***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: : Link to the code : C code

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig: Python code

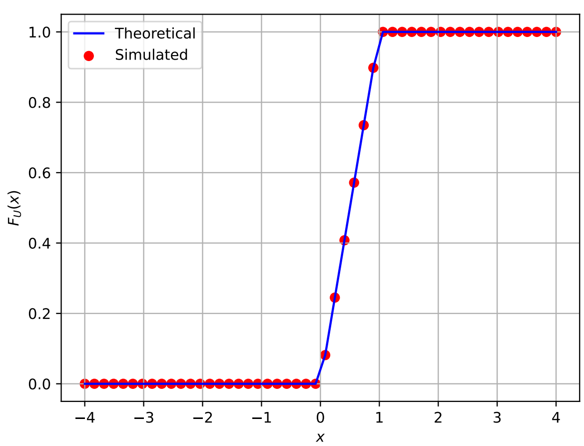

Fig. 1: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As U is uniformly distributed random variable in the interval (0,1) and

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{1.3}$$

$$F_U(x) = \begin{cases} 0, & x \in (-\infty,0) \\ x, & x \in (0,1) \\ 1, & x \in (1,\infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: Link to the code :C code

```
aniket@aniket-HP:~/Desktop/PR$ gcc code/code1_4.c
aniket@aniket-HP:~/Desktop/PR$ ./a.out
Mean is 0.500007
Variance is 0.083301
aniket@aniket-HP:~/Desktop/PR$
```

Fig. 1: Output

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution: we know that,

$$dF_U(x) = p_U(x)dx \tag{1.8}$$

also mean (μ) is E(U): Hence,

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x p_U(x)dx \\ &= \int_0^1 x dx \\ &= \left.\frac{x^2}{2}\right|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

$$\text{Variance}(\sigma^2) = E(U^2) - E(U)^2$$

$$\begin{aligned} E(U^2) &= \int_{-\infty}^{\infty} x^2 p_U(x)dx \\ &= \int_0^1 x^2 dx \\ &= \left.\frac{x^3}{3}\right|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

$$\sigma^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: link to the code :C code

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Link to Python code :Python code

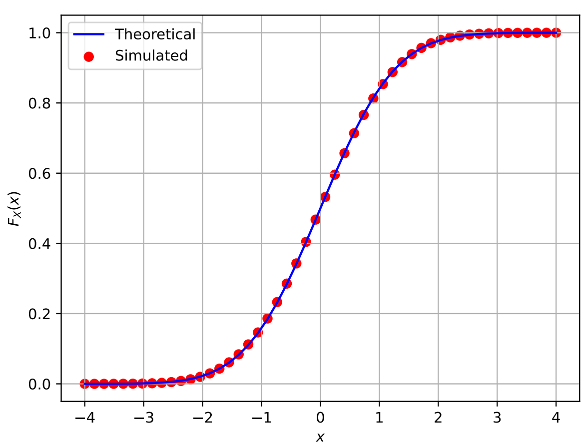

Fig. 2: The CDF of X

Properties:
(1) Graph is symmetric about a single point
(2) The $F_X(x)$ is non-decreasing function
(3) $\lim_{x\to\infty} F_X(x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: Link to the code :Python code

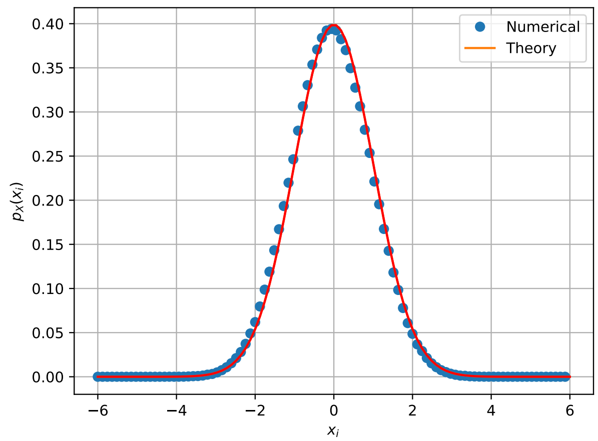

Fig. 2: The PDF of X

Properties :
(1) Area under the curve is One.
(2) Symmetric about line $x = \mu$.
(3) Increasing in first half and decreasing in other half.

2.4 Find the mean and variance of X by writing a C program.
Solution: Link to the code :C code

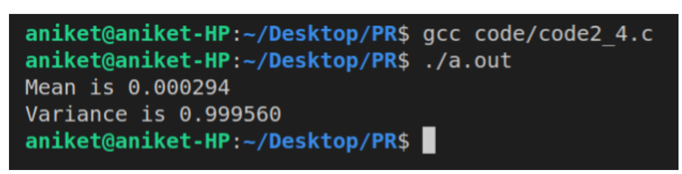

Fig. 2: Output

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically. **Solution:**
(1) CDF is

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx = 1$$

(2) Mean is

$$\mu = E(x) = \int_{-\infty}^{\infty} x p_X(x) dx$$

Due to symmetry clearly, $\mu = 0$

(3) Variance is

$$\begin{aligned} var[X] &= E(X^2) - E(X)^2 \\ &= \int_{-\infty}^{\infty} x^2 p_X(x) dx - 0 \\ &= 1 \end{aligned}$$

(4) Q function is

$$\begin{aligned} Q(x) &= Pr(x > X) \\ &= 1 - F_X(x) \end{aligned}$$

Then CDF is:

$$F_X(x) = 1 - Q(x)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Link to the code :Python code

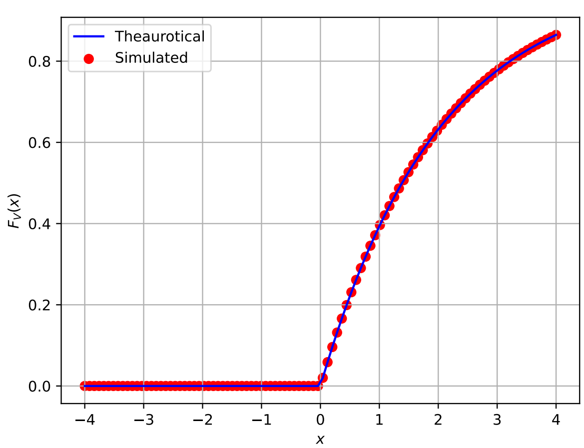

Fig. 3: CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$\begin{aligned} F_V(X) &= Pr(V \leq x) \\ &= Pr(-2\ln(1 - U) \leq x) \\ &= Pr((1 - U) \geq \exp(\frac{-x}{2}) \\ &= Pr(U \leq (1 - \exp(\frac{-x}{2})) \\ &= F_U(1 - \exp\frac{-x}{2}) \end{aligned}$$

from equation (1.3),

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ (1 - \exp\frac{-x}{2}), & x \in (0, \infty) \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Link to the code :C code

4.2 Find CDF of T
Solution: Link to the code :Pyhton code

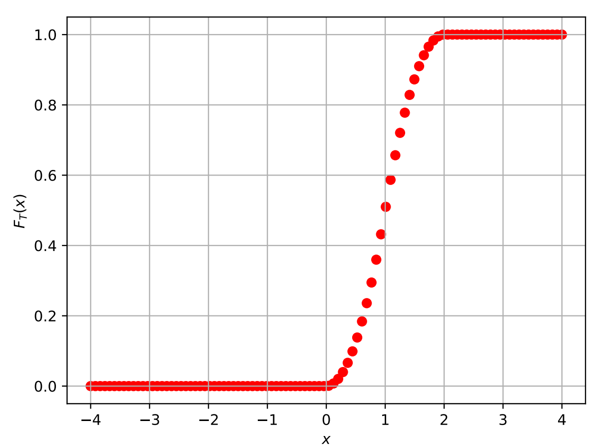

Fig. 4: CDF of T

4.3 Find the PDF of T.
Solution: Link to the code :Pyhton code

4.4 Find the theoretical expressions for the PDF and CDF of T.

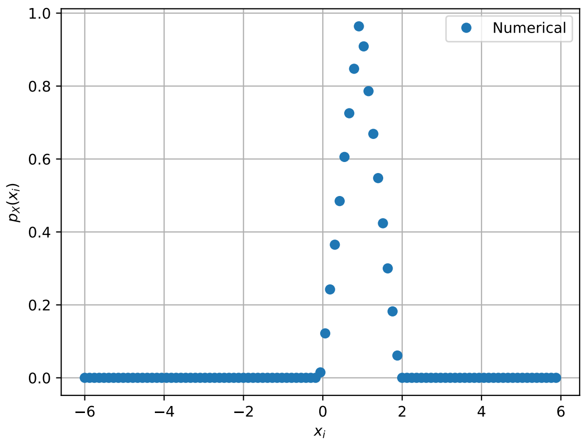

Fig. 4: PDF of T

Solution:

$$T = U_1 + U_2 \tag{4.2}$$
$$p_T(t) = p_{U_1} * p_{U_2} \tag{4.3}$$
$$p_T(t) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(t-\tau) d\tau \tag{4.4}$$
$$p_T(t) = \begin{cases} 0 & t \le 0 \\ \int_0^t d\tau & 0 \le t < 1 \\ \int_{t-1}^1 d\tau & 1 \le t < 2 \\ 0 & t > 2 \end{cases} \tag{4.5}$$

For CDF :

$$F_T(x) = \int_{-\infty}^{x} p_T(t) dt \tag{4.6}$$

From equation (4.5):

$$F_T(t) = \begin{cases} 0 & t \in (-\infty, 0) \\ \frac{x^2}{2} & t \in (0,1) \\ \frac{-t^2}{2} + 2t - 1 & t \in (1,2) \\ 1 & t \in (2, \infty) \end{cases} \tag{4.7}$$

4.5 Verify results through plot
Solution:
Link to the code for CDF:Pyhton code
Link to the code for PDF:Pyhton code

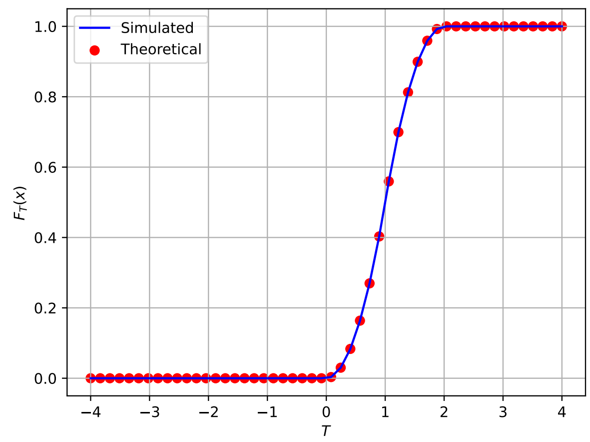

Fig. 4: Verification of CDF

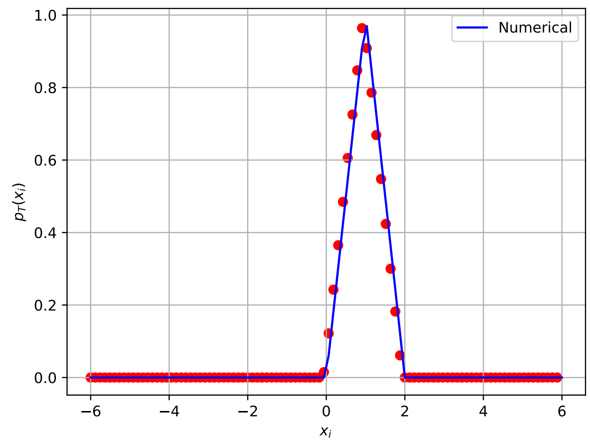

Fig. 4: verification of PDF

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$
Solution: Link to the code :C code

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where A = 5 dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Link to the code :C code

5.3 Plot Y using a scatter plot.
Solution: Link to the code :Python code

5.4 Guess how to estimate X from Y.

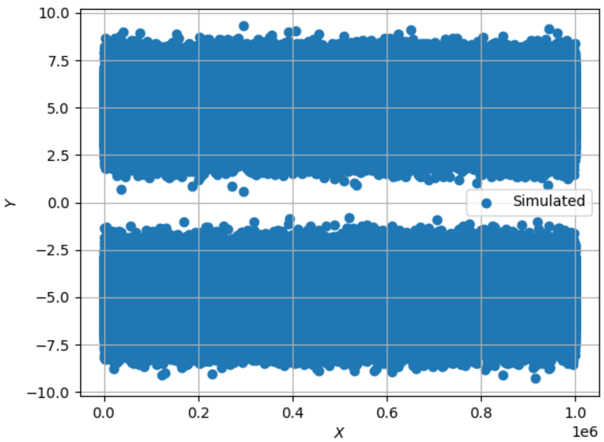

Fig. 5

Solution: From Data and Plot generally:

$$X = \begin{cases} 1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \tag{5.3}$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \tag{5.4}$$

Solution: Link to the code :C code

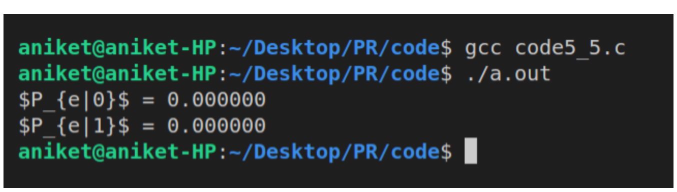

Fig. 5

From the Figure above :

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) = 0$$
$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) = 0$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Here.

$$\Pr(X = 1) = \Pr(X = -1) = \frac{1}{2} \tag{5.5}$$

Hence,

$$\begin{aligned} P_e &= \Pr(X = 1)P_{e|1} + \Pr(X = -1)P_{e|0} \\ &= \frac{1}{2}(P_{e|1} + P_{e|0}) \\ &= 0 \end{aligned}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 10dB.
Solution: Link to the code :Python code

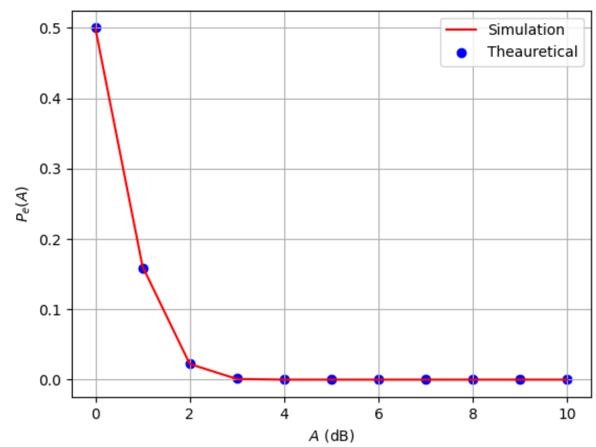

Fig. 5

5.8 Now consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e
Solution: From equation (5.2) Hence,

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) \\ &= \Pr\left(AX + N < \delta|X = 1\right) \\ &= \Pr\left(N < \delta - A\right) \\ &= Q_N(A - \delta) \end{aligned}$$

Where Q_N is the Q-function of the normal distribution. Similarly,

$$P_{e|1} = Q_N(A + \delta) \tag{5.6}$$

Therefore,

$$P_e = P_{e|0} \Pr\left(X = 1\right) + P_{e|1} \Pr\left(X = -1\right) \tag{5.7}$$
$$= \frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta)) \tag{5.8}$$
$$\tag{5.9}$$

To find minima by differentiate wrt δ

$$\frac{d(P_e)}{d\delta} = \frac{1}{2}\frac{d}{d\delta}(Q_N(A-\delta) + Q_N(A+\delta))$$
$$= \frac{1}{2\sqrt{2\pi}}(e^{-\frac{(\delta-A)^2}{2}} - e^{-\frac{(\delta+A)^2}{2}}) = 0$$

From above equation

$$(\delta - A)^2 = (\delta + A)^2$$

Hence,

$$\delta = 0$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.10}$$

Solution: From equation 5.8

$$P_e = P_{e|0}p + P_{e|1}(1-p)$$
$$= Q_N(A-\delta)p + Q_N(A+\delta)(1-p)$$

Again by differentiating,

$$p\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta+A)^2}{2}} = 0$$

By taking log,

$$\ln p - \frac{(\delta-A)^2}{2} = \ln(1-p) + \frac{(\delta+A)^2}{2}$$
$$\delta = \frac{1}{2A}\ln\frac{1-p}{p}$$

5.10 Repeat the above exercise using the MAP criterion.