JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about computing double integrals and about changing the order of integration. And how you can look at a given region and you can integrate over it by integrating dx dy or by integrating dy dx. So here I have some examples.

I have two regions. So one region is the triangle whose vertices are the origin, the point 0, 2, and the point minus 1, 2. And the other one is a sector of a circle. So the circle has a radius 2 and is centered at the origin. And I want the part of that circle that's above the x-axis and below the line y equals x.

So what I'd like you to do is I'd like you to write down what a double integral over these regions looks like, but I'd like you to do it two different ways. I'd like you to do it as an iterated integral in the order dx dy. And I'd also like you to do it as an iterated integral in the order dy dx. So I'd like you to express the integrals over these regions in terms of iterated integrals in both possible orders.

So why don't you pause the video, have a go at that, come back, and we can work on it together.

So the first thing to do whenever you're given a problem like this-- and in fact, almost anytime you have to do a double integral-- is to try and understand the region in question. It's always a good idea to understand the region in question. And by understand the region in question, really the first thing that I mean is draw a picture. All right.

So let's do part a first. So in part a, you have a triangle, it has vertices at the origin, at the point 0, 2, and at the point minus 1, 2. So this triangle is our region in question.

So now that we've got a picture of it, we can talk and we can say, what are the boundaries of this region, right? And we want to know what its boundaries are. So the top boundary is the line y equals 2, the right boundary is the line x equals 0, and this sort of lower left boundary-- the slanted line-- is the line y equals minus 2x. OK. So those are the boundary edges of this triangle.

And so now what we want to figure out is we want to figure out is, OK, if you're integrating this with respect to x and then y, or if you're integrating this with respect to y and then x, what does that integral look like when you set it up as a double integral. So let's start on one of them. It doesn't matter which one.

So let's try and write the double integral over this region R in the order dx dy. OK, so we have inside bounds dx dy. So OK. So we need to find the bounds on x first, and those bounds are going to be in terms of y.

So the bounds on x. So that means when we look at this region, what we want to figure out is we want to figure out for a given value y, what is the leftmost point and what is the rightmost point? What are the bounds on x?

So for given value y, the largest value x is going to take is along this line x equals 0. When you fix some value of y, the rightmost point that x can reach in this region is at this line x equals 0. So x is going to go up to 0. That's going to be its upper bound.

The lower bound is going to be the left edge of our region. For a given value of y, what is that leftmost boundary value? So what we want to do is we want to take that equation for that boundary and we want to solve it for x in terms of y. So that's not hard to do in this case. The line y equals minus 2x is also the line x equals minus 1/2 y. So that's that left boundary: minus 1/2 y. OK?

So then our outer bounds are dy. So we want to find the absolute bounds on y. What's the smallest value that y takes, and what's the largest value that y takes? So that means what's the lowest point of this region and what's the highest?

And so the lowest point here is the origin. So that's when y takes the value of 0. And the highest point-- the very top of this region-- is when y equals 2. OK. So this is what that double integral is going to become when we evaluate it in the order dx dy.

So now let's talk about evaluating it in the opposite order. So let's switch our bounds for dy dx. So we want the double integral over R, dy dx.

OK, so this is going to be an iterated integral. And this time the inner bounds are going to be for y in terms of x, and the outer bounds are going to be absolute bounds on x. So for y in terms of x, that means we look at this region-- we want to know for a fixed value of x-- what's the bottom boundary of this region, and what's the top boundary?

So here, it's easy to see that the bottom boundary is this line y equals minus 2x, and the top boundary is this line y equals 2. So y is going from minus 2x to 2. Yeah? So for a fixed value of x, the values of y that give you a point in this region are the values that y is at least minus 2x and at most 2. So OK.

And now we need the outer bounds. So the outer bounds have to be some real numbers, Those are the absolute bounds on x. So we need to know what the absolute leftmost point and the absolute rightmost point in this region are.

And so the absolute leftmost point is this point minus 1, 2. So that has an x-value of minus 1. And the absolute rightmost point is along this right edge at x equals 0. OK.

So here are the two integrals. The double integral with respect to x then y, and the double integral with respect to y and then x. OK. So that's the answer to part a. Let's go on to part b.

So for part b, our region is we take a circle of radius 2, and we take the line y equals x, and we take the line that's the x-axis. And so we want a circle, and we want this sector of the circle in here. So this region inside the circle, below the line y equal x, and above the x-axis. So this wedge of this circle.

Let's see. This value is at x equals 2, this is the origin, and this is the point square root of 2 comma square root of 2. A common point of the line y equals x in the circle x squared plus y squared equals 4. That's what this boundary curve is: x squared plus y squared equals 4.

And of course, this boundary curve is the line y equals x. And this boundary line is the x-axis, which has the equation y equals 0. So those are our boundary curves for our region. We've got this nice picture, so now we can talk about expressing it as an iterated integral in two different orders.

So let's again start off with this with respect to x first, and then with respect to y. So we want the double integral over R, dx dy. So this should be an iterated integral, something dx and then dy.

OK, so we need bounds on x, which means for a fixed value of y, we need to know what is the leftmost boundary and what's the rightmost bound. So for a fixed value of y, we want to know what the left edge is and the right edge is. And it's easy to see because we've drawn this picture, right? Drawing the picture makes this a much easier process.

The left edge is this line y equals x and the right edge is our actual circle. Yeah? So those are the left and right boundaries, so what we put here are just the equations of that left edge and the equation of that right edge. But we want their equations in the form x equals something. And that's the something that we put there.

So for this left edge, it's the line x equals y. So the left bound is y there. In this region, x is at least y.

And the upper bound here, which is going to be the rightmost bound-- the largest value that x takes-- is when x squared plus y squared equals 4. So when x is equal to the square root of 4 minus y squared. Now you might say to me, why do I know that it's the positive square root here and not the negative square root? And if you said that to yourself, that's a great question.

And the answer is that this part of the circle is the top half of the circle and it's also the right half of the circle. So here we have positive values of x. So it's the right half of the circle. We want the positive values of x, so we want the positive square root. OK. Good. And so those are the bounds on x.

Now we need the bounds on y. So the bounds on y, well, what are they? Well, we want the absolute bounds on y. y is the outermost variable that we're integrating with respect to, so we want the absolute bounds-- the absolute lowest value that y takes in this region, and the absolute largest value that y takes. So the smallest value that y takes in this region-- that's the lowest point-- that's along this line, and that's when y equals 0. And the largest value that y takes-- that's when y is as large as possible as it can get in this region-- is up at this point of intersection there, so that's when y is equal to the square root of 2.

OK, three quarters done. Yeah?

This is that iterated integral. So now, we want to do the same thing. R-- the integral over this region R-- dy dx. OK. So we're going to look at this region and we want to say-- dy is going to be on the inside-- so we're going to say, OK, so we need to know for a fixed value of x, what's the smallest value that y can take and what's the largest value that y can take? So what's the bottom boundary and what's the top boundary?

But if you look at this region-- right?-- life is a little complicated here. Because if you're in the left half of this region-- what do I mean by left half-- I mean if you're to the left of this point of intersection-- if you're at the left of this line x equals square root of 2-- when you're over there, y is going from 0 to x. But if you're over in the right part of this region, there's a different upper boundary. Right? It's a different curve that it came from. It has a different equation.

So over here, y is going from the x-axis up to the circle. So this is complicated, and what does this complication mean? Well, it means that it's not easy to write this as a single iterated integral. If you want to do this in this way, you have to break the region into two pieces, and write this double integral as a sum of two iterated integrals. OK? So one iterated integral will take care of the left part and one will take care of the right part.

So let's do the left part first. So here we're going to have a iterated integral integrating with respect to y first. So to fix the value of x, we want to know what the bounds on y are. And well, we can see from this picture-- when you're in this triangle-- that y is going from the x-axis up to the line y equals x.

So that means the smallest value that y can take is 0, and the largest value that y can take is x. So here it's from 0 to x. And when you're in this triangle, we need to know what the bounds on x are, then. We need to know the outer bounds. So we need to know the absolute largest and smallest values that x can take. Well, what does that mean?

We need to know the absolute leftmost and absolute rightmost points. So the absolute leftmost point is the origin. The absolute rightmost is this vertical line x equals square root of 2. So over here, the value of x is 0. And at the rightmost boundary of this triangle, the value of x is the square root of 2. OK. So that's going to give us the double integral just over this triangular part of the region. Yeah?

So now, we need to add to this-- but I'm going to put it down on this next line-- we need to add to this the part of the integral over this little segment of the circle here. The remainder of the region that's not in that triangle. So for that, again, we're going to write down two integrals, and it's going to be dy dx. Whew. We're nearly done, right?

So y is inside, so we need to know what the bounds on y are for a given value of x. So we need to know for a given value of x, what are the bottom and the topmost points of this region? So for a given value of x, that means that y is going here between the x-axis and between this circle. So the x-axis is y equals 0, so that's the lower bound.

So for the upper bound, we need to know this circle. What is y on this circle? Well, the equation of this circle is x squared plus y squared equals 4, so y is equal to the square root of the quantity 4 minus x squared. Where again here, we take the positive square root, because this is a part of the circle where y is positive. Yeah. If we were somehow on the bottom part of the circle, then we would have to take a negative square root there, but because we're on the top part of the circle where y is positive, we take a positive square root. OK, good.

So those are the bounds on y, and now we need to know the absolute bounds on x. Yeah? So those are the bounds on y in terms of x. And now because x is the outer thing we're integrating with respect to, we need the absolute bounds on x.

And you can see in this circular-- I don't really know what the name for a shape like that is-- but whatever that thing is, we need to know what its leftmost and rightmost points are. We need to know the smallest and largest values that x can take. And so its leftmost edge is this line x equals square root of 2. And its rightmost edge is that rightmost point on the circle there-- where the circle hit the x-axis-- and that's the value when x equals 2.

OK, so there you go. There's this last integral written in the dy dx order, but we can't write it as a single iterated integral. We need to write it as a sum of two iterated integrals because of the shape of this region. All right.

Let me just make one quick, summary comment. Which is that if you're doing this, one thing that should always be true, is that these integrals, when you evaluate them- so here, I haven't been writing an integrand. I guess the integrand has always been 1, or something. But for any integrand, the nature of this process is that it shouldn't matter which order you integrate. You should get the same answer if you integrate dx dy or dy dx.

So one very low-level check that you can make-- that you haven't done anything horribly, egregiously wrong when changing the bounds of integration-- is that you can check that actually these things evaluate the same. Yeah? Where you can choose any function that you happen to want to put in there-- function of x and y-- and evaluate this integral, and choose any function that you happen to want to put in there, and evaluate those integrals. And see that you actually get the same thing on both sides.

Now one simple example is that you could just evaluate the integral as written with a 1 written in there. And so in both cases, what you should get is the area of the region when you evaluate an integral like that. But you can also check with any other function if you wanted. It won't show that what you've done is right, but it will show if you've done something wrong. That method will sometimes pick it out, right? Because you'll actually be integrating over two different regions, and there's no reason you should get the same answer.

So if you were to compute these integrals and get different numbers, then you would know that something had gone wrong at some point for sure, and you'd have to go and figure out where it was. I think I'll end there.

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