DAVID JORDAN: Hello, and welcome back to recitation. In this question, we're going to be considering a contour plot, which is given to us as followed. The values are not indicated. So the first thing that we want to do is we want to identify-- on this contour plot, there is a unique saddle point, and we want to label that as point A, and there are two points which are either a maximum or a minimum. We can't actually tell because the labels aren't on this contour plot, but we want to go ahead and label those anyways: B and C. So they're either maximum or minimum, but we can still find them and we can still identify them. So that's the first part of the problem.

The second part is since this doesn't have the values entered onto the graph, we want to consider what possible configurations could we have? So the second question is: can B and C both be maximal? And can we have B maximal but C minimal? OK. And then in each of these two cases, we want to sketch the 3D graph. So why don't you take some time to work this out. Pause the video, and we'll check back, and I'll show you how I solve this.

Hello, and welcome back. So to get started, why don't we answer the first question by writing the points right on our original graph. So I'll just come over here. Now, when we're looking for a minimum or a maximum on a contour plot, you know, the thing that we should keep in mind is that a minimal or a maximal always is going to be contained in concentric contours that are either approaching the minimum from below or-- excuse me-- approaching the maximum from below or the minimum from above. And so if we look here, we see that these rings start to become concentric, and somewhere in here, there's got to be either a maximum or a minimum. Because, you know, inside this little region here, the function doesn't pass through another contour plot, so we have to find either a maximum or a minimum inside the innermost ring. And similarly, we have to find either a minimum or a maximum here. So let's just call this one B and this one C. OK.

Now, we also have a saddle point A in this problem, and it's a little bit hard to see in the contour plot. I think it'll be a little even clearer when we draw a 3D graph, but basically what's happening is the fact that you have these contours-- so this contour is, after all, the same as that contour. So the value of the function here and here are the same, and yet, if we look at this point and this point, the values, they'll either go up or down. We don't know. Let's assume that they go up. So here, in this direction, the values are going up, and in this direction, the values are determined by this contour curve, and so somewhere in this middle region here, there's got to be a saddle point. I think this'll be even clearer when we draw our graph. So we have a saddle point A in the middle there.

And, in fact, this is really, this is the general picture of what a saddle point is going to look like. It's going to be when you have two either maxima or minima rising out, and you have a contour which is containing the point in the middle. So those are our points A, B and C that we're going to be interested in.

So now, the second question that we have to consider is can B and C both be maximal? And the third question is can B be maximal and C minimal? And actually, we'll answer both of these questions by just sketching an example, so that's how will we'll understand this. So why don't we see if we can sketch an example where B and C are both maximal.

So here's the start of my graph in three dimensions, and if we want B and C to both be maximal, then let me go ahead and draw the contour lines that we have. So we have, first of all, we had this one, and then we had another one, and then we had a peak, and then we had a peak. So if we want to draw this in three dimensions, then what we just need to do is we just need to follow these contour plots up out of the plane and into space. So this goes up, and then there's a maximum, and then it comes back down along the contour lines, and then it goes back up, and then it goes back down. So that's just one of the curves that lies on the graph of the function, but then we need to flush out the contour lines, which they look like these sort of rings here. OK?

And so, indeed, we do see that it's possible for both B and C to be a maximal. Here's an example of such a thing. And so here's our point B, here's our point C, and as promised, I think it's much clearer now how A becomes a saddle point. Because you have these two mountains rising up, and the valley in between them is necessarily a little saddle here. It increases in this direction and it decreases in this direction.

Now, for the second one, we're asked can B be maximal and yet C be minimal? And the answer is still yes. And a nice way to think about this problem, to think about the graph that I'm going to draw, is imagine that we start over here and we dig a hole, and as we're digging, we throw the dirt over behind us. So we're going to have a hole, a dip here, and then we're going to pile that hole up over here.

And so then--now, notice that both of these that I've drawn, if we don't label the contour lines, these have the same contour plot, because the concentric rings on C, which are telling us that the function in increasing, they're the same below, because this is essentially the same thing. The concentric rings that

on this curve, on this surface, were telling us that B was a maximal point, now the same rings are telling us that B is a minimal point.

So this illustrates that a contour plot is really-- doesn't tell you everything about the graph unless you actually label the values of the contours. So here we see two examples where the sort of global behavior of B and C are very different even though they have the same contour plot. Now notice, in both cases, A is a saddle point. It's increasing in one direction and decreasing in the other. And I think I'll leave it at that.

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