Enumeration and Graph Theory

Started with basics: Cartesian product, basic arrangements, etc.

Lemma: Let A, B be disjoint finite. Then $|A \cup B| = |A| + |B|$.

Lemma: Let A, B be finite sets. Then:

1. $|A \cup B| = |A| + |B| - |A \cap B|$.   2. $|A \times B| = |A| \cdot |B|$.

Lemma: Let A be a finite set. Then $|P(A)| = 2^{|A|}$.

Proof: Binary choice to include/exclude each element. $\square$

Let S be a set with $|S| < \infty$.

A permutation of S is an ordered arrangement of S.

A partial permutation of S is a permutation of a subset $K \subseteq S$.

The number of partial permutations with k elements of an n-element set A is $\frac{n!}{(n-k)!}$.

"*Proof*": We have k slots and n choices. $\underset{n}{\square}\,\underset{n-1}{\square}\cdots\underset{n-(k-1)}{\square}$ so we multiply all. $\square$

Lemma: Let S be a set with $|S| = n$. Let $\mathcal{A}_k = \{A \in P(S) \mid |A| = k\}$ be the set of k-element subsets of S.

Then $|\mathcal{A}_k| = \frac{n!}{k!(n-k)!}$.

Proof: Notice each $\mathcal{A}_k$ defines exactly $k!$ partial permutations, so $|\mathcal{A}_k| = \frac{n!}{k!(n-k)!}$. $\square$

We define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ the binomial coefficient.

Theorem (Binomial): For $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof: Suffices to prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ since $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

$$\Rightarrow \left(1+\frac{z}{y}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{y}\right)^k \quad \text{using } x = \frac{z}{y}$$

$$\Rightarrow y^n\left(1+\frac{z}{y}\right)^n = \sum_{k=0}^{n} \binom{n}{k} z^k y^{n-k} = (y+z)^n.$$

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$: Consider $(1+x)(1+x)\cdots(1+x) = (1+x)^n$

For each term, we have n choices of either 1 or x $\Rightarrow \binom{n}{k}$ ways of choosing an x term. $\square$

Lemma: Let $n \in \mathbb{N}$. Then $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

Proof: Consider a set S with $|S| = n$. Clearly $|P(S)| = 2^n$ from before.

Now consider the set $S_k = \{X \subseteq S \mid |X| = k\}$ and notice $i \neq j \Rightarrow S_i \cap S_j = \emptyset$ disjoint.

For $k \in \mathbb{N}$, we have $|S_k| = \binom{n}{k}$ since we choose k elements from S,

and $\bigcup_{k=0}^{n} S_k = P(S)$ and $S_i \cap S_j$ when $i \neq j$ $\Rightarrow \left|\bigcup_{k=0}^{n} S_k\right| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n}{k} \Rightarrow |P(S)| = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\square$

Lemma: Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof: Similar combinatorial argument as above. $\square$

Alternatively we could form a bijection between two sets and show their sizes are equal.

Lemma: Let $n \in \mathbb{N}$ and let $0 \leq k \leq n$. Then $\binom{n}{k} = \binom{n}{n-k}$.

Proof: Let S be a set with $|S| = n$, and define $S_k = \{X \subseteq S \mid |X| = k\}$. Notice $|S_k| = \binom{n}{k}$ and $|S_{n-k}| = \binom{n}{n-k}$.

Define a function $\varphi: S_k \to S_{n-k}$, $X \mapsto S \setminus X$. Notice this is well-defined, and has an inverse $\varphi^{-1}: S_{n-k} \to S_k$, $X \mapsto S \setminus X$.

$\Rightarrow \varphi$ bijection and S_k, S_{n-k} finite cardinality $\Rightarrow |S_k| = |S_{n-k}| \Rightarrow \binom{n}{k} = \binom{n}{n-k}$. $\square$

Let S be a set. A function $\omega: S \to \mathbb{N}_0$ is a weight function if the preimage $|\omega^{-1}(\{n\})| < \infty$ has finite size.

We define the generating series of S with respect to ω by $\Phi_S^\omega(x) = \sum_{\sigma \in S} x^{\omega(\sigma)}$.

The coefficient of x^n in a generating series is denoted $[x^n]\,\Phi_S^\omega(x)$.

Then $[x^n]\,\Phi_S^\omega(x)$ is $|\omega^{-1}(\{n\})|$ the number of elements in S of weight n.

Theorem: If A, B are sets and $A, B, A \times B$ have weight functions $\omega_1, \omega_2, \omega$ respectively, and $\omega((a,b)) = \omega_1(a) + \omega_2(b) + c$ holds for all $(a,b) \in A \times B$, then $\Phi_{A\times B}^\omega(x) = x^c\, \Phi_A^{\omega_1}(x)\, \Phi_B^{\omega_2}(x)$.

Proof: Exercise. $\square$

Theorem (Product Rule for k-tuples): If $A_1, \dots, A_k, \bigtimes_{i=1}^k A_i$ have weight functions $\omega_1, \dots, \omega_k, \omega$ respectively, and $\omega((a_1, \dots, a_k)) = \omega_1(a_1) + \dots + \omega_k(a_k) + c$ for all $(a_1, \dots, a_k) \in \bigtimes_{i=1}^k A_i$, then $\Phi^\omega_{\bigtimes_{i=0}^k A_i}(x) = x^c \prod_{i=0}^k \Phi_{A_i}^{\omega_i}(x)$.

A formal power series is an object of the form $A(x) = \sum_{i \in \mathbb{N}} a_i x^i$ where $a_i \in \overset{\text{ring}}{R}$. Denote the set of these by $R[[x]]$.

We define the operations $(+, \cdot)$ for such objects by

$$\sum_{i \in \mathbb{N}} a_i x^i + \sum_{i \in \mathbb{N}} b_i x^i = \sum_{i \in \mathbb{N}} \underbrace{(a_i + b_i)}_{\text{ring addition}} x^i \quad \text{and} \quad \sum_{k \in \mathbb{N}} a_k x^k \sum_{k \in \mathbb{N}} b_k x^k = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \overset{\text{ring multiplication}}{a_i b_j}\, x^{i+j} = \sum_{i,j \in \mathbb{N}} a_i b_j\, x^{i+j}.$$

Lemma: The set $R[[x]]$ where R is a ring forms a ring.

Proof: Definitions. $\square$

Remark: We care about $R = \mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}$ mostly. We care about no notion of convergence in $R[[x]]$.

Theorem ($R[[x]]^\times$): Let R be a ring. Then $A \in R[[x]]^\times \Leftrightarrow A(0) \in R^\times$ (constant term invertible in R). If A^{-1} exists, it is unique.

Proof: Exercise. $\square$

The geometric series is $\sum_{i\geq 0} x^i = \frac{1}{1-x}$ for $x \in B(0,1) \subseteq \mathbb{C}$.

Henceforth, we default to formal power series in $\mathbb{C}[[x]]$.

A composition function $\circ$ can be defined by $A \circ B(x) = A(B(x)) = a_0 B(x)^0 + a_1 B(x)^1 + a_2 B(x)^2 + \ldots$

Remark: Not always well-defined.

Theorem (Composition): Let $A, B \in \mathbb{C}[[x]]$. Then $A \circ B \in \mathbb{C}[[x]] \iff B(0) = 0$.

Proof: Need $[x^n] A \circ B(x)$ to be finite, only occurs when $B(x)$ has constant term 0. $\square$

Theorem (Negative Binomial): For $n \in \mathbb{N}$, we have $(1-x)^{-n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$.

Proof: Induction. Let $k=1$, and statement trivially true.

For some $n \geq 1$, say that $(1-x)^{-n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$.

Consider $(1-x)^{-(k+1)} = (1-x)^{-k}(1-x)^{-1}$

$$= \sum_{k\geq 0} \binom{n+k-1}{k-1} x^k (1-x)^{-1}$$

$$\Rightarrow [x^n](1-x)^{-(k+1)} = \sum_{i=0}^{n} [x^i](1-x)^{-m} \underbrace{[x^{m-i}](1-x)^{-1}}_{1 \text{ by geometric}}$$

$$= \sum_{i=0}^{n} [x^i](1-x)^{-m} = \sum_{i=0}^{n} \binom{i+m-1}{m-1} \overset{*}{=} \binom{n+m}{m}.$$ Done by comparing coefficients. $\square$

Midterm Study

A1 1) Let $A = \{a_1, \dots, a_6\}$.

a) all containing a_1 = $\underbrace{\text{all}}_{P(A)=2^6} - \underbrace{\text{all not containing } a_1}_{2^5 \text{ subsets}}$ $\Rightarrow$ we have $\underbrace{2^6 - 2^5}_{2^5}$ such subsets.

b) contain a_2 and a_3 but not a_4.

Fix a_2, a_3, a_4. Then we choose to include/exclude $a_1, a_5, a_6 \Rightarrow 2^3$ subsets.

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

2) a) $(1+x)^5 = \sum_{k=0}^{5} \binom{5}{k} x^k = \sum_{k=0}^{5} \frac{5!}{k!(5-k)!} x^k = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

b) $(1+2x)^5 = \sum_{k=0}^{5} \binom{5}{k} 2^k x^k = 1 + 2\cdot 5x + 2^2 \cdot 10x^2 + 2^3 \cdot 10 x^3 + 2^4 \cdot 5x^4 + 2^5 \cdot x^5$

3) a) $[x^7](1+3x^3)^{10}$: $(1+3x^3)^{10} = \sum_{k=0}^{10} \binom{10}{k} 3^k x^{3k}$ so x^7 term has coeff. 0.

b) $[x^9](1+3x^3)^{10}$: $\sum_{k=0}^{10} \binom{10}{k} 3^k x^{3k}$ so $k=3$ gives $\binom{10}{3} 3^3 x^9 \rightsquigarrow \binom{10}{3} 3^3$.

A3 Blocks and compositions.

T1) k successive dice rolls, total of n $\longleftrightarrow$ compositions of k with size n.

Let $P = \{1, \dots, 6\}$ be the part set so we have $S = P^k$.

By product lemma, $\Phi_S(x) = \Phi_{P^k}(x) = (\Phi_P(x))^k$. Since $P = \{1,\dots,6\}$ we have

$$\Phi_P(x) = x^1 + x^2 + \dots + x^6 = x(1 + x + \dots + x^5) = x\,\frac{x^6 - 1}{x - 1} = x\,\frac{(1-x^6)}{(1-x)}$$

$\Rightarrow \Phi_S(x) = \dfrac{x^k (1-x^6)^k}{(1-x)^k}$. We want $[x^n]\,\Phi_S(x)$.

Intuition: we want $\underbrace{(x + x^2 + \dots + x^6)}_{1^{st}\text{ roll}} \underbrace{(x + x^2 + \dots + x^6)}_{2^{nd}\text{ roll}} \cdots \underbrace{(x + x^2 + \dots + x^6)}_{k^{th}\text{ roll}}$ value rolled

n^{th} power coeff. = number of ways to sum to n

A4 A4) Say $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = 4$.

a) Convert to generating series.

$a_2 - 3a_1 + 2a_0 = 0$
$a_3 - 3a_2 + 2a_1 = 0$
$a_4 - 3a_3 + 2a_2 = 0$

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

$\Rightarrow \sum_{n\geq 2} (a_n - 3a_{n-1} + 2a_{n-2}) x^n = 0$

$\Rightarrow \underbrace{\sum_{n\geq 2} a_n x^n}_{A(x) - a_0 - a_1 x} - 3\underbrace{\sum_{n\geq 2} a_{n-1} x^n}_{xA(x) - a_0} + 2\underbrace{\sum_{n \geq 2} a_{n-2} x^n}_{x^2 \sum_{n\geq 2} a_{n-2} x^{n-2} \,=\, x^2 A(x)} = 0$

$\Rightarrow A(x) - a_0 - a_1 x - 3(xA(x) - a_0) + 2x^2 A(x) = 0 \rightsquigarrow$ solve for $A(x)$

$\Rightarrow A(x) - 3x A(x) + 2x^2 A(x) = a_0 + a_1 x + 3a_0 = 4a_0 + a_1 x$

$\Rightarrow A(x) = \dfrac{4a_0 + a_1 x}{1 - 3x + 2x^2}$ and $1 - 3x + 2x^2 = (1-x)(1-2x)$.

Thus $A(x)$ has inverse roots 1 and 2

$$\Rightarrow [x^n]A(x) = p_1(n)1^n + p_2(n)2^n \quad \text{where } \deg(p_i) < d_i \Rightarrow \deg(p_i)=0$$

$$a_n = C_1 + 2^n C_2 \overset{\text{initial cond}}{\Rightarrow} C_1 = -2 \text{ and } C_2 = 3$$

$$\Rightarrow a_n = -2 + 3\cdot 2^n \text{ for } n \geq 0.$$

1) $b_n = a_n^2 = (-2+3\cdot 2^n)^2 = 4 - 2(2)(3)2^n + 9\cdot 4^n$

$$= 4 - 12\cdot 2^n + 9\cdot 4^n = 4\cdot \underbrace{1}_{}{}^n - 12\cdot \underbrace{2}_{}{}^n + 9\cdot \underbrace{4}_{}{}^n$$

inverse roots

$\Rightarrow G(x)$ for b_n has denominator

$$(1-x)(1-2x)(1-4x) = 1 - 7x + 14x^2 - 8x^3$$

$$\Rightarrow b_n - 7b_{n-1} + 14b_{n-2} - 8b_{n-3} = 0 \text{ for } n \geq 3.$$

Theorem: Let $\{a_n\}_{n\geq 0}$ be a sequence satisfying the recurrence $a_n + c_1 a_{n-1} + \dots + c_d a_{n-d} = 0$ for $n \geq d$. Let $A(x) = \sum_{n\geq 0} a_n x^n$. Then $A(x) = \dfrac{P(x)}{1 + \sum_{i=1}^{d} c_i x^i}$ and $\deg(P) \leq n-1$.

Proof We have $a_n + c_1 a_{n-1} + \dots + c_d a_{n-d} = 0 \Rightarrow \sum_{n\geq d}(a_n + c_1 a_{n-1} + \dots + c_d a_{n-d})x^n = 0$

$$\Rightarrow \sum_{n\geq d} a_n x^n + \sum_{n\geq d} c_1 a_{n-1} x^n + \dots + \sum_{n\geq d} c_d a_{n-d} x^n = 0$$

so $\displaystyle\sum_{j=0}^{d}\Big(c_j \underbrace{\sum_{n\geq d} a_{n-j} x^n}_{x^j \sum_{n\geq d} a_{n-j} x^{n-j}}\Big) = 0$ where $c_0 = 1$

$$x^j\Big(\underbrace{\sum_{n\geq j} a_{n-j} x^{n-j}}_{A(x)} - \underbrace{\sum_{k=j}^{d-1} a_{n-j} x^{n-j}}_{\substack{B_j(x) \\ \deg(B_j) \leq n-j-1}}\Big) \Rightarrow \sum_{j=0}^{d} c_j x^j (A(x) - B_j(x)) = 0$$

$$\sum_{j=0}^{d} c_j x^j A(x) - \sum_{j=0}^{d} c_j x^j B_j(x) = 0$$

$$\Rightarrow \sum_{j=0}^{d} c_j x^j A(x) = A(x)\sum_{j=0}^{d} c_j x^j = \sum_{j=0}^{d} c_j x^j B_j(x)$$

$$\Rightarrow A(x) = \frac{\overbrace{\sum_{j=0}^{d} c_j x^j B_j(x)}^{\deg \leq n-1}}{\sum_{j=0}^{d} c_j x^j} = \frac{P(x)}{1 + \sum_{j=1}^{d} c_j x^j} \quad \text{as required.} \quad \square$$

$c_0 = 1$

Theorem: Let $G(x) = \dfrac{P(x)}{Q(x)}$ where $P(x), Q(x) \in \mathbb{C}[x]$ and $\deg(P) < \deg(Q)$. Let $Q(0) = 1$.

Suppose $Q(x) = (1-\lambda_1 x)^{d_1} \dots (1-\lambda_m x)^{d_m} = \prod_{i=1}^{m}(1-\lambda_i x)^{d_i} \in \mathbb{C}[x]$ with λ_i distinct.

1) $\exists C_{i,j} \in \mathbb{C}$ such that $G(x) = \left(\dfrac{C_{11}}{1-\lambda_1 x} + \dots + \dfrac{C_{1,d_1}}{(1-\lambda_1 x)^{d_1}}\right) + \dots + \left(\dfrac{C_{m,1}}{1-\lambda_m x} + \dots + \dfrac{C_{m,d_m}}{(1-\lambda_m x)^{d_m}}\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{d_i} \dfrac{C_{i,j}}{(1-\lambda_i x)^j}\right)$.

2) $[x^n]G(x) = p_1(n)\lambda_1^n + \dots + p_m(n)\lambda_m^n$ for some $p_i(n) \in \mathbb{C}[n]$ and $\deg(p_i) < d_i$.

Proof (1): Write $G(x) = \dfrac{P(x)}{\prod_{i=1}^{m}(1-\lambda_i x)^{d_i}}$ and apply partial fractions to decompose this into $\sum_{i=1}^{m} \dfrac{f_i(x)}{(1-\lambda_i x)^{d_i}}$ where $\deg(f_i) < d_i$. Continue partial fractions to get result. $\triangle$

(2): Careful geometric series analysis. $\triangle\square$

Graph Theory

A **graph** $G=(V,E)$ is a tuple such that $V(G)=V$ is a set of vertices and $E(G)=G$ is a set of **edges**.

unordered pairs from vertices $V(G)$

$u,v\in V(G)$ are **adjacent** when $uv\in E(G)$ is an edge.

The **neighbours** of $v\in V(G)$ is the set $\{v_0\in V(G) \mid vv_0\in E(G)\}$.

If $v\in V$ and $e\in E$, we say **v is incident to e** if $e=uv$ for some $u\in V$.

A **simple graph** is a graph G such that:

1. each $u,v\in V$ have at most one edge between them,
2. $vv\notin E$ for all $v\in V$,
3. $|V|<\infty$.

all graphs henceforth are simple

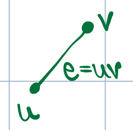
The **degree** of a vertex $v\in V$ is defined $\deg(v)=|\{e\in E \mid v\in E\}|$.

Theorem (Handshaking Lemma): G graph. Then $\sum_{v\in V}\deg(v)=2|E|$.

Proof: Each $e\in E$ is incident to exactly two vertices $v_i, v_j\in V$ such that $v_i\neq v_j$ (G simple).

Write $V=\{v_i\}_{i=1}^n$ so $\sum_{i=1}^n \deg(v_i)=\sum_{i=1}^n|\{e\in E \mid v_i\in e\}| = 2|E|$. $\square$

$e=v_iv_j$ gets counted twice



Corollary: G graph. Let $S_0=|\{v\in V \mid \deg(v) \text{ is odd}\}|$. Then $S_0\equiv 0 \bmod 2$.

Proof: Let $V_0=\{v\in V \mid \deg(v) \text{ odd}\}$. Then $\sum_{v\in V}\deg(v) = \sum_{v\in V_0}\deg(v)+\underbrace{\sum_{v\in V\setminus V_0}\deg(v)}_{\equiv 0 \bmod 2} = 2|E|\equiv 0 \bmod 2$

$$\Rightarrow \sum_{v\in V_0}\underbrace{\deg(v)}_{\equiv 1 \bmod 2}\equiv 0 \bmod 2$$

so $|V_0|=S_0\equiv 0 \bmod 2$. $\square$

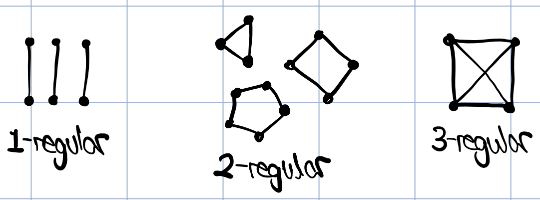
We define the **density** of a graph G by $\underbrace{\frac{1}{|V|}\sum_{v\in V}\deg(v)}_{\text{average degree of vertex}} = \frac{2|E|}{|V|}$. (Handshake)

Let $G_0=(V_0,E_0)$ and $G_1=(V_1,E_1)$ be graphs.

A function $f:V_0\to V_1$ is a **homomorphism** (written $f:G_0\to G_1$) when f **preserves edges** (so $uv\in V_0\Rightarrow f(u)f(v)\in V_1$).

If $f:V_0\to V_1$ is a bijection, and $f:G_0\to G_1$ is a homomorphism, then f is an **isomorphism** and $G_0\cong G_1$.

We say G is **k-regular** if $\deg(v)=k$ for all $v\in V$.



Naturally, define the **automorphism group** $\mathrm{Aut}(G)=(\{f:G\to G \mid f \text{ isomorphism}\}, \circ)$. (composition)

Lemma: G graph. Then $|\mathrm{Aut}(G)| \mid |V|!$.

Proof: Notice $\underset{\in\mathbb{Z}}{\mathrm{Aut}(G)}\subseteq F_G=\{\varphi:G\to G \mid \varphi \text{ bijection}\}$ and $|F_G|=|V|!\Rightarrow|\mathrm{Aut}(G)|\leq|V|!\Rightarrow|\mathrm{Aut}(G)| \mid |V|!$. $\square$

Naturally, we ask questions about subgraphs of G.

A subgraph $H \subseteq G$ is a graph such that $V(H) \overset{\neq\emptyset}{\subseteq} V(G)$ and $E(H) \subseteq E(G)$.

If $H \subseteq G$ with $V(H) = V(G)$, we say H spans G.

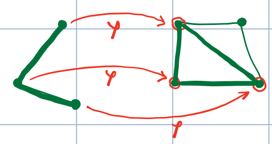
$E(H) = E(H)$, we say H is induced by G.

Theorem: G, H graphs. There exists an injective homomorphism $\varphi: G \to H \iff G$ is isomorphic to some subgraph $K \subseteq H$.

Proof ($\Rightarrow$): Say $\varphi: G \to H$ defines an injective homomorphism. Since G is simple, G is finite, so write $V(G) = \{g_i\}_{i=1}^n$.

Let $\varphi(g_i) = h_i \in H$ and notice $\varphi(V(G)) = \{\varphi(g_i)\}_{i=1}^n = \{h_i\}_{i=1}^n \subseteq V(H)$. Define $K = \varphi(G)$.

Notice that $\underset{h_i \mapsto g_i}{\varphi^{-1}: K \to G}$ defines an inverse for $\underset{g_i \mapsto h_i}{\varphi: G \to \varphi(G)}$ and thus $G \simeq K \subseteq H$. $\triangle$



($\Leftarrow$) Say $G \simeq K \subseteq H$ for some subgraph $K \subseteq H$, through $\varphi: G \to K$.

Extend φ to a function $\varphi: G \to H$ since $K \subseteq H$. Notice that $\varphi: G \to K$ injective $\Rightarrow \varphi: G \to H$ injective.

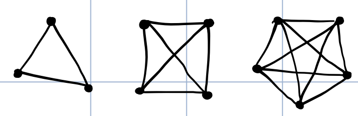
Since $K \subseteq H$, we have $V(K) \subseteq V(H)$ and $E(K) \subseteq E(H)$.

Thus $uv \in E(G) \Rightarrow \varphi(u)\varphi(v) \in E(K) \subseteq E(H) \Rightarrow \varphi(u)\varphi(v) \in E(H)$. Done. $\triangle\square$

Lemma: Let G be a k-regular graph. Then $k|V| = 2|E|$.

Proof. By Handshake, $\underbrace{\sum_{v \in V} \deg(v)}_{\sum_{v \in V} k \text{ k-regularity}} = 2|E| \Rightarrow k \sum_{v \in V} 1 = k|V| = 2|E|$. $\square$

A complete graph G is a graph where every pair of vertices is adjacent.



Lemma: Let G be complete with $|V| = n$. Then G is unique up to isomorphism.

Proof: Let G and H be complete graphs on n vertices. Write $V(G) = \{v_1, \dots, v_n\}$ and $V(H) = \{u_1, \dots, u_n\}$.

Define $\underset{v_i \mapsto u_i}{\varphi: G \to H}$ and notice $\varphi: V(G) \to V(H)$ bijective, and $v_i v_j \in E(G) \Rightarrow \varphi(v_i)\varphi(v_j) = u_i u_j \in E(H)$ (since H complete)

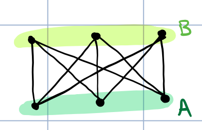
$\Rightarrow \varphi$ isomorphism. $\square$

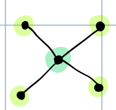
We denote the complete graph on n vertices by K_n.

Corollary: $|E(K_n)| = \frac{n(n-1)}{2}$.

Proof: K_n is $(n-1)$-regular, so $(n-1)\underbrace{|V|}_{n} = 2|E| \Rightarrow \frac{(n-1)(n)}{2} = |E(K_n)|$. $\square$

A graph is bipartite if there exists some bipartition $(\underset{\subseteq V}{A}, \underset{\subseteq V}{B})$ of $V(G)$ such that $uv \in E$ has one endpoint in A and the other in B.

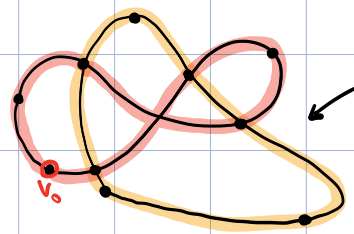




A walk along a graph G is a sequence of vertices $v_0 v_1 \cdots v_n$ such that $v_i v_{i+1} \in E$ for each i.

An Euler circuit (tour) is a closed walk along G such that each edge is traversed exactly once in the walk.

When does a graph G have an Euler tour?

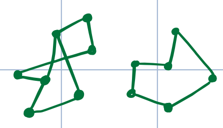


we must enter and exit each vertex $\Rightarrow$ each vertex must have even degree

Let G be a graph. If $v_0 v_1 \cdots v_n$ is a walk in G with $v_i \in V$ distinct, $v_i v_{i+1} \in E$, we say $v_0 v_1 \cdots v_n$ is a path ($v_0 v_n$-path).

If $v_0 = v_n$ and G is fully defined by the (closed) path above, then G is a cycle.

Theorem (Cycles): G graph with $\deg(v) \geq 2$ for all $v \in V$. Then $\exists C \subseteq G$ such that C is a cycle.



Proof: Let $P = v_0 v_1 \cdots v_n$ be a path of maximal length in G.

Since $\deg(v_0) \geq 2$, and v_i are distinct, there is some $u \overset{\neq v_1}{\in} V$ such that $uv_0 \in E$ but $uv_0 \notin P$.

If $u \notin P$, then $u\underbrace{v_0 \cdots v_n}_{P}$ is a path of greater length $\Rightarrow$ contradiction of maximality. ↯

Say $u \in P$. Then $u = v_j$ for some $j > 1 \Rightarrow v_0 v_1 \cdots v_j v_0$ is a cycle. Choose C accordingly. □

Let G be a graph. We say G is connected if $\exists uv$-path for all $u, v \in V$.

Now, we look at our first big theorem. We prove it two ways: one more elementary, and another using Theorem (Cycles).

Theorem (Euler Tour): G connected graph. Then G has an Euler tour $\Leftrightarrow$ every vertex has even degree.

Proof: ($\Rightarrow$) Say $v_0 v_1 \cdots v_{n-1} v_n v_0$ is an Euler tour on G, where v_i are not necessarily distinct.

However, notice that all edges are distinct $\Rightarrow v_{i-1}v_i \neq v_i v_{i+1}$ $\Rightarrow v_i$ has 2 distinct vertices for every
$\Rightarrow v_{i-1} \neq v_{i+1}$

occurrence. Thus any such vertex has even degree, namely all of them. △

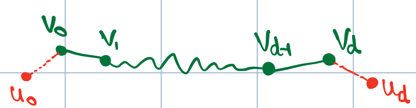
($\Leftarrow$) Suppose G is connected and has even degree vertices. Notice G connected $\Rightarrow \deg(v) \geq 2$, $\forall v \in V$.
Contradiction (1)

Let W be the longest walk in G not repeating an edge. We show W is an Euler tour.

STP that W covers every vertex and edge.

W is a closed walk.

SFAC W not closed. Write $W = v_0 v_1 \cdots v_d$ (where $v_0 \neq v_d$). Since $\deg(v_d) \equiv 0 \bmod 2$, we have used up to $\deg(v_d) - 1$ neighbours of v_d in W (since each occurrence of v_d accounts for 2 distinct neighbours, except the last). The same holds for v_0.



$\Rightarrow$ there are some vertices u_0, u_d such that the edges v_0, u_d are not in W

$\Rightarrow$ extend W to a walk $u_0 v_0 \cdots v_d u_d$. Contradiction of the maximality of W. ↯

W covers every edge.

SFAC there is an edge $e = v_a v_b \in E$ uncovered by W. We construct a longer walk.

WLOG let $v = v_a$ and let $u \overset{\neq v}{\in} V$ be some arbitrary vertex in W.

$\overset{\text{connectedness}}{\Rightarrow}$ there is a uv-path in G. Let P be a uv-path.

Get on the path P, until you hit W. Say this "intersecting" vertex is $v_x \in V$ in W.

Since W is closed, walk on W with start/end vertex v_x.

$\Rightarrow v \cdots \underbrace{v_x \cdots v_x}_{W}$ forms a longer walk. Contradiction. ↯

Since $\deg(v) \geq 2$, $\forall v \in V$, it follows that W closed and W covers every edge $\Rightarrow$ W covers every vertex.

Thus, W is an Euler tour. $\Delta\square$

We provide another proof below, using induction. First, we must introduce some additional notions.

G graph. We say $C \subseteq G$ is a connected component (or simply component) of G if $C \subseteq G$ is a maximal connected subgraph of G.

We say a vertex $v \in V$ is isolated if $\deg(v) = 0$.

Induction (2) ($\Leftarrow$) Induction on $|E|$. Let $|E| = m$. Base case $m = 0$ must be the isolated vertex, which has a trivial Euler tour.

Say G a connected graph where every vertex has even degree, with $|E| < m \Rightarrow G$ has an Euler tour.

Let G be a connected graph of even-degree vertices with $|E| = m$. We show that G has an Euler tour.

$\deg(v) \geq 2$ for all $v \in V \overset{\text{Cycles}}{\Longrightarrow} G$ has a cycle $C \subseteq G$.

Say $C = v_1 \dots v_k v_1$ and consider G' defined by removing all isolated vertices from $G - C$.

G' is connected with $\deg(v)$ even $\forall v \in V(G')$.

For $v_i \in C$, notice $\deg(v_i) \equiv 0 \bmod 2 \Rightarrow v_i$ was adjacent to an even number of vertices in $V(G')$.

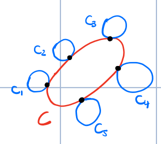
* how is this so trivial?

$\Rightarrow$ every vertex in G' has even degree.

Removing isolated vertices $\Rightarrow G'$ connected.

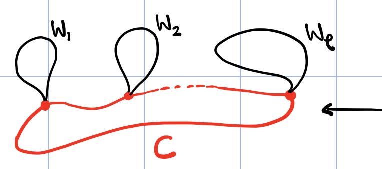
Decompose G' into components $C_1, \dots C_\ell$. Notice $|E(C_i)| < m$ for all C_i,

and that each component C_i must have a common vertex $v_{a_i} \in C_i$ with C



$\overset{\text{Induction}}{\Longrightarrow}$ each C_i has an Euler tour W_i.

We can now construct an Euler tour on G by chaining each C_i through C using $v_{a_i} \in C, C_i$.



$\leftarrow$ pictorial representation

$\Rightarrow G$ has an Euler tour. $\Delta\square$

How can we tell whether two graphs are $\underbrace{\text{not isomorphic}}$? (proving isomorphic $\Leftrightarrow$ provide isomorphism) We can consider some invariants under isomorphism.

Define $\delta(G)$ the minimum degree $\min_{v \in V}(\deg(v))$

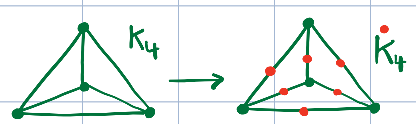
$\Delta(G)$ the maximum degree $\max_{v \in V}(\deg(v))$ } local properties

$d(G)$ the average degree $\frac{1}{|V|}\sum_{v \in V} \deg(v)$ (or the density)

$\varepsilon(G)$ the proportion $\frac{|E|}{|V|}$ (notice $\varepsilon(G) = \frac{1}{2}\delta(G)$ by Handshake) } global properties

What are the relations between these invariants?

$d(G)$ large $\not\Rightarrow \delta(H)$ large for some $H \subseteq G$?



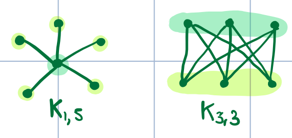
Consider K_n and notice K_n has large $d(G) = n-1$. Define $\dot{K}_n$ by inserting a vertex on every edge of K_n.

Notice there is no $H \subseteq \dot{K}_n$ with $\delta(H) > 2$. But what is $d(\dot{K}_n)$? Does $d(\dot{K}_n) \xrightarrow{n \to \infty} \infty$? (No!)

Say $d(G)$ large for a graph G. Intuitively, $\Rightarrow$ many vertices have large degree
OR few vertices have extremely large degree $\rightarrow$ are such vertices clumped together?

$\rightsquigarrow$ can we identify high density subgraphs?

A complete bipartite $K_{m,n}$ is a graph with a bipartition (A, B) such that $|A|=m$ and $|B|=n$, and every vertex in A is adjacent to every vertex in B.



Notice $|E(K_{m,n})| = mn$ since every $v\in A$ has $|B|$ neighbours.

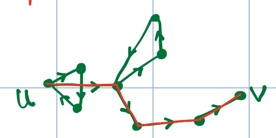
Lemma: Let G graph. Define $uPv := \exists uv$-path in G. Then P defines an equivalence relation.

Proof: Trivially, we have $uPv \Rightarrow vPu$ (reverse the path). Trivially vPv (empty path).

Notice uPv and $vPw \Rightarrow uPw$ by path concatenation. □

Lemma: G graph, and $u,v\in V$. Then $\exists uv$-walk $\Rightarrow \exists uv$-path.

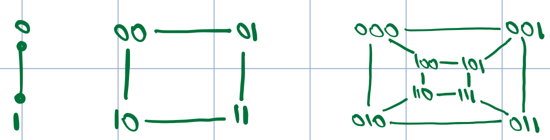
Proof: Reduce the walk to a path as follows:



Let $W = v_0 v_1 \dots v_n v$ be a uv-walk. Let v_{a_i} be the last occurrence of $v_i \in W$. Then v_{a_i} form a uv-path. □

length-n binary strings

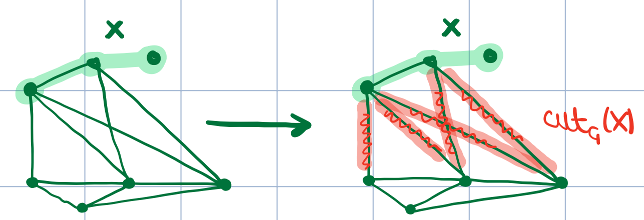
The n-cube is the graph defined by $V=\{\{0,1\}^n\}$ and $E=\{uv \mid u$ and v differ in exactly 1 bit$\}$.



Lemma (n-cube): Let G be an n-cube. Then we have:

1. G is regular — $\deg(v)=n, \forall v\in V$
2. $|V|=2^n$ — Trivially
3. $|E|=n2^{n-1}$ — Trivially
4. G is bipartite — Count number of 1s in a vertex (mod 2)
5. G is connected — Construct arbitrary paths

Let G be a graph and let $X\subseteq V(G)$ be some subset of its vertices.



The cut induced by X denoted $cut_G(X)$ is the set of all edges in $E(G)$ with exactly one endpoint in X.

Intuition: If we want to disconnect X and $V(G)\setminus X$, what edges must we "cut"?

Theorem (Cuts and Connectedness): G graph. Then G connected $\Leftrightarrow$ $cut_X(G)\neq\emptyset$ for all $X\subset V(G)$.
non-empty

if there is some $X\subseteq V(G)$ such that no edges need to be cut to disconnect X and $V(G)\setminus X$, then they were already disconnected.

Proof: ($\Rightarrow$) Say G connected. Let $X \overset{\neq\emptyset}{\subset} V$ be arbitrary. Since $X \overset{\neq\emptyset}{\subset} V$, we have $V\setminus X\neq\emptyset \Rightarrow$ let $u\in X$, $v\in V\setminus X$.

Since G connected, $\exists uv$-path $P=uv_1\cdots v_n v$. Let $v_k = v_{\underset{1\le i\le n}{\max(i)}}$ such that $v_k\in X$ and $v_{k+1}\in V$.

By definition, $v_k v_{k+1}\in cut_G(X) \Rightarrow cut_G(X)\neq\emptyset$. △

($\Leftarrow$): Suppose $cut_G(X)\neq\emptyset$ for all $X\subset V$.

SFAC G not connected, and let $C_1,\dots,C_d$ be the connected components of d with $d\geq 2$.

WLOG let $X=V(C_1)$. Since C_1 and $G\setminus C_1$ are not connected, there are no edges $e\in E(G)$ such that e has one endpoint in X and another in $V\setminus X \Rightarrow cut_G(X)=\emptyset$. ↯

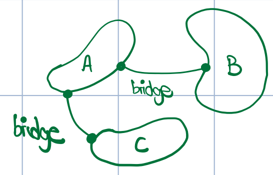
$\Rightarrow G$ must be connected. △□

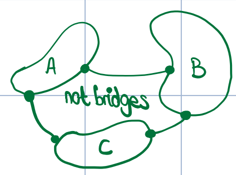
Corollary (Cuts and Disconnectedness): G graph. Then G disconnected $\Leftrightarrow \exists X \overset{\neq\emptyset}{\subset} V(G)$ such that $cut_G(X)\neq\emptyset$.

Proof: Equivalent to above. □

Let G be a graph. Define $comp(G) = \#$ distinct components of G.

An edge $e \in E$ is called a bridge (or cut-edge) if $comp(G-e) > comp(G)$.





Notice bridges cannot be in cycles (intuitively).

Lemma: G connected graph. If $e = uv \in E$ is a bridge, then $comp(G-e) = 2$, and u, v are in distinct components of $G-e$.

Proof: Let $G' = G - e$. G connected $\Rightarrow comp(G) = 1 \Rightarrow comp(G') \geq 2$.

We show $comp(G) \leq 2$.

SFAC $comp(G') = k > 2$. Then at least one component C_j of G' does not contain u or v

$\Rightarrow cut_{G'}(V(C_j)) = \emptyset$ and $u, v \notin V(C_j) \Rightarrow cut_G(C_j) = \emptyset \Rightarrow comp(G) \geq 2$. $\lightning$

Thus $comp(G') = comp(G-e) = 2$. $\triangle$

Similarly, if we SFAC that u and v are in the same component of G', we get a contradiction. $\triangle\square$

Theorem (Bridges): G graph. Then $e \in E$ is a bridge $\Leftrightarrow$ e is not in any cycles of G.

Proof: $(\Rightarrow)$ Say $e = uv \in E$ is a bridge.

SFAC e is in a cycle $uvv_1 \cdots v_d u$ of G. Then e bridge $\Rightarrow u$ and v are in distinct components of $G-e$.

However $vv_1 \cdots v_d u \in G-e$ is a uv-path $\Rightarrow u$ and v are in the same component of $G-e$. $\lightning$ Thus e not in cycle. $\triangle$

$(\Leftarrow)$ Suppose $e = uv \in E$ is not in any cycle of G. Let H be the component of G containing u, v.

SFAC e is not a bridge, so $H-e$ is connected $\Rightarrow \exists uv$-path $uv_1 \cdots v_d v \in H-e$. Construct a cycle $vuv_1 \cdots v_d v \in H \subseteq G$ containing e. $\lightning$

$\Rightarrow e$ not in any cycles of G. $\triangle\square$

What properties do acyclic graphs have?

A tree is a connected acyclic graph. A forest is an acyclic graph.

Let G be a forest. A vertex $v \in V$ is a leaf if $\deg(v) = 1$. Let $\ell(G)$ denote the number of leaves in G.

Lemma: G forest. Then all $e \in E$ are bridges.

Proof: Let $e \in E$ be arbitrary. G forest $\Rightarrow G$ acyclic $\Rightarrow e \in E$ not in any cycle of $G \overset{\text{Bridges}}{\Rightarrow} e$ is a bridge. $\square$

Remark: $\Rightarrow$ trees are minimally connected graphs.

Lemma: G tree, with $|V| \geq 2$. Then $\ell(G) \geq 2$.

Proof: Let $P = v_1 \ldots v_k$ be a longest path in G. G connected and $|V| \geq 2 \Rightarrow k \geq 2$ (nontrivial path exists).
(1)

v_1 and v_k must be leaves.

SFAC that at least one of v_1, v_k is not a leaf. WLOG say v_k not a leaf.

$\Rightarrow \deg(v_k) \geq 2 \Rightarrow$ there exists some neighbour $v_{k+1} \in V$ of v_k such that $v_{k+1} \notin P$ since G acyclic.

$\Rightarrow v_1 \ldots v_k v_{k+1}$ is a longer path. $\lightning$

$\Rightarrow \ell(G) \geq 2$. $\square$

Lemma: G tree. Then $|E| = |V|-1$.

Proof: Induction on the vertices. Let $|V|=n$. Clearly, base case $n=1$ holds trivially.

Say that if H is a tree with $|V(H)|<n$, then $|E(H)| = |V(H)|-1$.

Let G be a tree with $|V|=n$.

G tree $\Rightarrow \ell(G) \geq 2 \Rightarrow$ let $u, v \in V$ be leaves.

Define $G' = G \setminus \{u\}$ by removing one of the leaves of G. Notice $|V(G')| = n-1$ and that G' is a tree.

$\overset{IH}{\Rightarrow} \underbrace{|V(G')|}_{|V|-1} - 1 = \underbrace{|E(G')|}_{|E|-1} \Rightarrow |V|-2 = |E|-1 \Rightarrow |E| = |V|-1$. Done. $\square$

Corollary: G forest. Then $|V| = |E| - \text{comp}(G)$.

Proof: Apply Lemma above on each component. $\square$

Lemma (Leaves): G tree with $|E| \geq 1$. For $i \geq 1$, let $n_i = |\{v \in V \mid \deg(v) = i\}|$ be the number of degree-i vertices in G.

Then $\ell(G) = 2 + \sum_{i \geq 3} (i-2) n_i$.

Proof: Notice $V = \bigcup_{i \geq 1} V_i$ where $V_i = \{v \in V \mid \deg(v) = i\}$ is a disjoint union, so $|V| = |\bigcup_{i\geq 1} V_i| = \sum_{i \geq 1} |V_i| = \sum_{i \geq 1} n_i$.

$|E| = |V|-1 \Rightarrow |E| = -1 + \sum_{i \geq 1} n_i$ by Lemma above, and

$2|E| = \sum_{v \in V} \deg(v) = \sum_{i \geq 1} i \, n_i$ (degree, count) by Handshake $\Rightarrow -2 + 2\sum_{i \geq 1} n_i = \sum_{i \geq 1} i n_i = n_1 + 2n_2 + \sum_{i \geq 3} i n_i$

$= -2 + 2n_1 + 2n_2 + \sum_{i \geq 3} n_i$

$\Rightarrow \ell(G) = n_1 = 2 + \sum_{i \geq 3} (i-2) n_i$. $\square$

Lemma: G tree, and $u, v \in V$. There exists a unique uv-path in G.

Proof: G tree $\Rightarrow G$ connected $\Rightarrow \exists uv$-path. SFAC $u u_1 \ldots u_a v$ and $u v_1 \ldots v_b v$ are distinct uv-paths. Then $u u_1 \ldots u_a v v_b \ldots v_1 u$ contains a cycle. $\lightning$ $\square$

G graph. A spanning tree of G is a spanning subgraph $T \subseteq G$ such that T is a tree.

Theorem (Spanning Trees): G graph. Then G is connected $\Leftrightarrow$ G has a spanning subtree.

Proof ($\Leftarrow$): Say G has a spanning tree $T \subseteq G$. T spans $G \Rightarrow V(T) = V(G)$ and T tree $\Rightarrow T$ connected. $T \subseteq G \Rightarrow E(T) \subseteq E(G)$.

Let $u, v \in V(G)$. Then $u, v \in V(T) \Rightarrow \exists uv$-path in T (connectedness) $\Rightarrow \exists uv$-path in G (subgraph) $\Rightarrow G$ connected. Δ

($\Rightarrow$): Induct on the cycles. Idea: reduce k-cycled graphs to $<k$-cycled graphs, and use this to construct a spanning tree.

Let G be connected. Let $c(G) = \#$cycles in G. If $c(G) = 0$, then G is a tree $\Rightarrow G$ has a spanning tree $G \subseteq G$.

Suppose that for G connected, that $c(G) < k \Rightarrow G$ has a spanning tree, for some $k \geq 1$.

Let G be connected with $c(G) = k$. Since $c(G) \geq 1$, we have $\exists C \subseteq G$ some cycle C.

Choose $e \in E$ such that e is in the cycle C. Thus e is not a bridge (Bridges)

$\Rightarrow c(G-e) < c(G) = k$ and $\text{comp}(G-e) = \text{comp}(G) = 1 \Rightarrow G-e$ is connected with $V(G-e) = V(G)$.

$\overset{IH}{\Rightarrow}$ Let $T \subseteq G-e$ be a spanning tree for $G-e$

Then $T \subseteq G$ is a spanning tree for G. $\triangle\square$

Theorem (Trees): G graph. Then G tree $\Leftrightarrow$ G is connected and $|E| = |V| - 1$.

Proof: ($\Rightarrow$) Lemma above. $\triangle$

($\Leftarrow$) Say G is a connected graph with $|E| = |V| - 1$.

G connected $\Rightarrow$ G has a spanning tree $T \subseteq G$. T tree $\Rightarrow$ $|E(T)| = |V(T)| - 1$ but $|V(T)| = |V|$

$\Rightarrow |E(T)| = |V| - 1 = |E|$ so $E(T) \subseteq E \Rightarrow E(T) = E$ and $V(T) = V \Rightarrow T = G$.

Thus G tree. $\triangle\square$

Corollary (2/3): G graph. If any two of the following hold, then G is a tree:

1. G connected; 2. G has no cycles; 3. $|E| = |V| - 1$.

Proof: (1,3) Theorem above. $\triangle$

(1,2) Definition of a tree. $\triangle$

(2,3) Say G has no cycles, and $|E| = |V| - 1$.

STP that G is connected.

SFAC G not connected, so $\text{comp}(G) \geq 2$. G has no cycles $\Rightarrow$ G forest.

$\Rightarrow |E| = |V| - \text{comp}(G) \neq |V| - 1$ since $\text{comp}(G) \neq 1$. $\lightning$ $\triangle\square$

Let G graph. Define the girth $g(G)$ to be the minimum length of a cycle.
circumference of G to be the maximum length of a cycle.

Theorem (Erdős): For every $\delta, g \in \mathbb{N}$, there exists a graph G with $\delta(G) \geq \delta$ and $g(G) \geq g$.

Lemma: G connected graph, P longest path. If u is at one end of P, then $G-u$ is connected.

Proof SFAC $G-u$ not connected, so $G-u$ has ≥ 2 components but G has 1 component.

Let $\{v_1, \dots, v_d\}$ be the neighbours of u. Let $P = u_1 \dots u_k$ such that $u_k = u$.

If $v_i \notin P$ then $u_1 \dots u_k v_i$ forms a longer path $\Rightarrow$ $v_i = u_j$ for all neighbours.

$G-u$ not connected $\Rightarrow \exists X, Y \subset V(G-u)$ such that X and Y are not connected,

and $v_i \in Y$ for some i and $u_{k-1} \in X$

$\Rightarrow \nexists\, u_{k-1}v_i$-path in $G-u$. However, since $v_i = u_j$ we have a path $u_{k-1} \dots u_j$ $\lightning$

So $G-u$ not disconnected. $\square$

Lemma: G tree. Then G tree $\Rightarrow G$ bipartite.

Proof: Induction on $|V|$. Base case $|V|=1$ trivially bipartite.

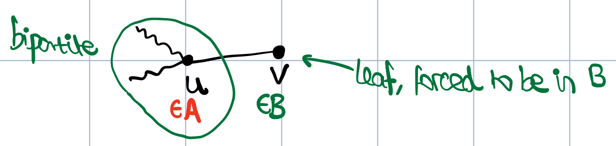
Suppose that if G is a tree with $|V(G)|<n$, then G is bipartite, for some $n \geq 2$.

Say G is a tree with $|V|=n$. We reduce $|V|$ and construct a bipartition on G.

G tree $\Rightarrow \ell(G) \geq 2$. Let v be a leaf of G. Say u is the (only) neighbour of v.

Let $G' = G - v$. Then $|V(G')| = |V|-1 < n \Rightarrow G'$ has a bipartition (A, B).

Since $u \in G'$, WLOG say $u \in A$. Extend (A,B) to a bipartition $(A, B \cup \{v\})$ on G. $\square$



Lemma: G graph. Then G odd cycle $\Rightarrow G$ not bipartite.

Proof: Let $u_1 u_2 \cdots u_k u_1$ be an odd cycle (so k odd).

SFAC $u_1 u_2 \cdots u_k u_1$ bipartite (A,B). WLOG say $v_1 \in A$. Then $v_{2i} \in B$ for $i \in \mathbb{N} \Rightarrow v_k \in A$.

But then $v_1 v_k$ has two endpoints in A. $\lightning$ $\Rightarrow G$ not bipartite. $\square$

Lemma: G graph. G bipartite $\Rightarrow H \subseteq G$ bipartite.

Proof: Let (A,B) be a bipartition on G. Then $(A \cap V(H), B \cap V(H))$ bipartition on H. $\square$

Theorem (Bipartiteness): G graph. Then G bipartite $\Leftrightarrow$ G contains no odd cycles.

Proof: $(\Rightarrow)$ Say G bipartite. $C \subseteq G$ is an odd cycle. Since C odd cycle $\Rightarrow C$ bipartite and $C \subseteq G$

$\Rightarrow G$ not bipartite. $\lightning$ $\triangle$

$(\Leftarrow)$ Suppose G not bipartite. We show that C contains an odd cycle (thus no odd cycle $\Rightarrow$ bipartite).

We construct an odd cycle

At least one component $H \subseteq G$ is not bipartite.

H connected $\Rightarrow H$ has a spanning tree T.

$T \subseteq H \subseteq G$ tree $\Rightarrow T$ bipartite, so let (A_T, B_T) be a bipartition on T.

H not bipartite $\Rightarrow$ there must be some edge $e \in E(H) \setminus E(T)$ in $H - T$ with both endpoints $e = uv$ in the same partition (since $V(T) = V(H)$, we have $u, v \in T$).

$\Rightarrow u, v$ in (A_T, B_T)

WLOG, say $u, v \in A_T$. Notice T connected $\Rightarrow \exists uv$-path in T.

Let $uv_1 \cdots v_k v \in T$ be a uv-path. Since $uv \notin E(T)$, we have $uv_1 \cdots v_k v u \in H$ a cycle, where $uv_1 \cdots v_k v u$ not bipartite $\Rightarrow$ the cycle is odd.

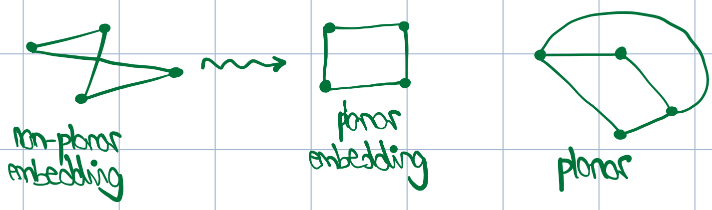
Thus G contains no odd cycles $\Rightarrow G$ bipartite. $\triangle \square$

Lemma: G graph. G has an Euler walk $\Rightarrow G$ has no bridges.

A planar embedding of a graph G is an embedding of G in $\mathbb{R}^2$ where no two edges intersect, and no two vertices coincide.

If a graph G has a planar embedding, we say G is planar.

When are graphs planar? What properties do these graphs have?

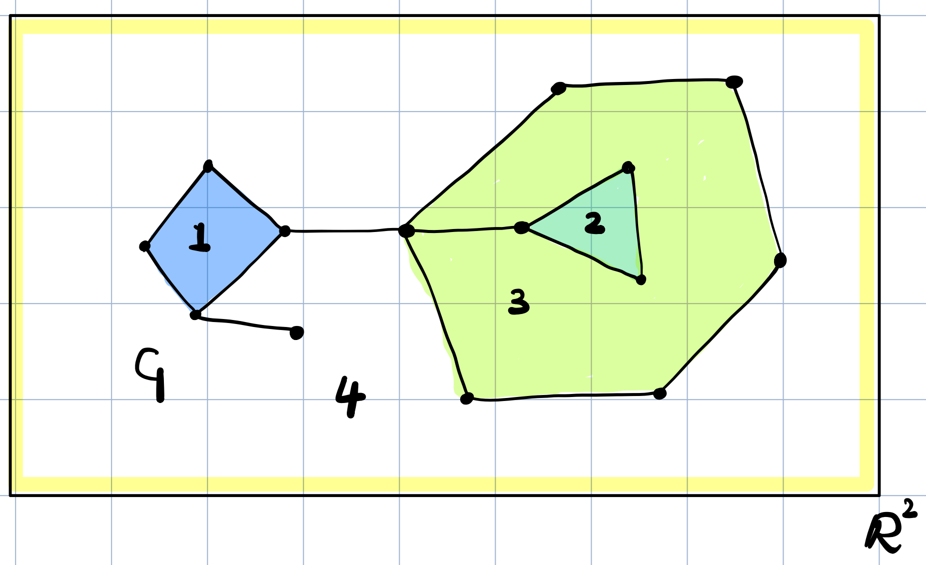


When we embed in $\mathbb{R}^2$, we naturally induce a topology and geometry on our graph, which was purely algebraic.

Thus, we can use these notions in our definitions. Let $(\mathbb{R}^2, \tau)$ be our topological space.

Let G be planar with some embedding $\mathcal{E}$.

A face in $\mathcal{E}$ is a maximally connected region (in τ) not separated by any edges. For a graph G with embedding $\mathcal{E}$, we will denote $F(\mathcal{E}) = F(G)$ to be the set of faces in $\mathcal{E}$.



The boundary of a face is a subgraph of all vertices and edges that touch the face.

Remark: The definitions above may seem informal or non-rigorous, but are in fact the formal definitions.

If we wanted, one formalization of an embedding could be to define a function such that

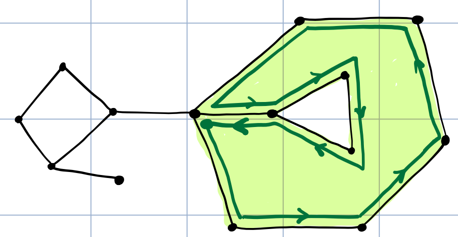
vertices $\rightsquigarrow$ points and edges $\rightsquigarrow$ piecewise smooth paths, so an embedding $\varphi_{\mathcal{E}}$ could be a pair of functions

$$\begin{cases} \varphi_V : V(G) \to \mathbb{R}^2 \text{ such that } \varphi_V \text{ injective,} \\ \varphi_E : E(G) \to \left(\phi_t : [0,1] \to \mathbb{R}^2,\ t \mapsto \phi_t\right) \text{ such that } \bigcap_{e \in E} \varphi_E(e)^* = \emptyset \end{cases}$$

where $\varphi_E(e)^*$ is the trace of the resultant path.

What if we only allowed straight edges? Would these definitions be equivalent?

Theorem (Fáry): Straight- and curved-edge planarity are equivalent.



Let G be connected. We define the boundary walk of a face $f \in F(G)$ to be a closed walk around the faces boundary.

The degree of a face is the length of its boundary walk.

Theorem (Faceshaking Lemma): G connected and planar. Then $\sum_{f \in F(G)} \deg(f) = 2|E|$.

Proof: When we sum degrees of faces, each edge gets counted twice. $\square$

Lemma: G planar with embedding $\mathcal{E}$. Then $e \in E$ is a bridge $\Leftrightarrow$ the two sides of e are the same face in $\mathcal{E}$.

Proof: By definition, a face is a closed region in $\mathbb{R}^2$. The idea is that $U \subseteq \mathbb{R}^2$ face $\Leftrightarrow$ ∂U is defined by edges in $\mathcal{E}$

$\Leftrightarrow$ we can build a Jordan curve.

$\Leftrightarrow$ cycles in G break $\mathbb{R}^2$ into closed regions. $\square$

Remark: Cycles in G define Jordan curves in $\mathcal{E}(G)$.

Theorem (Jordan Curve): Let γ be a Jordan curve. Then $\mathbb{R}^2 \setminus \gamma^*$ has two components: $\mathbb{R}^2 \setminus \gamma^* = I \cup O$, where I, O are connected, and $I \cap O = \emptyset$, and $\bar{I}$ is compact.

Theorem (Weak Euler): G connected and planar, with embedding $\mathcal{E}$. Then $|V|+|F|-|E| = 2$.

Proof: We fix $|V|$ and induct on $|E|$.

Idea: G connected $\Rightarrow \exists T \subset G$ spanning tree $\Rightarrow$ reduce $|E|$ and use induction.

Let $|V|$ be fixed and let G be an arbitrary connected graph on V.

Base case: fewest $|E|$ possible.

G connected $\Rightarrow \exists T \subseteq G$ spanning tree, and $\overset{|V|}{\overset{\parallel}{|V(T)|}}-1 = |E(T)| \leq |E|$

$\Rightarrow |E| = |V|-1$ must be our base case, since this is the smallest E can get.

Thus G must be a tree, so G has no cycles $\Rightarrow |F| = 1$ so $\underset{|E|+1}{\underset{\parallel}{|V|}}+\underset{1}{\underset{\parallel}{|F|}}-|E| = 2$. $\triangle$

(I.H.):
Now suppose G is connected and planar on $|V|$ vertices, and $|E| < E_0 \Rightarrow |V|+|F|-|E| = 2$.
$\hookrightarrow$ some $E_0 \geq |V|$

Let G on $|V|$ vertices be connected and planar, with $|E| = E_0$.

We reduce $|E|$ and show that $|V|+|F|-|E|=2$.

Notice $|E| = E_0 \Rightarrow |E| \geq |V| \Rightarrow G$ not a tree $\Rightarrow \exists C \subseteq G$ a cycle.

Let $e \in E(C)$ be an edge in the cycle.

Recall $e \in E(C) \Rightarrow e$ is not a bridge $\Rightarrow G-e$ connected.

Trivially, G planar $\Rightarrow G-e$ planar.

2 faces are merged into 1

Lastly e not a bridge $\Rightarrow$ two sides of e touch distinct faces, so $|F(G-e)| = |F|-1$.

We know $|V(G-e)| + |F(G-e)| - |E(G-e)| = 2$ by I.H.

$$\underset{|V|}{\underset{\parallel}{|V(G-e)|}} + \underset{(|F|-1)}{\underset{\parallel}{|F(G-e)|}} - \underset{(|E(G)|-1)}{\underset{\parallel}{|E(G-e)|}} = 2 \Rightarrow |V|+|F|-|E| = 2. \quad \square$$

Ex. Let G be connected, 3-regular, and planar. Suppose there exists an embedding $\mathcal{E}$ of G such that all faces have degree 4, 6, or 8. Say there are 6 faces of degree 8. How many faces have degree 4?

G planar $\Rightarrow |V|+|F|-|E| = 2$. Let $F_i = |\{f \in F \mid \deg(f) = i\}|$ be #faces of degree i.

We have $\sum_{f \in F} \deg(f) = 2|E|$ by Faceshaking

$$\overset{\parallel}{\sum_{\substack{f \in F \\ \deg 4}} \deg(f)} + \sum_{\substack{f \in F \\ \deg 6}} \deg(f) + \sum_{\substack{f \in F \\ \deg 8}} \deg(f) = 4F_4 + 6F_6 + 8\overset{6}{\cancel{F_8}} = 4F_4 + 6F_6 + 48$$

$$\Rightarrow 2F_4 + 3F_6 + 24 = |E|$$

Also $\sum_{v \in V} \deg(v) = 2|E| \overset{3\text{-reg.}}{\Rightarrow} 3|V| = 2|E|$ by Handshaking $\Rightarrow |V| = \frac{2}{3}|E|$.

Some ugly algebra combining these results yields $F_4 = 12$.

To show planarity, we provide an embedding $\mathcal{E}$ typically as a drawing.

How do we show non-planarity? Idea: too many edges ⇒ non-planarity. Eventually, we introduce the concept of minors to decisively characterize planarity.

Lemma: G planar. If $\exists\,\mathcal{E}$ embedding such that $\deg(f) \geq d \geq 3$ for all $f \in F(\mathcal{E})$, then $|E| \leq \frac{d(|V|-2)}{d-2}$.

Remark: faces large degree ⇒ upper bound for $|E|$.

Proof: Case 1: G connected $\overset{\text{Faceshake}}{\Rightarrow} 2|E| = \sum_{f\in F} \deg(f) \geq d|F|$

$\overset{\text{Euler}}{\Rightarrow} |V|+|F|-|E| = 2$ so $|F| = 2-|V|+|E|$

so $2|E| \geq d|F| = d(2-|V|+|E|) \Rightarrow 2|E| - d(2-|V|+|E|) \geq 0$

$\Rightarrow 2|E| - d|E| - 2d + d|V| \geq 0$

$\Rightarrow (d-2)|E| + 2d - d|V| \leq 0$ so $|E| \leq \frac{d(|V|-2)}{d-2}$. △

Case 2: G disconnected.

Exercise. Apply Case 1 on all components, accounting for the common outer (unbounded) face. △□

Theorem: G planar with $|V| \geq 3$. Then $|E| \leq 3|V|-6$.

Proof: Case 1: G forest (acyclic). Then $|E| = |V| - \text{comp}(G)$ by Lemma

$\Rightarrow |E| \leq |V|-1$ since $\text{comp}(G) \geq 1$

and $|V|-1 \leq 3|V|-6$ for $|V| \geq 3$ so $|E| \leq 3|V|-6$. △

Case 2: G not forest. (G has a cycle) G has a cycle and $\deg(f) \geq 3$ for all $f \in F$

$\overset{\text{Lemma}}{\Rightarrow} |E| \leq \frac{3(|V|-2)}{3-2} = 3|V|-6$. △□

Why do we need 2 cases? Case 2 seems to work in general.

Theorem: G planar with $|V| \geq 3$. Then G bipartite ⇒ $|E| \leq 2|V|-4$.

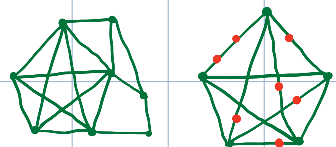
Proof: G bipartite ⇒ G has no odd cycles

$\Rightarrow \deg(f) \geq 4$ for all $f \in F \overset{\text{Lemma}}{\Rightarrow} |E| \leq \frac{4(|V|-2)}{4-2} = 2|V|-4$. □

Theorem (Strong Euler): G planar. Then $|V|+|F|-|E| = 1+\text{comp}(G)$ for any planar embedding of G.

Proof: Weak Euler on each component, accounting for the common outer (unbounded) face. □

Corollary: K_5 and $K_{3,3}$ are not planar.



If a graph has a non-planar subgraph, then the graph should also be non-planar.

However, we can extend this. Adding vertices on existing edges of a non-planar graph should maintain non-planarity.

We formalize this concept now.

Let G be a graph. An edge contraction is an operation on an edge $e = uv \in E$ removing the edge from G and merging vertices $u, v \in V$ into a single vertex. We denote the resulting graph G/e.

Formally, let $\varphi_{uv} : V \to V \setminus \{u,v\}$ be a function with $\varphi(x) = \begin{cases} x & \text{if } x \notin \{u,v\} \\ w & \text{otherwise} \end{cases}$ resulting in a graph

$G' = (V', E')$ where $V' = (V \setminus \{u,v\}) \cup \{w\}$ and $E' = E \setminus \{uv\}$ and $x' = f(x) \in V'$ incident to $e' \in E'$

$\Leftrightarrow x \in V$ incident to $e \in E$ in G.

corresponding edge

Then φ_{uv} is a contraction of G with respect to $uv \in E$.

G graph. A graph H is called a minor of G if H can be formed from G by deleting edges, deleting vertices, and by contracting edges.

We say H is a subdivision of G if H can be obtained by replacing edges in G with paths of length ≥ 1.

Remark: New vertices when subdividing a graph G always have degree 2, since they must lie only on some path that was added. We write $H \leq G$ to denote H is a minor of G.

Both definitions above are up to isomorphism.

Corollary: G graph, H graph. $G \leq H$ and $H \leq G \Rightarrow G \cong H$.

we introduce this notion soon

Conjecture (Hadwiger): G graph. If $K_n \not\leq G$, then G has a proper $(n-1)$-colouring.

We will work now to characterize all planar graphs.

Lemma: G graph. Then G planar $\Leftrightarrow$ every subdivision of G is planar.

Proof: Intuitively trivial.

($\Leftarrow$): G is a subdivision of itself, so all subdivisions of G planar $\Rightarrow$ G planar. $\triangle$

($\Rightarrow$): Let $\mathcal{E}$ planar embedding of G. Add vertices onto each edge (subdivide). No intersections will be created. $\triangle\square$

Lemma: G graph. G planar $\Rightarrow$ $H \subseteq G$ planar.

Proof: Let $\mathcal{E}$ be a planar embedding of G, so $\mathcal{E} : G \to \mathbb{R}^2$ is a function. Then $\mathcal{E}|_H : H \to \mathbb{R}^2$ is also a planar embedding. $\square$

We define a few more useful notions before proving Kuratowski's Theorem. G graph.

A cut-vertex is a vertex $v \in V$ such that $comp(G - v) > comp(G)$.

A block of G is a subgraph $B \subseteq G$ such that B maximally has no cut vertices. We can view any graph G as being built of blocks, pasted together at cut-vertices.

We say G is 2-connected if G is connected and has no cut-vertices.

In particular, G is k-connected if $G - S$ is connected for any $S \subseteq V$ with $|S| \leq k-1$. must remove k vertices to disconnect G

Lemma: Let G be 2-connected, and $u,v \in V$. Then $\exists C \subseteq G$ cycle such that $u,v \in V(C)$.

Proof: Induct on the distance between u and v. Denote this $\mathrm{dist}(u,v) \in \mathbb{N}$. ← well-defined since G 2-connected $\Rightarrow$ G connected

$\mathrm{dist}(u,v)=1$. Then u and v are adjacent. Notice that $\deg(v)=1 \Rightarrow v$ is a cut-vertex, but G is 2-connected $\lightning$

$\Rightarrow \deg(v) > 1$.

Let $w \neq u$ be a neighbour of v, and consider $G-v$.

Notice $G-v$ connected $\Rightarrow \exists uw$-path P such that $v \notin P$. Let $P = uv_1 \dots v_n w$. Then $vuv_1 \dots v_n wv$ cycle. $\triangle$

Suppose the result holds for any $u,v \in G$ such that $\mathrm{dist}(u,v) \leq d-1$ for some $d \in \mathbb{N}_{\geq 2}$.

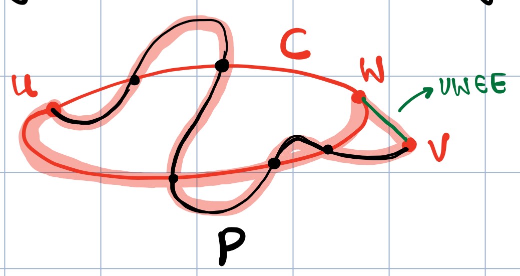
Let Q be a uv-path of length d. Say $Q = u \dots wv$ so $w \neq u$ (since $d \geq 2$).

Notice $\mathrm{dist}(u,w) = d-1$, so $\exists C \subseteq G$ cycle such that $u,w \in V(C)$ (I.H.). If $v \in V(C)$ we are done.

Say $v \notin C$. G 2-connected $\Rightarrow$ $G-w$ connected $\Rightarrow \exists uv$-path $P \subseteq G-w$ such that $w \notin P$.

Let $P = uv_1 \dots v_n v$. Extend C by starting at u, follow C until hitting P to v, go to w, follow C to u.

The picture shows this better.



$\Rightarrow$ there exists a cycle containing u and v. $\triangle \square$

Theorem (Kuratowski's): G graph. Then G non-planar $\Leftrightarrow \exists H \subseteq G$ such that H is a subdivision of K_5 or $K_{3,3}$.

Proof: ($\Leftarrow$) $\exists H \subseteq G$ such that H is a subdivision of $\underbrace{K_5 \text{ or } K_{3,3}}_{\text{non-planar}} \Rightarrow H$ non-planar $\Rightarrow G \supseteq H$ non-planar. $\triangle$

($\Rightarrow$) This direction is non-trivial.

We want to show every non-planar graph somehow contains $K_{3,3}$ or K_5.

SFAC that G is non-planar with neither a subdivision of $K_{3,3}$ or K_5 as a subgraph.

From all such counterexamples, let G be a minimal counterexample, such that

removing any vertices or edges from G causes it to satisfy the theorem. ← how do we know a minimal G exists?

1. G must be 2-connected.

G planar $\overset{*}{\Leftrightarrow}$ every block of G is planar. Thus $\exists B \subseteq G$ such that B non-planar, since G non-planar.

If $B \subset G$ then B contradicts minimality of G, so $G = B \overset{\text{def.}}{\Rightarrow} G$ is 2-connected. $\triangle$

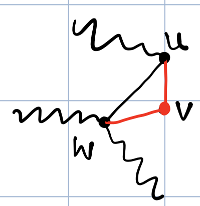
2. $\delta(G) > 2$ (minimum degree).

We know G 2-connected $\Rightarrow \delta(G) \geq 2$. We show $\delta(G)=2$ cannot happen.

SFAC $\exists v \in V$ with $\deg(v)=2$. Let $u,w \in V$ be the neighbours of v. We consider 2 cases.

Case 1: $uw \in E$. Consider $H = G-v$. By minimality of G, H is planar.

Let $\mathcal{E}$ be a planar embedding $\mathcal{E}: H \to \mathbb{R}^2$. Insert the path uvw next to uw:



This is always possible regardless of $\mathcal{E}$, since v only has 2 neighbours

$\Rightarrow$ we have constructed a planar embedding of G. $\lightning$

Case 2: $uw \notin E$. Remove v and replace it with edge uw to obtain H.

Notice H is smaller than $G \Rightarrow H$ planar $\Rightarrow$ let $\mathcal{E}: H \to \mathbb{R}^2$ be a planar embedding.

Subdivide $uw \in E(H)$ to produce G. Thus G planar. ↯

So $\delta(G) \neq 2 \Rightarrow \delta(G) \geq 3$. △

3. $\exists e \in E$ such that $G-e$ is 2-connected.

Exercise. △

Let $H = G - \overset{=uv}{e}$ such that H is 2-connected. By minimality of G, we have H planar.

$\Rightarrow \exists C \subseteq H$ cycle such that $u, v \in V(C)$ by 2-connectedness.

Form planar embedding $\mathcal{E}: H \to \mathbb{R}^2$ satisfying:

- $u, v \in V(C)$
- The number of faces inside C is maximal among all other embeddings
- If $C' \neq C$ is a cycle containing u and v, then the number of faces inside C' is $\leq$ number of faces inside C.

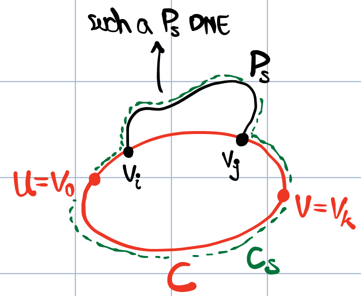
ie. we have chosen $C \overset{\ni u,v}{\subseteq} G$ such that $|F(C)|$ maximal across all cycles $C' \overset{\ni u,v}{\subseteq} G$ among all embeddings of H.

Write $C = \underset{u}{v_0} v_1 \dots v_{k-1} \underset{v}{v_k} v_{k+1} \dots v_\ell \underset{u}{v_0}$. Notice $uv \notin E(H)$ since $H = G - uv$, so $k \geq 2$.

Notice:

* • there is no path connecting two vertices in $\{v_0, \dots, v_k\}$ that lies exterior to $\mathcal{E}(C)$
or $\{v_k, \dots, v_\ell, v_0\}$

* We prove this for $S = \{v_0, \dots, v_k\}$ here.
(proof identical for other set)



SFAC there exists some path P_S between two vertices of S so $\mathcal{E}(P_S)$ is exterior to $\mathcal{E}(C)$.

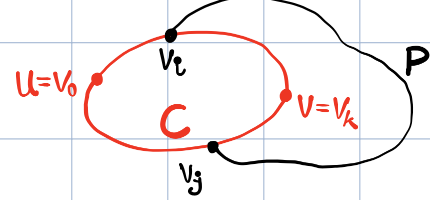
$\Rightarrow$ construct cycle C_S as $v_0 \dots v_i P_S v_j \dots v_\ell v_0$ "larger" than $C \Rightarrow$ contradiction of maximality ↯

larger in the sense that $\mathcal{E}(C)$ is entirely within region $\mathcal{E}(C_S)$

So we have $H = G - e$ a planar 2-connected graph, and some $\mathcal{E}: H \to \mathbb{R}^2$ such that some $C \overset{\ni u,v}{\subseteq} H$ is somehow maximal.

H is 2-connected $\Rightarrow \exists P \subseteq H$ path such that P connects some vertex in $\{v_1, \dots, v_{k-1}\}$ to some vertex in $\{v_{k+1}, \dots, v_\ell\}$.

$\Rightarrow$ Let $v_i \in \{v_1, \dots, v_{k-1}\}$ and $v_j \in \{v_{k+1}, \dots, v_\ell\}$ be such vertices.



notice this can exist since C is a cycle that maximally contains u, v, but P here does not violate this property!

$\Rightarrow$ edge $e = uv$ cannot be added exterior to C in $\mathcal{E}(H)$
since this would violate planarity.

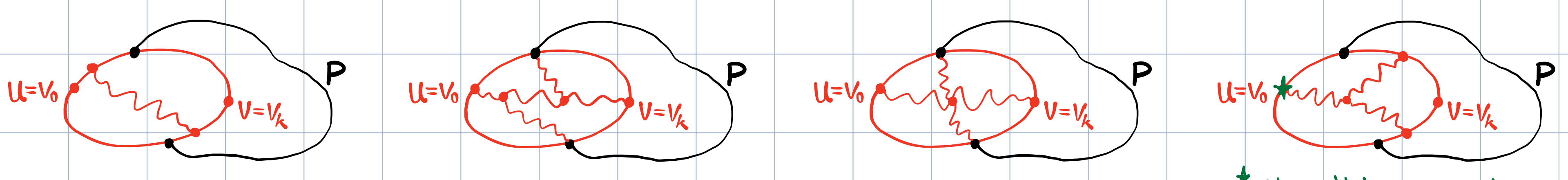
$\Rightarrow$ path P must be exterior to C, since then $H+e$ would be planar
(we could add e external to C in $\mathcal{E}$). **

Furthermore, * $\Rightarrow$ no vertex of P is adjacent to any other vertex in C (except v_i and v_j).

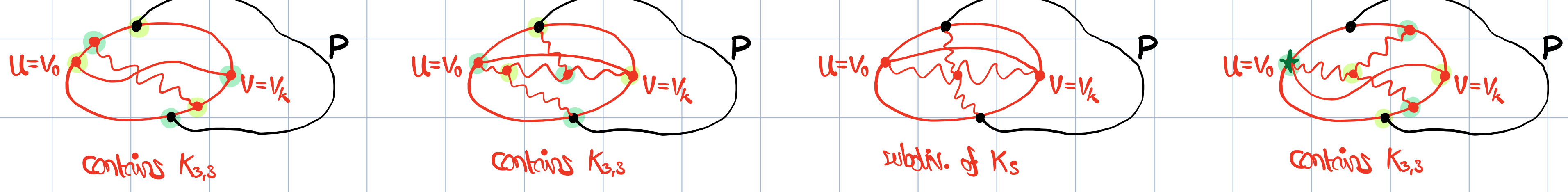
** C has some impedimentary structure that disallows P to be within $\mathcal{E}(C)$.

We know P is forced exterior to C. This means some structure internal to C prevents such a P (otherwise G would have been planar!).

This structure must take one of the following forms: → proof is an exercise*



* vertex could be v_0, v_i, v_k, v_j but we chose to show one such case



In all four cases, adding back $e=uv$ forms a subdivision of K_5 or $K_{3,3}$.

$\Rightarrow$ G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$. ↯ Done. $\square$

Kuratowski's allows us to determine whether a graph has an embedding $\mathcal{E}: G \to \mathbb{R}^2$ based on subdivisions.

What about other topological spaces? → a torus with g holes?

Theorem (Topological Minor): For any $g \geq 0$, there exists a finite list $\mathcal{G}_g$ of graphs such that a graph G embeds into a torus with g holes $\Leftrightarrow$ G contains no subgraph $H \subseteq G$ such that $H \leq K$ for any $K \in \mathcal{G}_g$.

Proof: Incredibly difficult but beautiful. Too hard. $\square$

Remark: We only know $\mathcal{G}_g$ for $g=0$, but we know $|\mathcal{G}_1| \geq 16000$. Unsolved problem.

Colourings

G graph. A *k-vertex-colouring* (*k-colouring*) of G is a partition $\overbrace{(C_1, \dots, C_k)}^{\text{colours}}$ of V such that no adjacent vertices are in the same partition. *Edge-colourings* and *face-colourings* are defined similarly.

If G has a k-colouring, we say G is *k-colourable*.

Notice G k-colourable $\Rightarrow$ G n-colourable for any $n \geq k$, so we care about how small we can make k.

Lemma: G planar $\Rightarrow$ $\exists v \in V$ with $\deg(v) \leq 5$.

Proof: Consider the average degree $d(G) = \dfrac{\sum_{v \in V} \deg(v)}{|V|} \overset{\text{Handshaking}}{=} \dfrac{2|E|}{|V|}$.

G planar $\Rightarrow$ $|E| \leq 3|V| - 6$

$\Rightarrow \dfrac{2|E|}{|V|} \leq \dfrac{6|V|-12}{|V|} = 6 - \dfrac{12}{|V|} < 6$ so $\overset{\text{average degree}}{d(G)} < 6$. Thus $\exists v \in V$ of degree ≤ 5. $\square$

Theorem (6-Colour): G planar $\Rightarrow$ G is 6-colourable.

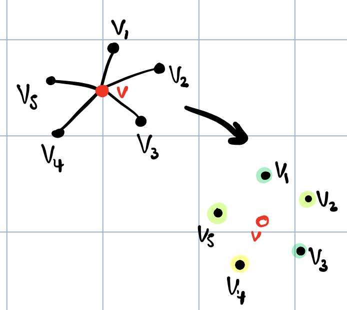
Proof: Induction on $|V|$. We will reduce $|V|$ by choosing v with small degree and expand our colouring inductively.

$|V|=1$: Trivially 6-colourable.

Suppose that G planar with $|V|<k \Rightarrow G$ is 6-colourable, for some $k \in \mathbb{N}_{\geq 2}$.

Let G be planar with $|V|=k$. We construct a 6-colouring on G.

G planar $\Rightarrow \exists v \in V$ with $\deg(v) \leq 5$. Let $v_1, \dots, v_t$ be the neighbours of v, where $t \leq 5$.



Consider $G-v$. Notice G planar $\Rightarrow$ $G-v$ planar and $|V(G-v)| = k-1 < k \overset{\text{I.H.}}{\Rightarrow} G-v$ is 6-colourable.

★ $\Rightarrow v_1, \dots, v_t$ have some colouring in $G-v$.

Extend the 6-colouring of $G-v$ to G by colouring v with any of the unused colours amongst neighbours. $\square$

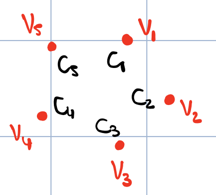
Theorem (5-Colour): G planar $\Rightarrow$ G is 5-colourable.

Proof: Similar to above, but with additional steps. Start at $*$ in 6-CT.

$v_1, \dots, v_t$ must only occupy 4 colours in the 5-colouring of $G-v$.

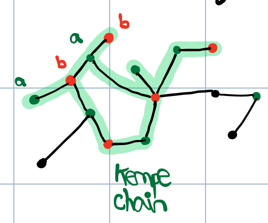
If $t \leq 4$, we are done. Say $t = 5$. We show that $v_1, \dots, v_5$ occupy at most 4 colours.

SFAC $v_1, \dots, v_5$ must occupy all 5 colours in all 5-colourings of $G-v$. We show this contradicts planarity.



Say v_i is coloured C_i. We form a topological contradiction.

A (a,b)-Kempe chain is a maximal subgraph of G such that the colours of vertices u_i in the subgraph alternate between a and b.



Case 1: Suppose there is no v_1v_3-path in $G-v$. Colour v_1 with C_3; we claim this works.

Let P_1 be the (C_1, C_3)-Kempe chain starting at v_1. Notice $v_3 \notin P_1$. The colouring on this path looks like

$C_1C_3C_1C_3 \dots C_3C_j$ or $C_1C_3C_1C_3 \dots C_1C_j$. (where $j \neq 1,3$) Switch C_1 and C_3 on P_1 such that v_1 is coloured C_3.

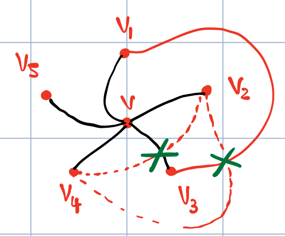
↓switch ↓switch

$C_3C_1C_3C_1 \dots C_1C_j$ $C_3C_1C_3C_1 \dots C_3C_j$. $v_3 \notin P_1 \Rightarrow v_3$ is still coloured C_3.

Colour v with C_1 in G to construct a 5-colouring on G. Δ

Case 2: There is a v_1v_3-path in $G-v$.

If there is no v_2v_4-path, then we reduce to Case 1 using those vertices.

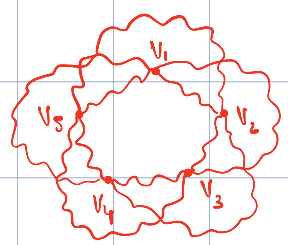


Thus we reduce to the case that there is a v_2v_4-path in $G-v$

$\Rightarrow \exists v_1v_3$-path in G and $\exists v_2v_4$-path in G.

SFAC
We repeat the process with all vertex pairs, and reduce to the case that $\exists v_iv_j$-path for every pair (i,j).

$\Rightarrow$ $G-v$ has a subgraph that looks like



but this is a subdivision of K_5

$\Rightarrow$ $G-v$ not planar. $\lightning$

So not all vertex pairs can be connected

as shown above, so we can always find some (C_i, C_j)-Kempe chain to switch along,

leaving colour C_i free for vertex v.

Use this to extend the 5-colouring on $G-v$ to one on G. $\square$

Theorem (4-Colour): G planar $\Rightarrow$ G is 4-colourable.

Proof: Solved by computers. $\square$

The above results are the main results on vertex colouring (for planar graphs) that are relatively elementary. We generalize more.

G graph. The chromatic number of a graph G is defined $\chi(G) = \min\{k \in \mathbb{N} \mid G \text{ is } k\text{-colourable}\}$.

Corollary: G planar $\Rightarrow \chi(G) \leq 4$.

Proof: Direct result of 4CT.

Lemma: G graph. Then $|E| \geq \binom{\chi(G)}{2}$.

Proof: Let $\mathcal{C} = (C_1, \dots, C_{\chi(G)})$ be a colouring on G of size $\chi(G)$. Notice $C_1, \dots, C_{\chi(G)}$ are disjoint and $\bigcup_{i=1}^{\chi(G)} C_i = V$.

Each edge has two vertices of distinct colours, so there are $\binom{\chi(G)}{2}$ possible edge colour pairs.

$|E| < \binom{\chi(G)}{2}$ would contradict the minimality of $\chi(G)$.

SFAC $|E| < \binom{\chi(G)}{2}$. Then some colour pair C_i, C_j was not used in E

$\Rightarrow v \in C_i \Rightarrow$ no neighbours of v are coloured $C_j \Rightarrow$ re-colour all $v \in C_i$ with C_j. $\lightning$ smaller colouring

So $|E| \geq \binom{\chi(G)}{2}$. $\square$

Corollary: G graph. Then $\chi(G) \leq \sqrt{2|E|} + 1$. ← upper bound for $\chi(G)$

Proof: $|E| \geq \binom{\chi(G)}{2} = \dfrac{\chi(G)(\chi(G)-1)}{2} \Rightarrow 2|E| \geq (\chi(G))(\chi(G)-1) \geq (\chi(G)-1)^2$

$\Rightarrow \sqrt{2|E|} \geq \chi(G) - 1 \Rightarrow \chi(G) \leq \sqrt{2|E|} - 1$. $\square$

G graph with $|V| = n$. We define the complement $\bar{G}$ by $(\bar{V}, \bar{E})$ where $\bar{V} = V$ and $\bar{E} = E(K_n) \setminus E$.

Lemma: G graph, $\bar{G}$ complement on n vertices. Then $n \leq \chi(G)\chi(\bar{G})$. ← lower bound

Proof: Let C_G be a $\chi(G)$-colouring of G and $C_{\bar{G}}$ be a $\chi(\bar{G})$-colouring of $\bar{G}$. Let $\chi(G) = k$ and $\chi(\bar{G}) = \ell$.

Define $S = \{(C_G(v), C_{\bar{G}}(v)) \mid v \in V\}$ be the set of colours of v in C_G and $C_{\bar{G}}$.

Notice $(C_G(v), C_{\bar{G}}(v)) \in \underbrace{\{1, \dots, k\}}_{\chi(G)} \times \underbrace{\{1, \dots, \ell\}}_{\chi(\bar{G})}$ so $|S| \leq k\ell$. Clearly $|S| \leq n$ as well.

$|S| = n$. Say $u, v \in V$ are such that $(C_G(u), C_{\bar{G}}(u)) = (C_G(v), C_{\bar{G}}(v))$ for $u \neq v$.

Then we have $C_G(u) = C_G(v) \Rightarrow u$ and v are not adjacent in $G \Rightarrow uv \notin E$

$C_{\bar{G}}(u) = C_{\bar{G}}(v) \Rightarrow u$ and v are not adjacent in $\bar{G} \Rightarrow uv \notin \bar{E}$

but $uv \in E(K_n) = E \cup \bar{E}$. $\lightning$

$\Rightarrow n = |S| \leq k\ell = \chi(G)\chi(\bar{G})$. $\square$

Corollary: G graph, $\bar{G}$ complement on n vertices. Then $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.

G graph, $H \subseteq G$ subgraph. We define the quotient graph (or subgraph contraction) G/H by identifying all $v \in V(H)$ to a single (new) vertex v_H. In particular, define a function $\varphi: G \to G/H$ by functions (φ_V, φ_E) such that

$\varphi_V: V(G) \to (V(G) \setminus V(H)) \cup \{v_H\}$ by $\varphi_V(v) = \begin{cases} v & \text{if } v \in V(G \setminus H); \\ v_H & \text{if } v \in V(H). \end{cases}$

$\varphi_E: E(H) \to \mathrm{Im}(\varphi_E)$ by $\varphi_E(uv) = \begin{cases} \varphi_V(u)\varphi_V(v) & \text{if at least one of } u, v \notin V(H) \\ \text{Null} & \text{otherwise} \end{cases}$

} intuitive definition

Formally all quotient maps are of the form $\varphi: A \to A/\sim$ where $\sim$ defines an equivalence relation on A.

We formalize φ in this way. We will denote $\pi: G \to G/H$ to be the (properly-defined) quotient map.

Define $\sim$ on V by $u \sim v \Leftrightarrow u = v$ or $u, v \in V(H)$. all vertices $\in V(H)$ fall into a class $[H]$; every vertex $\notin V(H)$ fall into singleton equiv. classes

Write $\pi_V: V \to V/\sim$ by $\pi_V(u) = [u]$ where $V/\sim = \{[H]\} \cup \{[w] \mid w \in V(G) \setminus V(H)\}$.

$\pi_E: E \to E(G/H)$ by $\pi_E(\overset{uv}{e}) = \pi_V(u)\pi_V(v)$ where $E(G/H) = \{\pi_V(u)\pi_V(v) \mid uv \in E, \pi_V(u) \neq \pi_V(v)\}$. ← delete all loops to keep G/H simple.

Then $\pi: \underset{G}{\underset{\parallel}{(V, E)}} \to \underset{G/H}{\underset{\parallel}{(V/\sim, E(G/H))}}$ defines a graph homomorphism and G/H is the quotient graph.

We can think of G/H as constantly contracting edges $e \in E(H)$ until none are left.

Lemma: G graph, $H \subseteq G$. Then G planar $\Rightarrow G/H$ planar.

Proof: Exercise.

G graph. A matching on G is a set of edges $M \subseteq E(G)$ such that no two edges in M share a common vertex.

A vertex $v \in V$ is saturated by M if $uv \in M$ for some $u \in V$ (i.e. v touches some edge in M). Else v is unsaturated.

We say a matching M is perfect if M saturates all $v \in V$.

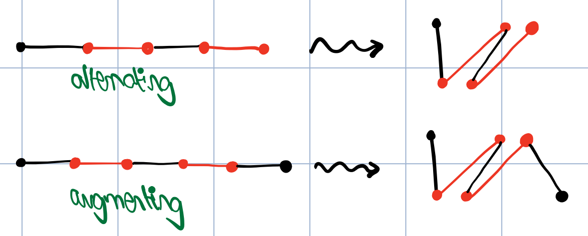
Remark: We can consider the notions of maximum and maximal matchings. Note that these are not equiv..

We can define the set of matchings M on G as a poset with order set inclusion.

Maximal ≈ cannot be further extended.

Maximum ≈ largest possible across all matchings M (non-unique).

Let M be a matching. An alternating path is a path $v_0 v_1 \dots v_n$ if edges alternate between being in M and not being in M.

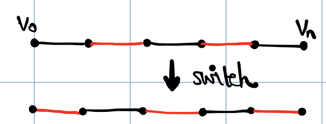


A path $v_0 v_1 \dots v_n$ is an augmenting path if it is alternating and v_0, v_n are unsaturated.

Lemma: G graph, M matching. Let $G = P$ be an augmenting path. Then M is not maximal.

Proof: P augmenting $\Rightarrow$ Let $P = v_0 v_1 \dots v_{n-1} v_n$ such that v_0 and v_n are unsaturated $\Rightarrow v_0 v_1 \notin M$ and $v_{n-1} v_n \notin M$.

Switch all unmatched and matched edges in M to construct M' so that $v_0 v_1, v_{n-1} v_n \in M'$.



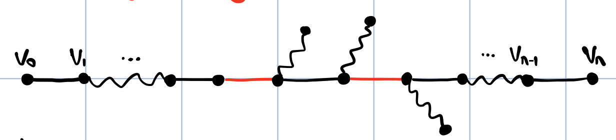
Then $|M'| \geq |M| + 1$ so M was not maximal. $\square$

Remark: Weak! Here, we need $P = G$. Otherwise, we don't know what a switch on M will do to other parts of G.

Lemma (Berge's): G graph, M matching. Then M is maximum $\Leftrightarrow$ there is no augmenting path for M.

Proof: We will prove that M is not maximum $\Leftrightarrow$ there is an augmenting path for M.

($\Leftarrow$): Say there is an augmenting path P for M.



Write $P = v_0 \dots v_n$ for this path, and notice that

any edge $e \in E$ incident to $v_i \in P_{\{v_0, v_n\}}$ must be unmatched (otherwise M not a matching).

Therefore, we can switch along P for a larger matching M', leaving all other edges unaffected.

$\Rightarrow M$ not maximal $\Rightarrow M$ not maximum. $\triangle$

($\Rightarrow$): Say M is not maximum. We must show that there is an augmenting path for M.

Let M' be a larger matching for G.

Define a graph G' by taking the symmetric difference of M and M' (i.e. $(M - M') \cup (M' - M)$).

G' consists of components that are one of the following: *Exercise

- an isolated vertex;
- an even cycle whose edges alternate between M and M';
- a path whose edges alternates between M and M' with distinct endpoints

M' larger than $M \Rightarrow G'$ contains a component that has more edges in M' than M.

Such a component must be a path in G with terminal edges in $M' \Rightarrow$ there is an augmenting path fo

G graph. A **cover** of G is a set $C \subseteq V$ such that $e = uv \in E \Rightarrow \{u,v\} \cap C \neq \emptyset$. Every $e \in E$ has at least one endpoint in C.

We cared about how large we can make matchings: making them small is trivial.

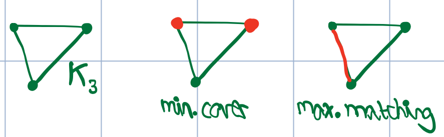
We care about how small we can make covers.

Lemma: G graph with matching $M \subseteq E$, cover $C \subseteq V$. Then $|M| \leq |C|$.

Proof: For each $e = uv \in E$, at least one of $u, v \in C$. Since no two edges in M share the same vertex, we have some injection $\varphi: M \to C$ by identifying $e \in M$ to a unique $v \in C \Rightarrow |M| \leq |C|$. $\square$

Corollary: G graph, C cover, M matching. If $|M| = |C|$ then C is minimum and M is maximum.

Remark: Converse not true, see K_3.



Lemma: G cycle, M maximum matching and C minimum cover. Then $|M| = \left\lfloor \frac{|V|}{2} \right\rfloor$ and $|C| = \left\lceil \frac{|V|}{2} \right\rceil$.

Proof: Exercise.

Notice if we exclude odd cycles in G, then we will have $|M| = |C| \Leftrightarrow$ extremality.

But G contains no odd cycles $\Leftrightarrow G$ bipartite.

Theorem (König): G bipartite graph, M matching, C cover. Then $|M| = |C| \Leftrightarrow$ M is maximum and C is minimum.

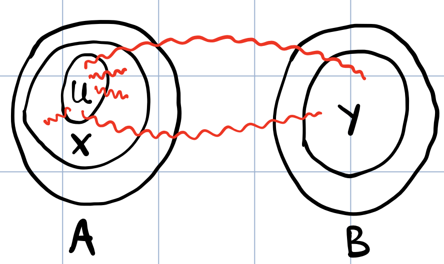
Proof: There are many proofs to this result. Notable ones are the constructive flow, constructive (our proof), and LP duality proofs.

(⇒): Done above (see Lemma).

(⇐): Let (A, B) be a bipartition of G. Consider $U = \{v \in A \mid v \text{ unsaturated by } M\} \subseteq A$

$X = \{v \in A \mid v \in U \text{ or there is an alternating path from } u \in U \text{ to } v\} \subseteq A$

$Y = \{v \in B \mid \text{there is an alternating path from } u \in U \text{ to } v\} \subseteq B$



$\Rightarrow v \in \underbrace{X \setminus U}_{\subseteq A}$ has an alternating path to some $u \in U$. Notice $v \in X \setminus U \Rightarrow v$ saturated by M.

Let $e = vw$ for some $w \in V$. We have $v \in X \setminus U \subseteq A \Rightarrow w \in B$ by bipartiteness, and by the path above

$\exists$ alternating path from w to $U \Rightarrow w \in Y$.

$\Rightarrow$ all edges from $X \setminus U$ have an endpoint in Y, and are in M.

$\Rightarrow$ there are no edges between $X \setminus U$ and Y that are not in M.

1. Any alternating path P_v starting at v and terminating in U has even length if $v \in X$ and odd length if $v \in Y$.

Result follows from bipartedness. $U \subseteq A$ and $X \subseteq A$, but $Y \subseteq B$. $\triangle$

$\Rightarrow v \in X \Rightarrow$ last edge of P_v is in M (vacuously true if $v \in U$, otherwise use length parity).
and $v \in Y \Rightarrow$ last edge of P_v is not in M.

2. There is no $e \in E$ from X to $B \setminus Y$.

SFAC such $e \in E$ exists. Let $u \in X$, $v \in B \setminus Y$, and $uv \in E$. Since $u \in X$ we have some alternating path P_u
ending in U, and P_u must have even length.

Add v to P_u $\Rightarrow$ we have an odd-length alternating path to $v \Rightarrow v \in Y$. $\lightning$ $\triangle$

3. $C = Y \cup (A \setminus X)$ is a cover of G.

The only edges that are not incident to C are edges from X to $B \setminus Y$. These don't exist. $\triangle$

4. $\nexists e \in M$ from Y to $A \setminus X$.

5. $|M| = |C| - |U_Y|$ where $U_Y = \{v \in Y \mid v \text{ unsaturated}\}$.

6. There is an augmenting path to each vertex in U_Y.

Exercise. Long and unintuitive. $\square$

Corollary: G bipartite, $\Delta(G)$ maximum degree. Then $\exists M \subseteq E$ matching with $|M| \geq \frac{|E|}{\Delta(G)}$.

Proof: Let $C \subseteq V$ be a minimum cover

$\Rightarrow$ each $v \in C$ covers at most $\Delta(G)$ edges so $|C| \geq |E| \cdot \frac{1}{\Delta(G)}$.

Let M be a max matching $\overset{\text{König}}{\Rightarrow} |M| = |C| \geq \frac{|E|}{\Delta(G)}$ since G bipartite. $\square$

König's Algorithm
$\begin{cases} \text{1. Start with a matching} \\ \text{2. Systematically switch augmenting/alternating} \\ \text{3. Generate max. matching} \end{cases}$

Let G be bipartite (A, B), and let $M \subseteq E$ be any matching on G.

1. Define $U := \{v \in A \mid v \text{ unsaturated by } M\} \subseteq A$ (all unsaturated vertices of A).

$X := U$
$Y := \emptyset$
$\Big\}$ we define U, X and Y as if M was maximum and construct iteratively

2. Find all neighbours of X in $B \setminus Y$

a) if any of these are unsaturated, we've found an augmenting path
$\Rightarrow$ update M by switching on this path and start over.

b) If all such vertices are saturated, add all vertices to Y and add the other endpoint of the matched edge to X. Repeat 2.

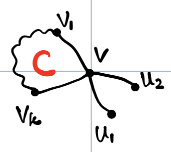
c) no such vertices exist $\Rightarrow$ M maximum, and $Y \cup (A \setminus X)$ min. cover.

Lemma: G graph. If $\delta(G) \geq 4$, then there are two cycles $C_1, C_2 \subseteq G$ such that $E(C_1) \cap E(C_2) = \emptyset$.

Proof: We know that $\delta(G) \geq 2 \Rightarrow G$ has a cycle $C \subseteq G$.

Let $v \in V(C)$ be in this cycle. Write $C = vv_1 \cdots v_k v$. Notice $v_1 \neq v_k$, and $\deg(v) \geq 4$

$\Rightarrow v$ has neighbours u_1, u_2 such that $u_1, u_2 \notin V(C)$.



Contract $C \setminus \{v\}$ into v, so all edges of C disappear.

The resultant graph G/C has $\delta(G/C) \geq 2 \Rightarrow G/C$ has a cycle C' (sharing no edges with C)

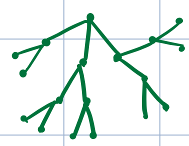
$\Rightarrow C, C'$ are such cycles by the above construction. $\square$

Exercise: T tree with $\ell(T) = 10$. Say T has no vertices of degree 2. Determine $\max_G(|V(G)|)$.

T tree $\Rightarrow |E| = |V| - 1$. We have $\ell(G) = 2 + \sum_{i \geq 3} (i-2) n_i$ where $n_i = |\{\deg^{-1}(i)\}|$.

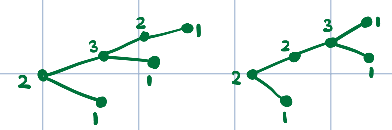
$$= 2 + \sum_{i \geq 3} (i-2) n_i = 10 \Rightarrow \sum_{i \geq 3} (i-2) n_i = 8$$

maximize $n_i \leftrightarrow$ minimize $i-2 \Rightarrow i = 3$ only, so consider $n_3 = 8$.



$\Rightarrow |V| = n_3 + \ell(G) = 18$ vertices.

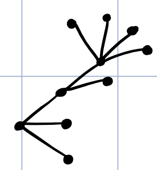
Exercise Find 2 non-isomorphic bipartite graphs with the same degree sequence.



Lemma: T tree. Then $\deg(v)$ odd $\forall v \in V \iff \forall e \in E$, we have odd #vertices in both components of $T - e$.

Proof: $(\Rightarrow)$ Say $\deg(v)$ odd, $\forall v \in V$. T tree $\Rightarrow$ every $e \in E$ bridge, so $\mathrm{comp}(T-e) = 2$.

Let T_1 and T_2 be the components of $T - e$.



We have $\sum_{v \in V} \deg(v) = 2|E|$ and T tree $\Rightarrow |E| = |V| - 1$.

$$\sum_{v \in V} \deg(v) = \underbrace{\sum_{v \in T_1} \deg(v)}_{d_1} + \underbrace{\sum_{v \in T_2} \deg(v)}_{d_2} + 2.$$ Notice Handshaking $\Rightarrow d_1, d_2 \equiv 0 \bmod 2$.

Say $e = v_1 v_2$ with $v_1 \in T_1$ and $v_2 \in T_2$. We have $\deg(v_1) \equiv 0 \bmod 2$ and $\deg(v_2) \equiv 0 \bmod 2$ due to the removed edge ($v_1, v_2 \in T$ had odd degree), so we have

$d_i \equiv (|V(T_i)| - 1) \bmod 2 \equiv 0 \bmod 2 \Rightarrow |V(T_i)| \equiv 1 \bmod 2$. $\triangle$

$(\Leftarrow)$: Let components T_1, T_2 of $T - e$ have odd #vertices for all $e \in E$.

SFAC there exists $v \in V$ with even degree $^{\neq 0}$. Choose $e = uv$ incident to v. Say $v \in T_1$.

Then $\deg_{\in T_1}(v_1) \equiv 1 \bmod 2$, so $d_1 \equiv |V(T_1)| \bmod 2 \equiv 1 \bmod 2$. $\lightning$ We had $d_1 \equiv 0 \bmod 2$.

Notice that adding any more even-degree vertices to T_1 or T_2 does not change the result. $\triangle$

Done. $\square$

Lemma: G graph with $|V| \geq 5$. Then at most one of $G, \bar{G}$ is bipartite.

Proof: Exercise.

Lemma: G connected graph. Then $e \in E$ belongs to every spanning tree $T \subseteq G \Leftrightarrow e$ is a bridge of G.

Proof: ($\Leftarrow$) Say $e \in E$ is a bridge. SFAC $\exists T \subseteq G$ spanning tree with $e \notin E(T)$.

Write $e = uv$ and notice u, v in distinct components C_1, C_2 of $G - e$.

We have $\mathrm{cut}_G(V(C_1)) = \{e\}$ and $\underbrace{V = V(T)}_{\Rightarrow V(C_1) \subseteq V(T)} \Rightarrow \mathrm{cut}_T(V(C_1)) = \{e\} \cap E(T) = \emptyset \Rightarrow T$ disconnected $\lightning$

Thus $e \in E$ bridge $\Rightarrow e \in E(T)$ for any spanning tree $T \subseteq G$. $\triangle$

($\Rightarrow$) Say $e \in E$ belongs to every spanning tree $T \subseteq G$. We show e is not in any cycles of G.

SFAC e is in some cycle $C = uv_1 \dots v_k vu$ where $e = uv$. Let T be a spanning tree with $e \in E(T)$.

T tree and $uv \in E(T) \Rightarrow$ path $uv_1 \dots v_k v \not\subseteq T$.

We can construct T' by removing e from T, and adding $uv_1 \dots v_k v$ instead, adjusting accordingly.

$\Rightarrow T' \subseteq G$ is a tree spanning $G \Rightarrow$ not all spanning trees of G contain e. $\lightning$

Therefore, $e \in E$ cannot be in any cycles of $G \Rightarrow e$ is a bridge. $\triangle\square$

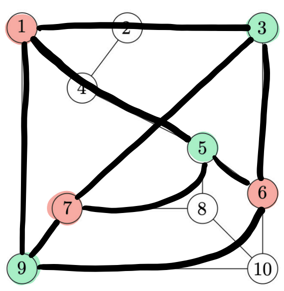
Lemma: G connected graph, containing $C \subseteq G$ a cycle of length k. Let $T \subseteq G$ spanning tree containing ℓ edges of C, for some $0 \leq \ell \leq k-2$. Then there exists a spanning tree containing $\ell+1$ edges of C.

these may not be consecutive!

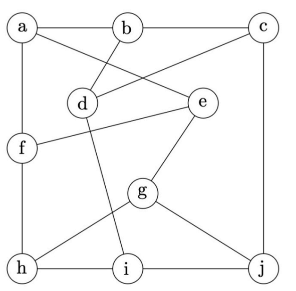
Proof: Say $e_1, \dots, e_\ell \in E(C)$ and $\in E(T)$, and $e_{\ell+1}, \dots, e_k \in E(C)$ but $\notin E(T)$.

Write $e_i = u_i v_i$. Exercise.

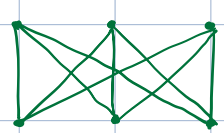
Exercise:

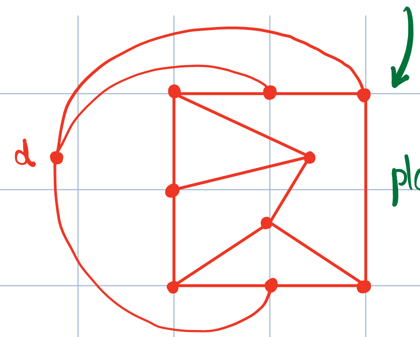


$K_{3,3}$



Determine (with proof) whether either graph is planar.





planar embedding

G graph, $D \subseteq V$. We define the neighbour set $N(D)$ to be $N(D) = \{v \in V \mid \exists u \in D \text{ with } uv \in E\}$.

If G bipartite (A, B), then if there was a matching saturating A, we must have $|N(D)| \geq |D|$.

Theorem (Hall's): G bipartite (A, B).

a matching that is A-saturating must be perfect

Then G has a matching saturating all $v \in A$ $\Leftrightarrow$ every $D \subseteq A$ satisfies $|N(D)| \geq |D|$.

Proof: $(\Rightarrow)$ Say G has a matching M saturating all $v \in A$.

$\Rightarrow$ For any $D \subseteq A$, we have $N(D)$ containing the other end of an edge of M incident with $v \in D$, for any $v \in D$. All these "other ends" must be distinct

$\Rightarrow |N(D)| \geq |D|$. $\triangle$

$(\Leftarrow)$ If there is no matching M saturating A completely then $\exists D \subseteq A$ with $|N(D)| < |D|$.

$\overset{\text{König's}}{\Rightarrow}$ $\exists M$ max matching, $\exists C$ min. cover such that $|M| = |C| < |A|$.

Consider partition $A \cap C$, $A \setminus C$, $B \cap C$, $B \setminus C$ of V (where some of these partitions may be empty!).

C cover $\Rightarrow$ no edge joins $A \setminus C$ to $B \setminus C$ (such an edge would be uncovered)

$\Rightarrow N(A \setminus C) \subseteq B \cap C$ (no edge joins $A \setminus C$ to $A \cap B$ because (A, B) bipartition).

Thus we have $|N(A \setminus C)| \leq \underbrace{|B \cap C|}_{= C \setminus (A \cap C)} = |C| - |A \cap C| < |A| - |A \cap C| = |A \setminus C|$

$\Rightarrow D = A \setminus C$ works. $\triangle \square$

Lemma: G bipartite (A, B). Say for any $X \subseteq A$, we have $|N(X)| \geq |X| - d$. Then $\exists \overset{\text{matching}}{M} \subseteq E$ with $|M| \geq |A| - d$.

Proof: We show that if there is no matching M with $|M| \geq |A| - d$, then $\exists X \subseteq A$ with $|N(X)| \leq |X| - d$.

G bipartite $\overset{\text{König's}}{\Rightarrow}$ $\exists M$ max matching, $\exists C$ min. cover with $|C| = |M| < |A| - d$.

Consider partition $A \cap C$, $A \setminus C$, $B \cap C$, $B \setminus C$.

C cover $\Rightarrow$ no edge joins $A \setminus C$ to $B \setminus C$ (such an edge would be uncovered).

(A, B) bipartition $\Rightarrow$ no edge joins $A \setminus C$ to $A \cap C$ or $A \setminus C$.

$\Rightarrow N(A \setminus C) \subseteq B \cap C$. Thus $|N(A \setminus C)| \leq |B \cap C| = |C| - |A \cap C| < |A| - d - |A \cap C| \leq |A \setminus C| - d$.

$\Rightarrow$ choose $X = A \setminus C$, satisfying our requirement. $\square$

Proof (2): We have $|N(X)| \geq |X| - d$ but want $|N(X)| \geq |X|$.

Let $G' = (V', E')$ where $V' = V \cup \{v_1, \dots, v_d\}$ and $E' = E \cup \{v_i a \mid i \in \{1, \dots, d\}, a \in A\}$.

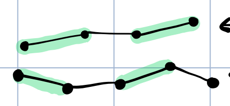
$\Rightarrow$ for any $X \subseteq A$, we have $N_{G'}(X) = N_G(X) \cup \{v_1, \dots, v_d\} \Rightarrow |N_{G'}(X)| = |N_G(X)| + d \geq |X| - d + d = |X|$.

$\Rightarrow G'$ satisfies hall condition $\Rightarrow \exists M$ matching that is A-saturating.

Let $M' := M \setminus \text{cut}(\{v_1, \dots, v_d\})$. This removes at most d edges from M $\Rightarrow |M'| \geq |M| - d = |A| - d$. $\square$

Remark: Unintuitive proof (2).

Lemma: G tree. Then G has at most 1 perfect matching.

Proof: Idea: depends on the length of any path.  leaf-edge fully determines the matching

Let P be a path in G, of length k.

If k is even, then any matching on P will result in at least one unsaturated vertex.

Say k is odd. We have a unique M that saturates P, and G tree $\Rightarrow$ M on P completely determines M on G by alternating edges.

$\Rightarrow$ if len(P) odd for all leaf-leaf paths P, we have a perfect matching, otherwise none. $\square$

Lemma: G bipartite, k-regular graph. Then $\exists M \subseteq E$ (matching) with $|M| \geq |E|/k$.

Proof: Let (A, B) bipartition. Let $X \subseteq A$. We prove $|N(X)| \geq |X|$.

Notice $|N(X)| = \left| \bigcup_{v \in X} N(\{v\}) \right|$.

We have $cut_G(X) \subseteq cut_G(N(X))$ since $e \in cut_G(X) \Rightarrow e$ has an endpoint in $N(X)$ and another out of $N(X)$ $\Rightarrow e \in cut_G(N(X))$.

By definition, $X \subseteq N(N(X))$.

$\Rightarrow |cut_G(X)| = k|X| \leq |cut(N(X))| = k|N(X)| \Rightarrow |X| \leq |N(X)|$.

Let M be a maximum matching on G. Then $|X| \leq |N(X)|$ for all $X \subseteq A \overset{\text{Hall's}}{\Longrightarrow} M$ is A-saturating

$\Rightarrow |M| = |A|$.

G bipartite $\Rightarrow |E| = \sum_{v \in A} deg(v) = k|A| = k|M| \Rightarrow |M| = \frac{|E|}{k}$. $\square$

Lemma: G bipartite k-regular, for $k \geq 0$.

1. E can be partitioned into k perfect matchings.
2. V can be partitioned into 2 minimum covers.

Exercise Generating series w.r.t. length for bin. strings not containing 01110.

1. Regex: start with $1^*(\underbrace{00^*}_{\{0,00,\ldots\}}\underbrace{11^*}_{\{1,11,\ldots\}})^* 0^* \longrightarrow \{\varepsilon, 0, 00, \ldots\}$

$$\|$$

$$1^*(00^*11^*)^*(\varepsilon \smile 00^*)$$

$$1^*((00^*11^*)^* \smile (00^*(1 \smile 11 \smile 1111)^* 00^*))$$

Exercise n even positive, m positive. $[x^n]\underbrace{(1-3x^2)^{-m}}\underbrace{(1+x)^2}$

$$\sum_{k\geq 0}\binom{m+k-1}{k}(3x^2)^k \qquad (1+2x+x^2)$$

$$= \sum_{k\geq 0}\binom{m+k-1}{k}3^k x^{2k}$$

Exercise $n \geq 0$. Determine #non-negative integer sols. to $a+2b+c=n$.

Consider representing the problem as a composition (r_1, r_2, r_3) where $r_1 \in \{0,1,2,\ldots\}$, $r_2 \in \{0,2,4,\ldots\}$, $r_3 \in \{0,1,2,\ldots\}$.

By product lemma we have

$$\Phi_S(x) = \Phi_{R_1}^2(x)\,\Phi_{R_2}(x) \text{ where } \Phi_{R_1}(x) = \frac{1}{1-x} \text{ since } R_1 = \{0,1,2,\ldots\} \Rightarrow \Phi_{R_1}(x) = \sum_k x^k = \frac{1}{1-x}$$

$$\Phi_{R_2}(x) = \frac{1}{1-x^2} \text{ since } R_2 = \{0,2,4,\ldots\} \Rightarrow \Phi_{R_2}(x) = \sum_k x^{2k} = \frac{1}{1-x^2}$$

$$\Rightarrow \Phi_S(x) = \frac{1}{1-x^2}\frac{1}{(1-x)^2}$$

$$= \sum_k x^{2k}\left(\sum_k x^k\right)^2 \rightsquigarrow \text{get } n^{th} \text{ coeff for answer.}$$

Exercise Generating series for compositions with even #parts each of which is even.

Consider $S = (c_1, \ldots, c_n)$ with n even. Each c_i has set $C = \{0, 2, \ldots\} \Rightarrow S = C^n$.

$$\overset{\text{prod. lemma}}{\Rightarrow} \Phi_S(x) = \Phi_{C^n}(x) = (\Phi_C(x))^n. \text{ We have } \Phi_C(x) = \sum_{k\geq 0} x^{2k} = \frac{1}{(1-x^2)} \Rightarrow (\Phi_C(x))^n = (1-x^2)^{-n}.$$

We sum over all n even

$$\rightsquigarrow \sum_{i\geq 0}(1-x^2)^{-2i} \Rightarrow \frac{1}{1-(1-x^2)^2} \rightsquigarrow \text{simplify to } \frac{1-2x^2+x^4}{1-2x^2}.$$

Exercise $c_n = 5c_{n-1} - 7c_{n-2} + 3c_{n-3}$ with $c_0 = 3,\ c_1 = 8,\ c_2 = 21$.

$$\Rightarrow c_n - 5c_{n-1} + 7c_{n-2} - 3c_{n-3} = 0 \rightsquigarrow 1 - 5x + 7x^2 - 3x^3 = 0.$$

Consider $C(x) = \sum_{n\geq 0} c_n x^n$. Then $C(x)$ is a rational power series with denom. $1-5x+7x^2-3x^3$.

$1-5x+7x^2-3x^3 = (1-x)^2(1-3x)$ with inverse roots 1 and 3.

$$\Rightarrow c_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n = \underbrace{A_1 3^n + (A_2 + A_3 n)}_{\text{solve } A_1, A_2, A_3 \text{ using initial conditions}}$$

Exercise $a_n = 3a_{n-1} - 2a_{n-2}$, $a_0 = 1$, $a_1 = 4$. $b_n = a_n a_{n+1}$, find linear recurrence.

$a_n = 3a_{n-1} - 2a_{n-2} \Rightarrow a_n - 3a_{n-1} + 2a_{n-2} = 0$. Consider $A(x) = \sum_{n\geq 0} a_n x^n$.

$\Rightarrow A(x)$ has denominator $1 - 3x + 2x^2 = (1-2x)(1-x) \Rightarrow$ inverse roots $\lambda_1 = 2$ and $\lambda_2 = 1$ with multiplicity 1

$\Rightarrow a_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n$ where $\deg(p_i) <$ multiplicity of λ_i

$\Rightarrow a_n = C_1 2^n + C_2$. $a_0 = 1 \Rightarrow C_1 + C_2 = 1$, $a_1 = 4 \Rightarrow 2C_1 + C_2 = 4$ $\Rightarrow C_1 = 3$, $C_2 = -2$ so $a_n = 3\cdot 2^n - 2$ for $n \geq 0$.

$$b_n = a_n a_{n+1} = (3\cdot 2^n - 2)(3\cdot 2^{n+1} - 2) = 9\cdot 2^{2n+1} - 2\cdot 3\cdot 2^n - 2\cdot 3\cdot 2^{n+1} + 4$$
$$= 18\cdot 4^n - 6\cdot 2^n - 12\cdot 2^n + 4\cdot 1^n$$
$$= 18\cdot 4^n - 18\cdot 2^n + 4\cdot 1^n \Rightarrow \text{inverse roots are } 4, 2, 1 \text{ with mult. } 1$$

$\Rightarrow$ denominator $(1-4x)(1-2x)(1-x) = 1 - 7x + 14x^2 - 8x^3$

$\Rightarrow b_n - 7b_{n-1} + 14b_{n-2} - 8b_{n-3} = 0$

$\Rightarrow b_n = 7b_{n-1} - 14b_{n-2} + 8b_{n-3}$. Then we compute init. conditions.

Exercise #compositions a_n in which every part odd $=$ #compositions b_{n-1} where every part $\in \{1,2\}$.

1) Let $A = \{1, 3, 5, \ldots\}$ so all parts are in A. We have $S = \bigcup_{k\geq 0} A^k = A^*$ so $\Phi_S(x) = \Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$.

We have $\Phi_A(x) = x + x^3 + \ldots = \frac{x}{1-x^2} \Rightarrow \Phi_S(x) = \frac{1}{1 - \frac{x}{1-x^2}} = \frac{1}{\frac{1-x^2-x}{1-x^2}} = \frac{1-x^2}{1-x-x^2}$.

2) Let $B = \{1, 2\}$ so $U = \bigcup_{k\geq 0} B^k = B^*$ and $\Phi_U(x) = \frac{1}{1-\Phi_B(x)}$. We have $\Phi_B(x) = x + x^2$

$\Rightarrow \Phi_U(x) = \frac{1}{1-x-x^2}$.

Exercise $p_n = 4p_{n-1} - 4p_{n-2}$, $p_0 = 2$, $p_1 = 5$.

$p_n - 4p_{n-1} + 4p_{n-2} = 0$. Let $P(x) = \sum_{n\geq 0} p_n x^n$. We have $P(x)$ a rational series with denominator

$1 - 4x + 4x^2 = (1-2x)^2 \Rightarrow$ inverse root $\lambda_1 = 2$ with multiplicity 2

$\Rightarrow p_n = f_1(n)\lambda_1^n = 2^n(a_1 n + a_2)$ since $\deg(f_1) < 2$, constants a_1, a_2.

We have $p_0 = a_2 = 2$, $p_1 = 2(a_1 + a_2) = 2(a_1 + 2) = 2a_1 + 4 = 5 \Rightarrow a_1 = \frac{1}{2}$.

$\Rightarrow p_n = 2^n(\frac{1}{2}n + 2)$.

Exercise Given $C(x) = \frac{1+2x}{(1-x)(1-3x)} = \sum_{n\geq 0} c_n x^n$, extract a closed-form linear recurrence for c_n with initial conds

We have denominator $(1-x)(1-3x) = 1 - 3x - x + 3x^2 = 1 - 4x + 3x^2 \Rightarrow c_n$ satisfies $c_n - 4c_{n-1} + 3c_{n-2} = 0$.

Want: $[x^0]C(x)$ and $[x^1]C(x)$.

$[x^0]C(x) = [x^0](1+2x)(1-x)^{-1}(1-3x)^{-1} = [x^0](1\cdot 1\cdot (-3)) = -3$

$[x^1]C(x) = [x^1]((1+2x)(1+x+O(x^2))(1-3x+O(x^2)))$

$= [x^1]((1+2x)(1+x)(1-3x)) = [x^1]((1+3x)(1-3x)) = 0$

Lemma: G planar with embedding $\mathcal{E}$ such that $\mathcal{E}(G)$ has no face of degree 3. Then $\exists v \in V$ with $\deg(v) \leq 3$.

Proof: We have $\deg(f) \geq d = 4$ for all $f \in F$

$\Rightarrow |E| \leq \frac{4(|V|-2)}{4-2} = 2|V| - 4$. We have average degree $\frac{\sum_{v \in V} \deg(v)}{|V|} = \frac{2|E|}{|V|} \leq \frac{4|V|-8}{|V|} = 4 - \frac{8}{|V|} < 4$.

$\Rightarrow \exists v \in V$ with $\deg(v) \leq 3$. $\square$

Exercise: A sequence of k dice rolls can be represented by $(r_1, \dots, r_k)$ where $r_i \in \{1, \dots, 6\}$ with a sum $\sum_{i=1}^{k} r_i$.

$\Rightarrow$ sequence of rolls is a composition of k parts with each part in $R = \{1, \dots, 6\}$.

$\Rightarrow S = A^k \Rightarrow \Phi_S(x) = \Phi_{R^k}(x) = (\Phi_R(x))^k$. We have $\Phi_R(x) = x + x^2 + \dots + x^6 = x\left(\frac{x^6-1}{x-1}\right)$.

$\Rightarrow \Phi_S(x) = x^k \left(\frac{x^6-1}{x-1}\right)^k = x^k (1-x^6)^k (1-x)^{-k}$.