Extrapolation Methods for Accelerating PageRank Computations

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Abstract

The original PageRank algorithm uses the Power Method to compute successive iterates that converge to the principal eigenvector of the Markov matrix representing the Web link graph.It computes the principal eigenvector of the Markov matrix representing the hyperlink structure of the Web.For Web graphs containing a billion nodes, computing a PageRank vector can take several days.Therefore this paper presents an algorithm called **quadratic extrapolation** that accelerates the pagerank computation. All results and formulations are derived from [1]. Empirically, this paper shows that using Quadratic Extrapolation speeds up PageRank computation by 25–300% on a Web graph of 80 million nodes, with minimal overhead.

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1 Introduction

PageRank (PR) is an algorithm used by Google Search to rank web pages in their search engine results. PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is. PageRank is a link analysis algorithm and it assigns a numerical weighting to each element of a hyperlinked set of documents, such as the World Wide Web, with the purpose of "measuring" its relative importance within the set. The algorithm may be applied to any collection of entities with reciprocal quotations and references.
Initially Power Method was used to calculate the principal eigen vector of the matrix. A short proof of the power method is presented below.

$$\vec{x}^{(0)} = \vec{u}_1 + \alpha_2\vec{u}_2 + \alpha_3\vec{u}_3..... + \alpha_{m-1}\vec{u}_{m-1} + \alpha_m\vec{u}_m$$
$$\vec{x}^{(1)} = A\vec{x}^{(0)} = \vec{u}_1 + \alpha_2\lambda_2\vec{u}_2 + ... + \lambda_m\alpha_m\vec{u}_m$$
$$\vec{x}^{(n)} = A^n\vec{x}^{(0)} = \vec{u}_1 + \alpha_2\lambda_2^n\vec{u}_2 + ... + \lambda_m^n\alpha_m\vec{u}_m$$

Algorithm for power method

function $\vec{x}^n$ = powermethod(){
$\vec{x}^{(0)}= \vec{v}$
k=1
repeat
$\vec{x}^{(k)} = \mathrm{A}\vec{x}^{(k-1)}$
$\delta = ||\vec{x}^{(k)} - \vec{x}^{(k-1)}||_1$
k=k+1
until $\delta < \epsilon$
}

Operation Count
A single iteration of the Power Method consists of the single matrix-vector multiply $A\vec{x}^{(k)}$. Generally, this is an O(n^2) operation. now for a Markov matrix the largest eigen value is 1. so if the 2nd largest eigen value is is close to 1,then the power method is slow to converge, because n must be large

before λ_2 is close to 0, and vice versa.

2 Main Results

Quadratic Extrapolation it is a fast method for determining the dominant eigenvector of a matrix that is too large for standard fast methods to be practical.Quadratic Extrapolation, accelerates the convergence of the Power Method by periodically subtracting off estimates of the nonprincipal eigenvectors from the current iterate. One may think of Quadratic Extrapolation as using successive iterates generated by the Power Method to extrapolate the value of the principal eigenvector.

The formulation of the method.

We assume that the Markov matrix A has only 3 eigenvectors, and that the iterate $\vec{x}^{(k-3)}$ can be expressed as a linear combination of these 3 eigenvectors.

$$\vec{x}^{(k-3)} = \vec{u_1} + \alpha_2\vec{u_2} + \alpha_3\vec{u_3}$$

We then define the successive iterates as

$$\vec{x}^{(k-2)} = A\vec{x}^{(k-3)} \tag{1}$$

$$\vec{x}^{(k-1)} = A\vec{x}^{(k-2)} \tag{2}$$

$$\vec{x}^{(k)} = A\vec{x}^{(k-1)} \tag{3}$$

Since we assume that A has 3 eigenvectors the characteristic polynomial is given by

$$p_A(\lambda) = \gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3$$

Since A is a Markov matrix $\lambda_1 = 1$

$$p_A(1) = 0 \implies \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 0 \tag{4}$$

Using Cayley-Hamilton Theorem where let z be any arbitrary vector we get:

$$p_A(A)z = 0 \implies [\gamma_0 I + \gamma_1 A + \gamma_2 A^2 + \gamma_3 A^3]z = 0$$

Letting $z = \vec{x}^{(k-3)}$,

$$[\gamma_0 I + \gamma_1 A + \gamma_2 A^2 + \gamma_3 A^3]\vec{x}^{(k-3)} = 0$$

From equations 1 , 2 and 3 we get

$$\gamma_0\vec{x}^{(k-3)} + \gamma_1\vec{x}^{(k-2)} + \gamma_2\vec{x}^{(k-1)} + \gamma_3\vec{x}^{(k)} = 0$$

Using equation 4 we get

$$(-\gamma_1 - \gamma_2 - \gamma_3)\vec{x}^{(k-3)} + \gamma_1\vec{x}^{(k-2)} + \gamma_2\vec{x}^{(k-1)} + \gamma_3\vec{x}^{(k)} = 0$$

It can be rewritten as:

$$(\vec{x}^{(k-2)} - \vec{x}^{(k-3)})\gamma_1 + (\vec{x}^{(k-1)} - \vec{x}^{(k-3)})\gamma_2 + (\vec{x}^{(k)} - \vec{x}^{(k-3)})\gamma_3 = 0$$

Let us define a few new terms as following:

$$\vec{y}^{(k-2)} = \vec{x}^{(k-2)} - \vec{x}^{(k-3)}$$

$$\vec{y}^{(k-1)} = \vec{x}^{(k-1)} - \vec{x}^{(k-3)}$$

$$\vec{y}^{(k)} = \vec{x}^{(k)} - \vec{x}^{(k-3)}$$

We write equation in the form of a matrix:

$$(\ \vec{y}^{(k-2)} \qquad \vec{y}^{(k-1)} \qquad \vec{y}^{(k)}\ )\vec{\gamma} = 0$$

We want to solve for $\vec{\gamma}$. Since we are not interested in the trivial solution $\vec{\gamma} = 0$, we constraint the leading term of the characteristic polynomial:

$$\gamma_3 = 1$$

Doing this does not affect the zeroes of the polynomial.
Therefore equation can be rewritten as:

$$\begin{pmatrix}\vec{y}^{(k-2)} & \vec{y}^{(k-1)}\end{pmatrix}\begin{pmatrix}\gamma_1 \\ \gamma_2\end{pmatrix} = -\vec{y}^{(k)}$$

As this is an overdetermined system of equations we solve the corresponding least-square problem.

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = Y^{+}\vec{y}^{(k)}$$

Where Y^{+} is the psuedoinverse of the matrix Y.
We divide the characteristic polynomial by $\lambda - 1$ to get a new polynomial $q_A(\lambda)$ where:

$$q_A(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$$

Polynomial Division gives the following values:

$$\begin{aligned}\beta_0 &= \gamma_1 + \gamma_2 + \gamma_3\\ \beta_1 &= \gamma_2 + \gamma_3\\ \beta_2 &= \gamma_3\end{aligned}$$

Again using Cayley-Hamilton let z be any vector then:

$$q_A(A)z = \vec{u}_1$$

This happens because:

$$\begin{aligned}q_A(\lambda)(\lambda - 1) &= p_A(\lambda)\\ q_A(A)(A - 1) &= p_A(A)\\ q_A(A)(A - 1)z &= p_A(A)z\end{aligned}$$

As $p_A(A)z = 0$

$$\begin{aligned}q_A(A)Az - q_A(A)z &= 0\\ A(q_A(A)z) &= q_A(A)z\end{aligned}$$

It is of the form

$$A\vec{v} = \lambda\vec{v}$$

Here $\vec{u}_1$ is the eigenvector of A corresponding to the eigenvalue 1. Let z $= \vec{x}^{(k-2)}$ we get:

$$\vec{u}_1 = q_A(A)\vec{x}^{(k-2)} = [\beta_0 I + \beta_1 A + \beta_2 A^2]\vec{x}^{(k-2)}$$

Using equations 1,2 and 3 we get

$$\vec{u}_1 = \beta_0\vec{x}^{(k-2)} + \beta_1\vec{x}^{(k-1)} + \beta_2\vec{x}^{(k)}$$

Since this solution is based on the assumption that A has only 3 eigenvectors, equation 6 gives only an approximation to $\vec{u}_1$ which is the principal eigen vector.

Algorithm for Quadratic Extrapolation:

function $\vec{x}^* = QuadraticExtrapolation(\vec{x}^{(k-3)}.....\vec{x}^{(k)})\{$
for j=k-2:k do
$\vec{y}^{(j)} = \vec{x}^{(j)} - \vec{x}^{(k-3)}$;
end
$Y = (\vec{y}^{(k-2)} \quad \vec{y}^{(k-1)})$
$\gamma_3 = 1$
$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$ = -$Y^+\vec{y}^{(k)}$
$\gamma_0 = -(\gamma_1 + \gamma_2 + \gamma_3)$;
$\beta_0 = \gamma_1 + \gamma_2 + \gamma_3$;
$\beta_1 = \gamma_2 + \gamma_3$;
$\beta_2 = \gamma_3$;
$^* = \beta_0\vec{x}^{(k-2)} + \beta_1\vec{x}^{(k-1)} + \beta_2\vec{x}^{(k)}$;
}

function $\vec{x}^n$ = quadraticpowermethod(){
$\vec{x}^{(0)} = \vec{v}$
k=1
repeat
$\vec{x}^{(k)} = A\vec{x}^{(k-1)}$
$\delta = ||\vec{x}^{(k)} - \vec{x}^{(k-1)}||_1$
periodically,
$\vec{x}^k = QuadraticExtrapolation(\vec{x}^{(k-3)}.....\vec{x}^{(k)})$
k=k+1
until
$\delta < \epsilon$
}

3 Examples

3.1 Example for Power Method

$$\text{A}=\begin{bmatrix} \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix} \qquad \vec{x}^{(0)}=\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\vec{x}^{(1)} = A\vec{x}^{(0)} = \begin{bmatrix} 0.385 \\ 0.416 \\ 0.194 \end{bmatrix}$$

$$\vec{x}^{(2)} = A^2\vec{x}^{(0)} = \begin{bmatrix} 0.324 \\ 0.513 \\ 0.162 \end{bmatrix}$$

$$\vec{x}^{(3)} = A^3\vec{x}^{(0)} = \begin{bmatrix} 0.210 \\ 0.594 \\ 0.135 \end{bmatrix}$$

.....

$$\vec{x}^{(18)} = A^{18}\vec{x}^{(0)} = \begin{bmatrix} 0.017 \\ 0.973 \\ 0.008 \end{bmatrix}$$

$$\vec{x}^{(19)} = A^{19}\vec{x}^{(0)} = \begin{bmatrix} 0.014 \\ 0.999 \\ 0.007 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3.2 For Quadratic Extrapolation

$$\text{A}=\begin{bmatrix} \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix} \qquad \vec{x}^{(0)}=\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\vec{x}^{(1)} = A\vec{x}^{(0)} = \begin{bmatrix} 0.385 \\ 0.416 \\ 0.194 \end{bmatrix}$$

$$\vec{x}^{(2)} = A^2\vec{x}^{(0)} = \begin{bmatrix} 0.324 \\ 0.513 \\ 0.162 \end{bmatrix}$$

$$\vec{x}^{(3)} = A^3\vec{x}^{(0)} = \begin{bmatrix} 0.270 \\ 0.594 \\ 0.135 \end{bmatrix}$$

.....

$$\vec{y}^{(1)} = \vec{x}^{(1)} - \vec{x}^{(0)} = \begin{bmatrix} 0.055 \\ 0.083 \\ -0.138 \end{bmatrix}$$

$$\vec{y}^{(2)} = \vec{x}^{(2)} - \vec{x}^{(0)} = \begin{bmatrix} -0.009 \\ 0.180 \\ -0.171 \end{bmatrix}$$

$$\vec{y}^{(3)} = \vec{x}^{(3)} - \vec{x}^{(0)} = \begin{bmatrix} -0.063 \\ 0.261 \\ -0.198 \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \\ -1 \end{bmatrix}$$

$$\beta_0 = -6.661 * 10^{-16} \approx 0 \qquad \beta_1 = 2.2204 * 10^{-16} \approx 0 \qquad \beta_2 = 1$$

$$\vec{x}^{(3)} = quadraticextrapolation(\vec{x}^{(0)}, \vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}) = \begin{bmatrix} 0.270 \\ 0.594 \\ 0.135 \end{bmatrix}$$

iteration 2

$\vec{x}^{(4)} = A\vec{x}^{(3)} = \begin{bmatrix} 0.225 \\ 0.662 \\ 0.112 \end{bmatrix}$

$\gamma = \begin{bmatrix} -0.475 \\ -0.655 \end{bmatrix}$

$\beta = \begin{bmatrix} -0.130 \\ -0.344 \\ 1 \end{bmatrix}$

$\vec{x}^{(4)} = quadraticextrapolation(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}, \vec{x}^{(4)}) = \begin{bmatrix} 0.275 \\ 0.800 \\ 0.137 \end{bmatrix}$

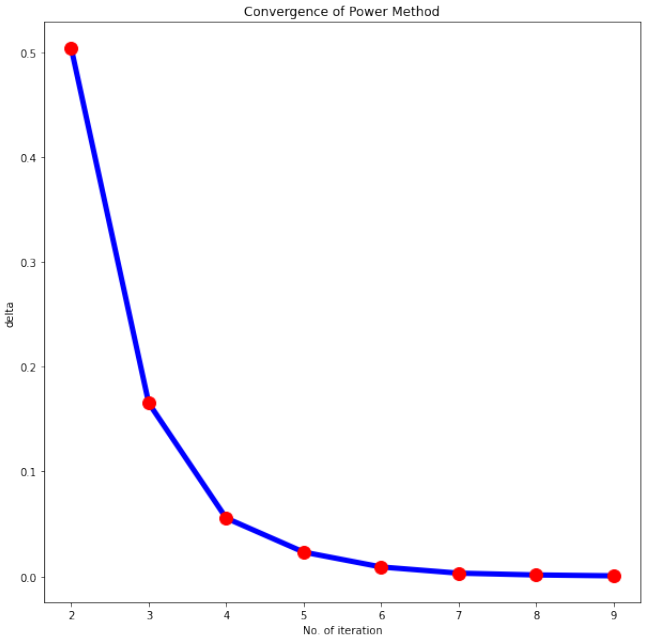
.....

iteration 17:

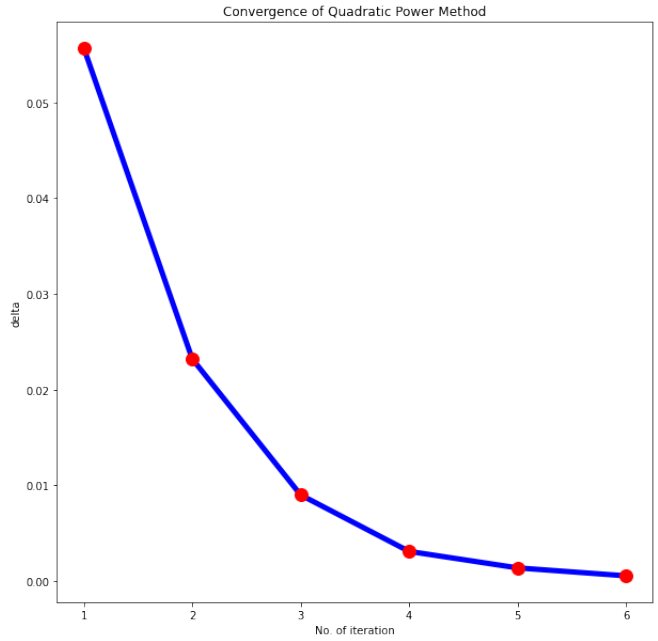
$= \begin{bmatrix} 0.01 \\ 1.017 \\ 0.006 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

4 Observations

Quadratic Extrapolation considerably improves convergence relative to the Power Method.



The Graph above shows the convergence rate of the Power Method. It shows the value of delta or the difference between successive iterates against the number of iterations.



The Graph above shows the convergence rate of the Quadratic Extrapolation Method. It shows the value of delta or the difference between successive iterates against the number of iterations.
As evident from the graphs Quadratic Extrapolation converges at a rate faster than Power Method.

5 Conclusion

Web search has become an integral part of modern information access, posing many interesting challenges in developing effective and efficient strategies for ranking search results. Although PageRank is an offline computation it has become increasingly desirable to speed up this computation. Quadratic Extrapolation is an implementation- ally simple technique that requires little additional infrastructure to integrate into the standard Power Method. In particular, Quadratic Extrapolation works by eliminating the bottleneck for the Power Method, namely the second and third eigenvector components in the current iterate, thus boosting the effectiveness of the simple Power Method itself.

6 References

[1] Sepandar D. Kamvar, Taher H. Haveliwala, Christopher D. Manning, and Gene H. Golub. Extrapolation methods for accelerating pagerank computations. In *Proceedings of the 12th International Conference on World Wide Web*, WWW '03, page 261–270, New York, NY, USA, 2003. Association for Computing Machinery.

Work Contribution	
Member	contribution
Ankush Dey	Beamer slides and power Method
Anjali Pugalia	Report and formulation of quadratic extrapolation
Ananya S	Formulation of quadratic extrapolation, examples and algorithm
Aniket Santra	Code and conclusion