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## 4.3 MINIMUM SPANNING TREES

---

- ▶ *introduction*
- ▶ *greedy algorithm*
- ▶ *edge-weighted graph API*
- ▶ *Kruskal's algorithm*
- ▶ *Prim's algorithm*
- ▶ *context*

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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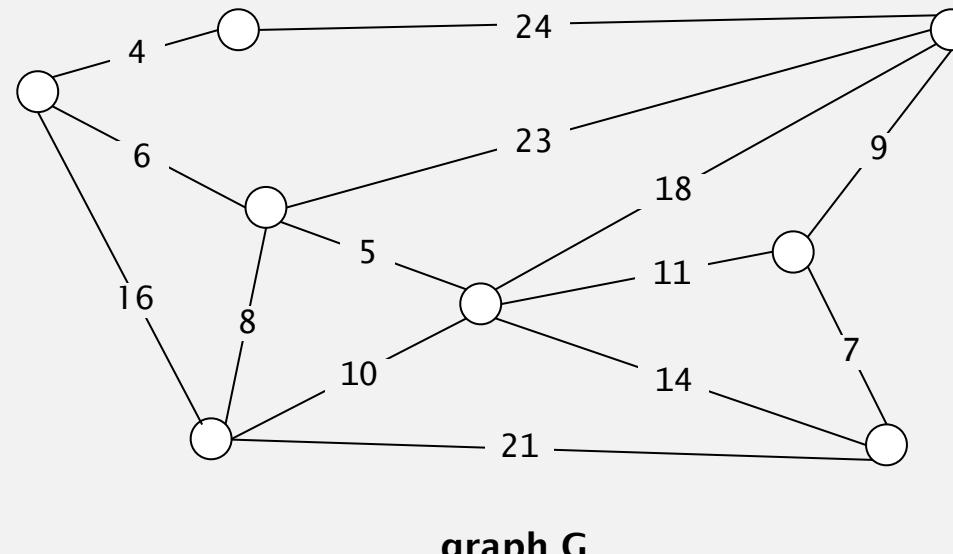
# Minimum spanning tree

---

**Given.** Undirected graph  $G$  with positive edge weights (connected).

**Def.** A **spanning tree** of  $G$  is a subgraph  $T$  that is connected and acyclic.

**Goal.** Find a min weight spanning tree.



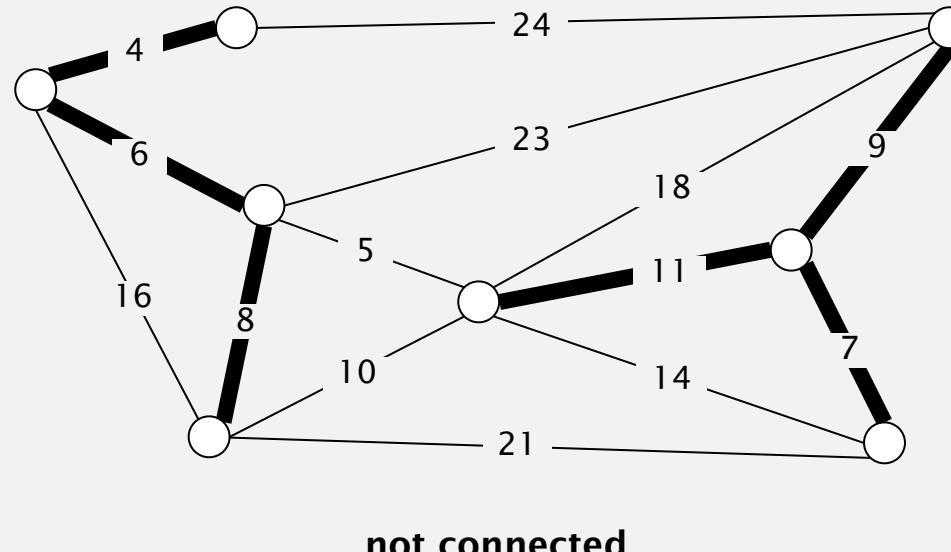
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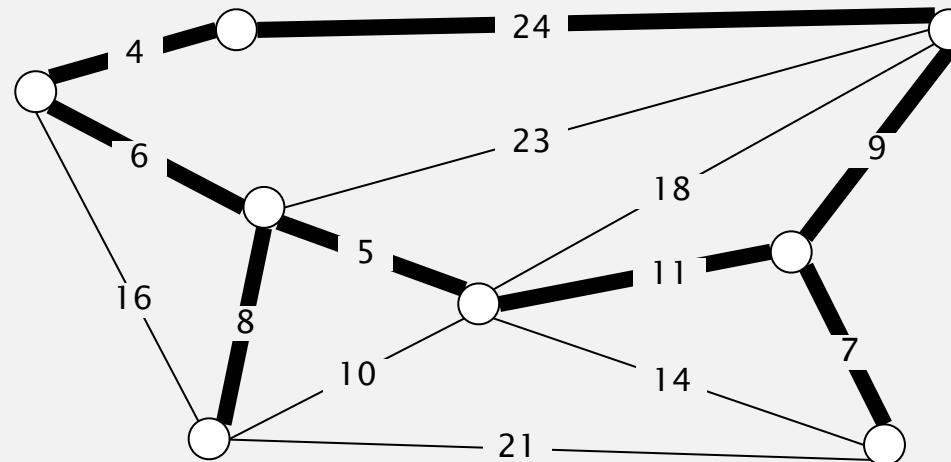
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Goal. Find a min weight spanning tree.



not acyclic

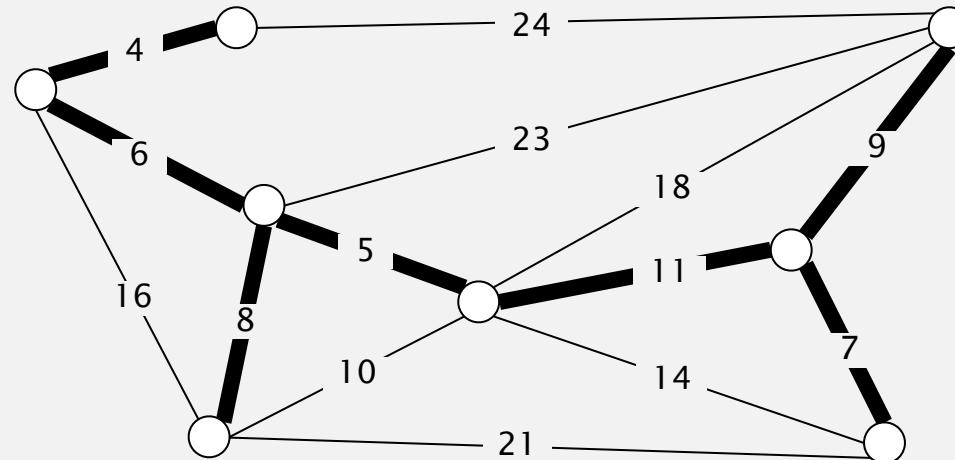
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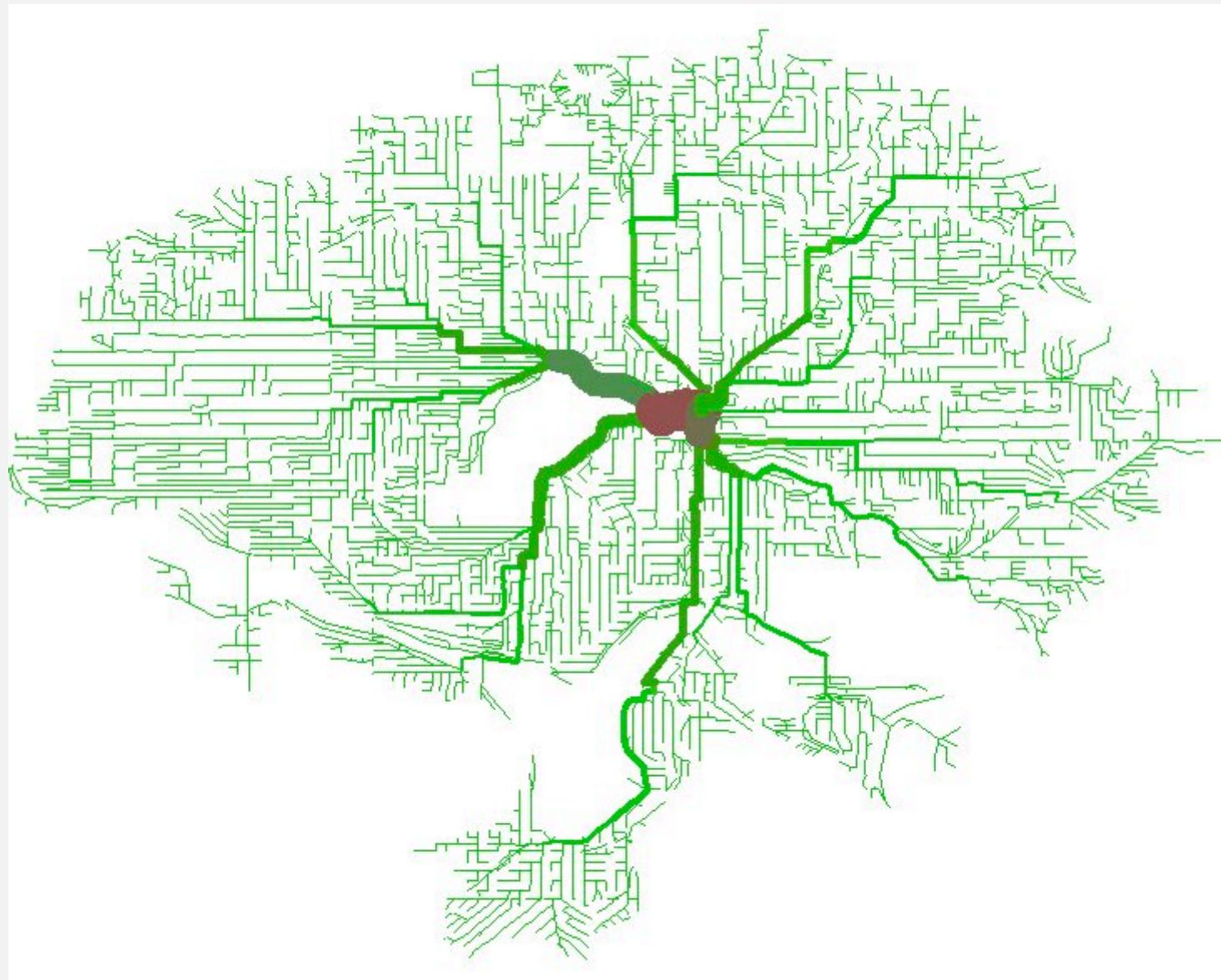
**spanning tree T: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7**

Brute force. Try all spanning trees?

# Network design

---

MST of bicycle routes in North Seattle

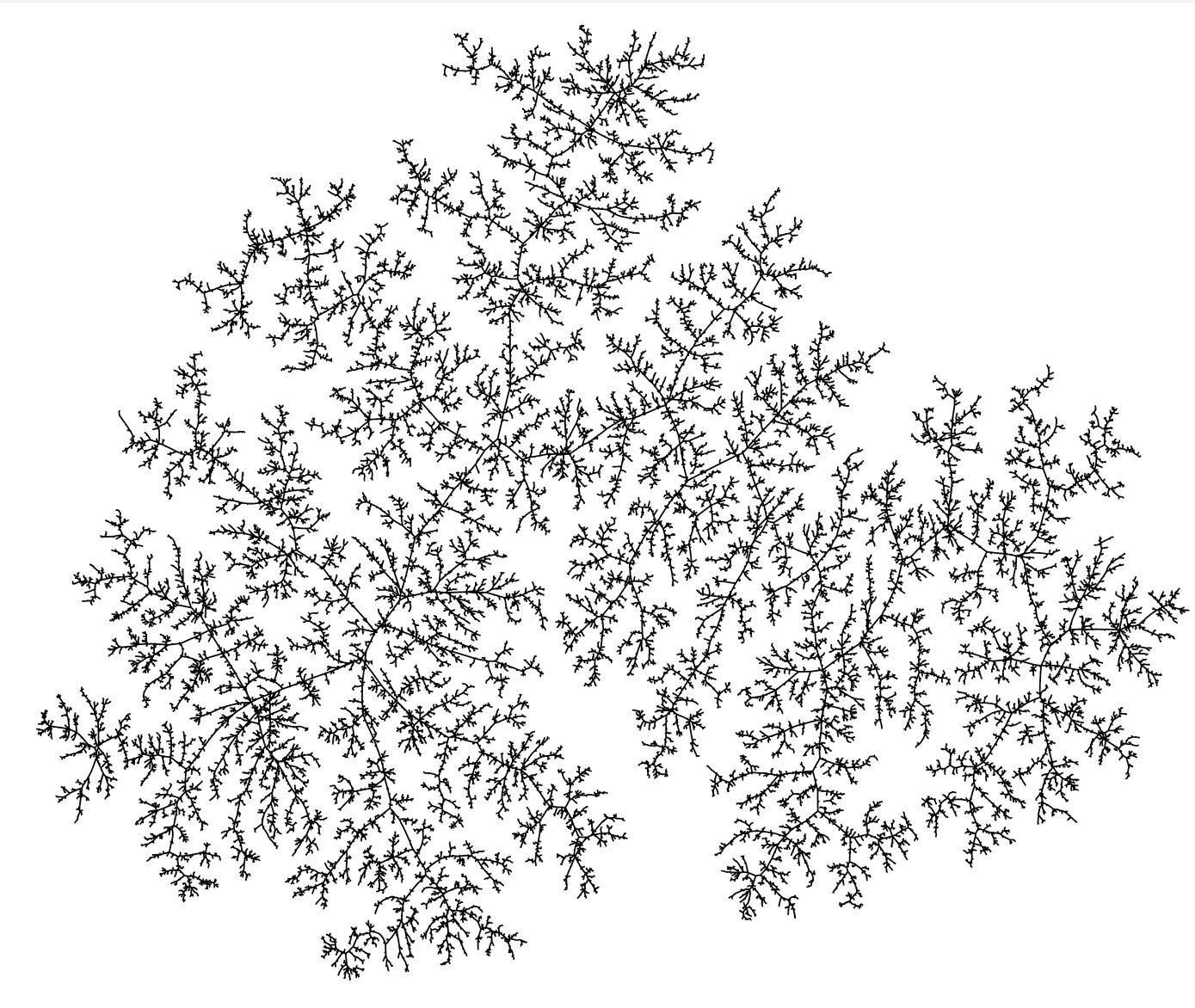


<http://www.flickr.com/photos/ewedistrict/21980840>

# Models of nature

---

MST of random graph

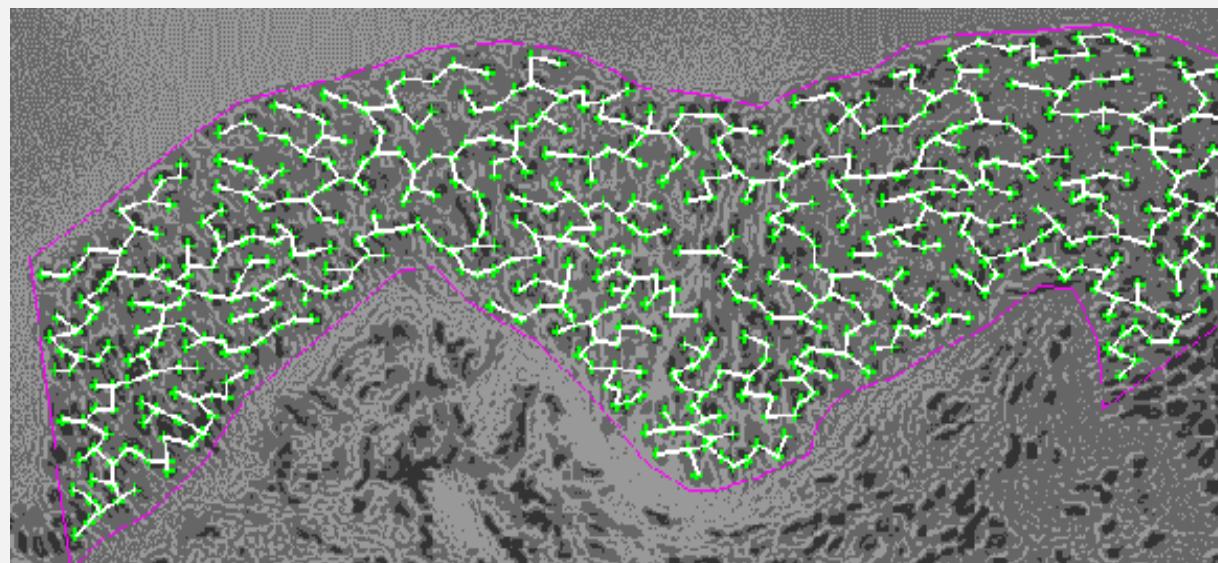
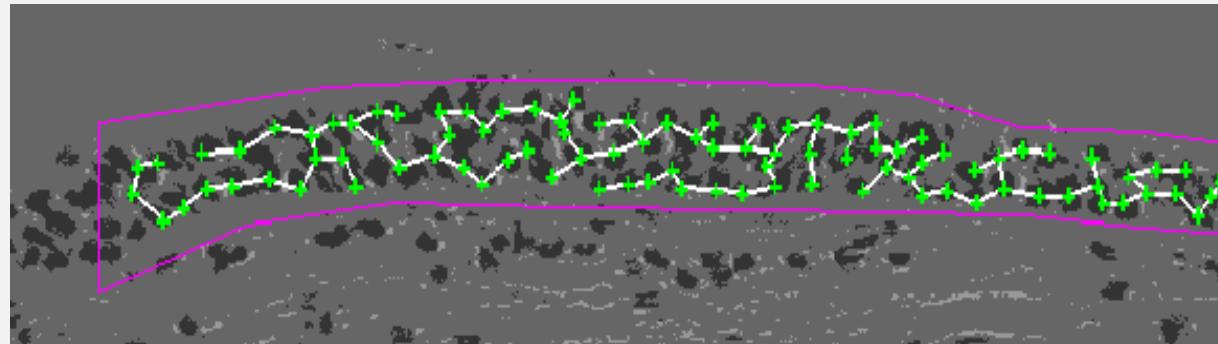


<http://algo.inria.fr;broutin/gallery.html>

# Medical image processing

---

MST describes arrangement of nuclei in the epithelium for cancer research

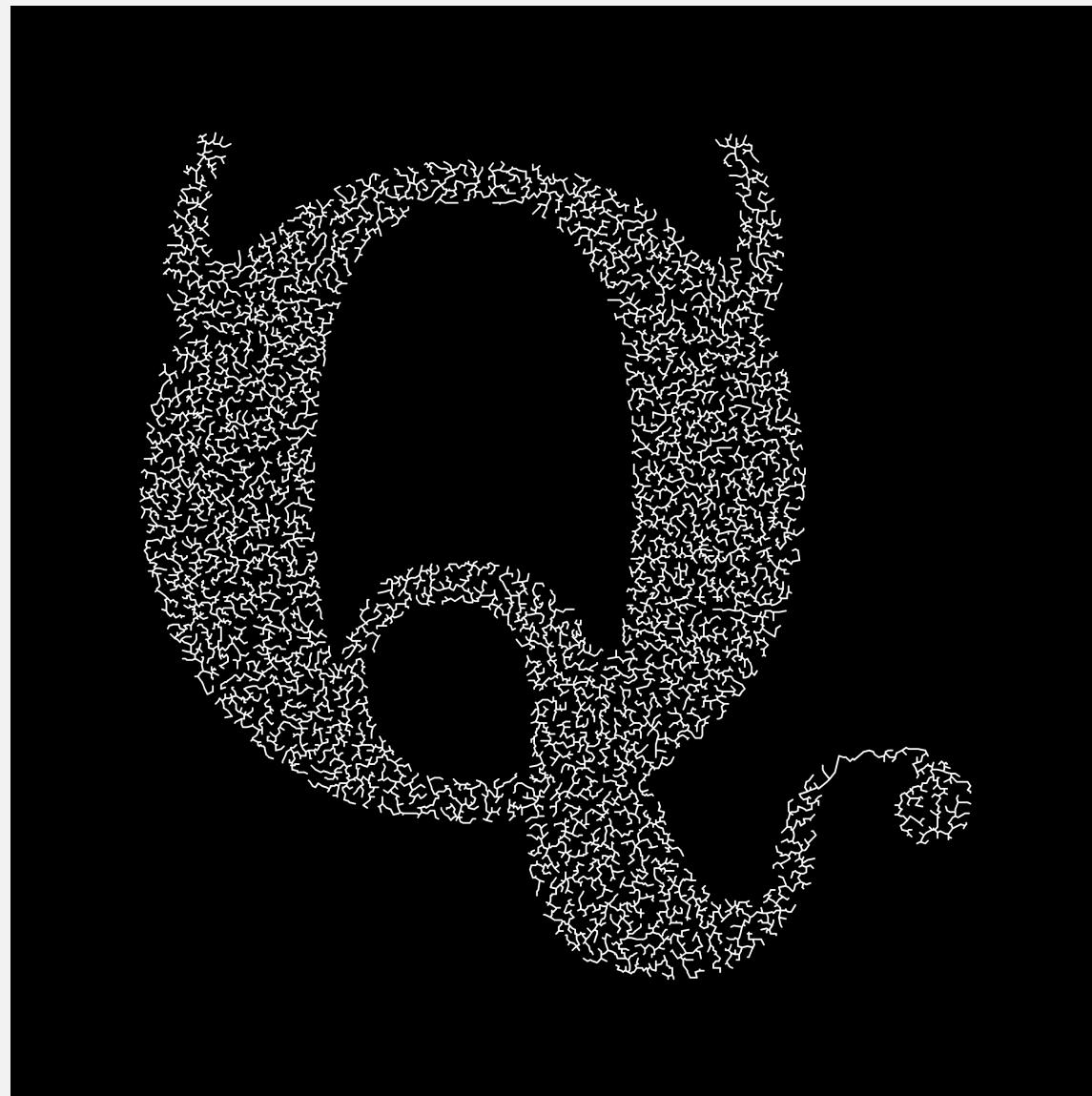


[http://www.bccrc.ca/ci/ta01\\_archlevel.html](http://www.bccrc.ca/ci/ta01_archlevel.html)

# Medical image processing

---

MST dithering



<http://www.flickr.com/photos/quasimondo/2695389651>

# Applications

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MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).

<http://www.ics.uci.edu/~eppstein/gina/mst.html>

# Algorithms

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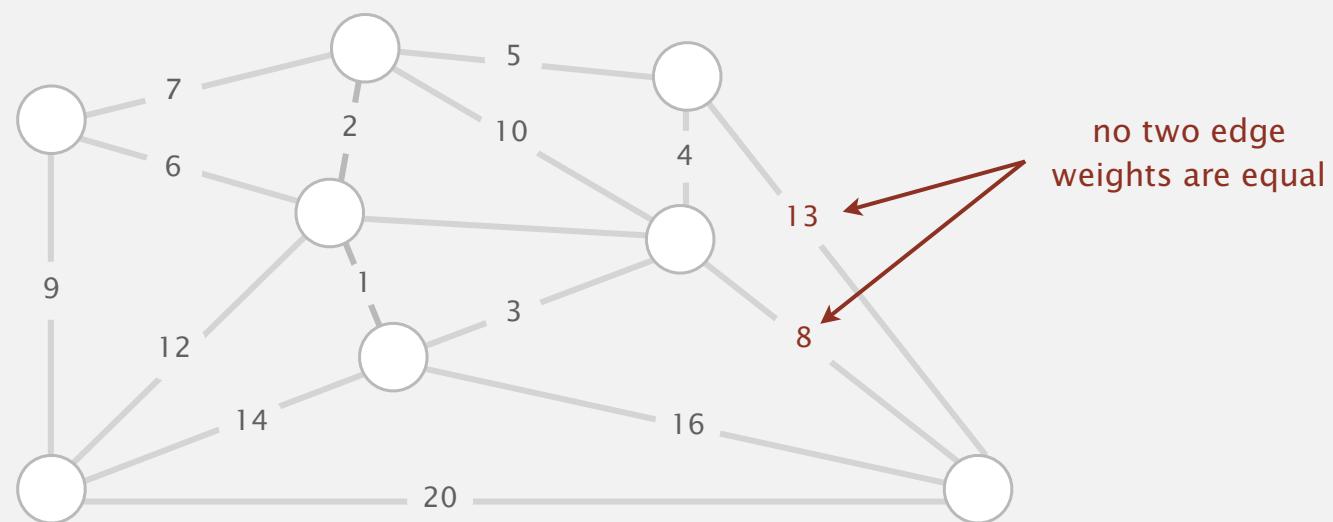
# Simplifying assumptions

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Simplifying assumptions.

- Edge weights are distinct.
- Graph is connected.

Consequence. MST exists and is unique.



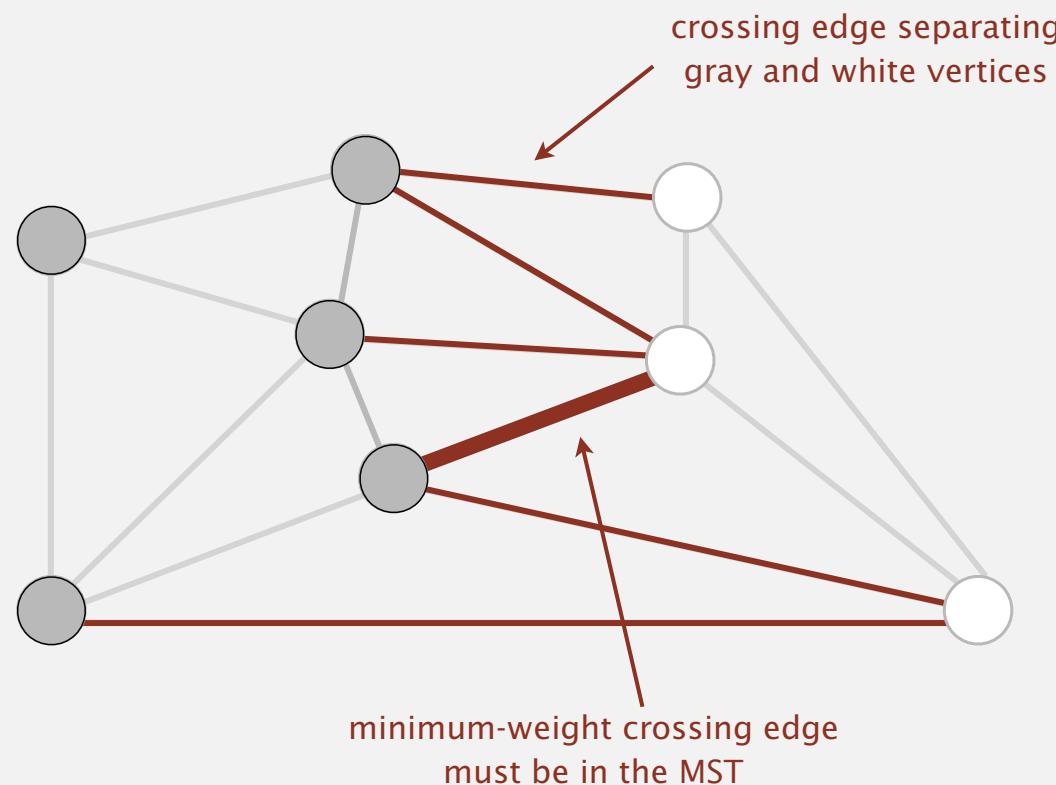
## Cut property

---

Def. A **cut** in a graph is a partition of its vertices into two (nonempty) sets.

Def. A **crossing edge** connects a vertex in one set with a vertex in the other.

**Cut property.** Given any cut, the crossing edge of min weight is in the MST.



## Cut property: correctness proof

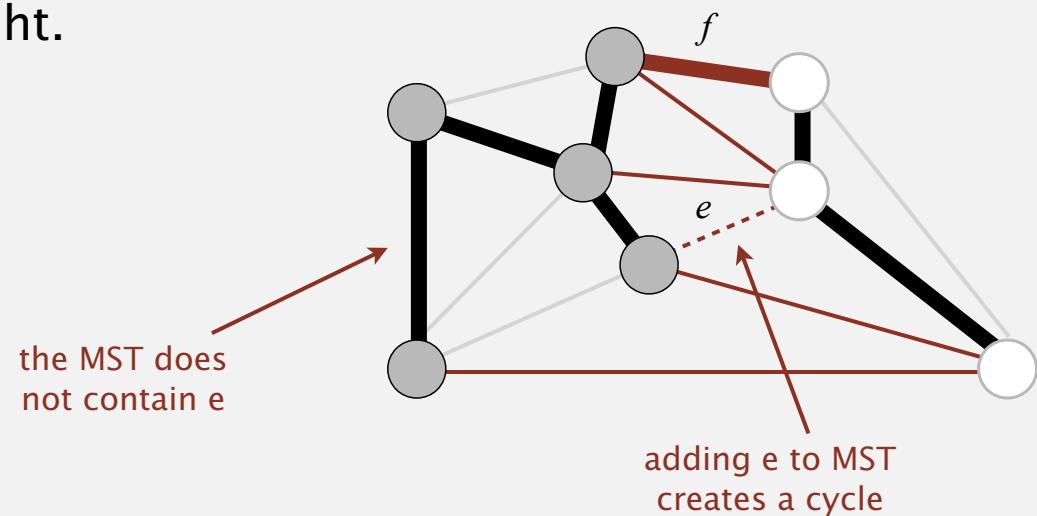
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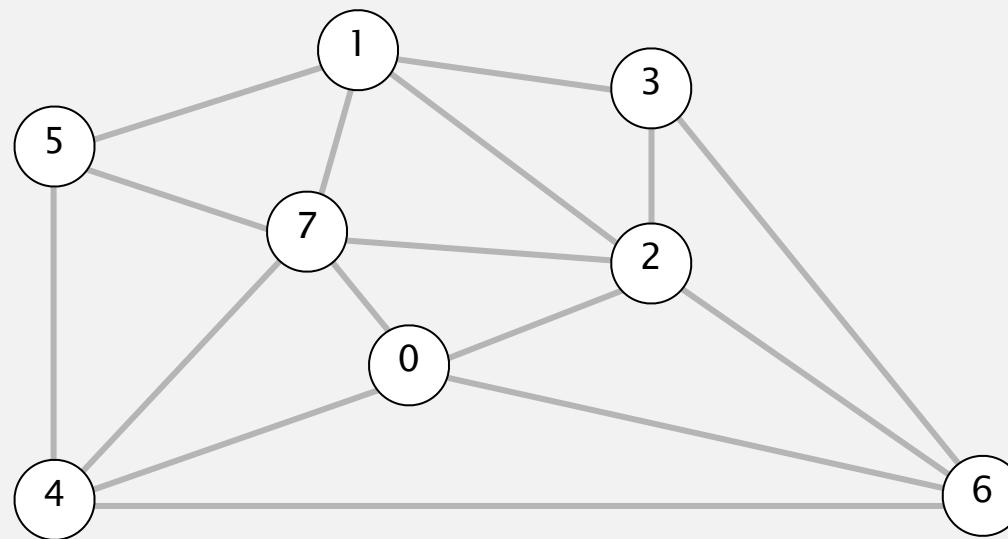
Pf. Suppose min-weight crossing edge  $e$  is not in the MST.

- Adding  $e$  to the MST creates a cycle.
- Some other edge  $f$  in cycle must be a crossing edge.
- Removing  $f$  and adding  $e$  is also a spanning tree.
- Since weight of  $e$  is less than the weight of  $f$ ,  
that spanning tree is lower weight.
- Contradiction. ▀



# Greedy MST algorithm demo

- Start with all edges colored gray.
- Find cut with no black crossing edges; color its min-weight edge black.
- Repeat until  $V - 1$  edges are colored black.

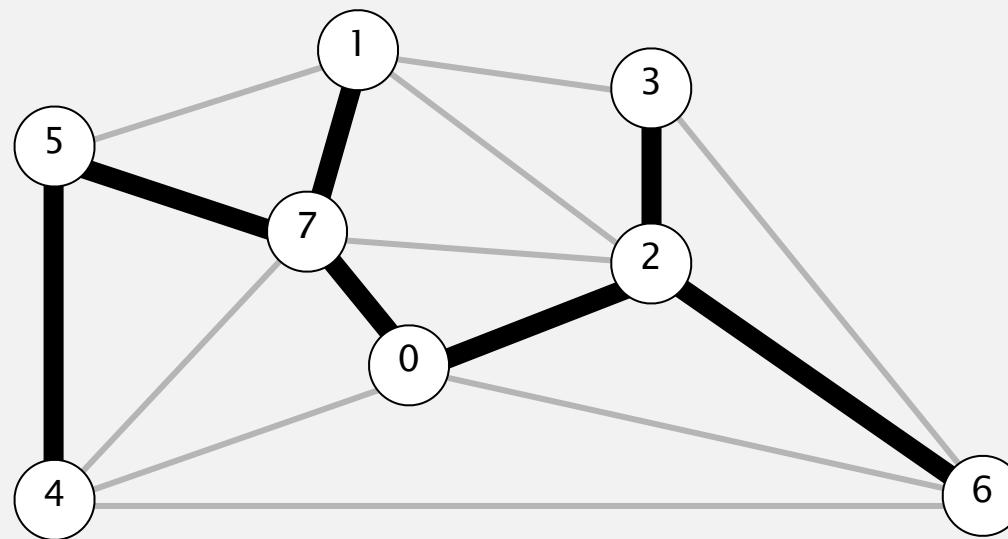


0-7	0.16
2-3	0.17
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0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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3-6	0.52
6-0	0.58
6-4	0.93

## Greedy MST algorithm demo

---

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MST edges

0-2    5-7    6-2    0-7    2-3    1-7    4-5

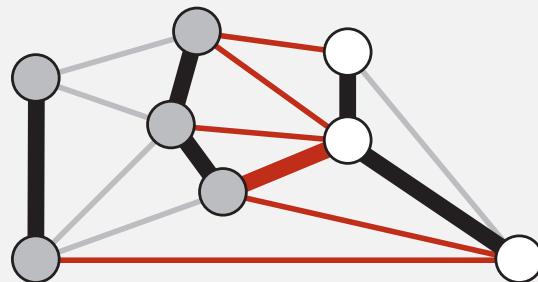
# Greedy MST algorithm: correctness proof

---

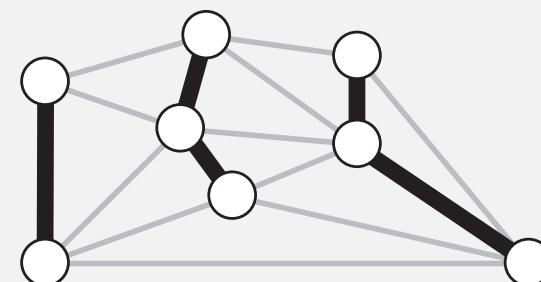
**Proposition.** The greedy algorithm computes the MST.

Pf.

- Any edge colored black is in the MST (via cut property).
- Fewer than  $V - 1$  black edges  $\Rightarrow$  cut with no black crossing edges.  
(consider cut whose vertices are one connected component)



*a cut with no black crossing edges*



*fewer than  $V - 1$  edges colored black*

## Greedy MST algorithm: efficient implementations

---

Proposition. The greedy algorithm computes the MST.

Efficient implementations. Choose cut? Find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

Ex 2. Prim's algorithm. [stay tuned]

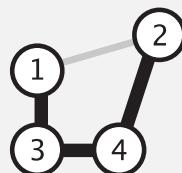
Ex 3. Borüvka's algorithm.

## Removing two simplifying assumptions

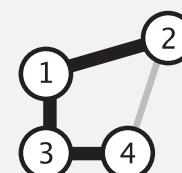
---

Q. What if edge weights are not all distinct?

A. Greedy MST algorithm still correct if equal weights are present!  
(our correctness proof fails, but that can be fixed)



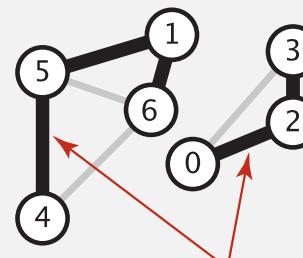
1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50



1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50

Q. What if graph is not connected?

A. Compute minimum spanning forest = MST of each component.



*can independently compute  
MSTs of components*

4	5	0.61
4	6	0.62
5	6	0.88
1	5	0.11
2	3	0.35
0	3	0.6
1	6	0.10
0	2	0.22

# Greed is good

---



**Gordon Gecko (Michael Douglas) address to Teldar Paper Stockholders in Wall Street (1986)**

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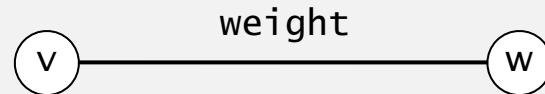
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# Weighted edge API

Edge abstraction needed for weighted edges.

public class Edge implements Comparable<Edge>	
Edge(int v, int w, double weight)	<i>create a weighted edge v-w</i>
int either()	<i>either endpoint</i>
int other(int v)	<i>the endpoint that's not v</i>
int compareTo(Edge that)	<i>compare this edge to that edge</i>
double weight()	<i>the weight</i>
String toString()	<i>string representation</i>



Idiom for processing an edge e: `int v = e.either(), w = e.other(v);`

# Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
    private final int v, w;
    private final double weight;

    public Edge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int either()
    {   return v;   }

    public int other(int vertex)
    {
        if (vertex == v) return w;
        else return v;
    }

    public int compareTo(Edge that)
    {
        if      (this.weight < that.weight) return -1;
        else if (this.weight > that.weight) return +1;
        else                                return 0;
    }
}
```

The diagram illustrates the Java code for a weighted edge with the following annotations:

- constructor**: Points to the constructor `public Edge(int v, int w, double weight)`.
- either endpoint**: Points to the method `public int either()`.
- other endpoint**: Points to the method `public int other(int vertex)`.
- compare edges by weight**: Points to the `compareTo` method, specifically to the comparison logic involving `this.weight` and `that.weight`.

# Edge-weighted graph API

```
public class EdgeWeightedGraph
```

```
    EdgeWeightedGraph(int V)
```

*create an empty graph with  $V$  vertices*

```
    EdgeWeightedGraph(In in)
```

*create a graph from input stream*

```
    void addEdge(Edge e)
```

*add weighted edge  $e$  to this graph*

```
    Iterable<Edge> adj(int v)
```

*edges incident to  $v$*

```
    Iterable<Edge> edges()
```

*all edges in this graph*

```
    int V()
```

*number of vertices*

```
    int E()
```

*number of edges*

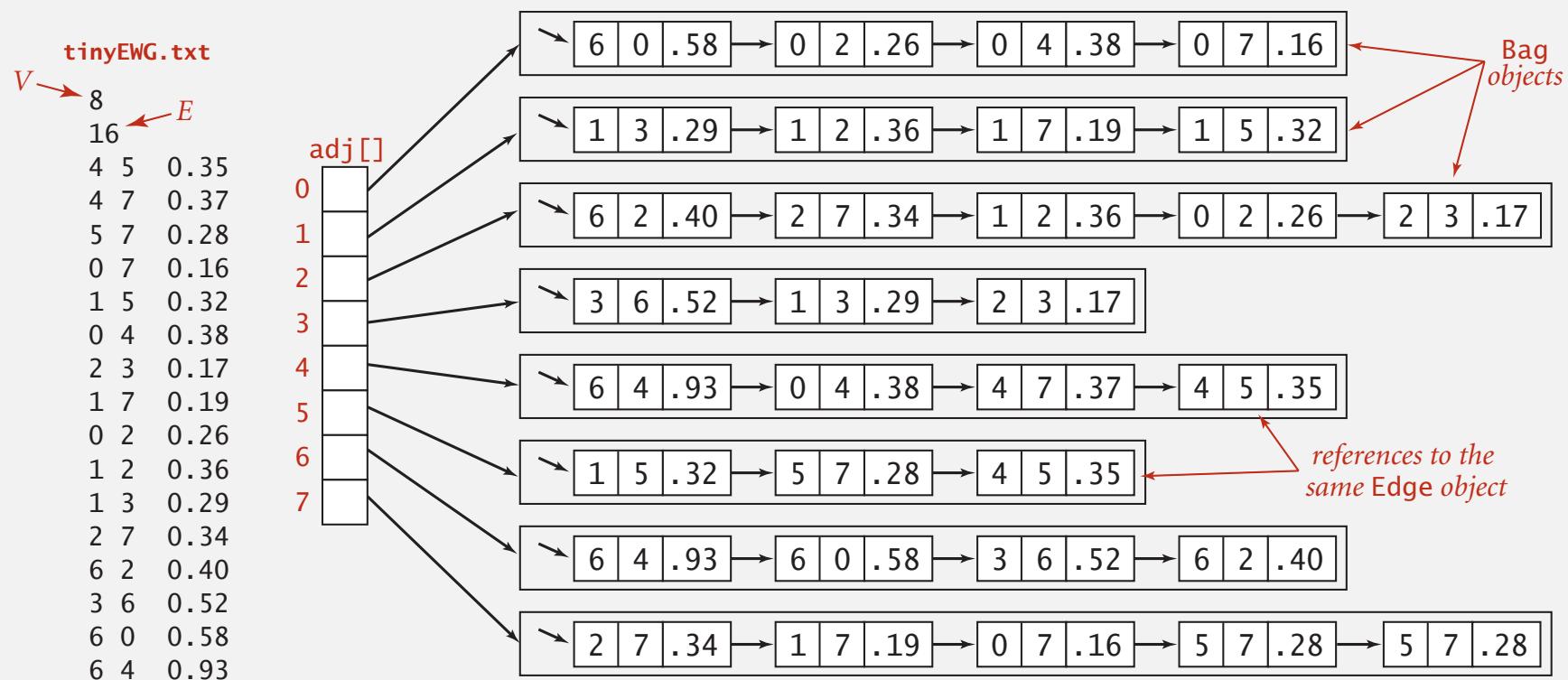
```
    String toString()
```

*string representation*

**Conventions.** Allow self-loops and parallel edges.

# Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of Edge lists.



# Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
    private final int V;
    private final Bag<Edge>[] adj;

    public EdgeWeightedGraph(int V)
    {
        this.V = V;
        adj = (Bag<Edge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Edge>();
    }

    public void addEdge(Edge e)
    {
        int v = e.either(), w = e.other(v);
        adj[v].add(e);
        adj[w].add(e);
    }

    public Iterable<Edge> adj(int v)
    { return adj[v]; }
}
```

same as Graph, but adjacency lists of Edges instead of integers

constructor

add edge to both adjacency lists

# Minimum spanning tree API

Q. How to represent the MST?

```
public class MST
```

```
MST(EdgeWeightedGraph G)
```

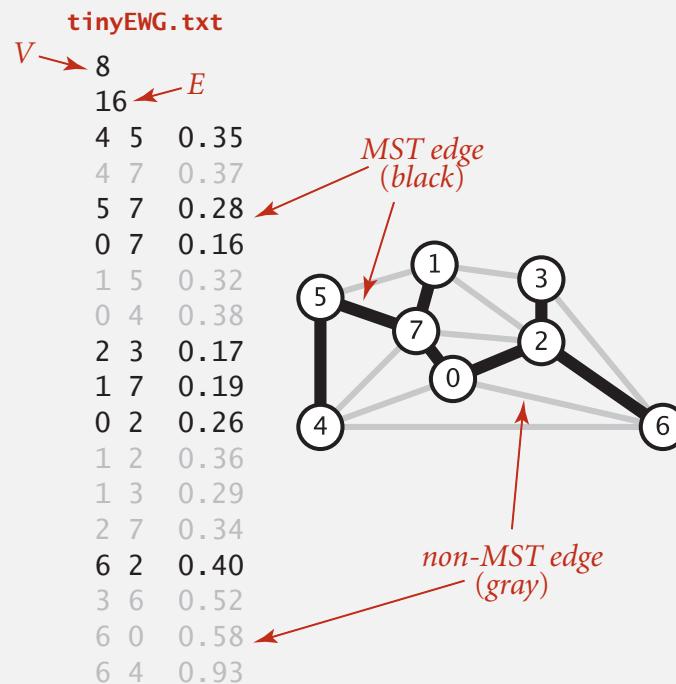
*constructor*

```
Iterable<Edge> edges()
```

*edges in MST*

```
double weight()
```

*weight of MST*



```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

# Minimum spanning tree API

Q. How to represent the MST?

```
public class MST
```

```
    MST(EdgeWeightedGraph G)
```

*constructor*

```
    Iterable<Edge> edges()
```

*edges in MST*

```
    double weight()
```

*weight of MST*

```
public static void main(String[] args)
{
    In in = new In(args[0]);
    EdgeWeightedGraph G = new EdgeWeightedGraph(in);
    MST mst = new MST(G);
    for (Edge e : mst.edges())
        StdOut.println(e);
    StdOut.printf("%.2f\n", mst.weight());
}
```

```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
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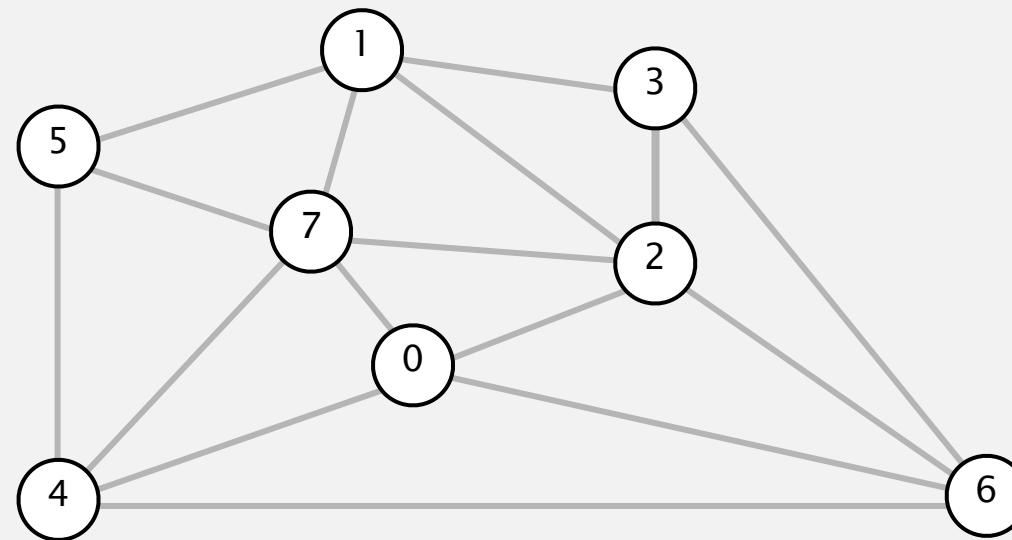
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# Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree  $T$  unless doing so would create a cycle.



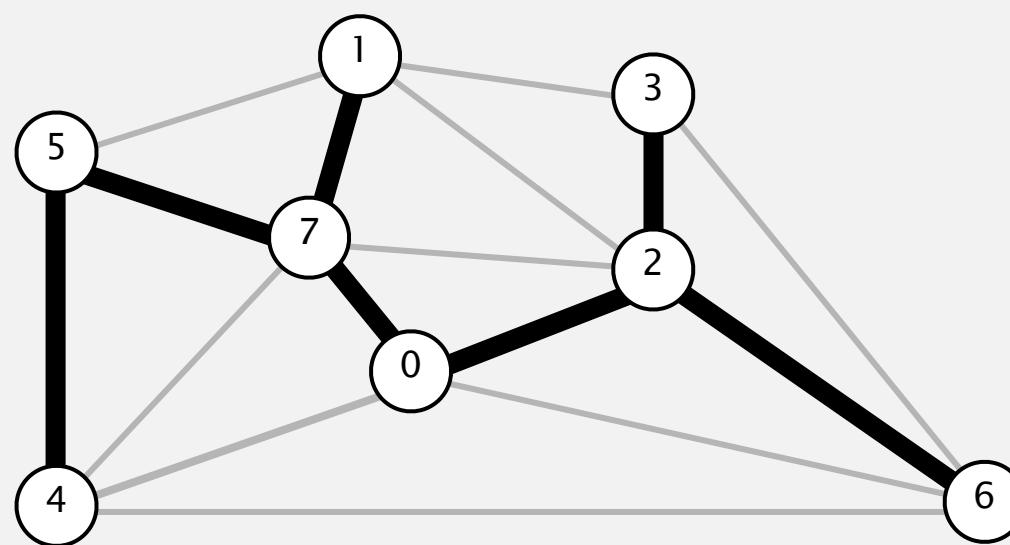
an edge-weighted graph

graph edges sorted by weight	
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
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4-7	0.37
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6-0	0.58
6-4	0.93

## Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree  $T$  unless doing so would create a cycle.

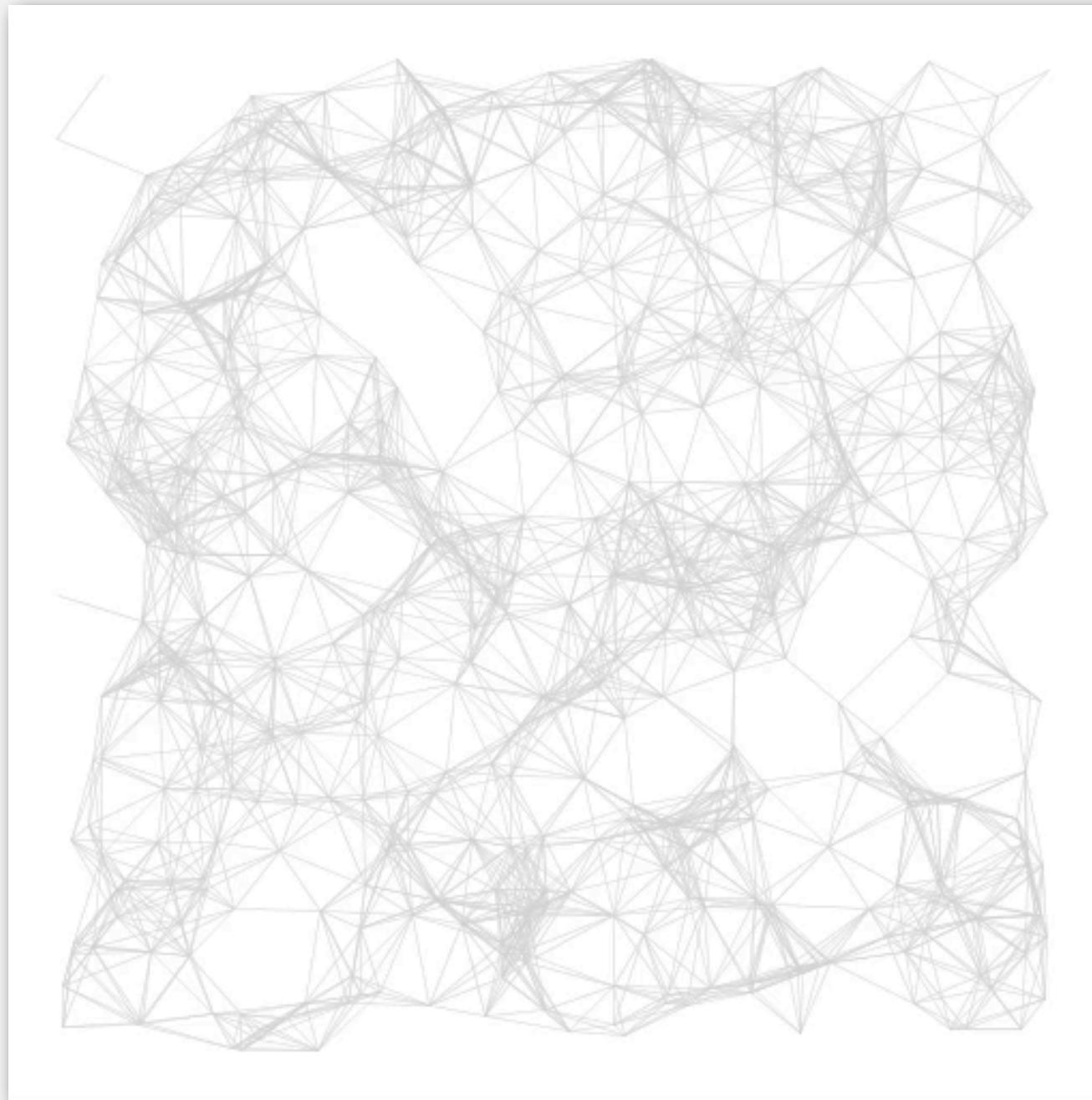


a minimum spanning tree

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## Kruskal's algorithm: visualization

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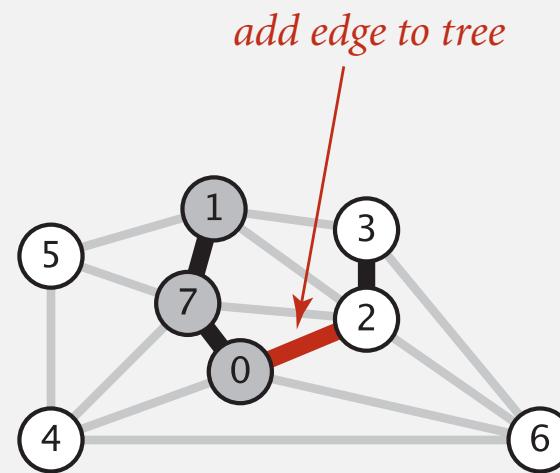
## Kruskal's algorithm: correctness proof

---

**Proposition.** [Kruskal 1956] Kruskal's algorithm computes the MST.

**Pf.** Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors the edge  $e = v-w$  black.
- Cut = set of vertices connected to  $v$  in tree  $T$ .
- No crossing edge is black.
- No crossing edge has lower weight. Why?

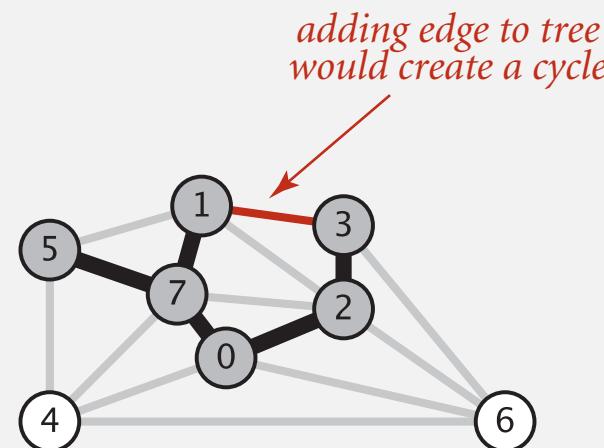
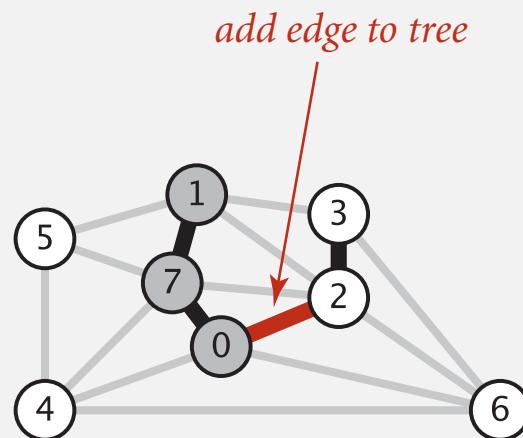


# Kruskal's algorithm: implementation challenge

**Challenge.** Would adding edge  $v-w$  to tree  $T$  create a cycle? If not, add it.

How difficult?

- $E + V$
- $V$  ← run DFS from  $v$ , check if  $w$  is reachable  
( $T$  has at most  $V - 1$  edges)
- $\log V$
- $\log^* V$  ← use the union-find data structure !
- 1



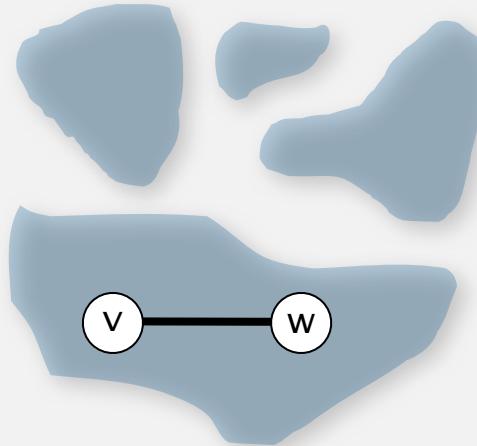
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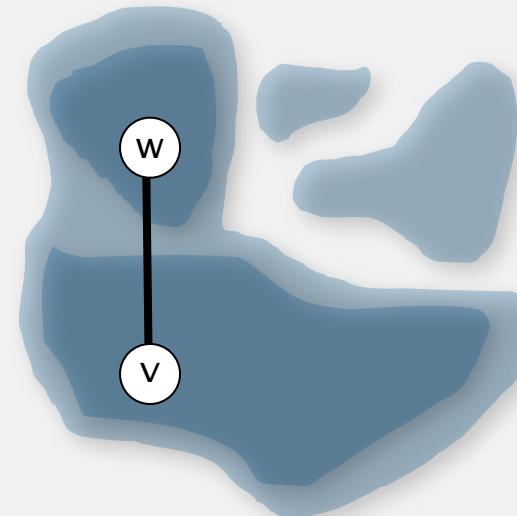
**Challenge.** Would adding edge  $v-w$  to tree  $T$  create a cycle? If not, add it.

**Efficient solution.** Use the **union-find** data structure.

- Maintain a set for each connected component in  $T$ .
- If  $v$  and  $w$  are in same set, then adding  $v-w$  would create a cycle.
- To add  $v-w$  to  $T$ , merge sets containing  $v$  and  $w$ .



Case 1: adding  $v-w$  creates a cycle



Case 2: add  $v-w$  to  $T$  and merge sets containing  $v$  and  $w$

# Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
    private Queue<Edge> mst = new Queue<Edge>();

    public KruskalMST(EdgeWeightedGraph G)
    {
        MinPQ<Edge> pq = new MinPQ<Edge>();
        for (Edge e : G.edges())
            pq.insert(e);

        UF uf = new UF(G.V());
        while (!pq.isEmpty() && mst.size() < G.V()-1)
        {
            Edge e = pq.delMin();
            int v = e.either(), w = e.other(v);
            if (!uf.connected(v, w))
            {
                uf.union(v, w);
                mst.enqueue(e);
            }
        }
    }

    public Iterable<Edge> edges()
    {   return mst;   }
}
```

← build priority queue

← greedily add edges to MST

← edge v-w does not create cycle

← merge sets

← add edge to MST

## Kruskal's algorithm: running time

**Proposition.** Kruskal's algorithm computes MST in time proportional to  $E \log E$  (in the worst case).

Pf.

operation	frequency	time per op
build pq	1	$E \log E$
delete-min	$E$	$\log E$
union	$V$	$\log^* V \dagger$
connected	$E$	$\log^* V \dagger$

$\dagger$  amortized bound using weighted quick union with path compression

recall:  $\log^* V \leq 5$  in this universe



**Remark.** If edges are already sorted, order of growth is  $E \log^* V$ .

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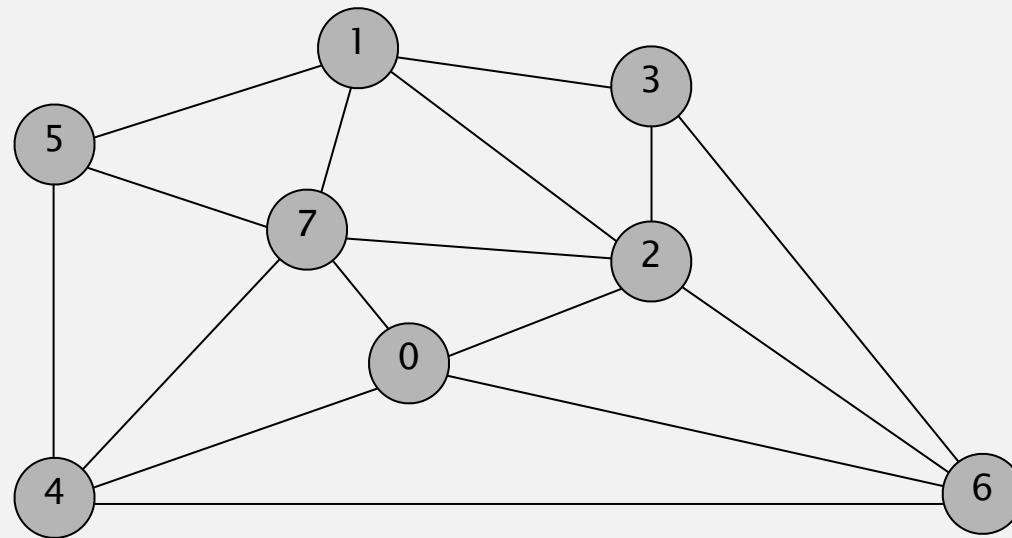
## 4.3 MINIMUM SPANNING TREES

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- ▶ *introduction*
- ▶ *greedy algorithm*
- ▶ *edge-weighted graph API*
- ▶ *Kruskal's algorithm*
- ▶ ***Prim's algorithm***
- ▶ *context*

## Prim's algorithm demo

- Start with vertex 0 and greedily grow tree  $T$ .
- Add to  $T$  the min weight edge with exactly one endpoint in  $T$ .
- Repeat until  $V - 1$  edges.



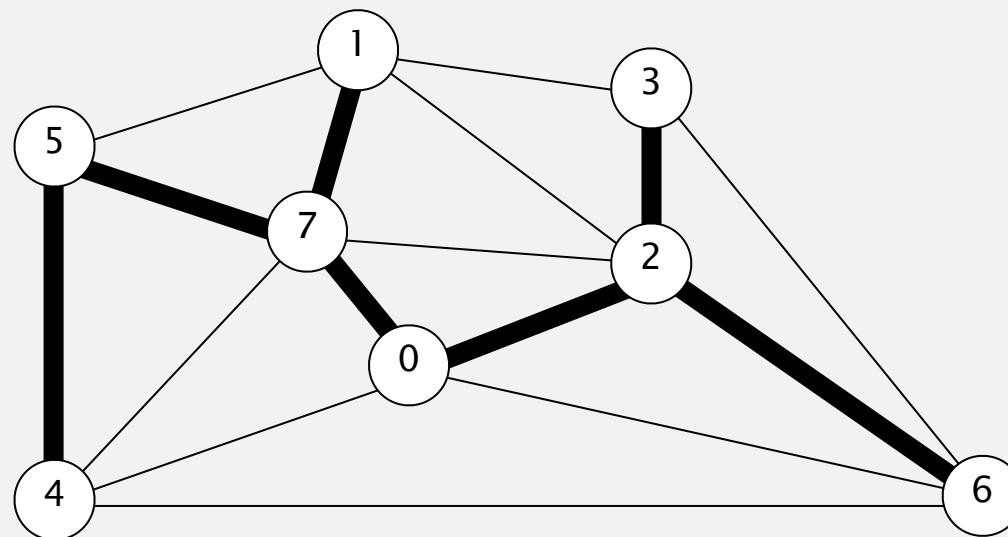
an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

## Prim's algorithm demo

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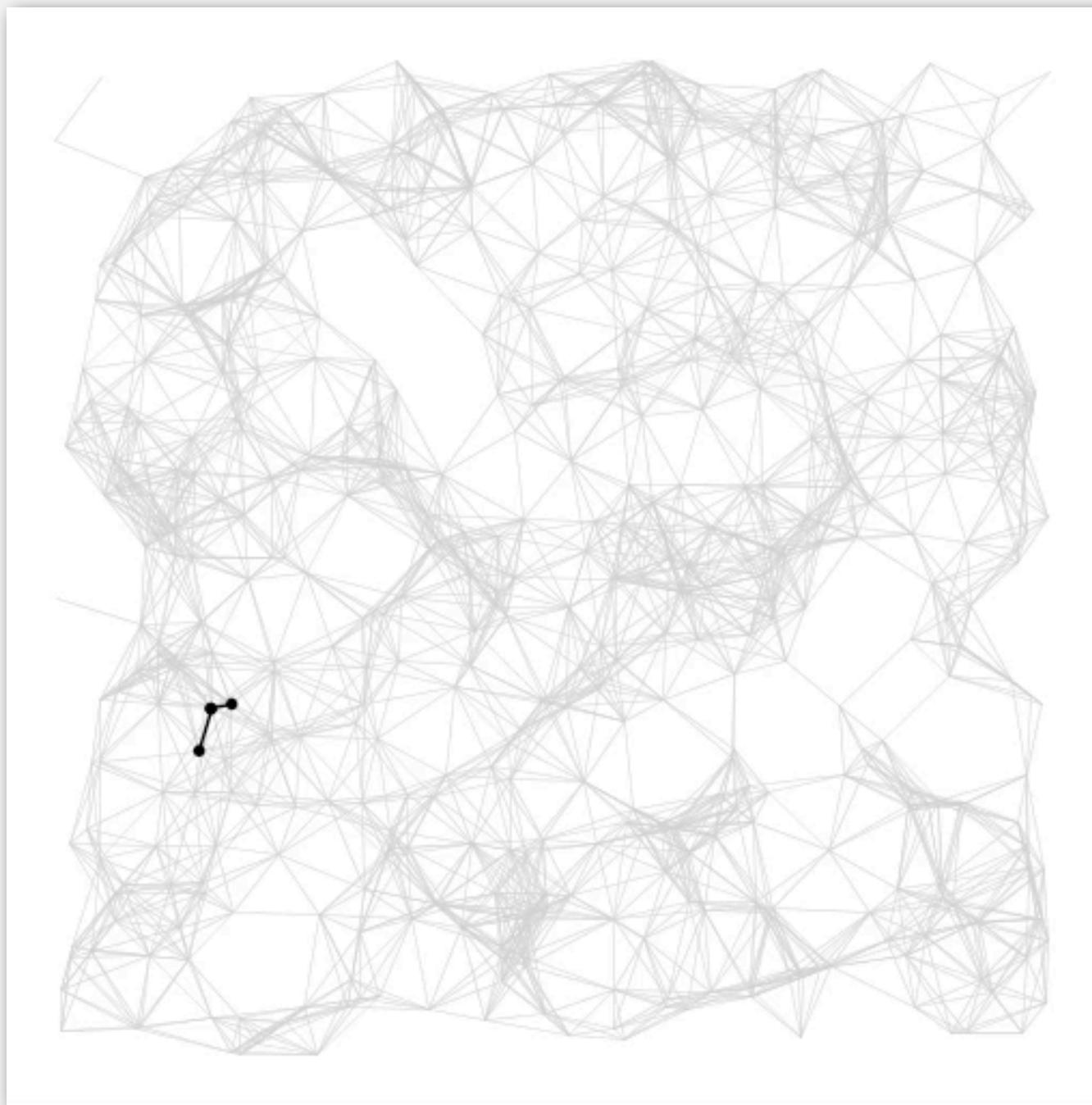


MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

## Prim's algorithm: visualization

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## Prim's algorithm: proof of correctness

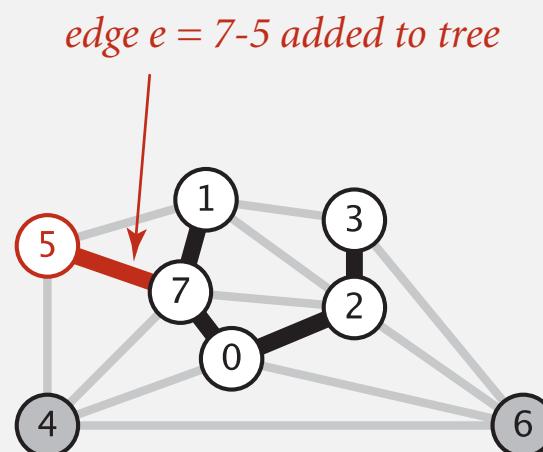
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**Proposition.** [Jarník 1930, Dijkstra 1957, Prim 1959]

Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.

- Suppose edge  $e = \min$  weight edge connecting a vertex on the tree to a vertex not on the tree.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
- No crossing edge has lower weight.



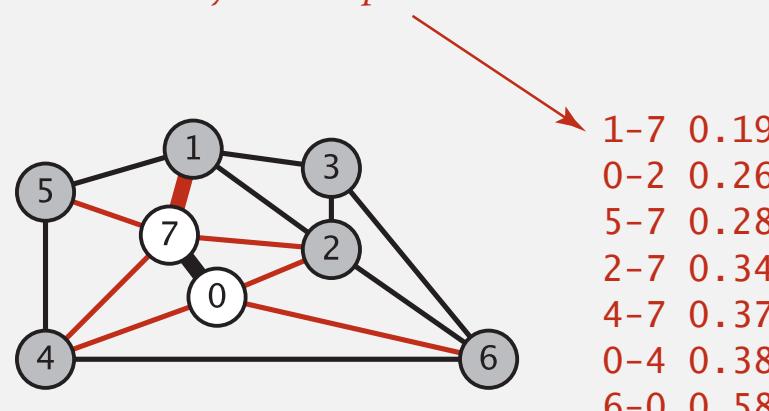
# Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in  $T$ .

How difficult?

- $E$       ← try all edges
- $V$
- $\log E$       ← use a priority queue!
- $\log^* E$
- 1

1-7 is min weight edge with  
exactly one endpoint in  $T$

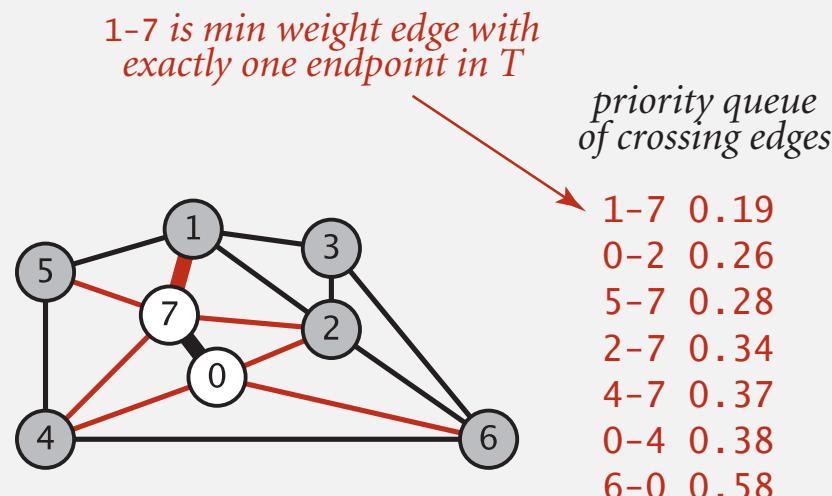


## Prim's algorithm: lazy implementation

**Challenge.** Find the min weight edge with exactly one endpoint in  $T$ .

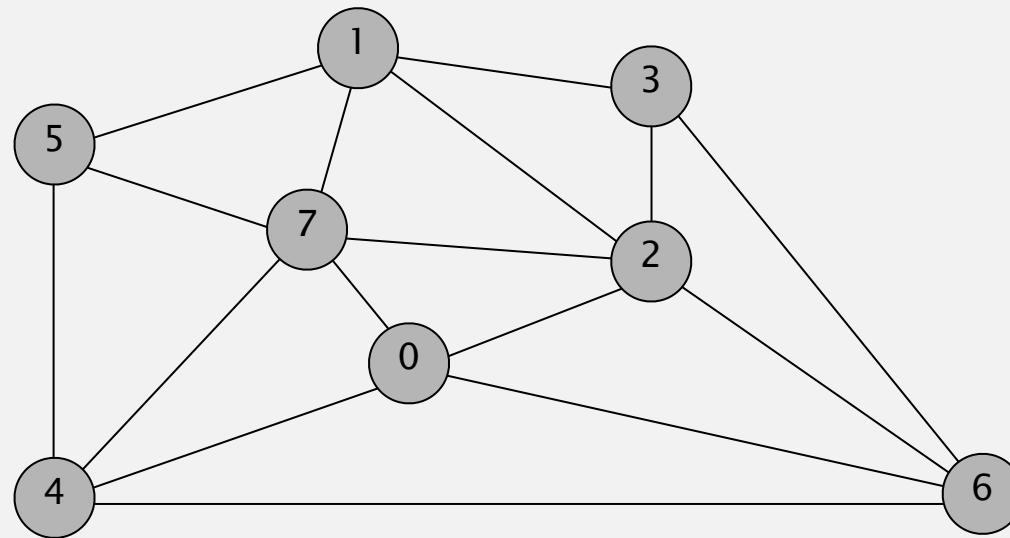
**Lazy solution.** Maintain a PQ of edges with (at least) one endpoint in  $T$ .

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge  $e = v-w$  to add to  $T$ .
- Disregard if both endpoints  $v$  and  $w$  are in  $T$ .
- Otherwise, let  $w$  be the vertex not in  $T$ :
  - add to PQ any edge incident to  $w$  (assuming other endpoint not in  $T$ )
  - add  $w$  to  $T$



## Prim's algorithm (lazy) demo

- Start with vertex 0 and greedily grow tree  $T$ .
- Add to  $T$  the min weight edge with exactly one endpoint in  $T$ .
- Repeat until  $V - 1$  edges.



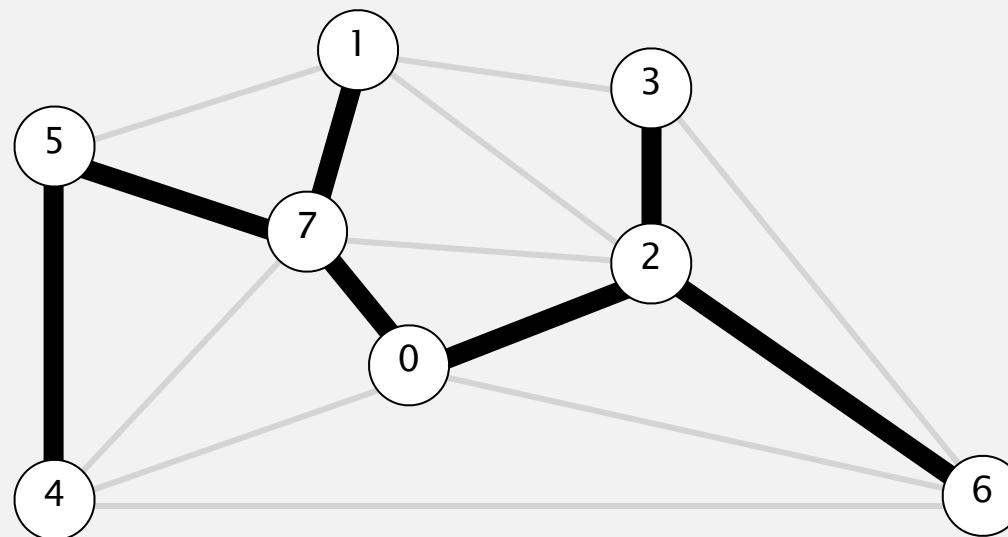
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## Prim's algorithm (lazy) demo

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- Start with vertex 0 and greedily grow tree  $T$ .
- Add to  $T$  the min weight edge with exactly one endpoint in  $T$ .
- Repeat until  $V - 1$  edges.



MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

# Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
    private boolean[] marked;      // MST vertices
    private Queue<Edge> mst;      // MST edges
    private MinPQ<Edge> pq;       // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);
    }

    while (!pq.isEmpty() && mst.size() < G.V() - 1)
    {
        Edge e = pq.delMin();
        int v = e.either(), w = e.other(v);
        if (marked[v] && marked[w]) continue;
        mst.enqueue(e);
        if (!marked[v]) visit(G, v);
        if (!marked[w]) visit(G, w);
    }
}
```

- ← assume  $G$  is connected
- ← repeatedly delete the min weight edge  $e = v-w$  from PQ
- ← ignore if both endpoints in  $T$
- ← add edge  $e$  to tree
- ← add  $v$  or  $w$  to tree

## Prim's algorithm: lazy implementation

```
private void visit(WeightedGraph G, int v)
{
    marked[v] = true;
    for (Edge e : G.adj(v))
        if (!marked[e.other(v)])
            pq.insert(e);
}

public Iterable<Edge> mst()
{   return mst; }
```

← add v to T

← for each edge  $e = v-w$ , add to PQ if w not already in T

## Lazy Prim's algorithm: running time

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**Proposition.** Lazy Prim's algorithm computes the MST in time proportional to  $E \log E$  and extra space proportional to  $E$  (in the worst case).

Pf.

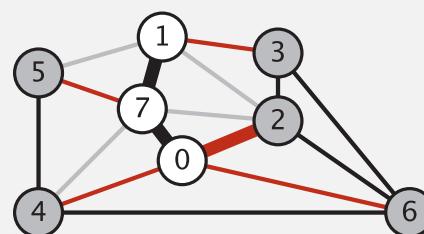
operation	frequency	binary heap
delete min	$E$	$\log E$
insert	$E$	$\log E$

## Prim's algorithm: eager implementation

Challenge. Find min weight edge with exactly one endpoint in  $T$ .

Eager solution. Maintain a PQ of vertices connected by an edge to  $T$ , where priority of vertex  $v$  = weight of shortest edge connecting  $v$  to  $T$ .

- Delete min vertex  $v$  and add its associated edge  $e = v-w$  to  $T$ .
- Update PQ by considering all edges  $e = v-x$  incident to  $v$ 
  - ignore if  $x$  is already in  $T$
  - add  $x$  to PQ if not already on it
  - decrease priority of  $x$  if  $v-x$  becomes shortest edge connecting  $x$  to  $T$

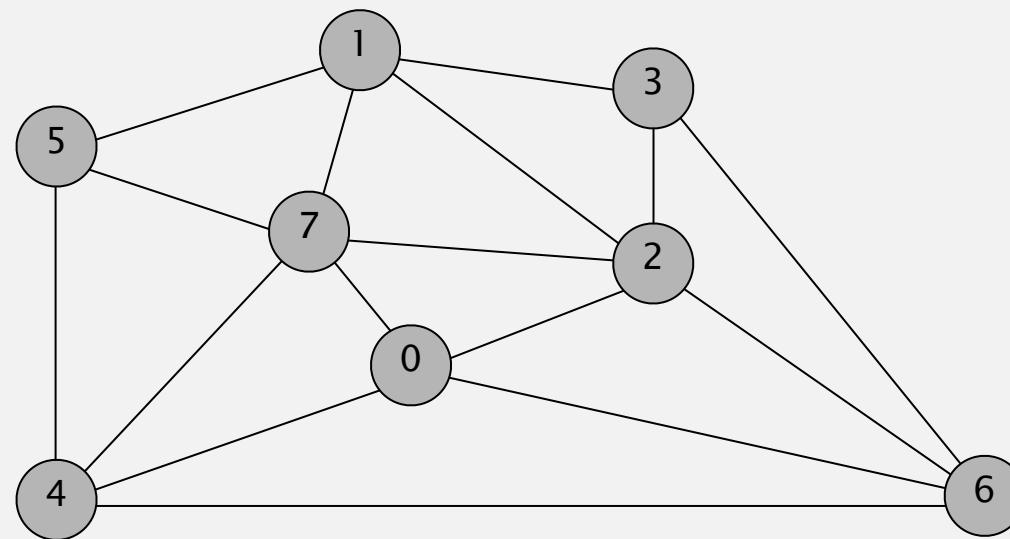


0		
1	1-7	0.19
2	0-2	0.26
3	1-3	0.29
4	0-4	0.38
5	5-7	0.28
6	6-0	0.58
7	0-7	0.16

black: on MST  
red: on PQ

## Prim's algorithm (eager) demo

- Start with vertex 0 and greedily grow tree  $T$ .
- Add to  $T$  the min weight edge with exactly one endpoint in  $T$ .
- Repeat until  $V - 1$  edges.

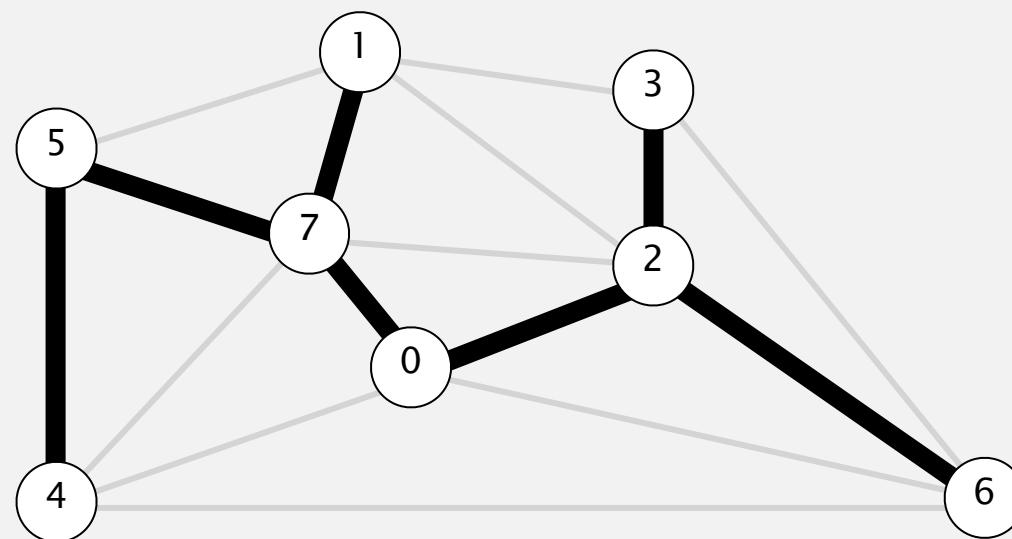


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## Prim's algorithm (eager) demo

- Start with vertex 0 and greedily grow tree  $T$ .
- Add to  $T$  the min weight edge with exactly one endpoint in  $T$ .
- Repeat until  $V - 1$  edges.



v	edgeTo[]	distTo[]
0	-	-
7	0-7	0.16
1	1-7	0.19
2	0-2	0.26
3	2-3	0.17
5	5-7	0.28
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MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

## Indexed priority queue

---

Associate an index between 0 and  $N - 1$  with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

public class IndexMinPQ<Key extends Comparable<Key>>	
IndexMinPQ(int N)	<i>create indexed priority queue with indices 0, 1, ..., N-1</i>
void insert(int i, Key key)	<i>associate key with index i</i>
void decreaseKey(int i, Key key)	<i>decrease the key associated with index i</i>
boolean contains(int i)	<i>is i an index on the priority queue?</i>
int delMin()	<i>remove a minimal key and return its associated index</i>
boolean isEmpty()	<i>is the priority queue empty?</i>
int size()	<i>number of entries in the priority queue</i>

# Indexed priority queue implementation

## Implementation.

- Start with same code as MinPQ.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
- Use `swim(qp[i])` implement `decreaseKey(i, key)`.

i	0	1	2	3	4	5	6	7	8
keys[i]	A	S	O	R	T	I	N	G	-
pq[i]	-	0	6	7	2	1	5	4	3
qp[i]	1	5	4	8	7	6	2	3	-

Diagram of a binary heap structure:

```
graph TD; A((A)) -- 1 --> N((N)); A -- 1 --> G((G)); N((N)) -- 2 --> O((O)); N((N)) -- 2 --> S((S)); G((G)) -- 3 --> I((I)); G((G)) -- 3 --> T((T)); O((O)) -- 4 --> R((R));
```

The heap has 9 nodes labeled A through T. Node N is highlighted with a red oval and has a red number 2 above it, indicating it is the current node being processed. Node A is the root with a red number 1 above it. Node O has a red number 4 above it. Node R has a red number 8 above it. Node N has a red number 2 above it. Node G has a red number 3 above it. Node I has a red number 6 above it. Node T has a red number 7 above it.

## Prim's algorithm: running time

---

Depends on PQ implementation:  $V$  insert,  $V$  delete-min,  $E$  decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
<b>array</b>	1	$V$	1	$V^2$
<b>binary heap</b>	$\log V$	$\log V$	$\log V$	$E \log V$
<b>d-way heap (Johnson 1975)</b>	$d \log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
<b>Fibonacci heap (Fredman-Tarjan 1984)</b>	$1^\dagger$	$\log V^\dagger$	$1^\dagger$	$E + V \log V$

$\dagger$  amortized

### Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

# Algorithms

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- ▶ *context*

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# Does a linear-time MST algorithm exist?

## deterministic compare-based MST algorithms

year	worst case	discovered by
1975	$E \log \log V$	Yao
1976	$E \log \log V$	Cheriton-Tarjan
1984	$E \log^* V, E + V \log V$	Fredman-Tarjan
1986	$E \log (\log^* V)$	Gabow-Galil-Spencer-Tarjan
1997	$E \alpha(V) \log \alpha(V)$	Chazelle
2000	$E \alpha(V)$	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	$E$	???

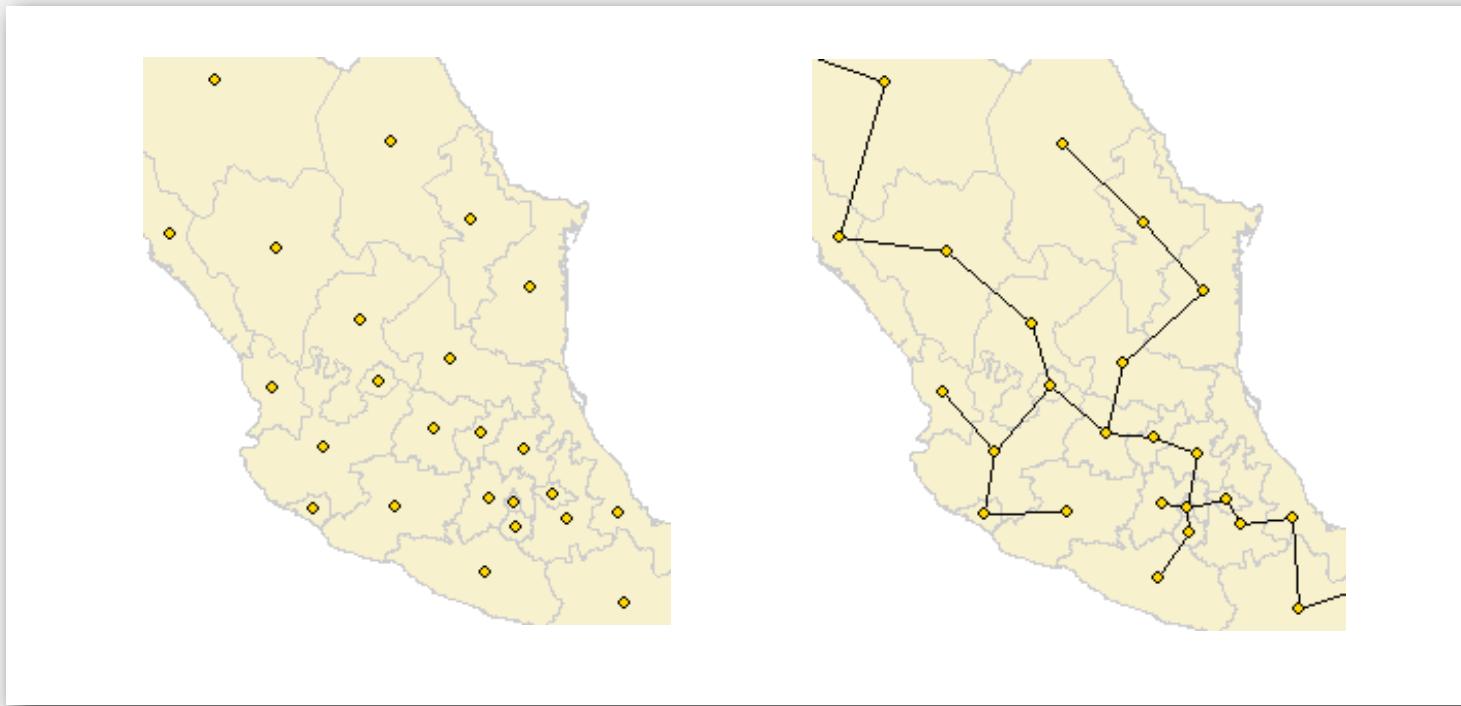


**Remark.** Linear-time randomized MST algorithm (Karger-Klein-Tarjan 1995).

## Euclidean MST

---

Given  $N$  points in the plane, find MST connecting them, where the distances between point pairs are their **Euclidean** distances.



Brute force. Compute  $\sim N^2 / 2$  distances and run Prim's algorithm.

Ingenuity. Exploit geometry and do it in  $\sim c N \log N$ .

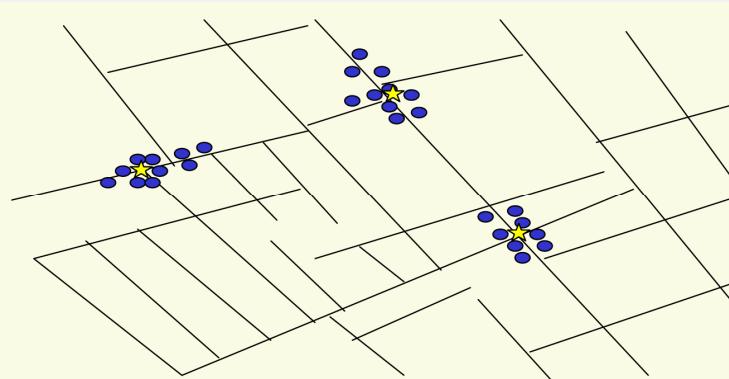
## Scientific application: clustering

---

**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Goal.** Divide into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

## Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases.
- Skycat: cluster  $10^9$  sky objects into stars, quasars, galaxies.

# Single-link clustering

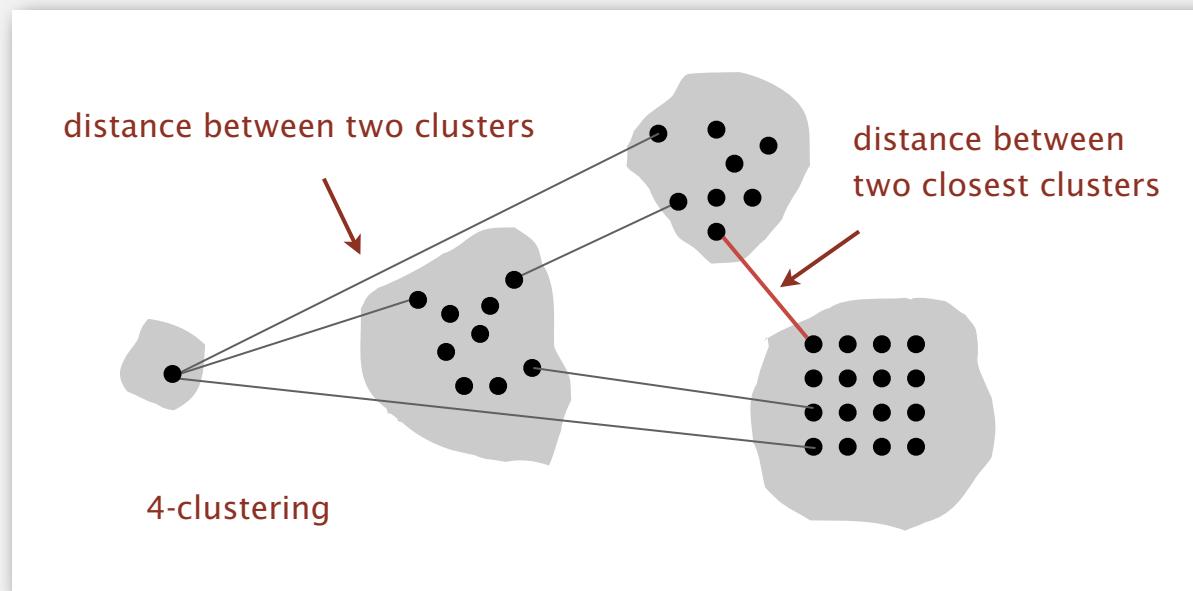
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**k-clustering.** Divide a set of objects classify into k coherent groups.

**Distance function.** Numeric value specifying "closeness" of two objects.

**Single link.** Distance between two clusters equals the distance between the two closest objects (one in each cluster).

**Single-link clustering.** Given an integer k, find a k-clustering that maximizes the distance between two closest clusters.



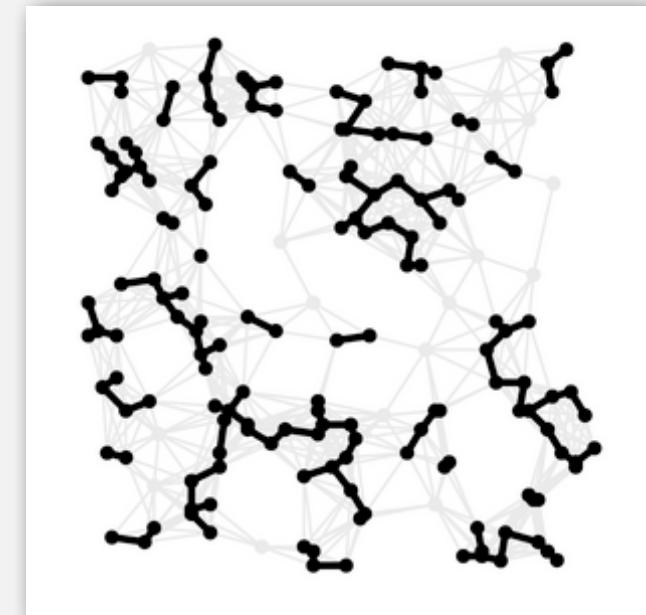
# Single-link clustering algorithm

---

“Well-known” algorithm in science literature for single-link clustering:

- Form  $V$  clusters of one object each.
- Find the closest pair of objects such that each object is in a different cluster, and merge the two clusters.
- Repeat until there are exactly  $k$  clusters.

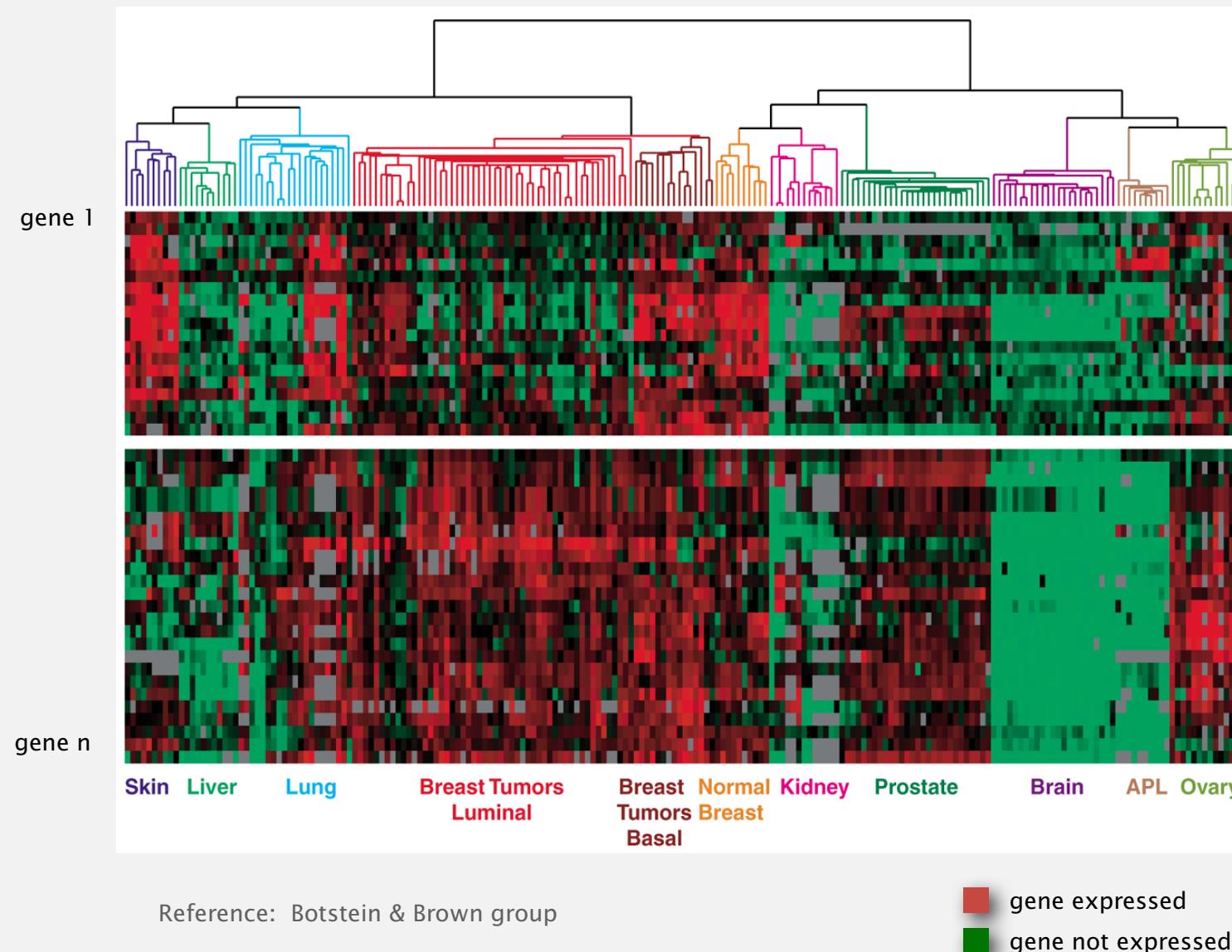
**Observation.** This is Kruskal's algorithm  
(stop when  $k$  connected components).



**Alternate solution.** Run Prim's algorithm and delete  $k-1$  max weight edges.

# Dendrogram of cancers in human

Tumors in similar tissues cluster together.



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