3D-Parabolised Stability Equations

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Chapter 1

Parabolized Stability Equations

Incompressible fluid flow is governed by mass conservation (continuity equation) and momentum equations (Navier-stokes equations) which are written below[1].

Continuity Equation

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0$$

and Navier Stokes equations

$$\rho\left[\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}\right] = -\frac{\partial p^*}{\partial x^*} + \mu\left[\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}}\right]u^*$$

$$\rho\left[\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}\right] = -\frac{\partial p^*}{\partial y^*} + \mu\left[\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}}\right]v^*$$

$$\rho\left[\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right] = -\frac{\partial p^*}{\partial z^*} + \mu\left[\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}}\right]w^*$$

Non-dimensional form of these equations can be obtained by specifying non dimensional quantities as shown below

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{L}, \quad z = \frac{z^*}{L},$$

$$u = \frac{u^*}{U}, \quad v = \frac{v^*}{U}, \quad w = \frac{w^*}{U},$$

$$t = \frac{Ut^*}{L}, \quad p = \frac{p^*}{\rho U^2}, \quad Re = \frac{\rho U L}{\mu},$$

Substituting in above continuity and momentum equations we get non-dimensional equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

[1]* marked quantities are dimensional

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]$$

1.1 Linearized Stability Equations

Let

$$q(x,y,z,t)=\bar{q}(x,y,z)+q'(x,y,z,t)$$

Above equations are satisfied by instantaneous quantities $q(x,y,z,t)$ and base flow $\bar{q}(x,y,z,t)$. Writing continuity and momentum for both and subtracting one from other give following linearized stability equations.

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

$$\frac{\partial u'}{\partial t}+u'\frac{\partial \bar{u}}{\partial x}+v'\frac{\partial \bar{u}}{\partial y}+w'\frac{\partial \bar{u}}{\partial z}+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}+\bar{w}\frac{\partial u'}{\partial z}=-\frac{\partial p'}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right]$$

$$\frac{\partial v'}{\partial t}+u'\frac{\partial \bar{v}}{\partial x}+v'\frac{\partial \bar{v}}{\partial y}+w'\frac{\partial \bar{v}}{\partial z}+\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}+\bar{w}\frac{\partial v'}{\partial z}=-\frac{\partial p'}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}+\frac{\partial^2 v'}{\partial z^2}\right]$$

$$\frac{\partial w'}{\partial t}+u'\frac{\partial \bar{w}}{\partial x}+v'\frac{\partial \bar{w}}{\partial y}+w'\frac{\partial \bar{w}}{\partial z}+\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}+\bar{w}\frac{\partial w'}{\partial z}=-\frac{\partial p'}{\partial z}+\frac{1}{Re}\left[\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}+\frac{\partial^2 w'}{\partial z^2}\right]$$

where

$$q'(x,y,z,t)=\tilde{q}(x,y,z)\exp i\Theta$$

1.2 Parabolized Stability Equations

For different analysis type $\tilde{q}$ and Θ take different form. For example, for a bi-global analysis $\tilde{q}(x,y,z)=\tilde{q}(x,y)$ and $\Theta=\beta z-\omega t$. For Orr-sommerfeld equations $\tilde{q}(x,y,z)=\tilde{q}(y)$ and $\Theta=\alpha x+\beta z-\omega t$ .The introduction of a harmonic decomposition in x and z implies homogeneity of the base flow in these directions. These conditions, however, are too restrictive for complicated vortical flows and those around two-dimensional bodies. Consequently, the flow is assumed to be general in a two-dimensional plane, and mildly inhomogeneous in the third direction which is permitted by allowing the axial wavenumber to vary slowly. This leads to the parabolised stability equation concept, formally introduced by Herbert[**?**].

$$q'(x,y,z,t) = \tilde{q}(x,y,z)\exp i\Theta$$

where $\Theta = \int_{z_0}^{z} \beta(\xi)d\xi - \omega t$

$$\frac{\partial q'}{\partial z} = \left\{ i\beta\tilde{q} + \frac{\partial \tilde{q}}{\partial z} \right\} \exp i\Theta$$

and

$$\frac{\partial^2 q'}{\partial z^2} = \left\{ \frac{\partial^2 \tilde{q}}{\partial z^2} + 2i\beta\frac{\partial \tilde{q}}{\partial z} - \beta^2\tilde{q} + i\frac{d\beta}{dz}\tilde{q} \right\} \exp i\Theta$$

β is streamwise wavenumber complex in nature given as $\beta(\xi) = \beta_r(\xi) + i\beta_i(\xi)$.

It should be noted that $\tilde{q}(x,y,z)$ depends very weakly on z coordinate hence, second derivatives w.r.t z will be neglected. Therefore,

$$\frac{\partial^2 q'}{\partial z^2} \approx \left\{ 2i\beta\frac{\partial \tilde{q}}{\partial z} - \beta^2\tilde{q} + i\frac{d\beta}{dz}\tilde{q} \right\} \exp i\Theta$$

Above form of perturbations if fetched in linearized stability equations to derive PSE as shown below.

Continuity equation

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} + i\beta\tilde{w} = 0 \tag{1.2.1}$$

x momentum equation

$$-i\omega\tilde{u} + \frac{\partial \bar{u}}{\partial x}\tilde{u} + \frac{\partial \bar{u}}{\partial y}\tilde{v} + \frac{\partial \bar{u}}{\partial z}\tilde{w} + \bar{u}\frac{\partial \tilde{u}}{\partial x} + \bar{v}\frac{\partial \tilde{u}}{\partial y} + \bar{w}\frac{\partial \tilde{u}}{\partial z} + i\beta\bar{w}\tilde{u} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 + 2i\beta\frac{\partial}{\partial z} + i\frac{d\beta}{dz} \right\}\tilde{u}$$

Calling

$$\mathcal{L} \equiv \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 \right] - \bar{u}\frac{\partial}{\partial x} - \bar{v}\frac{\partial}{\partial y} - i\beta\bar{w}$$

Chapter 2

Change from here

Above equation can is written as

$$\left(\mathcal{L} - \frac{\partial \bar{u}}{\partial x} - \bar{u}\frac{\partial}{\partial x} + i\omega\right)\tilde{u} - \frac{\partial \bar{u}}{\partial y}\tilde{v} - \frac{\partial \bar{u}}{\partial z}\tilde{w} - \frac{\partial \tilde{p}}{\partial x} - i\alpha\tilde{p} = -\frac{2i\alpha}{Re}\frac{\partial \tilde{u}}{\partial x} - \frac{i}{Re}\frac{d\alpha}{dx}\tilde{u} \tag{2.0.1}$$

Similar manipulation give following equations for y and z linearized stability equations

$$-\frac{\partial \bar{v}}{\partial x}\tilde{u} + \left(\mathcal{L} - \frac{\partial \bar{v}}{\partial y} - \bar{u}\frac{\partial}{\partial x} + i\omega\right)\tilde{v} - \frac{\partial \bar{v}}{\partial z}\tilde{w} - \frac{\partial \tilde{p}}{\partial y} = -\frac{2i\alpha}{Re}\frac{\partial \tilde{v}}{\partial x} - \frac{i}{Re}\frac{d\alpha}{dx}\tilde{v} \tag{2.0.2}$$

$$-\frac{\partial \bar{w}}{\partial x}\tilde{u} - \frac{\partial \bar{w}}{\partial y}\tilde{v} + \left(\mathcal{L} - \frac{\partial \bar{w}}{\partial z} - \bar{u}\frac{\partial}{\partial x} + i\omega\right)\tilde{w} - \frac{\partial \tilde{p}}{\partial z} - i\alpha\tilde{p} = -\frac{2i\alpha}{Re}\frac{\partial \tilde{w}}{\partial x} - \frac{i}{Re}\frac{d\alpha}{dx}\tilde{w} \tag{2.0.3}$$

Above four equations are called parabolised stability equation.
In matrix form they are written as

$$\left(\mathcal{L}_0 + \mathcal{L}_1\right)\tilde{q} + \mathcal{L}_2\frac{\partial \tilde{q}}{\partial x} + i\frac{d\alpha}{dx}\mathcal{L}_3\tilde{q} = 0 \tag{2.0.4}$$

where $\tilde{q} = [\tilde{u} \quad \tilde{v} \quad \tilde{w} \quad \tilde{p}]^T$ and operators $\mathcal{L}_i,\ i = 0,\ 1,\ 2,\ 3$ are

$$\mathcal{L}_0 = \begin{bmatrix} \mathcal{L} + i\omega & -\frac{\partial \bar{u}}{\partial y} & -\frac{\partial \bar{u}}{\partial z} & -i\alpha \\ 0 & \mathcal{L} - \frac{\partial \bar{v}}{\partial y} + i\omega & -\frac{\partial \bar{v}}{\partial z} & -\frac{\partial}{\partial y} \\ 0 & -\frac{\partial \bar{w}}{\partial y} & \mathcal{L} - \frac{\partial \bar{w}}{\partial z} + i\omega & -\frac{\partial}{\partial z} \\ i\alpha & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix}$$

$$\mathcal{L}_1 = \begin{bmatrix} -\frac{\partial \bar{u}}{\partial x} & 0 & 0 & 0 \\ -\frac{\partial \bar{v}}{\partial x} & 0 & 0 & 0 \\ -\frac{\partial \bar{w}}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{L}_2 = \begin{bmatrix} -\bar{u} + \frac{2i\alpha}{Re} & 0 & 0 & -1 \\ 0 & -\bar{u} + \frac{2i\alpha}{Re} & 0 & 0 \\ 0 & 0 & -\bar{u} + \frac{2i\alpha}{Re} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathcal{L}_3 = \begin{bmatrix} \frac{1}{Re} & 0 & 0 & 0 \\ 0 & \frac{1}{Re} & 0 & 0 \\ 0 & 0 & \frac{1}{Re} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix $\mathcal{L}_0$ is the only matrix present for parallel flow triglobal analysis.

2.0.1 Normalization condition

Implicit in this derivation is that the disturbance takes the form of a rapidly varying phase function and a slowly varying shape function. Under this assumption second derivatives w.r.t x can be neglected. For these properties to be satisfied, and to resolve the ambiguity of representing the disturbance component as the product of two functions of x, a normalization condition restricting rapid streamwise changes must be applied. In a manner analogous to that used by Bertolotti[**?**] in their analysis of the two-dimensional flat-plate boundary layer, a suitable normalization condition is

$$\int\limits_{y_{min}}^{y_{max}} \int\limits_{z_{min}}^{z_{max}} \tilde{u}^* \frac{\partial \tilde{u}}{\partial x} dydz = \int\limits_{y_{min}}^{y_{max}} \int\limits_{z_{min}}^{z_{max}} \frac{1}{2} \frac{\partial}{\partial x} \left| \tilde{u} \right|^2 dydz = 0$$

* represents the complex conjugate.

2.1 Solution methodology for PSE

The solution to PSE can be done by treating x_0 as time and marching in x direction using Euler scheme as described below.

Equation to be solved is

$$(\mathcal{L}_0 + \mathcal{L}_1)\, \tilde{q} + \mathcal{L}_2 \frac{\partial \tilde{q}}{\partial x} + i \frac{d\alpha}{dx} \mathcal{L}_3 \tilde{q} = 0$$

with initial conditions,

$$\tilde{q}(x_0, y, z) = q_0(y, z), \qquad \alpha(x_0) = \alpha_0$$

and boundary conditions, (Not finding a way to write it better)

$$\tilde{u}(x, y_b, z_b) = \tilde{v}(x, y_b, z_b) = \tilde{w}(x, y_b, z_b) = 0 \qquad \forall \quad y_b \cup z_b \in boundaries$$

It can be identified that LHS of normalization condition is a inner product of $\tilde{\mathbf{u}}$ and $\frac{\partial \tilde{\mathbf{u}}}{\partial x}$[1]. Defining b as

$$b = \frac{< \tilde{\mathbf{u}}, \frac{\partial \tilde{\mathbf{u}}}{\partial x} >}{< \tilde{\mathbf{u}}, \tilde{\mathbf{u}} >}$$

From the definition b represent component of $\tilde{\mathbf{u}}_x$ along $\tilde{\mathbf{u}}$. Therefore we can write ($\frac{\partial \tilde{\mathbf{u}}}{\partial x}$ is written concisely as $\tilde{\mathbf{u}}_x$)

$$\tilde{\mathbf{u}}_x = b\tilde{\mathbf{u}} + r(x, y, z)$$

where $r(x, y, z)$ represent component of $\tilde{\mathbf{u}}_x$ orthogonal to $\tilde{\mathbf{u}}$.

Homogeneous part of above differential equation yield solution $\tilde{\mathbf{u}} \sim \exp(bx)$, which suggests $\tilde{\mathbf{u}}$ may vary rapidly if $b > 0$; contrary to assumption about $\tilde{\mathbf{u}}$.

Above orthogonal decomposition helps us understand normalization condition as follows. Given the values of α_j and $\tilde{\mathbf{u}}_j$, we solve equation (1.2.5) for $\tilde{\mathbf{u}}_{j+1}$. Now b_{j+1} can be obtained from definition of b as

$$b_{j+1} = \frac{< \tilde{\mathbf{u}}_j, \frac{\tilde{\mathbf{u}}_{j+1} - \tilde{\mathbf{u}}_j}{\triangle x} >}{< \tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_j >}$$

Now in case b_{j+1} is non-zero growth rate of $\tilde{\mathbf{u}}$ is of concern to us. Hence we update α_j as $\alpha_j^{(1)} = \alpha_j^{(0)} + b_{j+1}^{(0)}$. Bertolotti has shown in his paper[**?**] that such a procedure leads to reduction in updated value of b_{j+1}. We repeat this procedure unless b_{j+1} acquires a sufficiently small value within prescribed tolerances. Thus, normalization condition provides the unique value of $\alpha(x)$ that removes any exponential change in $\tilde{\mathbf{u}}$ measured by b. Then, $\tilde{\mathbf{u}}$ equals $r(x, y, z)$ that captures the slow, streamwise variation of $\tilde{\mathbf{u}}$. Once convergence for b is achieved we march to next step.

[1] Bold u represent vector velocity perturbation

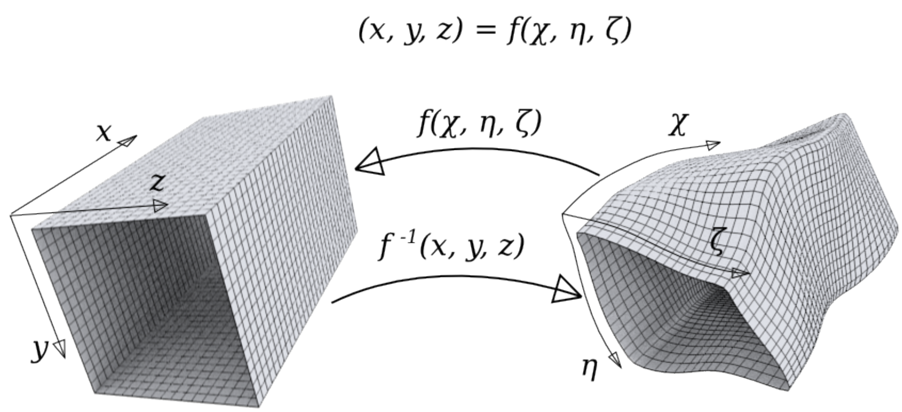
Chapter 3

Extension to generic geometries

The core code "pse" will be written to work on a coordinate system (x, y, z) wherein a very standard geometry is present as shown in the diagram below. x will be the streamwise direction. The y and z boundaries are restricted to $y = \pm 1$ and $z = \pm 1$. Any different geometry if required to be solved using "pse.c" first must be transformed to standard geometry using some transformation f given as

$$(x, y, z) = f(\chi, \eta, \zeta)$$

After transformation to standard geometry, the solution is done and result is transformed back to original geometry.



3.1 About pse

Bibliography

[1] Michael S. Broadhurst, Spence J. Sherwin, The Parabolised Stability Equations for 3D-Flows

[2] F. P. Bertolotti, Herbert and P. R. Spalart, Linear and nonlinear stability of the Blasius boundary layer