2-D HEAT CONDUCTION EQUATION WITH VARIABLE COEFFICIENTS

1. Discretization of 2-D Heat Conduction Equation with variable coefficients

The discretization is shown in figure below,

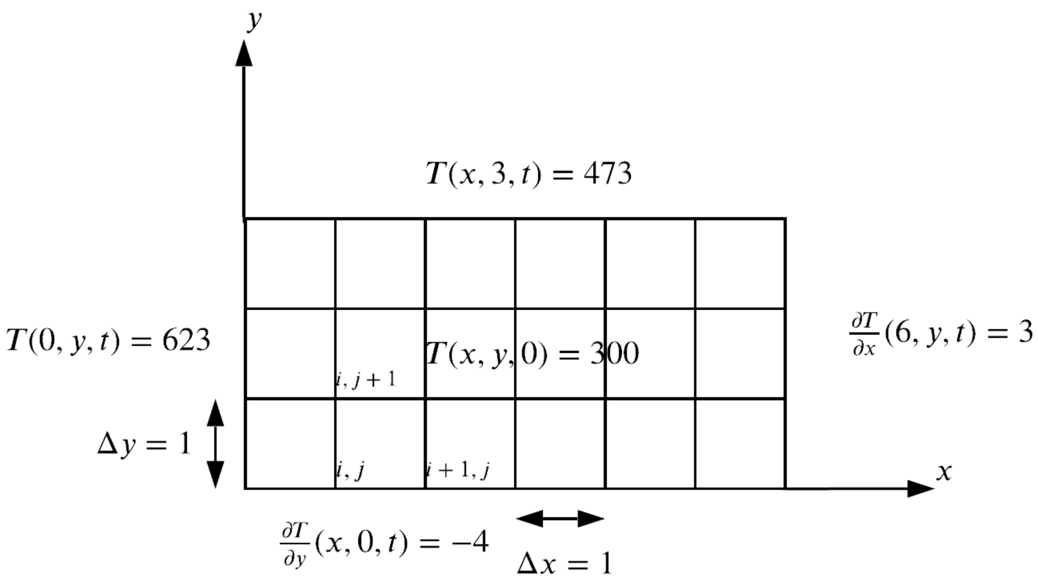

FIGURE 1. Illustration of the domain and the corresponding boundary conditions where i subscript denotes grid index in x direction and j subscript denotes grid index in y direction.

As given in the problem, 2-D Heat equation with variable coefficients reads:

$$\frac{\partial T}{\partial t} = \frac{(1+\sqrt{y})}{2}\frac{\partial^2 T}{\partial x^2} + \frac{(1+\sqrt{x})}{2}\frac{\partial^2 T}{\partial y^2} \tag{1.1}$$

By finite differences, the first derivative can be approximated as:

$$\frac{\partial T}{\partial x} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x}. \tag{1.2}$$

Similarly, the second derivative is computed as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\Delta x}\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x} - \frac{T_{i,j} - T_{i-1,j}}{\Delta x}\right) = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}. \tag{1.3}$$

Exact same discretization can be applied to the second derivative in y direction. As a result, the governing equation 1.4 can be discretized as:

$$\frac{\partial T_{i,j}}{\partial t} = \frac{(1+\sqrt{j\Delta y})}{2\Delta x^2}(T_{i+1,j} - 2T_{i,j} + T_{i+1,j}) + \frac{(1+\sqrt{i\Delta x})}{2\Delta y^2}(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}) \tag{1.4}$$

However, it should be noted that i and j indices do not cover the grid points along $x = 0$ and $y = 3$, as temperature of these nodes are given as Dirichlet boundary conditions. Similarly, Temperatures of nodes at $y = 0$ and $x = 3$ are computed as follows:

$$\frac{T_{i+1,j} - T_{i,j}}{\Delta x} = 3, \tag{1.5}$$

$$\frac{T_{i,j} - T_{i,j-1}}{\Delta y} = -4. \tag{1.6}$$

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Therefore,

$$T_{i+1,j} = T_{i,j} + 3\Delta x, \tag{1.7}$$

$$T_{i,j-1} = T_{i,j} + 4\Delta y. \tag{1.8}$$

These values are replaced into the governing equation with their corresponding i and j indices.

2. Time Integration with Alternating Direction Implicit Method for 2-D Heat Conduction

As for the time integration, by ADI method time integration can be applied to the 2-D conduction problem as:

$$[M]([T]_{n+1/2} - [T]_n) = \frac{\Delta t}{2}([K_x][T]_{n+1/2} + [K_x][T]_n) + \Delta t[\dot{Q}_x]_n \tag{2.1}$$

where n stands for the index of the time step. For, the conduction in y direction, ADI is formulated as:

$$[M]([T]_{n+1} - [T]_{n+1/2}) = \frac{\Delta t}{2}([K_y][T]_{n+1} + [K_y][T]_{n+1/2}) + \Delta t[\dot{Q}_y]_n \tag{2.2}$$

3. Results

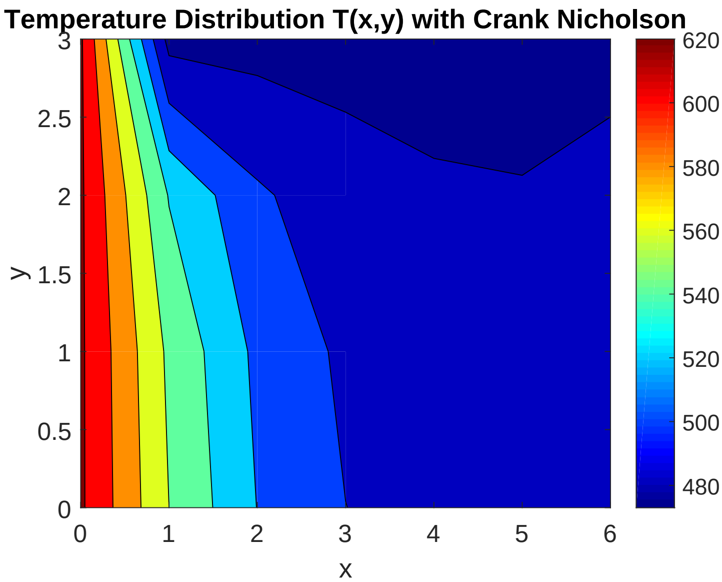

Figure 2. Terminal temperature distribution by Crank-Nicholson time integration.

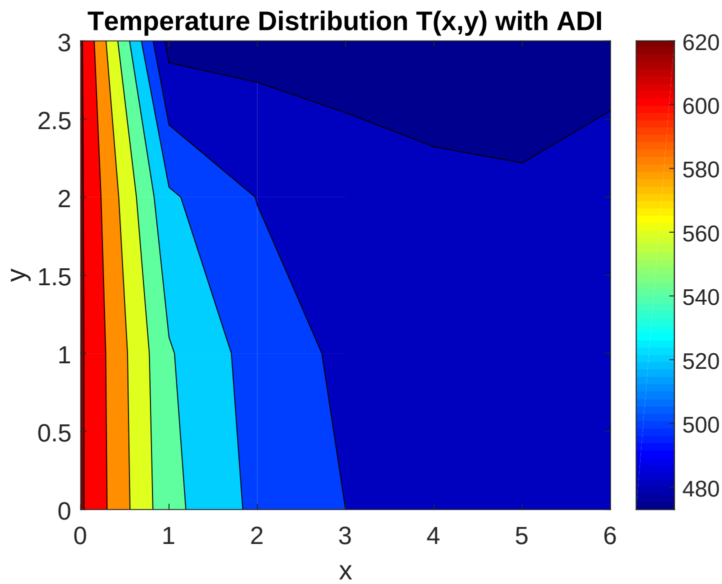

Figure 3. Terminal temperature distribution by Alternating Direction Implicit time integration.

Appendix A. Crank-Nicholson Time Integration Algorithm

After rearranging the expression above, the temperature distribution at the next step $[T]_{n+1}$ can be computed as:

$$[T]_{n+1} = ([M] - \frac{\Delta t}{2}[K])^{-1}(\frac{\Delta t}{2}[K] + [M])[T]_n + ([M] - \frac{\Delta t}{2}(K])^{-1}\Delta t[\dot{Q}]_n) \tag{A.1}$$