Biglobal Stability Analysis

Anil Kumar Singh

May 20, 2015

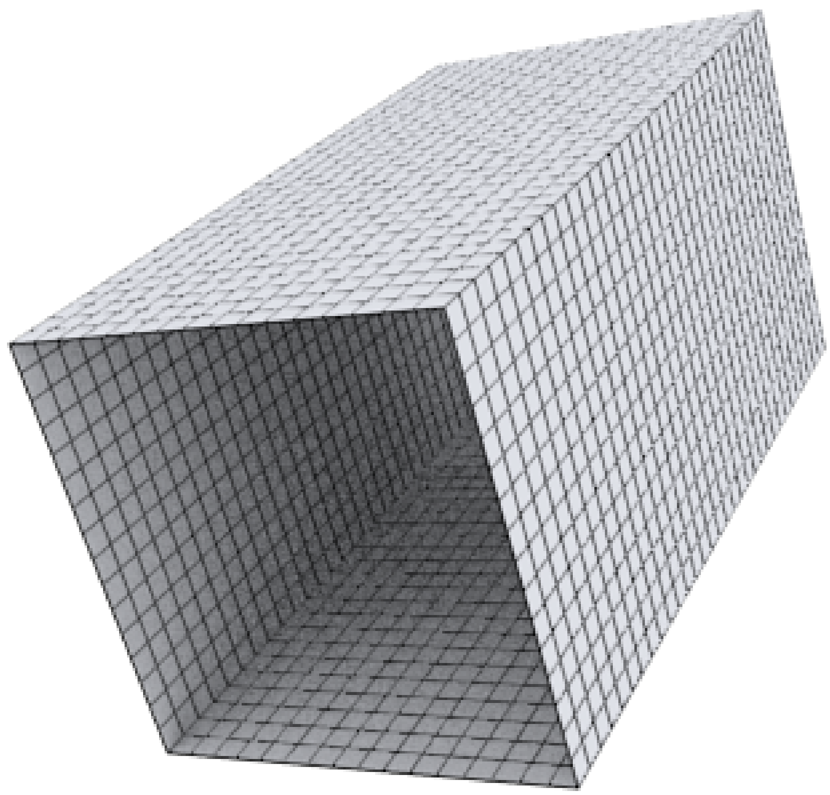
List of Figures

Contents

1 Discretization of space

We consider 2-d space to bre represented by $x-y$ cartesian coordinate system where $|x|=1$ and $|y|=1$ represent the boundaries. Let us have nx points along $x-$axis and ny points along $y-$axis. This forms the discretized space. Any physical quantity is represented as $q=q(x,y)$. On the discretized space such a quantity will be represented by its value at m^{th} point in x and n^{th} point in y. i.e. $q_{mn}=q(x_m,y_n)$ where $m\in\{0,1,2,...nx\}$ and $n\in\{0,1,2,...ny\}$. The values of q on entire grid can be arranged as a vector in orderly fashion as shown below. It is noted that first ny entries correspond to first x location; next ny entries to next x location and so on.

$$q = \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ . \\ . \\ . \\ q_{1(ny+1)} \\ q_{21} \\ q_{22} \\ . \\ . \\ . \\ q_{(nx+1)1} \\ q_{(nx+1)2} \\ . \\ . \\ . \\ q_{(nx+1)ny} \end{bmatrix}$$



2 Chebyshev Differentiation Matrices

$$D_{i,j} = \begin{cases} \frac{2N^2+1}{6} & i = j = 0 \\ -\frac{y_j}{2\sin^2\left(\frac{j\pi}{N}\right)} & i = j \neq 0, N \\ -\frac{c_i}{2c_j}\frac{(-1)^{i+j}}{\sin\left[\frac{(i+j)\pi}{2N}\right]\sin\left[\frac{(i-j)\pi}{2N}\right]} & i \neq j \\ -\frac{2N^2+1}{6} & i = j = N \end{cases}$$

where

$$c_j = \begin{cases} 2 & j = 0, N \\ 1 & 1 \leq j \leq N-1 \end{cases}$$

3 Governing Equations (Tensor)

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0$$

Momentum Equation

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \left(\mu_v + \frac{\mu}{3}\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

Energy Equation

$$\rho \frac{De}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu\left(S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)^2 + \mu_v\left(\frac{\partial u_j}{\partial x_j}\right)^2 + \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stokes' Assumption

$$\mu_v = \lambda + \frac{2}{3}\mu = 0$$

Hence momentum and energy equations are as follows

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

$$\rho \frac{De}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu\left(S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)^2 + \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

Consider term

$$\left(S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)^2$$

$$= S_{ij}S_{ij} + \frac{1}{9}\frac{\partial u_m}{\partial x_m}\frac{\partial u_m}{\partial x_m}\delta_{ij}\delta_{ij} - \frac{2}{3}\frac{\partial u_m}{\partial x_m}S_{ij}\delta_{ij}$$

$$= S_{ij}S_{ij} + \frac{1}{3}\frac{\partial u_m}{\partial x_m}\frac{\partial u_m}{\partial x_m} - \frac{2}{3}\frac{\partial u_m}{\partial x_m}S_{ii}$$

$$= S_{ij}S_{ij} + \frac{1}{3}\frac{\partial u_m}{\partial x_m}\frac{\partial u_m}{\partial x_m} - \frac{2}{3}\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_i}$$

$$= S_{ij}S_{ij} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2$$

$$= \frac{1}{2}S_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2$$

$$= \frac{1}{2}S_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}S_{ji}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2$$

Since S_{ij} is symmetric tensor

$$= S_{ij}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2$$

Hence energy equation is

$$\rho\frac{De}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu\left[S_{ij}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2\right] + \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

For a Calorically perfect gas

$$\rho c_v\frac{DT}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu\left[S_{ij}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2\right] + \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

4 Non dimensional form of governing equations

Let us create non dimension quantities as written below

$$\bar{x}_i = \frac{x_i}{L_o}, \quad \bar{u}_i = \frac{u_i}{U_o}, \quad \bar{t} = \frac{U_o t}{L_o}$$

$$\bar{\rho} = \frac{\rho}{\rho_o}, \quad \bar{p} = \frac{p}{\rho_o R T_o}, \quad \bar{T} = \frac{T}{T_o},$$

Using in governing equations
Continuity Equation

$$\frac{\rho_o U_o}{L_o}\frac{\partial\bar{\rho}}{\partial\bar{t}} + \frac{\rho_o U_o}{L_o}\frac{\partial\left(\bar{\rho}\bar{u}_i\right)}{\partial\bar{x}_i} = 0$$

$$\frac{\partial\bar{\rho}}{\partial\bar{t}} + \frac{\partial\left(\bar{\rho}\bar{u}_i\right)}{\partial\bar{x}_i} = 0$$

Momentum Equation (Body forces left out)

$$\rho_o\bar{\rho}\frac{U_o^2}{L_o}\frac{D\bar{u}_i}{D\bar{t}} = -\frac{\rho_o R T_o}{L_o}\frac{\partial\bar{p}}{\partial\bar{x}_i} + \frac{\mu U_o}{L_o^2}\frac{\partial}{\partial\bar{x}_j}\left(\frac{\partial\bar{u}_i}{\partial\bar{x}_j}\right) + \frac{\mu}{3}\frac{U_o}{L_o^2}\frac{\partial}{\partial\bar{x}_i}\left(\frac{\partial\bar{u}_j}{\partial\bar{x}_j}\right)$$

$$\bar{\rho}\frac{D\bar{u}_i}{D\bar{t}} = -\frac{\gamma R T_o}{\gamma U_o^2}\frac{\partial\bar{p}}{\partial\bar{x}_i} + \frac{\mu}{\rho_o U_o L_o}\frac{\partial}{\partial\bar{x}_j}\left(\frac{\partial\bar{u}_i}{\partial\bar{x}_j}\right) + \frac{1}{3}\frac{\mu}{\rho_o U_o L_o}\frac{\partial}{\partial\bar{x}_i}\left(\frac{\partial\bar{u}_j}{\partial\bar{x}_j}\right)$$

Note that

$$Re = \frac{\rho_o U_o L_o}{\mu}, \qquad \frac{\gamma R T_o}{U_o^2} = \frac{1}{M_o^2}$$

Hence

$$\bar{\rho}\frac{D\bar{u}_i}{D\bar{t}} = -\frac{1}{\gamma M_o^2}\frac{\partial\bar{p}}{\partial\bar{x}_i} + \frac{1}{Re}\frac{\partial}{\partial\bar{x}_j}\left(\frac{\partial\bar{u}_i}{\partial\bar{x}_j}\right) + \frac{1}{3Re}\frac{\partial}{\partial\bar{x}_i}\left(\frac{\partial\bar{u}_j}{\partial\bar{x}_j}\right)$$

Energy Equation

$$\rho c_v \frac{DT}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu \left[S_{ij}\frac{\partial u_i}{\partial x_j} - \frac{1}{3}\left(\frac{\partial u_m}{\partial x_m}\right)^2 \right] + \frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

$$\frac{\rho_o c_v T_o U_o}{L_o}\bar{\rho}\left(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u}_j\frac{\partial \bar{T}}{\partial \bar{x}_j}\right) = -\frac{\rho_o R T_o U_o}{L_o}\bar{p}\frac{\partial \bar{u}_m}{\partial \bar{x}_m} + 2\frac{\mu U_o^2}{L_o^2}\left[\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right)\frac{\partial \bar{u}_i}{\partial \bar{x}_j} - \frac{1}{3}\left(\frac{\partial \bar{u}_m}{\partial \bar{x}_m}\right)^2\right] + \frac{T_o}{L_o^2}\frac{\partial}{\partial x_j}\left(K\frac{\partial T}{\partial x_j}\right)$$

Assuming thermal conductivity to be constant and using $c_v = \frac{R}{\gamma - 1}$

$$\bar{\rho}\left(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u}_j\frac{\partial \bar{T}}{\partial \bar{x}_j}\right) = -\left(\gamma - 1\right)\bar{p}\frac{\partial \bar{u}_m}{\partial \bar{x}_m} + \frac{M_o^2\gamma\left(\gamma - 1\right)}{Re}\left[\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right)\frac{\partial \bar{u}_i}{\partial \bar{x}_j} - \frac{2}{3}\left(\frac{\partial \bar{u}_m}{\partial \bar{x}_m}\right)^2\right] + \frac{\gamma}{Pr\,.\,Re}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_j}$$

Equation of state

$$p = \rho R T$$

$$\rho_o R T_o \bar{p} = \rho_o \bar{\rho} R T_o \bar{T}$$

$$\bar{p} = \bar{\rho}\bar{T}$$

5 Non dimensional governing equations (Vector)

5.1 Continuity

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial\left(\bar{\rho}\bar{u}\right)}{\partial \bar{x}} + \frac{\partial\left(\bar{\rho}\bar{v}\right)}{\partial \bar{y}} + \frac{\partial\left(\bar{\rho}\bar{w}\right)}{\partial \bar{z}} = 0$$

5.2 Momentum

$$\bar{\rho}\frac{D\bar{u}_i}{D\bar{t}} = -\frac{1}{\gamma M_o^2}\frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{1}{Re}\frac{\partial}{\partial \bar{x}_j}\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j}\right) + \frac{1}{3Re}\frac{\partial}{\partial \bar{x}_i}\left(\frac{\partial \bar{u}_j}{\partial \bar{x}_j}\right)$$

$x-$ momentum

$$\bar{\rho}\left(\frac{\partial}{\partial \bar{t}} + \bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}} + \bar{w}\frac{\partial}{\partial \bar{z}}\right)\bar{u} = -\frac{1}{\gamma M_o^2}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re}\left[\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right]\bar{u} + \frac{1}{3Re}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial^2 \bar{w}}{\partial \bar{x}\partial \bar{z}}\right]$$

$y-$ momentum

$$\bar{\rho}\left(\frac{\partial}{\partial \bar{t}} + \bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}} + \bar{w}\frac{\partial}{\partial \bar{z}}\right)\bar{v} = -\frac{1}{\gamma M_o^2}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re}\left[\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right]\bar{v} + \frac{1}{3Re}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}\partial \bar{z}}\right]$$

$z-$ momentum

$$\bar{\rho}\left(\frac{\partial}{\partial \bar{t}} + \bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}} + \bar{w}\frac{\partial}{\partial \bar{z}}\right)\bar{w} = -\frac{1}{\gamma M_o^2}\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{1}{Re}\left[\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right]\bar{w} + \frac{1}{3Re}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{z}} + \frac{\partial^2 \bar{v}}{\partial \bar{y}\partial \bar{z}} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right]$$

5.3 Energy

$$\bar{\rho}\left(\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}_j\frac{\partial \bar{T}}{\partial \bar{x}_j}\right)=-\left(\gamma-1\right)\bar{p}\frac{\partial \bar{u}_m}{\partial \bar{x}_m}+\frac{M_o^2\gamma\left(\gamma-1\right)}{Re}\left[\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j}+\frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right)\frac{\partial \bar{u}_i}{\partial \bar{x}_j}-\frac{2}{3}\left(\frac{\partial \bar{u}_m}{\partial \bar{x}_m}\right)^2\right]+\frac{\gamma}{Pr\,.\,Re}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_j}$$

$$\bar{\rho}\left(\frac{\partial}{\partial \bar{t}}+\bar{u}\frac{\partial}{\partial \bar{x}}+\bar{v}\frac{\partial}{\partial \bar{y}}+\bar{w}\frac{\partial}{\partial \bar{z}}\right)\bar{T}=-\left(\gamma-1\right)\bar{p}\left(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}\right)+\frac{M_o^2\gamma\left(\gamma-1\right)}{Re}\text{Term}+\frac{\gamma}{Pr\,.\,Re}\left[\frac{\partial^2 \bar{T}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right]$$

Consider the term

$$\text{Term}=\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j}+\frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right)\frac{\partial \bar{u}_i}{\partial \bar{x}_j}-\frac{2}{3}\left(\frac{\partial \bar{u}_m}{\partial \bar{x}_m}\right)^2$$

$$\begin{aligned}\frac{4}{3}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{z}}\right)+\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2+\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2+\left(\frac{\partial \bar{v}}{\partial \bar{x}}\right)^2+\left(\frac{\partial \bar{v}}{\partial \bar{z}}\right)^2+\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2+\left(\frac{\partial \bar{w}}{\partial \bar{y}}\right)^2\\ +\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \bar{u}}{\partial \bar{z}}+\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{x}}+\frac{\partial \bar{w}}{\partial \bar{y}}\frac{\partial \bar{v}}{\partial \bar{z}}+\frac{\partial \bar{u}}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial \bar{y}}\\ -\frac{4}{3}\left[\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{w}}{\partial \bar{z}}+\frac{\partial \bar{v}}{\partial \bar{y}}\frac{\partial \bar{w}}{\partial \bar{z}}\right]\end{aligned}$$

6 Governing Equation (low mach no. limit)

We expand flow parameters as a function of asymptotic parameter M_o i.e.

$$\bar{u}(x,y,t)=\bar{u}_o+M_o\bar{u}_1+M_o^2\bar{u}_2+...$$

$$\bar{v}(x,y,t)=\bar{v}_o+M_o\bar{v}_1+M_o^2\bar{v}_2+...$$

$$\bar{w}(x,y,t)=\bar{w}_o+M_o\bar{w}_1+M_o^2\bar{w}_2+...$$

$$\bar{p}(x,y,t)=\bar{p}_o+M_o\bar{p}_1+M_o^2\bar{p}_2+...$$

$$\bar{\rho}(x,y,t)=\bar{\rho}_o+M_o\bar{\rho}_1+M_o^2\bar{\rho}_2+...$$

$$\bar{T}(x,y,t)=\bar{T}_o+M_o\bar{T}_1+M_o^2\bar{T}_2+...$$

6.1 Continuity

$$\frac{\partial \bar{\rho}}{\partial \bar{t}}+\frac{\partial\left(\bar{\rho}\bar{u}\right)}{\partial \bar{x}}+\frac{\partial\left(\bar{\rho}\bar{v}\right)}{\partial \bar{y}}+\frac{\partial\left(\bar{\rho}\bar{w}\right)}{\partial \bar{z}}=0$$

$$\frac{\partial \bar{\rho}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{\rho}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{\rho}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{\rho}}{\partial \bar{z}}+\bar{\rho}\left(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}\right)=0$$

$$\frac{\partial \bar{\rho}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{\rho}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{\rho}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{\rho}}{\partial \bar{z}}+\bar{\rho}\left(\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}\right)=0$$

Collecting the First order terms

$$\frac{\partial \bar{\rho}_o}{\partial \bar{t}}+\bar{u}_o\frac{\partial \bar{\rho}_o}{\partial \bar{x}}+\bar{v}_o\frac{\partial \bar{\rho}_o}{\partial \bar{y}}+\bar{w}_o\frac{\partial \bar{\rho}_o}{\partial \bar{z}}+\bar{\rho}_o\left(\frac{\partial \bar{u}_o}{\partial \bar{x}}+\frac{\partial \bar{v}_o}{\partial \bar{y}}+\frac{\partial \bar{w}_o}{\partial \bar{z}}\right)=0$$

6.2 Momentum

6.2.1 x-Momentum Equation

$$\bar{\rho}\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}+\bar{w}\frac{\partial}{\partial\bar{z}}\right)\bar{u}=-\frac{1}{\gamma M_o^2}\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{u}+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\partial^2\bar{v}}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{z}}\right]$$

Collecting highest order terms (i.e. coefficients of M_o^{-2})

$$\frac{\partial\bar{p}_o}{\partial\bar{x}}=0$$

Collecting coefficients of M_o^{-1}

$$\frac{\partial\bar{p}_1}{\partial\bar{x}}=0$$

Collecting coefficients of M_o^0

$$\bar{\rho}_o\left(\frac{\partial}{\partial\bar{t}}+\bar{u}_o\frac{\partial}{\partial\bar{x}}+\bar{v}_o\frac{\partial}{\partial\bar{y}}+\bar{w}_o\frac{\partial}{\partial\bar{z}}\right)\bar{u}_o=-\frac{1}{\gamma}\frac{\partial\bar{p}_2}{\partial\bar{x}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{u}_o+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}_o}{\partial\bar{x}^2}+\frac{\partial^2\bar{v}_o}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{w}_o}{\partial\bar{x}\partial\bar{z}}\right]$$

6.2.2 y-Momentum Equation

$$\bar{\rho}\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}+\bar{w}\frac{\partial}{\partial\bar{z}}\right)\bar{v}=-\frac{1}{\gamma M_o^2}\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{v}+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{v}}{\partial\bar{y}^2}+\frac{\partial^2\bar{w}}{\partial\bar{y}\partial\bar{z}}\right]$$

Collecting highest order terms (i.e. coefficients of M_o^{-2})

$$\frac{\partial\bar{p}_o}{\partial\bar{y}}=0$$

Collecting coefficients of M_o^{-1}

$$\frac{\partial\bar{p}_1}{\partial\bar{y}}=0$$

Collecting coefficients of M_o^0

$$\bar{\rho}_o\left(\frac{\partial}{\partial\bar{t}}+\bar{u}_o\frac{\partial}{\partial\bar{x}}+\bar{v}_o\frac{\partial}{\partial\bar{y}}+\bar{w}_o\frac{\partial}{\partial\bar{z}}\right)\bar{v}_o=-\frac{1}{\gamma}\frac{\partial\bar{p}_2}{\partial\bar{y}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{v}_o+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}_o}{\partial\bar{x}\partial\bar{y}}+\frac{\partial^2\bar{v}_o}{\partial\bar{y}^2}+\frac{\partial^2\bar{w}_o}{\partial\bar{y}\partial\bar{z}}\right]$$

6.2.3 z-Momentum Equation

$$\bar{\rho}\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}+\bar{w}\frac{\partial}{\partial\bar{z}}\right)\bar{w}=-\frac{1}{\gamma M_o^2}\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{w}+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}}{\partial\bar{x}\partial\bar{z}}+\frac{\partial^2\bar{v}}{\partial\bar{y}\partial\bar{z}}+\frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right]$$

Collecting highest order terms (i.e. coefficients of M_o^{-2})

$$\frac{\partial\bar{p}_o}{\partial\bar{z}}=0$$

Collecting coefficients of M_o^{-1}

$$\frac{\partial\bar{p}_1}{\partial\bar{z}}=0$$

Collecting coefficients of M_o^0

$$\bar{\rho}_o\left(\frac{\partial}{\partial\bar{t}}+\bar{u}_o\frac{\partial}{\partial\bar{x}}+\bar{v}_o\frac{\partial}{\partial\bar{y}}+\bar{w}_o\frac{\partial}{\partial\bar{z}}\right)\bar{w}_o=-\frac{1}{\gamma}\frac{\partial\bar{p}_2}{\partial\bar{z}}+\frac{1}{Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{w}_o+\frac{1}{3Re}\left[\frac{\partial^2\bar{u}_o}{\partial\bar{x}\partial\bar{z}}+\frac{\partial^2\bar{v}_o}{\partial\bar{y}\partial\bar{z}}+\frac{\partial^2\bar{w}_o}{\partial\bar{z}^2}\right]$$

From above equations, it is concluded that $\bar{p}_o$ and $\bar{p}_1$ are constants.

6.3 Energy

$$\bar{\rho}\left(\frac{\partial}{\partial\bar{t}}+\bar{u}\frac{\partial}{\partial\bar{x}}+\bar{v}\frac{\partial}{\partial\bar{y}}+\bar{w}\frac{\partial}{\partial\bar{z}}\right)\bar{T}=-\left(\gamma-1\right)\bar{p}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)+\frac{M_o^2\gamma\left(\gamma-1\right)}{Re}\text{Term}+\frac{\gamma}{Pr\,.\,Re}\left[\frac{\partial^2\bar{T}}{\partial\bar{x}^2}+\frac{\partial^2\bar{T}}{\partial\bar{y}^2}+\frac{\partial^2\bar{T}}{\partial\bar{z}^2}\right]$$

Collecting the highest order terms i.e. coefficients of M_o^0

$$\bar{\rho}_o\left(\frac{\partial}{\partial\bar{t}}+\bar{u}_o\frac{\partial}{\partial\bar{x}}+\bar{v}_o\frac{\partial}{\partial\bar{y}}+\bar{w}_o\frac{\partial}{\partial\bar{z}}\right)\bar{T}_o=-\left(\gamma-1\right)\bar{p}_o\left(\frac{\partial\bar{u}_o}{\partial\bar{x}}+\frac{\partial\bar{v}_o}{\partial\bar{y}}+\frac{\partial\bar{w}_o}{\partial\bar{z}}\right)+\frac{\gamma}{Pr\,.\,Re}\left[\frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right]\bar{T}_o$$

6.4 Equation of State

$$\bar{p}=\bar{\rho}\bar{T}$$

Collecting the highest order terms

$$\bar{p}_o=\bar{\rho}_o\bar{T}_o$$

Substituting $\bar{T}_o$ from above equation in energy equation

$$\bar{\rho}_o\left(\frac{\partial\bar{T}_o}{\partial\bar{t}}+\bar{u}_o\frac{\partial\bar{T}_o}{\partial\bar{x}}+\bar{v}_o\frac{\partial\bar{T}_o}{\partial\bar{y}}+\bar{w}_o\frac{\partial\bar{T}_o}{\partial\bar{z}}\right)=-\left(\gamma-1\right)\bar{p}_o\left(\frac{\partial\bar{u}_o}{\partial\bar{x}}+\frac{\partial\bar{v}_o}{\partial\bar{y}}+\frac{\partial\bar{w}_o}{\partial\bar{z}}\right)+\frac{\gamma}{Pr\,.\,Re}\left[\frac{\partial^2\bar{T}_o}{\partial\bar{x}^2}+\frac{\partial^2\bar{T}_o}{\partial\bar{y}^2}+\frac{\partial^2\bar{T}_o}{\partial\bar{z}^2}\right]$$

Consider LHS

$$\begin{aligned}\text{LHS}&=\bar{\rho}_o\left(\frac{\partial\bar{T}_o}{\partial\bar{t}}+\bar{u}_o\frac{\partial\bar{T}_o}{\partial\bar{x}}+\bar{v}_o\frac{\partial\bar{T}_o}{\partial\bar{y}}+\bar{w}_o\frac{\partial\bar{T}_o}{\partial\bar{z}}\right)\\&=\bar{\rho}_o\left(\frac{\partial\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{t}}+\bar{u}_o\frac{\partial\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{x}}+\bar{v}_o\frac{\partial\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{y}}+\bar{w}_o\frac{\partial\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{z}}\right)\\&=\left(\frac{\partial\bar{p}_o}{\partial\bar{t}}-\frac{\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{t}}-\frac{\bar{u}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{x}}-\frac{\bar{v}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{y}}-\frac{\bar{w}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{z}}\right)\end{aligned}$$

Assuming that $\bar{p}_o$ is not a function of time LHS is

$$=\left(-\frac{\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{t}}-\frac{\bar{u}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{x}}-\frac{\bar{v}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{y}}-\frac{\bar{w}_o\bar{p}_o}{\bar{\rho}_o}\frac{\partial\bar{\rho}_o}{\partial\bar{z}}\right)$$

Consider last term of RHS

$$\begin{aligned}\text{RHS}&=\frac{\gamma}{Pr\,.\,Re}\left[\frac{\partial^2\bar{T}_o}{\partial\bar{x}^2}+\frac{\partial^2\bar{T}_o}{\partial\bar{y}^2}+\frac{\partial^2\bar{T}_o}{\partial\bar{z}^2}\right]\\&=\frac{\gamma}{Pr\,.\,Re}\left[\bar{p}_o\frac{\partial}{\partial\bar{x}}\left(-\frac{1}{\bar{\rho}_o^2}\frac{\partial\bar{\rho}_o}{\partial\bar{x}}\right)+\frac{\partial^2\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{y}^2}+\frac{\partial^2\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{z}^2}\right]\\&=\frac{\gamma}{Pr\,.\,Re}\left[\bar{p}_o\left(-\frac{1}{\bar{\rho}_o^2}\frac{\partial^2\bar{\rho}_o}{\partial\bar{x}^2}+\frac{2}{\bar{\rho}_o^3}\left(\frac{\partial\bar{\rho}_o}{\partial\bar{x}}\right)^2\right)+\frac{\partial^2\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{y}^2}+\frac{\partial^2\left(\frac{\bar{p}_o}{\bar{\rho}_o}\right)}{\partial\bar{z}^2}\right]\end{aligned}$$

Hence energy equation is

$$\begin{aligned}\bar{\rho}_o^2\left(\frac{\partial\bar{\rho}_o}{\partial\bar{t}}+\bar{u}_o\frac{\partial\bar{\rho}_o}{\partial\bar{x}}+\bar{v}_o\frac{\partial\bar{\rho}_o}{\partial\bar{y}}+\bar{w}_o\frac{\partial\bar{\rho}_o}{\partial\bar{z}}\right)&=\left(\gamma-1\right)\bar{\rho}_o^3\left(\frac{\partial\bar{u}_o}{\partial\bar{x}}+\frac{\partial\bar{v}_o}{\partial\bar{y}}+\frac{\partial\bar{w}_o}{\partial\bar{z}}\right)\\&\quad-\frac{\gamma}{Pr\,.\,Re}\left[2\left\{\left(\frac{\partial\bar{\rho}_o}{\partial\bar{x}}\right)^2+\left(\frac{\partial\bar{\rho}_o}{\partial\bar{y}}\right)^2+\left(\frac{\partial\bar{\rho}_o}{\partial\bar{z}}\right)^2\right\}-\bar{\rho}_o\left\{\frac{\partial^2\bar{\rho}_o}{\partial\bar{x}^2}+\frac{\partial^2\bar{\rho}_o}{\partial\bar{y}^2}+\frac{\partial^2\bar{\rho}_o}{\partial\bar{z}^2}\right\}\right]\end{aligned}$$

7 Biglobal Stability Equations

We now drop superscripts, subscripts and bars from variables so that equations look easier to read. (i.e. we replace $\bar{q}_o$ by q ; $\bar{p}_2$ by p)

Flow variables are written as vector $q = [u \quad v \quad w \quad \rho \quad p]$.Base flow variables $\bar{q}(x,y)$ are functions of two spatial coordinates only. Instantaneous flow variables are function of three spatial directions and time i.e. $q = q(x,y,z,t)$

Now we write instantaneous quantities as

$$q(x,y,z,t) = \bar{q}(x,y) + q'(x,y,z,t)$$

where q' is perturbation. Perturbations are assumed to be of form

$$q'(x,y,z,t) = \tilde{q}(x,y)\exp\left[i\left(\beta z - \omega t\right)\right]$$

7.1 Continuity Equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Subtracting the base flow equation from instantaneous flow

$$\frac{\partial \rho'}{\partial t} + u'\frac{\partial \bar{\rho}}{\partial x} + v'\frac{\partial \bar{\rho}}{\partial y} + w'\frac{\partial \bar{\rho}}{\partial z} + \bar{u}\frac{\partial \rho'}{\partial x} + \bar{v}\frac{\partial \rho'}{\partial y} + \bar{w}\frac{\partial \rho'}{\partial z} + \bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + \rho'\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) = 0$$

$$-i\omega\tilde{\rho} + \tilde{u}\frac{\partial \bar{\rho}}{\partial x} + \tilde{v}\frac{\partial \bar{\rho}}{\partial y} + \bar{u}\frac{\partial \tilde{\rho}}{\partial x} + \bar{v}\frac{\partial \tilde{\rho}}{\partial y} + i\beta\bar{w}\tilde{\rho} + \bar{\rho}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\beta\tilde{w}\right) + \tilde{\rho}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0$$

$$\left(-i\omega + \bar{u}\mathcal{D}_x + \bar{v}\mathcal{D}_y + i\beta\bar{w} + \bar{u}_x + \bar{v}_y\right)\tilde{\rho} + \left(\bar{\rho}_x + \bar{\rho}\mathcal{D}_x\right)\tilde{u} + \left(\bar{\rho}_y + \bar{\rho}\mathcal{D}_y\right)\tilde{v} + i\beta\bar{\rho}\tilde{w} = 0$$

7.2 x-momentum Equation

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = -\frac{1}{\gamma}\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u + \frac{1}{3Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right]$$

Subtracting the base flow equation from instantaneous flow

$$\begin{aligned}\rho'\frac{\partial \bar{u}}{\partial t} + \bar{\rho}\frac{\partial u'}{\partial t} + \bar{\rho}\bar{u}\frac{\partial u'}{\partial x} + \bar{\rho}u'\frac{\partial \bar{u}}{\partial x} + \rho'\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial u'}{\partial y} + \bar{\rho}v'\frac{\partial \bar{u}}{\partial y} + \rho'\bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{\rho}\bar{w}\frac{\partial u'}{\partial z} + \bar{\rho}w'\frac{\partial \bar{u}}{\partial z} + \rho'\bar{w}\frac{\partial \bar{u}}{\partial z}\\ = -\frac{1}{\gamma}\frac{\partial p'}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u' + \frac{1}{3Re}\left[\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial^2 w'}{\partial x \partial z}\right]\end{aligned}$$

$$\begin{aligned}-i\omega\tilde{u}\bar{\rho} + \bar{\rho}\bar{u}\frac{\partial \tilde{u}}{\partial x} + \bar{\rho}\tilde{u}\frac{\partial \bar{u}}{\partial x} + \tilde{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \tilde{u}}{\partial y} + \bar{\rho}\tilde{v}\frac{\partial \bar{u}}{\partial y} + \tilde{\rho}\bar{v}\frac{\partial \bar{u}}{\partial y} + i\beta\bar{\rho}\bar{w}\tilde{u}\\ = -\frac{1}{\gamma}\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2\right]\tilde{u} + \frac{1}{3Re}\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial x \partial y} + i\beta\frac{\partial \tilde{w}}{\partial x}\right]\end{aligned}$$

$$\begin{aligned}\tilde{\rho}\bar{u}\frac{\partial \bar{u}}{\partial x} + \tilde{\rho}\bar{v}\frac{\partial \bar{u}}{\partial y} - i\omega\tilde{u}\bar{\rho} + \bar{\rho}\bar{u}\frac{\partial \tilde{u}}{\partial x} + \bar{\rho}\tilde{u}\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\bar{v}\frac{\partial \tilde{u}}{\partial y} + i\beta\bar{\rho}\bar{w}\tilde{u} + \bar{\rho}\tilde{v}\frac{\partial \bar{u}}{\partial y}\\ -\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2\right]\tilde{u} - \frac{1}{3Re}\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial x \partial y} + i\beta\frac{\partial \tilde{w}}{\partial x}\right] + \frac{1}{\gamma}\frac{\partial \tilde{p}}{\partial x} = 0\end{aligned}$$

7.3 y-momentum Equation

$$\rho\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)v=-\frac{1}{\gamma}\frac{\partial p}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]v+\frac{1}{3Re}\left[\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right]$$

Subtracting the base flow equation from instantaneous flow

$$\begin{aligned}\bar{\rho}\frac{\partial v'}{\partial t}+\bar{\rho}\bar{u}\frac{\partial v'}{\partial x}+\bar{\rho}u'\frac{\partial\bar{v}}{\partial x}+\rho'\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial v'}{\partial y}+\bar{\rho}v'\frac{\partial\bar{v}}{\partial y}+\rho'\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{\rho}\bar{w}\frac{\partial v'}{\partial z}+\bar{\rho}w'\frac{\partial\bar{v}}{\partial z}+\rho'\bar{w}\frac{\partial\bar{v}}{\partial z}\\=-\frac{1}{\gamma}\frac{\partial p'}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]v'+\frac{1}{3Re}\left[\frac{\partial^2 u'}{\partial x\partial y}+\frac{\partial^2 v'}{\partial y^2}+\frac{\partial^2 w'}{\partial y\partial z}\right]\end{aligned}$$

$$\begin{aligned}-i\omega\bar{\rho}\tilde{v}+\bar{\rho}\bar{u}\frac{\partial\tilde{v}}{\partial x}+\bar{\rho}\tilde{u}\frac{\partial\bar{v}}{\partial x}+\tilde{\rho}\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial\tilde{v}}{\partial y}+\bar{\rho}\tilde{v}\frac{\partial\bar{v}}{\partial y}+\tilde{\rho}\bar{v}\frac{\partial\bar{v}}{\partial y}+i\beta\bar{\rho}\bar{w}\tilde{v}\\-\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2\right]\tilde{v}-\frac{1}{3Re}\left[\frac{\partial^2\tilde{u}}{\partial x\partial y}+\frac{\partial^2\tilde{v}}{\partial y^2}+i\beta\frac{\partial\tilde{w}}{\partial y}\right]+\frac{1}{\gamma}\frac{\partial\tilde{p}}{\partial y}=0\end{aligned}$$

$$\begin{aligned}\tilde{\rho}\bar{v}\frac{\partial\bar{v}}{\partial y}+\tilde{\rho}\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{\rho}\tilde{u}\frac{\partial\bar{v}}{\partial x}-i\omega\bar{\rho}\tilde{v}+\bar{\rho}\bar{u}\frac{\partial\tilde{v}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial\tilde{v}}{\partial y}+\bar{\rho}\tilde{v}\frac{\partial\bar{v}}{\partial y}+i\beta\bar{\rho}\bar{w}\tilde{v}\\-\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2\right]\tilde{v}-\frac{1}{3Re}\left[\frac{\partial^2\tilde{u}}{\partial x\partial y}+\frac{\partial^2\tilde{v}}{\partial y^2}+i\beta\frac{\partial\tilde{w}}{\partial y}\right]+\frac{1}{\gamma}\frac{\partial\tilde{p}}{\partial y}=0\end{aligned}$$

7.4 z-momentum Equation

$$\rho\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)w=-\frac{1}{\gamma}\frac{\partial p}{\partial z}+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]w+\frac{1}{3Re}\left[\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right]$$

Subtracting the base flow equation from instantaneous flow

$$\begin{aligned}\bar{\rho}\frac{\partial w'}{\partial t}+\bar{\rho}\bar{u}\frac{\partial w'}{\partial x}+\bar{\rho}u'\frac{\partial\bar{w}}{\partial x}+\rho'\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial w'}{\partial y}+\bar{\rho}v'\frac{\partial\bar{w}}{\partial y}+\rho'\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{\rho}\bar{w}\frac{\partial w'}{\partial z}+\bar{\rho}w'\frac{\partial\bar{w}}{\partial z}+\rho'\bar{w}\frac{\partial\bar{w}}{\partial z}\\=-\frac{1}{\gamma}\frac{\partial p'}{\partial z}+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]w'+\frac{1}{3Re}\left[\frac{\partial^2 u'}{\partial x\partial z}+\frac{\partial^2 v'}{\partial y\partial z}+\frac{\partial^2 w'}{\partial z^2}\right]\end{aligned}$$

$$\begin{aligned}-i\omega\bar{\rho}\tilde{w}+\bar{\rho}\bar{u}\frac{\partial\tilde{w}}{\partial x}+\bar{\rho}\tilde{u}\frac{\partial\bar{w}}{\partial x}+\tilde{\rho}\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial\tilde{w}}{\partial y}+\bar{\rho}\tilde{v}\frac{\partial\bar{w}}{\partial y}+\tilde{\rho}\bar{v}\frac{\partial\bar{w}}{\partial y}+i\beta\bar{\rho}\bar{w}\tilde{w}\\-\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2\right]\tilde{w}-\frac{1}{3Re}\left[i\beta\frac{\partial\tilde{u}}{\partial x}+i\beta\frac{\partial\tilde{v}}{\partial y}-\beta^2\tilde{w}\right]+\frac{1}{\gamma}\frac{\partial\tilde{p}}{\partial z}=0\end{aligned}$$

$$\begin{aligned}\tilde{\rho}\bar{u}\frac{\partial\bar{w}}{\partial x}+\tilde{\rho}\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{\rho}\tilde{u}\frac{\partial\bar{w}}{\partial x}+\bar{\rho}\tilde{v}\frac{\partial\bar{w}}{\partial y}-i\omega\bar{\rho}\tilde{w}+\bar{\rho}\bar{u}\frac{\partial\tilde{w}}{\partial x}+\bar{\rho}\bar{v}\frac{\partial\tilde{w}}{\partial y}+i\beta\bar{\rho}\bar{w}\tilde{w}\\-\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\beta^2\right]\tilde{w}-\frac{1}{3Re}\left[i\beta\frac{\partial\tilde{u}}{\partial x}+i\beta\frac{\partial\tilde{v}}{\partial y}-\beta^2\tilde{w}\right]+\frac{i\beta}{\gamma}\tilde{p}=0\end{aligned}$$

7.5 Energy Equation

$$\begin{aligned}\rho^2\left(\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right) &= (\gamma-1)\,\rho^3\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\&-\frac{\gamma}{Pr\,.\,Re}\left[2\left\{\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2+\left(\frac{\partial\rho}{\partial z}\right)^2\right\}-\rho\left\{\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2}\right\}\right]\end{aligned}$$

Subtracting the base flow equation from instantaneous flow

$$\bar{\rho}^2\frac{\partial\rho'}{\partial t}+\bar{\rho}^2\bar{u}\frac{\partial\rho'}{\partial x}+\bar{\rho}^2u'\frac{\partial\bar{\rho}}{\partial x}+2\bar{\rho}\rho'\bar{u}\frac{\partial\bar{\rho}}{\partial x}+\bar{\rho}^2\bar{v}\frac{\partial\rho'}{\partial y}+\bar{\rho}^2v'\frac{\partial\bar{\rho}}{\partial y}+2\bar{\rho}\rho'\bar{v}\frac{\partial\bar{\rho}}{\partial y}+\bar{\rho}^2\bar{w}\frac{\partial\rho'}{\partial z}+\bar{\rho}^2w'\frac{\partial\bar{\rho}}{\partial z}+2\bar{\rho}\rho'\bar{w}\frac{\partial\bar{\rho}}{\partial z}$$
$$=(\gamma-1)\left\{\left(\bar{\rho}^3\frac{\partial u'}{\partial x}+3\bar{\rho}^2\rho'\frac{\partial\bar{u}}{\partial x}\right)+\left(\bar{\rho}^3\frac{\partial v'}{\partial y}+3\bar{\rho}^2\rho'\frac{\partial\bar{v}}{\partial y}\right)+\left(\bar{\rho}^3\frac{\partial w'}{\partial z}+3\bar{\rho}^2\rho'\frac{\partial\bar{w}}{\partial z}\right)\right\}$$
$$-\frac{\gamma}{Pr\,.\,Re}\left[2\left\{\frac{\partial\bar{\rho}}{\partial x}\frac{\partial\rho'}{\partial x}+\frac{\partial\bar{\rho}}{\partial y}\frac{\partial\rho'}{\partial y}+\frac{\partial\bar{\rho}}{\partial z}\frac{\partial\rho'}{\partial z}\right\}-\left\{\bar{\rho}\frac{\partial^2\rho'}{\partial x^2}+\rho'\frac{\partial^2\bar{\rho}}{\partial x^2}+\bar{\rho}\frac{\partial^2\rho'}{\partial y^2}+\rho'\frac{\partial^2\bar{\rho}}{\partial y^2}+\bar{\rho}\frac{\partial^2\rho'}{\partial z^2}+\rho'\frac{\partial^2\bar{\rho}}{\partial z^2}\right\}\right]$$

$$\Rightarrow -i\omega\bar{\rho}^2\tilde{\rho}+i\beta\bar{\rho}^2\bar{w}\tilde{\rho}+\bar{\rho}^2\bar{u}\frac{\partial\tilde{\rho}}{\partial x}+\bar{\rho}^2\bar{v}\frac{\partial\tilde{\rho}}{\partial y}+2\bar{\rho}\tilde{\rho}\bar{u}\frac{\partial\bar{\rho}}{\partial x}+2\bar{\rho}\tilde{\rho}\bar{v}\frac{\partial\bar{\rho}}{\partial y}+\bar{\rho}^2\tilde{u}\frac{\partial\bar{\rho}}{\partial x}+\bar{\rho}^2\tilde{v}\frac{\partial\bar{\rho}}{\partial y}$$
$$=(\gamma-1)\left\{\left(\bar{\rho}^3\frac{\partial\tilde{u}}{\partial x}+3\bar{\rho}^2\tilde{\rho}\frac{\partial\bar{u}}{\partial x}\right)+\left(\bar{\rho}^3\frac{\partial\tilde{v}}{\partial y}+3\bar{\rho}^2\tilde{\rho}\frac{\partial\bar{v}}{\partial y}\right)+i\beta\bar{\rho}^3\tilde{w}\right\}$$
$$-\frac{\gamma}{Pr\,.\,Re}\left[2\left\{\frac{\partial\bar{\rho}}{\partial x}\frac{\partial\tilde{\rho}}{\partial x}+\frac{\partial\bar{\rho}}{\partial y}\frac{\partial\tilde{\rho}}{\partial y}\right\}-\left\{\bar{\rho}\frac{\partial^2\tilde{\rho}}{\partial x^2}+\tilde{\rho}\frac{\partial^2\bar{\rho}}{\partial x^2}+\bar{\rho}\frac{\partial^2\tilde{\rho}}{\partial y^2}+\tilde{\rho}\frac{\partial^2\bar{\rho}}{\partial y^2}-\beta^2\bar{\rho}\tilde{\rho}\right\}\right]$$

Now the system of five equations can be written together in a matrix form as following

$$Aq=\omega Bq$$

$$A=\begin{pmatrix} \mathcal{L}_{2D}+\bar{\rho}\bar{u}_x-\frac{1}{3Re}\mathcal{D}_{xx} & \bar{\rho}\bar{u}_y-\frac{1}{3Re}\mathcal{D}_{xy} & -\frac{i\beta}{3Re}\mathcal{D}_x & \bar{u}\bar{u}_x+\bar{v}\bar{u}_y & \frac{1}{\gamma}\mathcal{D}_x \\ \bar{\rho}\bar{v}_x-\frac{1}{3Re}\mathcal{D}_{xy} & \mathcal{L}_{2D}+\bar{\rho}\bar{v}_y-\frac{1}{3Re}\mathcal{D}_{yy} & -\frac{i\beta}{3Re}\mathcal{D}_y & \bar{u}\bar{v}_x+\bar{v}\bar{v}_y & \frac{1}{\gamma}\mathcal{D}_y \\ \bar{\rho}\bar{w}_x-\frac{i\beta}{3Re}\mathcal{D}_x & \bar{\rho}\bar{w}_y-\frac{i\beta}{3Re}\mathcal{D}_y & \mathcal{L}_{2D}+\frac{\beta^2}{3Re} & \bar{u}\bar{w}_x+\bar{v}\bar{w}_y & \frac{i\beta}{\gamma} \\ \bar{\rho}\bar{\rho}_x+\bar{\rho}^2\mathcal{D}_x & \bar{\rho}\bar{\rho}_y+\bar{\rho}^2\mathcal{D}_y & i\beta\bar{\rho}^2 & \bar{\rho}\bar{u}\mathcal{D}_x+\bar{\rho}\bar{v}\mathcal{D}_y+i\beta\bar{\rho}\bar{w}+\bar{\rho}\bar{u}_x+\bar{\rho}\bar{v}_y & 0 \\ \bar{\rho}\bar{\rho}_x-(\gamma-1)\,\bar{\rho}^2\mathcal{D}_x & \bar{\rho}\bar{\rho}_y-(\gamma-1)\,\bar{\rho}^2\mathcal{D}_y & -i\beta\,(\gamma-1)\,\bar{\rho}^2 & \mathcal{M} & 0 \end{pmatrix}$$

where $\mathcal{L}_{2D}=\bar{\rho}\bar{u}\mathcal{D}_x+\bar{\rho}\bar{v}\mathcal{D}_y+i\beta\bar{\rho}\bar{w}-\frac{1}{Re}\left[\mathcal{D}_{xx}+\mathcal{D}_{yy}-\beta^2\right]$,
$\mathcal{M}=\bar{\rho}\bar{u}\mathcal{D}_x+\bar{\rho}\bar{v}\mathcal{D}_y+i\beta\bar{\rho}\bar{w}+2\bar{\rho}_x\bar{u}+2\bar{\rho}_y\bar{v}-3\,(\gamma-1)\,(\bar{\rho}\bar{u}_x+\bar{\rho}\bar{v}_y)+\frac{\gamma}{Pr\,.Re}\left[2\left(\frac{\bar{\rho}_x}{\bar{\rho}}\mathcal{D}_x+\frac{\bar{\rho}_y}{\bar{\rho}}\mathcal{D}_y\right)-\left(\mathcal{D}_{xx}+\mathcal{D}_{yy}-\beta^2+\frac{\bar{\rho}_{xx}}{\bar{\rho}}+\frac{\bar{\rho}_{yy}}{\bar{\rho}}\right)\right]$
and

$$B=\begin{pmatrix} i\bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & i\bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & i\bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & i\bar{\rho} & 0 \\ 0 & 0 & 0 & i\bar{\rho} & 0 \end{pmatrix}$$

References

[1] @misc{cminpack, title={C/C++ Minpack}, author={Devernay, Fr{\'e}d{\'e}ric}, year={2007}, howpublished = "\url{http://devernay.free.fr/hacks/cminpack/}", }