***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: : Link to the code : C code

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig: Python code

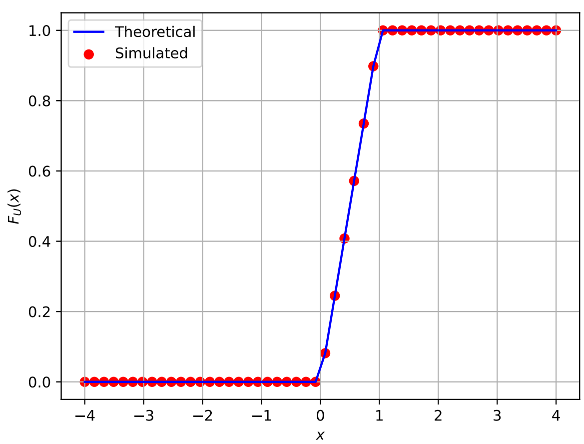


Fig. 1: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As U is uniformly distributed random variable in the interval (0,1) and

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0,1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: Link to the code :C code

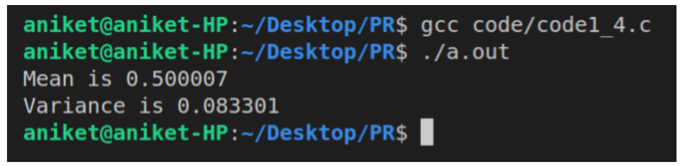


Fig. 1: Output

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: we know that,

$$dF_U(x) = p_U(x)dx \tag{1.8}$$

also mean (μ) is E(U): Hence,

$$\mu = \int_{-\infty}^{\infty} x p_U(x)dx$$
$$= \int_{0}^{1} x dx$$
$$= \left.\frac{x^2}{2}\right|_0^1$$
$$= \frac{1}{2}$$

$$\text{Variance}(\sigma^2) = E(U^2) - E(U)^2$$

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)dx$$
$$= \int_{0}^{1} x^2 dx$$
$$= \left.\frac{x^3}{3}\right|_0^1$$
$$= \frac{1}{3}$$

$$\sigma^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: link to the code :C code

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Link to Python code :Python code

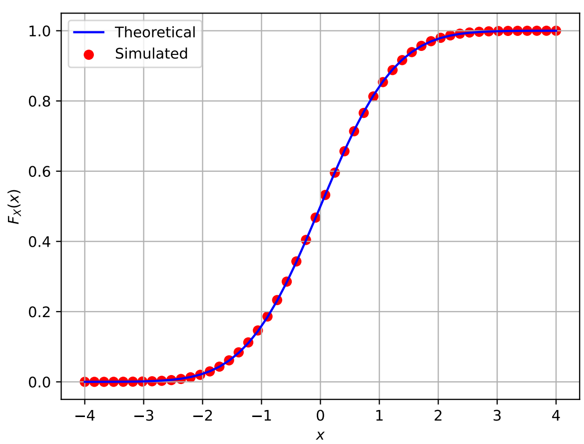


Fig. 2: The CDF of X

Properties:
(1) Graph is symmetric about a single point
(2) The $F_X(x)$ is non-decreasing function
(3) $\lim_{x\to\infty} F_X(x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: Link to the code :Python code

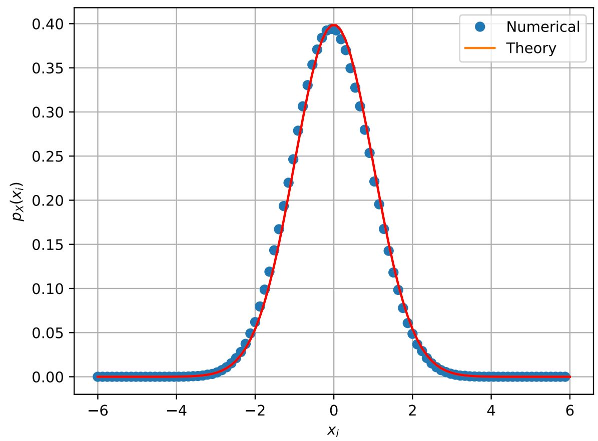


Fig. 2: The PDF of X

Properties :
(1) Area under the curve is One.
(2) Symmetric about line $x = \mu$.
(3) Increasing in first half and decreasing in other half.

2.4 Find the mean and variance of X by writing a C program.
Solution: Link to the code :C code

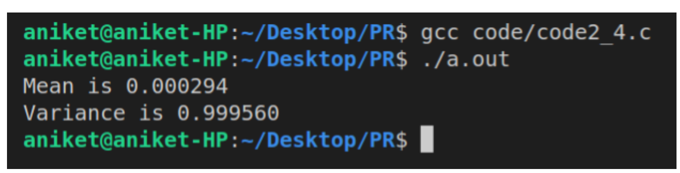


Fig. 2: Output

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically. **Solution:**
(1) CDF is

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx = 1$$

(2) Mean is

$$\mu = E(x) = \int_{-\infty}^{\infty} x p_X(x) dx$$

Due to symmetry clearly, $\mu = 0$

(3) Variance is

$$\begin{aligned} var[X] &= E(X^2) - E(X)^2 \\ &= \int_{-\infty}^{\infty} x^2 p_X(x) dx - 0 \\ &= 1 \end{aligned}$$

(4) Q function is

$$\begin{aligned} Q(x) &= Pr(x > X) \\ &= 1 - F_X(x) \end{aligned}$$

Then CDF is:

$$F_X(x) = 1 - Q(x)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Link to the code :Python code

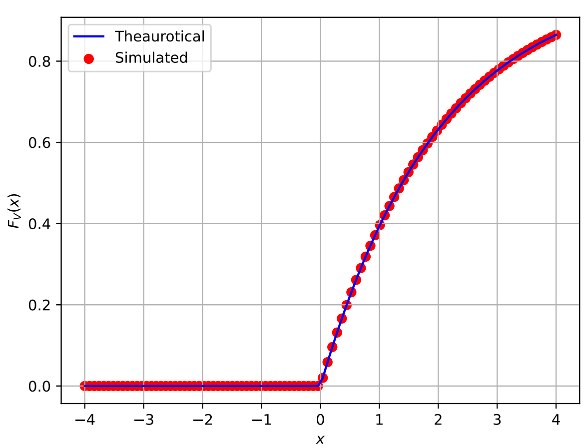


Fig. 3: CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$\begin{aligned} F_V(X) &= Pr(V \le x) \\ &= Pr(-2\ln(1-U) \le x) \\ &= Pr((1-U) \ge \exp(\frac{-x}{2})) \\ &= Pr(U \le (1 - \exp(\frac{-x}{2})) \\ &= F_U(1 - \exp\frac{-x}{2}) \end{aligned}$$

from equation (1.3),

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ (1 - \exp\frac{-x}{2}), & x \in (0, \infty) \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Link to the code :C code

4.2 Find CDF of T
Solution: Link to the code :Pyhton code

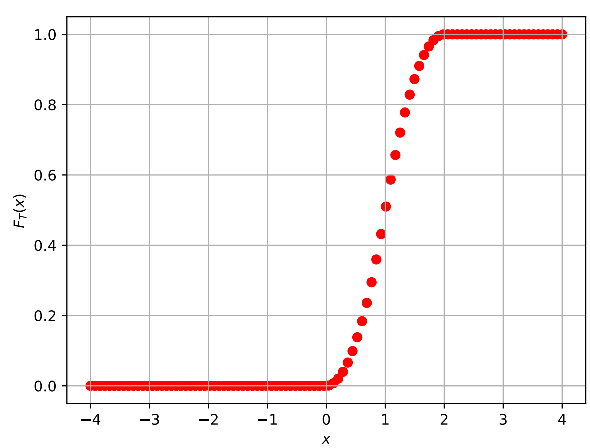


Fig. 4: CDF of T

4.3 Find the PDF of T.
Solution: Link to the code :Pyhton code

4.4 Find the theoretical expressions for the PDF and CDF of T.

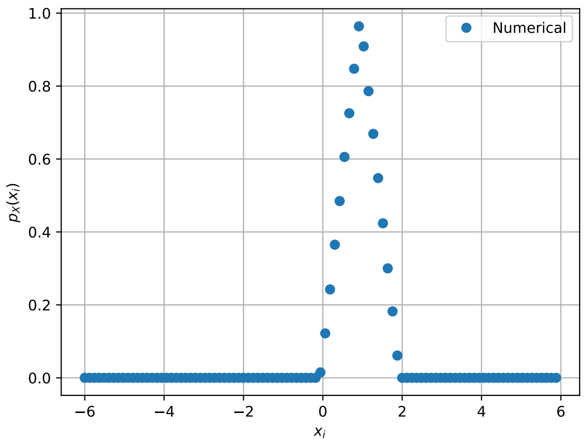


Fig. 4: PDF of T

Solution:

$$T = U_1 + U_2 \tag{4.2}$$
$$p_T(t) = p_{U_1} * p_{U_2} \tag{4.3}$$
$$p_T(t) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(t-\tau) d\tau \tag{4.4}$$
$$p_T(t) = \begin{cases} 0 & t \le 0 \\ \int_0^t d\tau & 0 \le t < 1 \\ \int_{t-1}^1 d\tau & 1 \le t < 2 \\ 0 & t > 2 \end{cases} \tag{4.5}$$

For CDF :

$$F_T(x) = \int_{-\infty}^{x} p_T(t) dt \tag{4.6}$$

From equation (4.5):

$$F_T(t) = \begin{cases} 0 & t \in (-\infty, 0) \\ \frac{x^2}{2} & t \in (0,1) \\ \frac{-t^2}{2} + 2t - 1 & t \in (1,2) \\ 1 & t \in (2, \infty) \end{cases} \tag{4.7}$$

4.5 Verify results through plot
Solution:
Link to the code for CDF:Pyhton code
Link to the code for PDF:Pyhton code

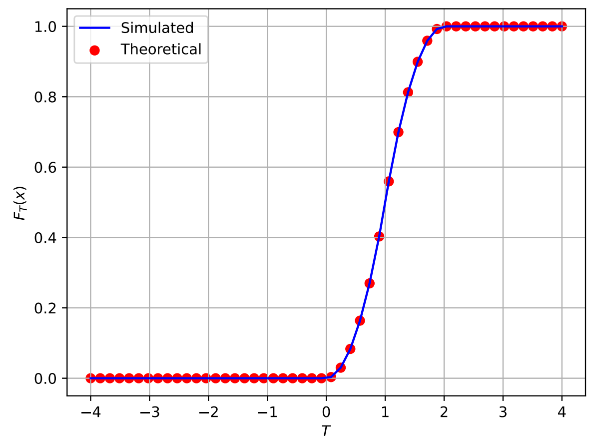


Fig. 4: Verification of CDF

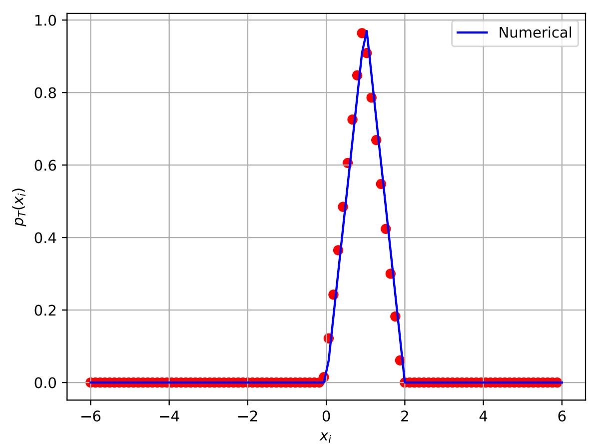


Fig. 4: verification of PDF

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$
Solution: Link to the code :C code

5.2 Generate
$$Y = AX + N, \tag{5.1}$$
where A = 5 dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Link to the code :C code

5.3 Plot Y using a scatter plot.
Solution: Link to the code :Python code

5.4 Guess how to estimate X from Y.

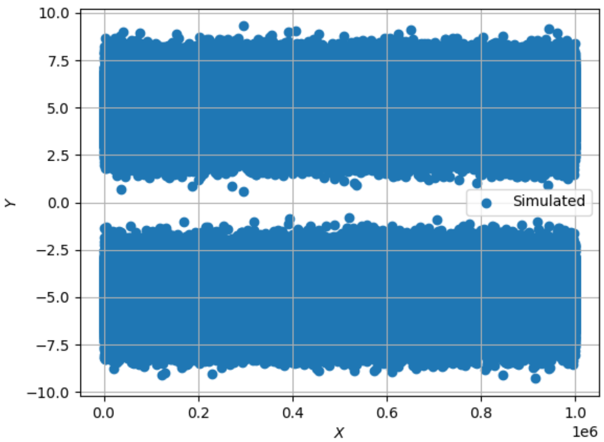


Fig. 5

Solution: From Data and Plot generally:

$$X = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.2}$$

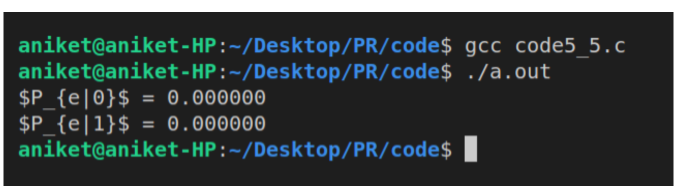
5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \tag{5.3}$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \tag{5.4}$$

Solution: Link to the code :C code



```
aniket@aniket-HP:~/Desktop/PR/code$ gcc code5_5.c
aniket@aniket-HP:~/Desktop/PR/code$ ./a.out
$P_{e|0}$ = 0.000000
$P_{e|1}$ = 0.000000
aniket@aniket-HP:~/Desktop/PR/code$
```

Fig. 5

From the Figure above :

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) = 0$$
$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) = 0$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Here.

$$\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2} \tag{5.5}$$

Hence,

$$\begin{aligned} P_e &= \Pr(X = 1)P_{e|1} + \Pr(X = -1)P_{e|0} \\ &= \frac{1}{2}(P_{e|1} + P_{e|0}) \\ &= 0 \end{aligned}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 10dB.
Solution: Link to the code :Python codeSemilog code

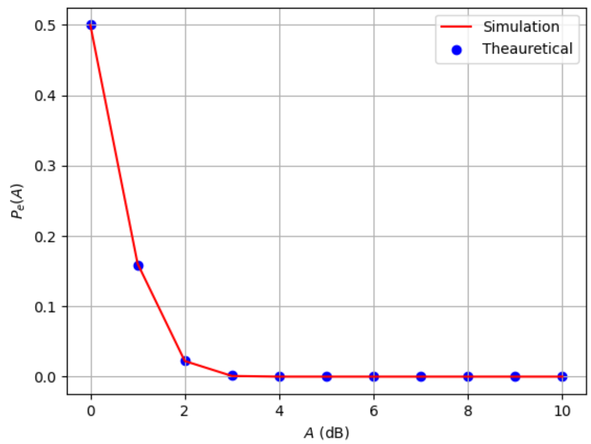


Fig. 5

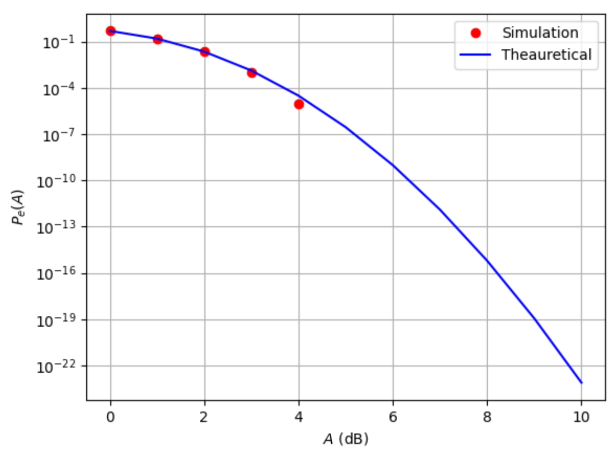


Fig. 5

5.8 Now consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e

Solution: From equation (5.2) Hence,

$$\begin{aligned}P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right)\\ &= \Pr(AX + N < \delta|X = 1)\\ &= \Pr(N < \delta - A)\\ &= Q_N(A - \delta)\end{aligned}$$

Where Q_N is the Q-function of the normal distribution. Similarly,

$$P_{e|1} = Q_N(A + \delta) \tag{5.6}$$

Therefore,

$$\begin{aligned}P_e &= P_{e|0}\Pr(X = 1) + P_{e|1}\Pr(X = -1) && (5.7)\\ &= \frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta)) && (5.8)\\ & && (5.9)\end{aligned}$$

To find minima by differentiate wrt δ

$$\begin{aligned}\frac{d(P_e)}{d\delta} &= \frac{1}{2}\frac{d}{d\delta}(Q_N(A - \delta) + Q_N(A + \delta))\\ &= \frac{1}{2\sqrt{2\pi}}(e^{-}\frac{(\delta - A)^2}{2} - e^{-}\frac{(\delta + A)^2}{2}) = 0\end{aligned}$$

From above equation

$$(\delta - A)^2 = (\delta + A)^2$$

Hence,

$$\delta = 0$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.10}$$

Solution: From equation 5.8

$$\begin{aligned}P_e &= P_{e|0}p + P_{e|1}(1 - p)\\ &= Q_N(A - \delta)p + Q_N(A + \delta)(1 - p)\end{aligned}$$

Again by differentiating,

$$p\frac{1}{\sqrt{2\pi}}e^{-}\frac{(\delta - A)^2}{2} - (1 - p)\frac{1}{\sqrt{2\pi}}e^{-}\frac{(\delta + A)^2}{2} = 0$$

By taking log,

$$\begin{aligned}\ln p - \frac{(\delta - A)^2}{2} &= \ln(1 - p) + \frac{(\delta + A)^2}{2}\\ \delta &= \frac{1}{2A}\ln\frac{1 - p}{p}\end{aligned}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: Using Bayes' Theorem, we get

$$\begin{aligned}&\Pr(X = 1|Y = y)\\ &= \frac{\Pr(N = y - A|X = 1)\Pr(X = 1)}{\Pr(Y = y)} && (5.11)\\ &= \frac{pf_N(y - A)}{pf_N(y - A) + (1 - p)f_N(y + A)} && (5.12)\\ &= \frac{p}{p + (1 - p)e^{-2yA}} && (5.13)\end{aligned}$$

and

$$\begin{aligned}&\Pr(X = -1|Y = y)\\ &= \frac{\Pr(N = y + A|X = -1)\Pr(X = -1)}{\Pr(Y = y)} && (5.14)\\ &= \frac{(1 - p)f_N(y + A)}{pf_N(y - A) + (1 - p)f_N(y + A)} && (5.15)\\ &= \frac{1 - p}{(1 - p) + pe^{2yA}} && (5.16)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{p}{p + (1 - p)e^{-2yA}} &\gtrless \frac{1 - p}{(1 - p) + pe^{2yA}} && (5.17)\\ \implies p^2e^{2yA} &\gtrless (1 - p)^2 e^{-2yA} && (5.18)\\ \implies y &\gtrless \frac{1}{2A}\ln\left(\frac{1 - p}{p}\right) && (5.19)\end{aligned}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$ Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution:

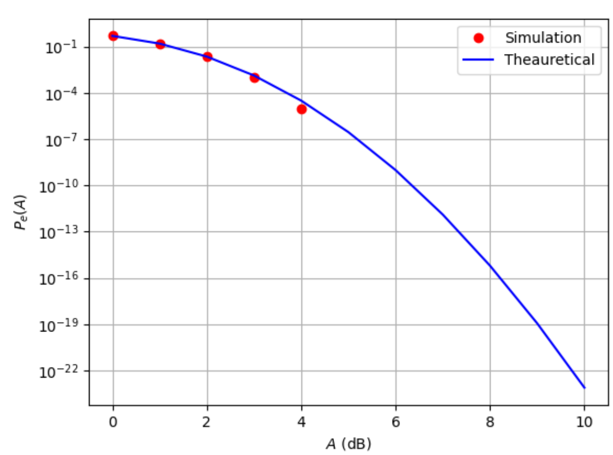


Fig. 6: CDF of V

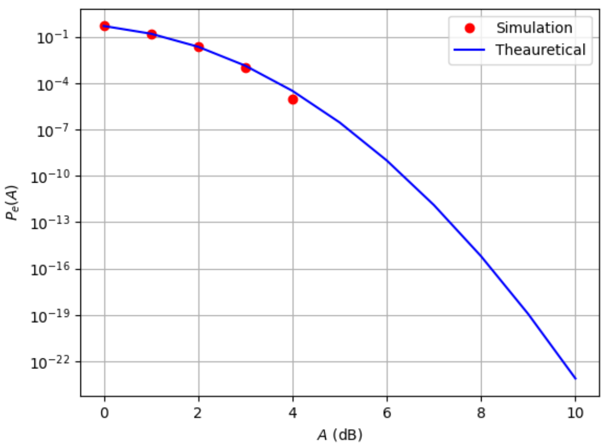


Fig. 6: PDF of V

We can take,

$$X_1 = Rsin\phi$$
$$X_2 = Rsin\phi$$

Then the Jacobian Matrix is given by

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \phi} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \phi} \end{bmatrix} \tag{6.2}$$

$$J = \begin{bmatrix} \cos\phi & -R\sin\phi \\ \sin\phi & R\cos\phi \end{bmatrix} \tag{6.3}$$

$$|J| = R \tag{6.4}$$

Also , sincce X_1 X_2 are i.i.d

$$\begin{aligned} p_{R,\phi}(r,\theta) &= p_{X_1}(x_1)p_{X_2}(x_2) \\ &= \frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}} \\ &= \frac{1}{2\pi}e^{-\frac{-r^2}{2}} \end{aligned}$$

Now since

$$p_{R,\theta}(r,\theta) = |J|P_{X_1,X_2}(x_1,x_2) \tag{6.5}$$

From (6.5) and (6.4), we get

$$\begin{aligned} p_{R,\phi}(r,\theta) &= \frac{r}{2\pi}e^{\frac{r^2}{2}} \\ p_R(r) &= \int_0^{2\pi} p_{R,\phi}(r,\theta)d\theta \\ &= re^{-\frac{r^2}{2}} \end{aligned}$$

Then we get CDF as

$$\begin{aligned} F_R(r) &= \int_0^r p_R(r) \\ &= \int_0^r re^{-\frac{r^2}{2}}dr \\ &= 1 - e^{-\frac{r^2}{2}} \end{aligned}$$

Finally,

$$F_V(x) = \Pr(V \le x) \tag{6.6}$$
$$= \Pr\left(R^2 \le x\right) \quad = F_R(\sqrt{x}) \tag{6.7}$$
$$= 1 - e^{-\frac{x}{2}} \tag{6.8}$$
$$\tag{6.9}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{6.10}$$

Solution: Since,

$$F_V(x) = 1 - e^{-\frac{x}{2}} \tag{6.11}$$
$$\therefore \alpha = \frac{1}{2} \tag{6.12}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.13}$$

Solution: Python code

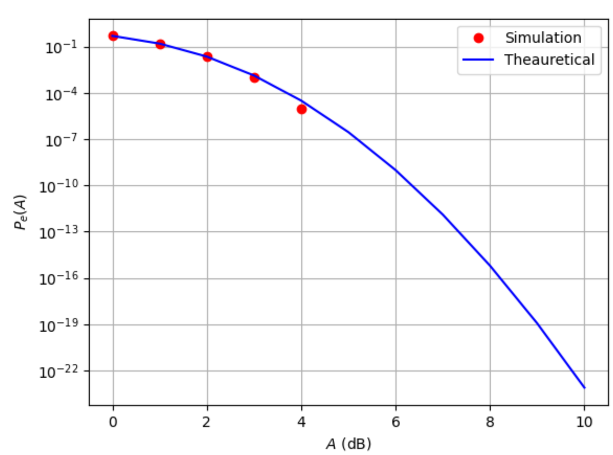


Fig. 6: CDF of V

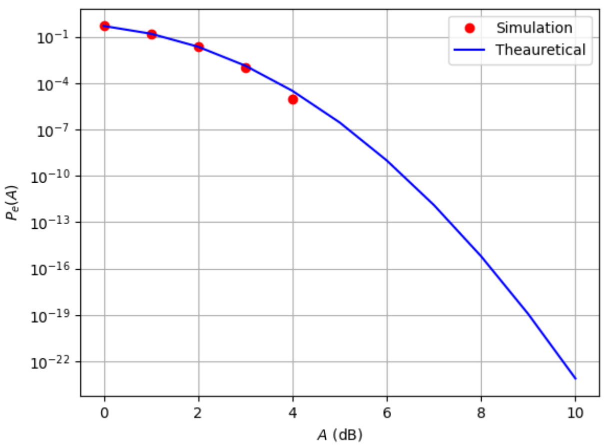


Fig. 6: PDF of V

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim N(0,1), X \in \{1,-1\}$ for $0 \leq \gamma 10$ dB.
Solution: Python code

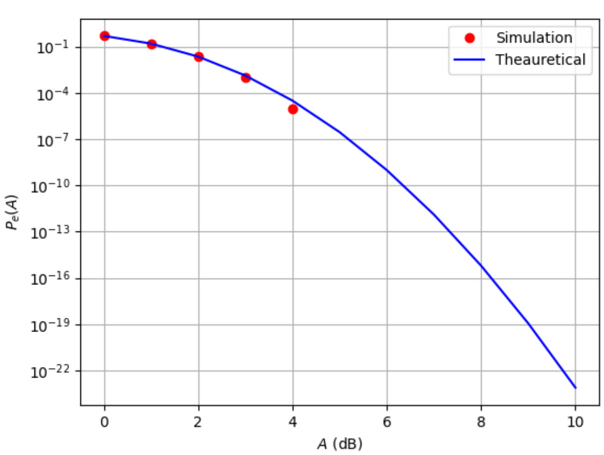


Fig. 7: P_e as a function of γ (rectangular axes)

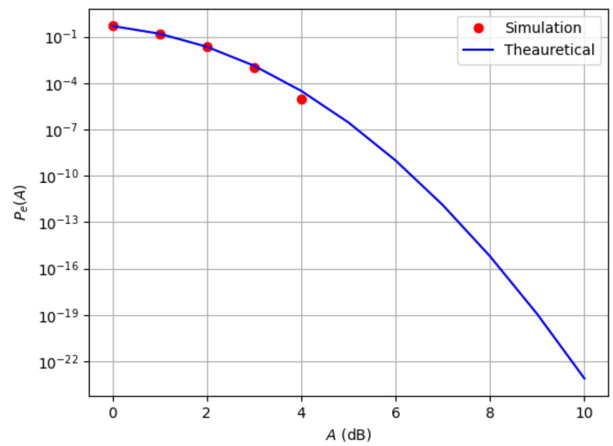


Fig. 7: P_e as a function of γ (semilog-y axes)

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A < -N) = F_A(-N) \tag{7.3}$$

$$= \begin{cases} 1 - e^{-N^2\gamma} & N \leq 0 \\ 0 & N > 0 \end{cases} \tag{7.4}$$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(X) p_X(x) dx \tag{7.5}$$

Find $P_e = E[P_e(N)]$
Solution:

$$P_e = \int_0^{\infty} F_A(x) f_N(x) dx \tag{7.6}$$

$$= \int_0^{\infty} (1 - e^{-\frac{x^2}{2}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{7.7}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} exp\left(-\frac{x^2}{\frac{2\gamma}{2+\gamma}}\right) dx \tag{7.8}$$

Since,

$$\int_0^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{4a}} \tag{7.9}$$

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{\gamma+2}}\right) \tag{7.10}$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t. γ, Comment.
Solution:

8 Two Dimensions

Let

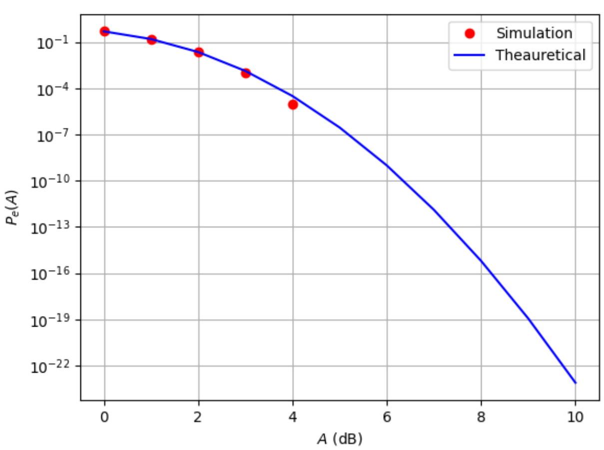
$$\mathbf{y} = A\mathbf{x} + \mathbf{n} \tag{8.1}$$

where

$$\mathbf{x} \in (\mathbf{s_0}, \mathbf{s_1}), \mathbf{s_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim N(0, 1) \tag{8.3}$$

8.1 Plot $\mathbf{y}|\mathbf{s_1}$ on the same graph using a scatter plot.
Solution: Python code



8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s_0}$ and $\mathbf{s_1}$.
Solution: The decision rule is,

$$\hat{x} = \begin{cases} \mathbf{s_0} & y_1 > y_2 \\ \mathbf{s_1} & y_1 < y_2 \end{cases} \tag{8.4}$$

where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s_1} | \mathbf{x} = \mathbf{s_0}\right) \tag{8.5}$$

with respect to the SNR from 0 to 10dB.
Solution: Python code

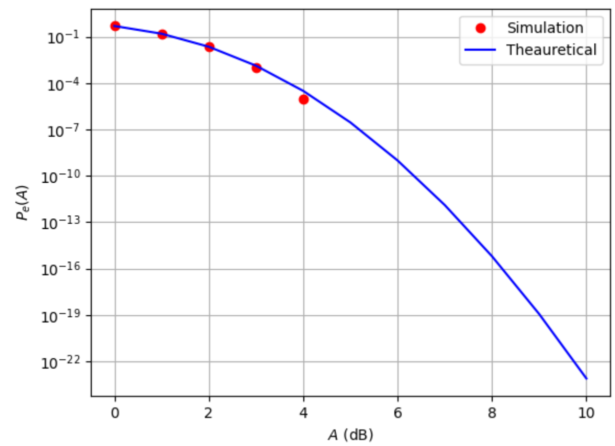


Fig. 8: P_e as a function of SNR(rectangular axes)

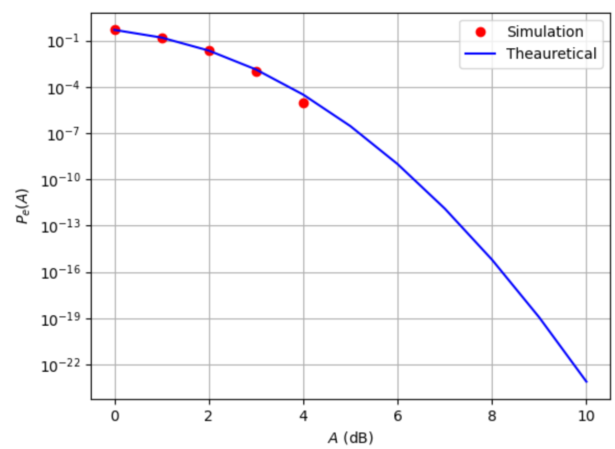


Fig. 8: P_e as a function of SNR(semilog axes)

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution: We have,

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s_1} | \mathbf{x} = \mathbf{s_0}\right) \tag{8.6}$$
$$= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s_0}\right) \tag{8.7}$$
$$= \Pr\left(A + n_1 < -A + n_2\right) \tag{8.8}$$
$$= \Pr\left(n_2 - n_1 > 2A\right) \tag{8.9}$$
$$= \Pr\left(N > 2A\right) = Q\left(\sqrt{2}SNR\right) \tag{8.10}$$

where $N = n_2 - n_1 \sim N(0, 2)$