}\int_0^{2\pi}\beta(e^{i\theta})\int_{\partial T}\frac{z+e^{i\theta}}{z-e^{i\theta}}\,dz\,d\theta$$

rational functions are holomorphic

$$= \frac{1}{2\pi}\int_0^{2\pi}\beta(e^{i\theta})\cdot 0\,d\theta = 0 \text{ by Cauchy's Theorem.}$$

By Morera's Theorem, we have $f\in\mathcal{H}(\mathbb{D})$. Thus $u = Re(f) \Rightarrow u\in Har(\mathbb{D})$. $\triangle$

2 $u|_{\partial\mathbb{D}} = \beta$. Consider $z\in\partial\mathbb{D}$ so $|z| = z\bar{z} = 1$.

occurs at one point

Then $\Psi_z(e^{it}) = \dfrac{z-e^{it}}{1-\bar{z}e^{it}} = \dfrac{e^{it}-z}{\bar{z}e^{it}-1} = z\,\dfrac{e^{it}-z}{e^{it}-z} = z$ unless $z = e^{it}$.

so $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\beta(\Psi_z(e^{it}))\,dt \overset{\text{except at one point}}{=} \frac{1}{2\pi}\int_0^{2\pi}\beta(z)\,dt = \beta(z).$ Singular points have measure 0.

$\triangle$

3 $u\in C^1(\mathbb{D})$ continuous.

A uniform continuity argument that uses boundedness. Note that $\Psi_z(e^{it})$ may not always be continuous (above).

$\triangle$ $\square$

Let $\mathcal{U}\subseteq\mathbb{C}$ be open. We will say $u:\mathcal{U}\to\mathbb{R}$ has Mean Value Property if for any $z\in\mathcal{U}$ and $R>0$ so $\overline{D}(z,R)\subset\mathcal{U}$,

that $u(z)=\frac{1}{2\pi}\int_0^{2\pi}u(z+Re^{it})\,dt$. Recall that $u\in Har(\mathcal{U})\Rightarrow u$ has Mean Value Property.

Lemma: Let $v:\overline{\mathbb{D}}\to\mathbb{R}$ be continuous (so v is bounded) such that

- v has Mean Value Property;
- $v|_{\partial\mathbb{D}}=0$.

Then $v=0$ on $\overline{\mathbb{D}}$.

Proof: Recall Maximum Principle.

v non-constant $\Rightarrow$ v admits no maximum in $\mathbb{D}$ but $v|_{\partial\mathbb{D}}=0$ so $v=0$ constant. $\square$

Theorem (Mean Value Property $\Rightarrow$ Harmonicity): Let $\mathcal{U}\subseteq\mathbb{C}$ be open. Let $u:\mathcal{U}\to\mathbb{R}$ be continuous with the Mean Value Property. Then $u\in Har(\mathcal{U})$.

Proof: 1 Suppose $\overline{\mathbb{D}}\subset\mathcal{U}$. We show that $u|_{\mathbb{D}}\in Har(\mathbb{D})$.

We have $\partial\mathbb{D}\subset\overline{\mathbb{D}}\subset\mathcal{U}\Rightarrow u|_{\partial\mathbb{D}}$ is continuous.

Consider the Poisson kernel $\tilde{u}(z)=\frac{1}{2\pi}\int_0^{2\pi}u(\Psi_z(e^{it}))\,dt$. Then $\tilde{u}|_{\mathbb{D}}\in Har(\mathbb{D})$ and $\tilde{u}|_{\partial\mathbb{D}}=u$

and $\tilde{u}\in C^1(\overline{\mathbb{D}})$ continuous.

Let $v=u-\tilde{u}$ so $v|_{\partial\mathbb{D}}=0$ and v has Mean Value Property on $\mathbb{D}$ $\overset{\text{Lemma}}{\Longrightarrow}$ $v=0$ on $\overline{\mathbb{D}}$

$\Rightarrow\tilde{u}=u\Rightarrow u\in Har(\mathbb{D})$. $\triangle$

2 Extending using Möbius maps.

Let $z_0\in\mathcal{U}$ and $R>0$ such that $\overline{D}(z_0,R)\subset\mathcal{U}$. We show that $u|_{D(z_0,R)}$ harmonic $\Rightarrow u\in Har(\mathcal{U})$.

We use conformal mapping. Let $\mu:\overline{\mathbb{D}}\to\overline{D}(z_0,R)$, $z\mapsto Rz+z_0$ defining a Möbius map.

If $z\in\mathbb{D}$ and $r\in(0,1)$ we have $\frac{1}{2\pi}\int_0^{2\pi}u\circ\mu(z+re^{it})\,dt=\frac{1}{2\pi}\int_0^{2\pi}u(Rz+Rre^{it}+z_0)\,dt$ in $D(z_0,R)$.

$$\Rightarrow\frac{1}{2\pi}\int_0^{2\pi}u\circ\mu(z+re^{it})\,dt=\frac{1}{2\pi}\int_0^{2\pi}u(Rz+z_0+Rre^{it})\,dt$$

$=u(Rz+z_0)=u\circ\mu(z)\Rightarrow u\circ\mu\in Har(\mathbb{D})$ by (1).

But Möbius maps are automorphisms!

Thus $u\circ\mu\circ\mu^{-1}\in Har(D(z_0,R))$ since μ^{-1} is holomorphic too. $\triangle$ $\square$

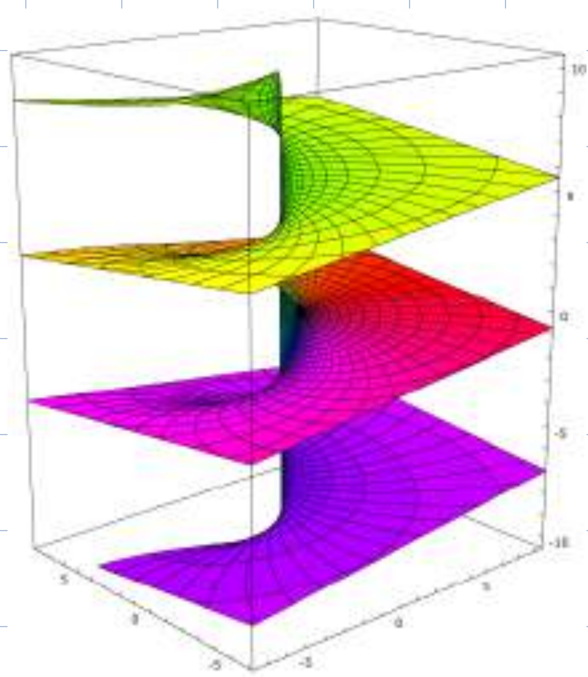
Riemann Surfaces and Log

The various branches of log cannot be glued to give a continuous function $\log : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ because two branches may give different values at points where both are defined.

Consider two copies: $\mathrm{Log}(z)$ on $\mathbb{C} - \mathbb{R}_{\leq 0}$ so $\theta \in (-\pi, \pi)$. Notice they agree on the upper half plane but
$\mathrm{Log}(z)$ on $\mathbb{C} - \mathbb{R}_{\geq 0}$ so $\theta \in (0, 2\pi)$. not on the lower half plane.

Thus we can glue along the upper half plane, but we will have two copies of the lower half.

We can continue by gluing more branches, which gives us a connected surface.



This gives a Riemann surface R associated with $\log z$.

A point on $\mathbb{R}$ can be thought of as a pair $(z, \theta) \in \mathbb{C} \times \mathbb{R}$ so the surface

R can be embedded in $\mathbb{C} \times \mathbb{R} \approx \mathbb{R}^3$.

This way, the branches glue to give a well-defined function $\log_R : R \to \mathbb{C}$.

$$(z, \theta) \mapsto \ln|z| + i\theta$$

This is the analytic continuation of Log.

There is a projection map $R \to \mathbb{C}\setminus\{0\}$ that "flattens" the Riemann surface,

sending $(z, \theta) \mapsto z$. The projection map $R \to \mathbb{C}\setminus\{0\}$ realizes R as a covering space of $\mathbb{C}\setminus\{0\}$. In fact, it is a Galois covering with deck transformation group isomorphic to $\mathbb{Z}$, generated by the homeomorphism $(z, \theta) \mapsto (z, \theta + 2\pi)$.

As a complex manifold, R is biholomorphic with $\mathbb{C}$ via $\log_R$, since the inverse $z \mapsto (e^z, \mathrm{Im}(z))$ is holomorphic.

Thus, R is simply connected, so R is the universal cover of $\mathbb{C}\setminus\{0\}$.

A manifold is a topological space that locally resembles Euclidean space. More precisely, an n-dimensional manifold is a topological space such that each point has a neighbourhood homeomorphic to some open subset of n-dimensional Euclidean space.

Circles, lines and curves are 1-manifolds. Surfaces are 2-manifolds.

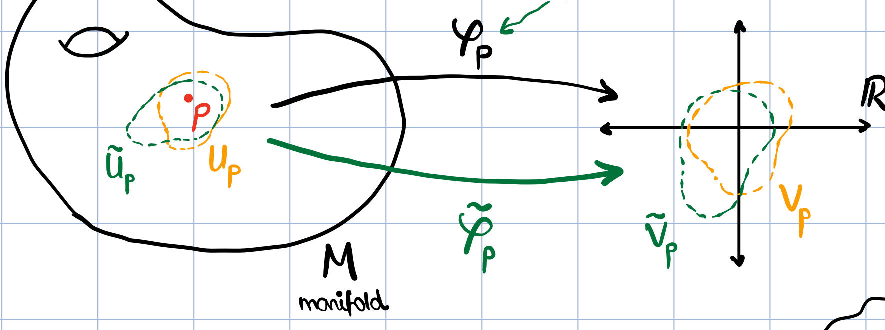
A Riemann surface X is a complex manifold of complex dimension one. This means that X is a connected Hausdorff space endowed with an atlas of charts to $\mathbb{D} \subset \mathbb{C}$. For every $x \in X$, there is a neighbourhood of x homeomorphic to $\mathbb{D}$, and the transition maps between two overlapping charts are required to be holomorphic.

Complex Manifolds : Brief Overview

A manifold is a topological space that is Hausdorff and locally Euclidean.

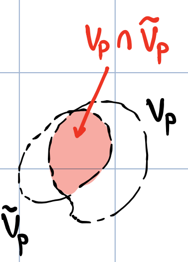
In other words, every $p \in X$ has a neighbourhood $U_p \underset{Top}{\cong} V_p \subseteq \mathbb{R}^n$ homeomorphic to some open set V_p in $\mathbb{R}^n$ with $p \mapsto 0 \in \mathbb{R}^n$.

$\varphi_p : U_p \to V_p$ with $\varphi_p(p) = 0$ homeomorphism



If neighbourhoods of p can be identified with $V_p \subseteq \mathbb{R}^n$, ← has coordinates

we must verify these coordinates are well-defined by looking at overlaps. Consider another neighbourhood $\tilde{U}_p$.



If we consider the map $\underset{\subseteq \mathbb{R}^n}{V_p \cap \tilde{V}_p} \xrightarrow{\tilde{\varphi}_p^{-1}} \tilde{\varphi}_p^{-1}(V_p \cap \tilde{V}_p) = U_p \cap \tilde{U}_p \xrightarrow{\varphi_p} V_p \cap \tilde{V}_p \subseteq \mathbb{R}^n$

between open subsets of $\mathbb{R}^n$, we have a meaning for calculus. $\varphi_p \circ \tilde{\varphi}_p^{-1}$

we can specify the regularity of these manifolds by the regularity of these maps.

A chart for a manifold M is an open set $\mathcal{U} \subseteq M$ together with a homeomorphism $\varphi : \mathcal{U} \to V \subseteq \mathbb{R}^n$.

gives a sensible way of adding coordinates to or pulling back coordinates from a manifold.

An atlas (of charts) for M is a collection of charts $\underbrace{\{U_\alpha, \varphi_\alpha\}}_{=: A}$ such that M is covered by U_α (so $M = \bigcup_\alpha U_\alpha$ defines an open covering of M)

and we say that A is smooth if the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \underset{\subseteq \mathbb{R}^n}{\varphi_\alpha(U_\alpha \cap U_\beta)} \longrightarrow \underset{\subseteq \mathbb{R}^n}{\varphi_\beta(U_\alpha \cap U_\beta)}$ are C^∞-smooth.

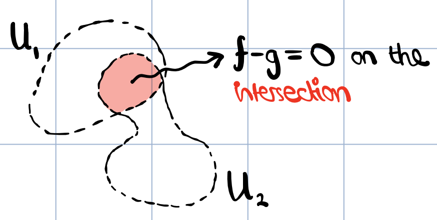
A complex manifold is a smooth manifold X (with a smooth atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$) locally homeomorphic to $\mathbb{C}^n$.

Therefore, our charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}^n$ and our transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \underset{\subseteq \mathbb{C}^n}{\varphi_\alpha(U_\alpha \cap U_\beta)} \longrightarrow \underset{\subseteq \mathbb{C}^n}{\varphi_\beta(U_\alpha \cap U_\beta)}$ are holomorphic. (in several $\mathbb{C}$-variables)

map to open subsets of $\mathbb{C}^n$

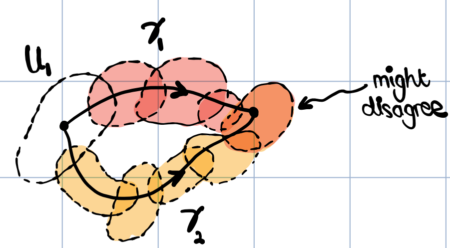
Theorem (Hartog's Fundamental Theorem): A function $f : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}^n$-holomorphic at a point $\zeta = (z_1, \dots, z_n) \in \mathbb{C}^n$ if f is holomorphic with respect to each variable $z_1, \dots, z_n$ at ζ.

Let $U_1, U_2 \subseteq \mathbb{C}$ be open and connected, and let $f \in \mathcal{H}(U_1)$ and $g \in \mathcal{H}(U_2)$. If $V := \overset{\text{open}}{U_1 \cap U_2} \neq \emptyset$ and $f|_V = g|_V$, then we say g is an analytic continuation of f to U_2. This makes sense from Theorem (Zero Sets).



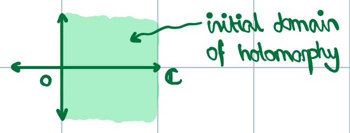
Given some set U, we can show that analytic continuation is unique.

We can define analytic continuation along a path γ this way, but notice that two continuations along γ_1 and γ_2 may not agree. They agree when γ_1 and γ_2 are contour homotopic in U.



(i) Continuation of $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt$. Notice $\Gamma(s+1) = s\Gamma(s)$ and we have the integral converging for $Re(s) > 0$.

We can show $\Gamma(s)$ is holomorphic for $Re(s) > 0$.



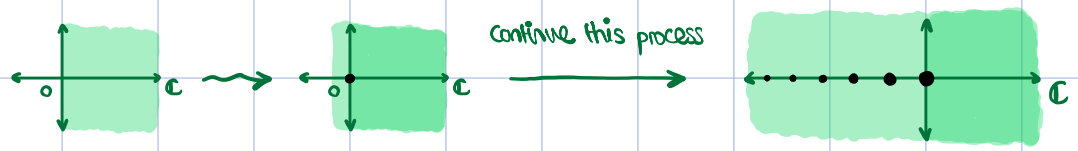
1 $\Gamma \in \mathcal{H}(\{s \in \mathbb{C} : Re(s) > 0\})$.

We must show that $\frac{d}{ds}\Gamma(s)$ exists, so it suffices to show that $\frac{d}{ds}\int_0^\infty e^{-t}t^{s-1}\,dt = \int_0^\infty \frac{\partial}{\partial s} e^{-t}t^{s-1}\,dt$.

Intuition: split into compact and unbounded parts $\int_0^\alpha + \int_\alpha^\infty$ and show that the unbounded part is small.

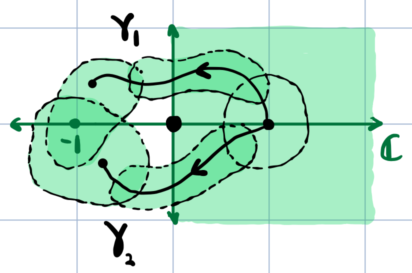
use Leibniz on compact part

Notice $\Gamma(s) = s^{-1}\Gamma(s+1)$ so we can define a new $\Gamma(s)$ for $Re(s) > -1$. This gives a singularity at $s = 0 \in \mathbb{C}$.



We get singularities for any $s \in \mathbb{Z}_{\leq 0}$.

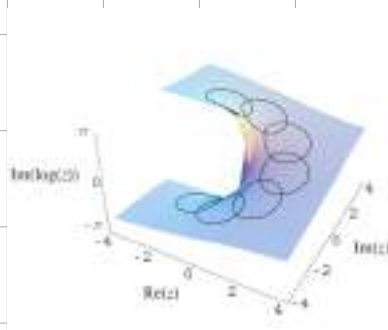
(ii) Continuation of $Log(z)$. We'll notice that continuing in two non-homotopic contours yields different results:



Continuation by γ_1 says $Log(-1) = \pi i$.

Continuation by γ_2 says $Log(-1) = -\pi i$.

we are really traversing the Riemann surface of $Log(z)$.



Let γ be a contour with start point $\gamma(0) = z_0$ and let $f \in \mathcal{H}(D(z_0, R))$ ($=: U \subseteq \mathbb{C}$) for some $R > 0$. An analytic continuation of (f, U) along γ is a collection of pairs (f_t, U_t) for $t \in [0,1]$ such that

- $(f_0, U_0) = (f, U)$;
- For each t, the set $U_t = D(\gamma(t), R_t)$ is an open disk and $f_t \in \mathcal{H}(U_t)$;
- For t, t' sufficiently close, the sets U_t and $U_{t'}$ coincide, and $f_t, f_{t'}$ agree on this intersection. $(U_t \cap U_{t'} \neq \emptyset)$

Theorem (Monodromy): Let $U \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(U)$. Let γ_0, γ_1 be contours that are contour homotopic. Then any analytic continuation along γ_0 agrees with any analytic continuation along γ_1 at the endpoint $p = \gamma_0(1) = \gamma_1(1) \in \mathbb{C}$.

Let $U \subseteq \mathbb{C}$ be open and connected. If $f \in \mathcal{A}(U) = \mathcal{H}(U)$ we call the pair (f, U) a function element.

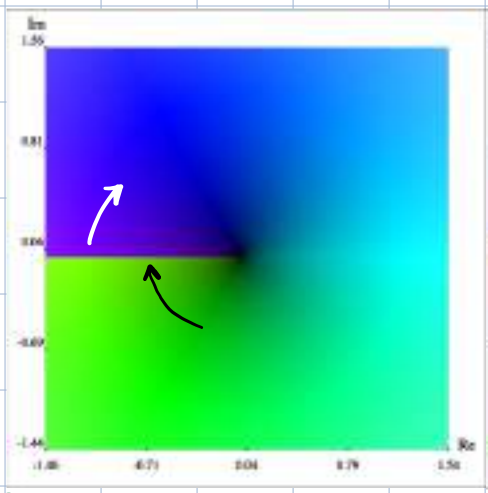
If f_2 is an analytic continuation of f_1 from U_1 to U_2, we can define an equivalence relation and write $(f_1, U_1) \sim (f_2, U_2)$.

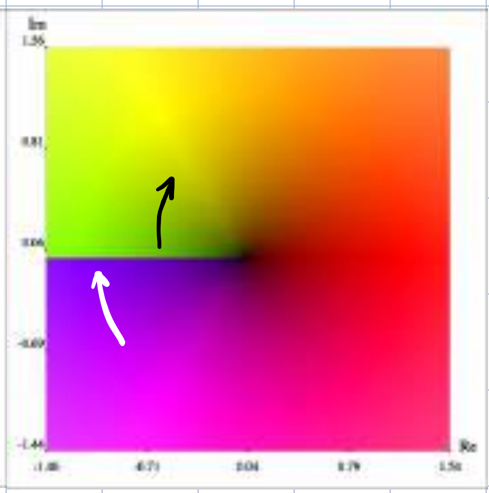
A complete analytic function is an equivalence class under the equivalence relation defined above, i.e. $[(f, U)]$, of function elements.

If there exist function elements (f_1, U_1) and (f_2, U_2) of a complete analytic function F such that $f_1|_{U_1\cap U_2} \neq f_2|_{U_1 \cap U_2}$ (disagreement on intersection), we say F is multiform. Otherwise, we say F is uniform.

(i) Consider $f(z) = \sqrt{z}$. Clearly f is multiform since we can choose either $r^{\frac{1}{2}}e^{\frac{i\theta}{2}}$ or $r^{\frac{1}{2}}e^{\frac{i\theta}{2}+\pi}$ for $f(re^{i\theta})$. This is because $z \mapsto z^2$ is two-to-one.

$\Rightarrow$ we can construct a Riemann surface for $\sqrt{z}$.



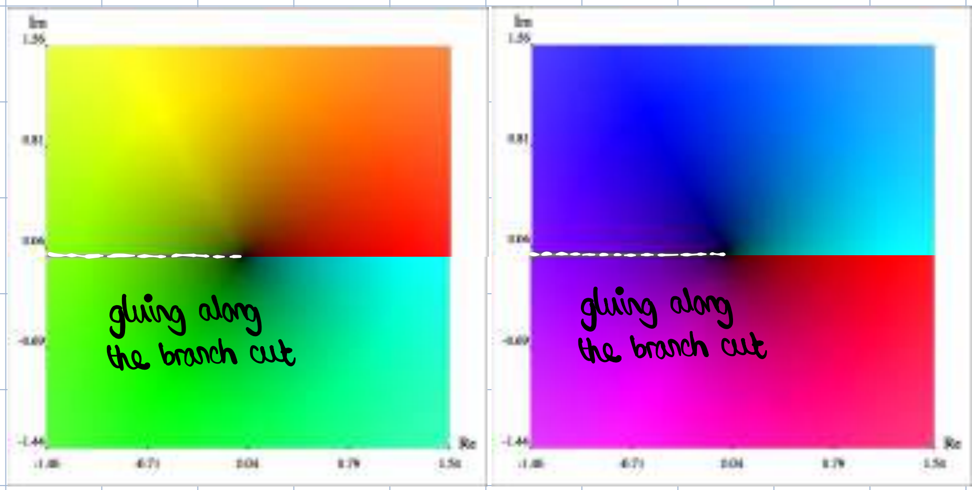


$\mathbb{C}_1$ the negative sheet of $\sqrt{z}$

$re^{i\theta} \longmapsto r^{\frac{1}{2}}e^{\frac{i\theta}{2}+\pi}$

$\mathbb{C}_2$ the positive sheet of $\sqrt{z}$

$re^{i\theta} \longmapsto r^{\frac{1}{2}}e^{\frac{i\theta}{2}}$



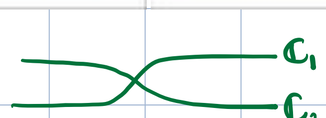
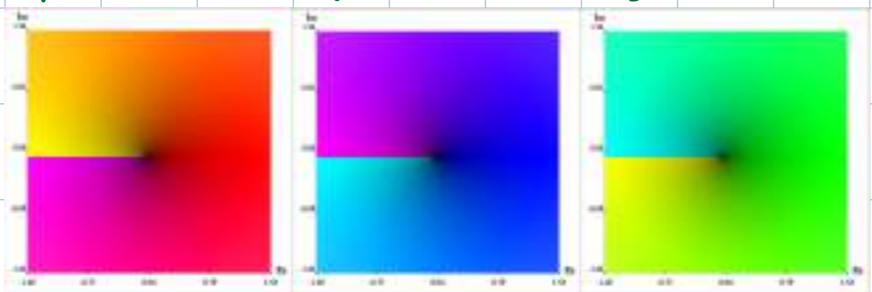


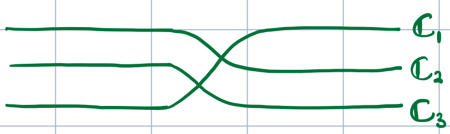
diagram of gluing sheets $\mathbb{C}_1$ and $\mathbb{C}_2$

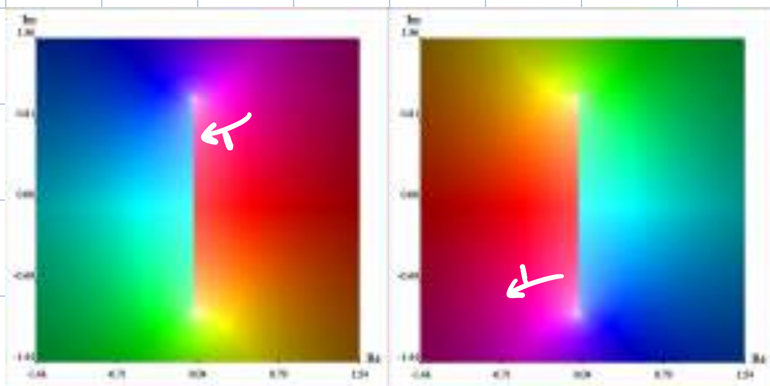
Each sheet comes from a different function element equivalence class of f.

$\mathbb{C}_1$ $\mathbb{C}_2$ $\mathbb{C}_3$



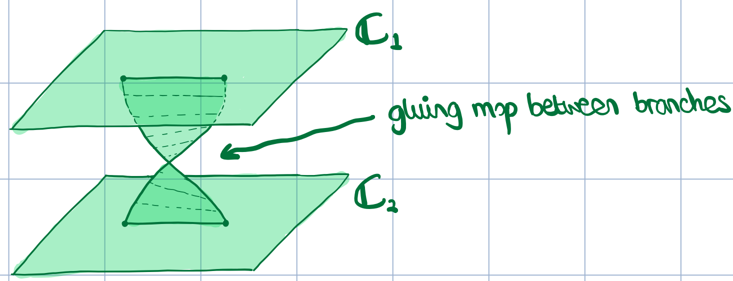
Sheets of $f(z) = z^{\frac{1}{3}}$ (cube root)





$\arctan\left(\frac{1}{z}\right)$

construct Riemann surface



The multiform nature of these functions is somehow "fixed" by the multiplicity of sheets of these functions. Even the geometry of the singularity is obvious.

We can redefine contour integration on these Riemann surfaces instead.

For instance, define $\gamma(t) = e^{it}$ for $t \in [0, 4\pi]$. This winds $0 \in \mathbb{C}$ twice.

winding number seems to be related to keeping track of which sheet we are in.

This gives a natural segué into homology. We're secretly been exploring these ideas as cycles Γ and winding number.

For later:

- Analytic continuation $\rightsquigarrow$ Riemann surfaces $\rightsquigarrow$ Covering spaces (algebraic topology)
  - $\rightsquigarrow$ germs $\downarrow$ define a topology on $\mathcal{G}$ set of germs $\int$ sheaf theory
  - charts and atlases
  - monodromy theory
  - $\rightarrow$ multivariate $\mathbb{C}$-analysis $\Rightarrow$ sheaf cohomology
- Analytic number theory: proof of Dirichlet's theorem $\rightsquigarrow$ analytic theory of $\zeta(s)$.
  - $\rightsquigarrow$ Dirichlet characters $\rightsquigarrow$ non-vanishing
  - $\downarrow$ L-functions and Dirichlet series
- Investigation of $\Gamma(s)$ gamma function

Topology $\rightsquigarrow$ Smooth manifolds $\rightarrow$ Complex manifolds

+ Differential Geometry

Hausdorff spaces

Algebraic topology