# Degree Sets: Realizability and Extension Problems

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# Talk Plan

- Degree Set Basics
- Extension Problem for Trees
- Extension Problems for Undirected Graphs
- Tree Realizability under Multiplicity Constraints
- Degree Set Variants for Directed Graphs
- Asymmetric Directed Graph Realization of Degree Set
- Directed Tree Realization of Degree Set
- Conclusion and Future Work

# Degree Set : Definition

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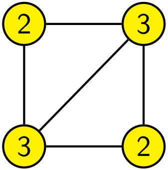

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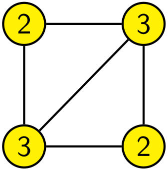


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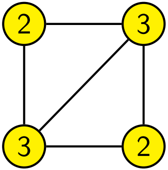


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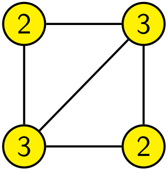


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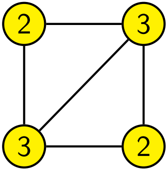


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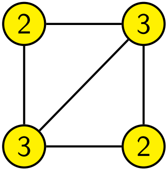


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- <u>*Convention*</u> : elements in $S$ are written in increasing order.
- $S$ is *realizable* if there is a graph respecting $S$.
- Interested in *Simple graph* realizations only.

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When $S$ - realized by a tree then we use $\mu_T(S)$ instead of $\mu(S)$.

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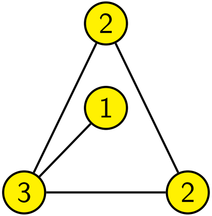

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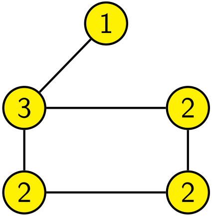

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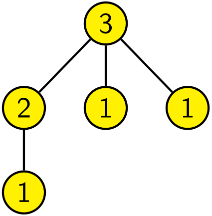

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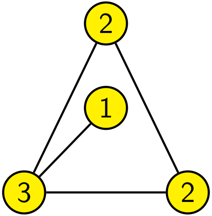

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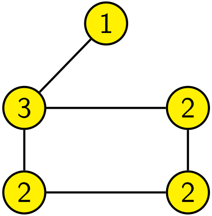

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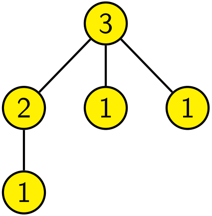

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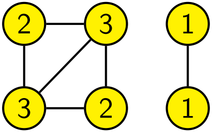

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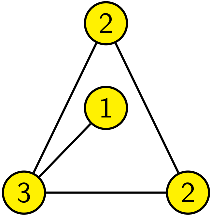

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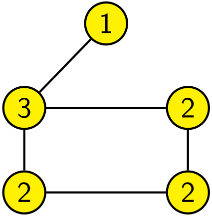

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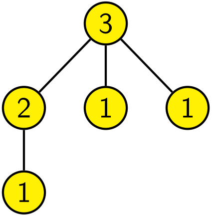

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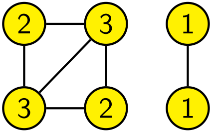

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$$\boxed{\mu(S) = 4}$$

# Integer Set Realization

**Theorem - Kapoor, Polimeni, Wall(1977)**

Any finite set $S = \{a_1 < a_2 < \ldots < a_n\}$,$a_i \in \mathbb{Z}^+$ is always realizable and $\mu(S) = a_n + 1$.

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- $t < \mu(S)$ - no realization exists
- $t \geq \mu(S)$ then $t = \mu(S) + r$ where $r \in \mathbb{Z} \bigcup \{0\}$

# Integer Set Realization of Trees and TEP

**Theorem - Kapoor, Polimeni, Wall(1977)**

Any finite set $S = \{a_1 < a_2 < \ldots < a_n\}$,$a_i \in \mathbb{Z}^+$ is realized by a tree *if and only if* $a_1 = 1$ and $\mu_T(S) = \sum_{i=1}^{n}(a_i - 1) + 2$

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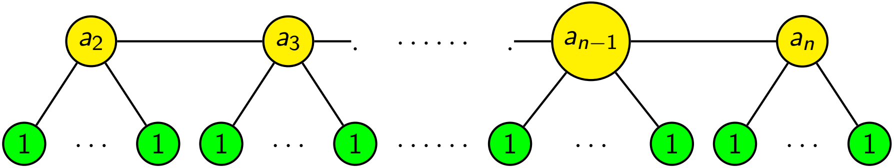

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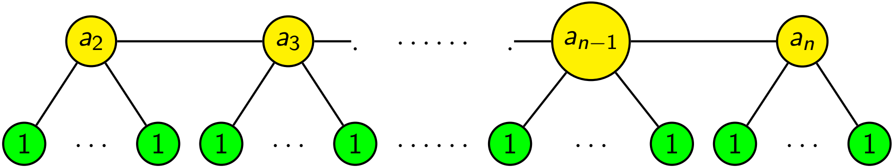


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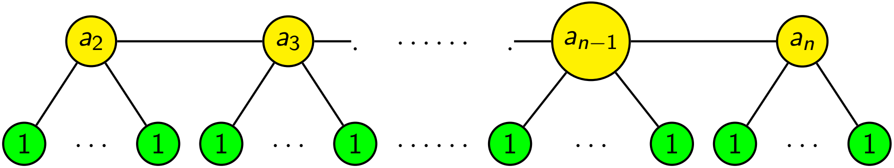


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**Tree Extension Problem(TEP)**

*Given a degree set $S$ with $a_1 = 1$, and an integer $r$, check if there is a tree $T'(V', E') \in \Gamma(S)$ such that $|V'| = \mu_T(S) + r$.*

# Tree Extension Characterization

## Characterization

If the degree set $S = \{1 = a_1 < a_2 < \cdots < a_n\}$ is realized by a tree $T(V, E)$ then there is another tree realization $T' = (V', E')$ where $|V'| = |V| + r, r \in \mathbb{Z}^+$, *if and only if*

$r = k_2(a_2 - 1) + k_3(a_3 - 1) + \cdots + k_n(a_n - 1) \qquad \forall i, k_i \in \mathbb{Z}^+ \bigcup \{0\}$

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*Proof* (Aliter): ($\Rightarrow$) W.l.o.g. assume $T(V,E)$ is a minimum order tree realizing $S$ and let out of $r$ vertices, $k_i$ vertices are there with degree $a_i \in S$.

# Tree Extension Characterization

Hence

$$(1^{\sum_{i=1}^{n} a_i - 2n + 3 + k_1}, {a_2}^{1+k_2}, \ldots, {a_i}^{1+k_i}, \ldots, {a_n}^{1+k_n})$$

is the degree sequence of $T'$ and $r = \sum_{i=1}^{n} k_i$.

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*lemma*(AM96): For the sequence $d = (d_1 \geq d_2 \geq \cdots \geq d_n)$, if $\sum_{i=1}^{n} d_i = 2(n-1)$ then the number of pendant vertices in any tree realization of $d$ is $\sum_{i=1}^{k}(d_i - 2) + 2$ where $k = max\{i | d_i \geq 3\}$

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Solving these we get $r = \sum_{i=2}^{n} k_i(a_i - 1)$.

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*(TEP is NP-Complete.)*

# Unary Version of TEP

## Unary Tree Extension Problem(UTEP)

Given a tree $T$ on $\ell$ vertices and a string $1^r$, test if there is another tree $T'$ having exactly $\ell + r$ vertices and the degree set same as that of $T$.

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## UNARY SUBSET SUM PROBLEM

Given a multiset $A$ of $m$ integers $b_1, b_2, \ldots, b_m$ and a value $c$ (all inputs in unary), test if there is a subset $A'$ of these integers such that $\sum_{i \in A'} b_i = c$

# Reduction from UTEP to UNARY SUBSET SUM

*Reduction*: Given a tree $T$ and $r$ in unary

- write down the set $A = \bigcup_{i=2,j=1}^{i=n,j=t_i} \{(a_i - 1)j\}$ where $t_i = \lceil \frac{r}{a_i - 1} \rceil$ and $r$ in unary, choose $c = r$.

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Hence the corresponding terms $k_i(a_i - 1)$ will appear in the set $A$ as well. Choosing these terms in $A'$ ensures $\sum_{i \in A'} b_i = r = c$.

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$\nearrow$ *graphic??* $\nwarrow$ *graphic*

# Integer Sequence Realization

## Erdös - Gallai Theorem (1960)

The positive integer sequence $d = (d_1 \geq d_2 \geq \cdots \geq d_i \geq \cdots \geq d_n)$, where $d_1 \leq n-1$, is realized by a simple graph if

- $\sum_{i=1}^{n} d_i$ is even, and
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Idea : (*can add* $d_{j'} < j'$ *to get another graphic sequence* $d'$)

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The corresponding degree sequence of $G$ will be

$$\mathbf{D} = (d_1 \geq d_2 \geq \cdots \geq d_j \geq \cdots \geq d_{a_n+1}) \text{ where}$$

$$\mathbf{d_j} = \begin{cases} a_n & \text{if } 1 \leq j \leq a_1 \\ a_{n-i} & \text{if } 1 + a_i \leq j \leq a_{i+1}, \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1 \\ a_{\lceil \frac{n}{2} \rceil} & \text{if } 1 + a_{\lfloor \frac{n}{2} \rfloor} \leq j \leq 1 + a_{\lfloor \frac{n}{2} \rfloor + 1} \\ a_{n-i} & \text{if } 2 + a_i \leq j \leq 1 + a_{i+1}, \text{ for } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 1 \end{cases}$$

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**Theorem**

For every $r \in \mathbb{Z}^+$, there exists an undirected simple graph with $\mu(S) + r$ vertices having degree set $S = \{a_1 < a_2 < \cdots < a_n\}$ except the case when $a_1 = 1$, $r$ is odd and only $a_n$ is even.

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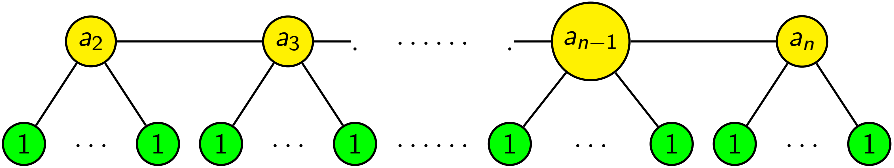
Minimum order tree construction



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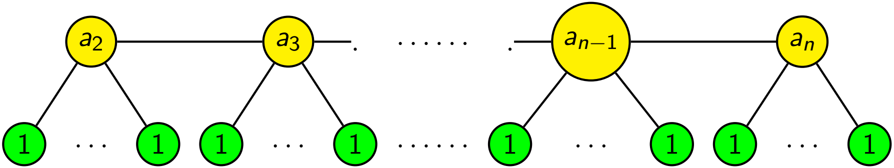


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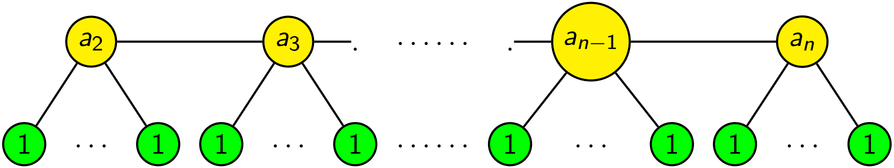


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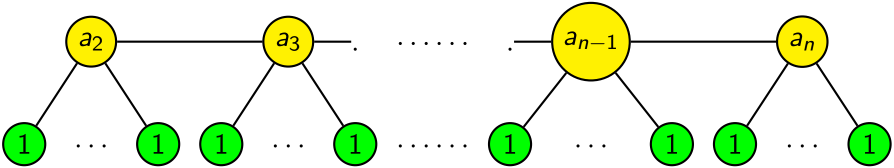


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*Q.* Can we have another tree realization with smaller no of pendant vertices?

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**Lemma**

*The minimum multiplicity of pendant vertices in any tree realization for the degree set* $S = \{1 = a_1 < a_2 < \cdots < a_n\}$ *is* $\sum_{i=1}^{n} a_i - 2n + 3$

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$$m_1 = 2 + (a_2 - 2)m_2 + (a_3 - 2)m_3 + \cdots + (a_n - 2)m_n \qquad \forall i, m_i \geq 1$$

$m_1$ will be minimum if $m_i = 1$ for each $i = 2, 3, \ldots, n$

## Tree Realizability under Multiplicity Constraints

Minimum order tree described earlier meets exactly this requirement.

So minimum value of

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\begin{aligned}
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This result can be generalized in case of *Degree Multiset*.

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if $\forall v \in V, d^+(v) \in S$ or $d^-(v) \in S$, and for each $a_i \in S$ there is a vertex $u \in V$ such that $d^+(u) = a_i$ or $d^-(u) = a_i$.

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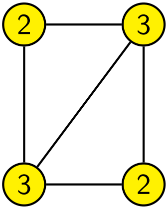
if $\forall v \in V, d^+(v) \in S$ and $d^-(v) \in S$, and for each $a_i \in S$ there is a vertex $v_p$ and $v_q \in V$ such that $d^+(v_p) = a_i$ and $d^-(v_q) = a_i$.

$S = \{2, 3\}$✓



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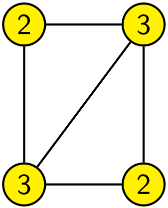


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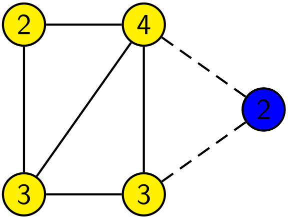

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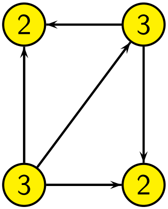


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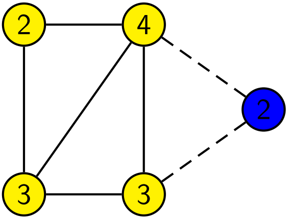


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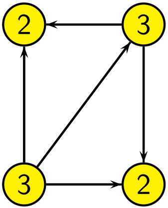


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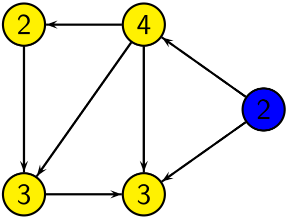


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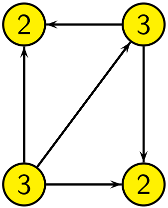


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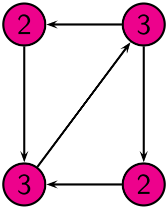
$S = \{1, 2\}$

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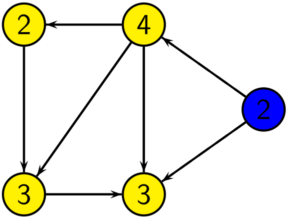


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# Digraph Realization for Degree Set

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*More feasible to study an asymmetric realization under $\wedge$-realizability constraints*.

# Asymmetric Digraph $\wedge$-realization

If $\mu_A(S)$ denotes the minimum order of any asymmetric directed graph $\wedge$-realizing $S = \{a_1 < a_2 < \ldots < a_n\}, n \geq 2, a_i \in \mathbb{Z}^+$ then

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*Extension* : Always possible if $r \geq a_n + a_{n-1} + 1$

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**Theorem**

*For the degree set $S = \{1 = a_1 < a_2 < \ldots < a_n\}$, the minimum order of a directed tree which $\vee$-realizes the degree set $S$, is $\sum_{i=1}^{n}(a_i - 1) + 2$.*

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For each $i$, $a_i \in S$ will appear as $(a_i, b_j)$ or $(b_k, a_i)$ at least once, where $b_j, b_k \in \mathbb{Z}^+ \bigcup \{0\}$. Thus

$$\sum_{v \in V}(d^-(v) + d^+(v)) = 2|E| = 2(V-1) \geq \sum_{i=1}^{n} a_i + (V - n)$$

This implies the lower bound $|V| \geq 2 + \sum_{i=1}^{n}(a_i - 1)$

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*For the degree set $S = \{0 < 1 < a_2 < \ldots < a_n\}$, the minimum order of a directed tree which $\wedge$-realizes the degree set s, is* $2\left(\sum_{i=1}^{n}(a_i - 1)\right) + 2$.

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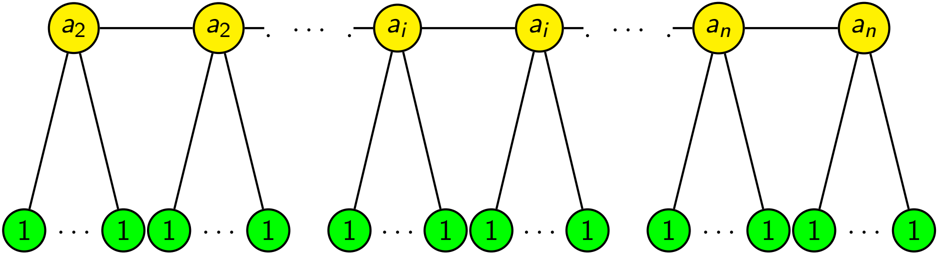

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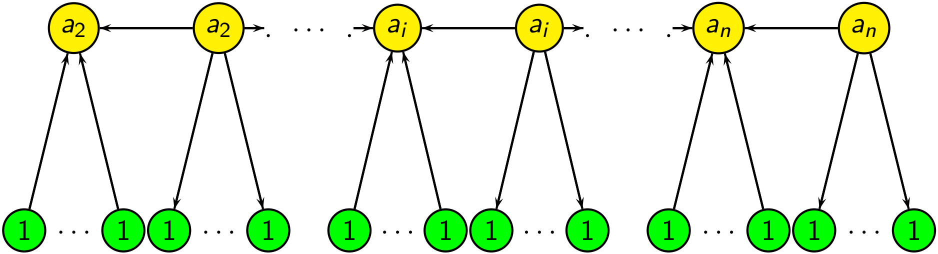

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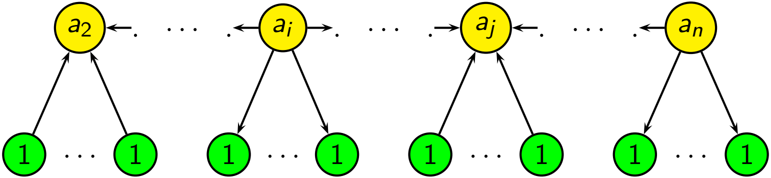
Minimum order Directed tree $\wedge$-realizing($\vee$-realizing) the degree set $S = \{0 < 1 < a_2 < \ldots < a_n\}$



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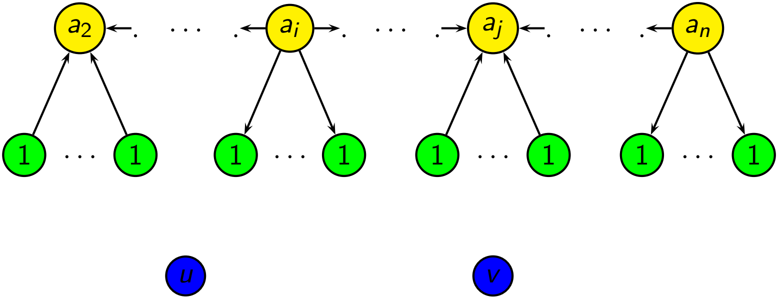
Every vertex to be added is added to the pendant vertex by an incoming or outgoing edge depending on whether the pendant vertex is a sink or source vertex respectively.



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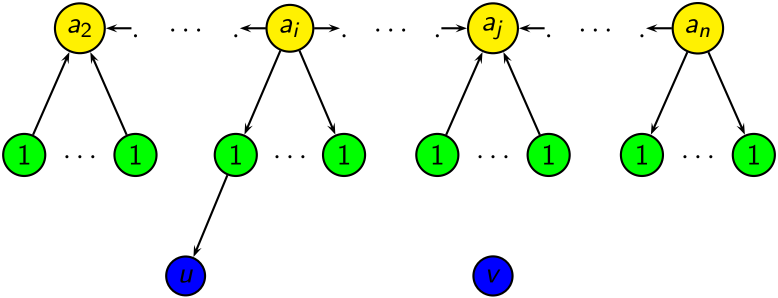

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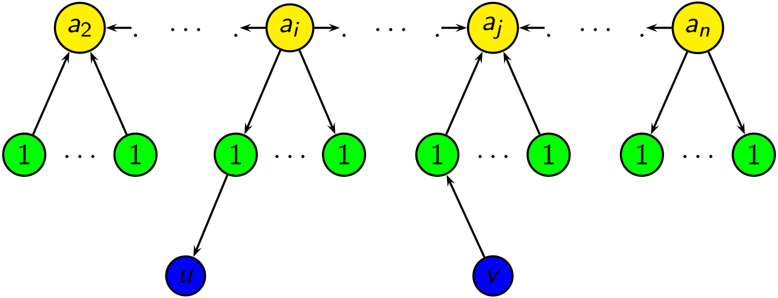

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Questions..??

Thank You.

# Asymmetric Digraph $\wedge$-realization

For $D = \{a_1\}$, $\mu_A(S) = 2a_1 + 1$ (Chartrand *et al*, 1976)

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- To prove the upper bound we construct an asymmetric graph $G_1$ with $2a_1 + 1$ vertices.

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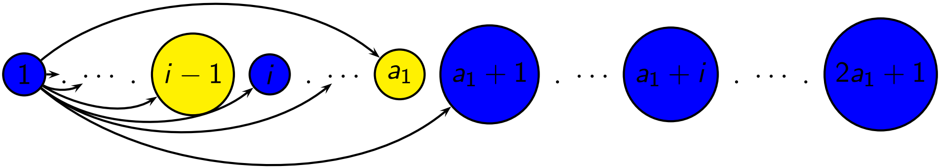

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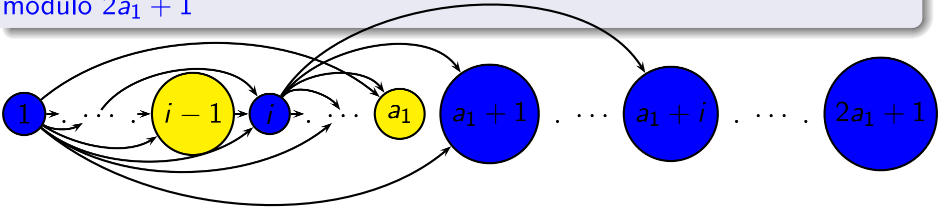

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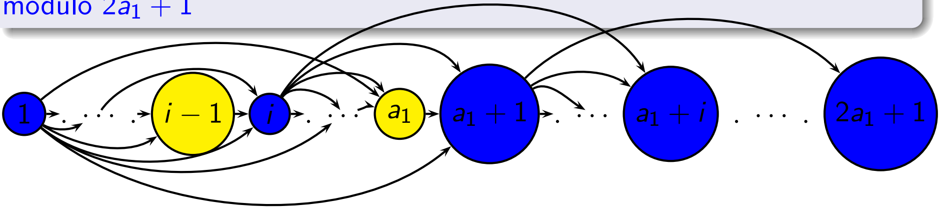

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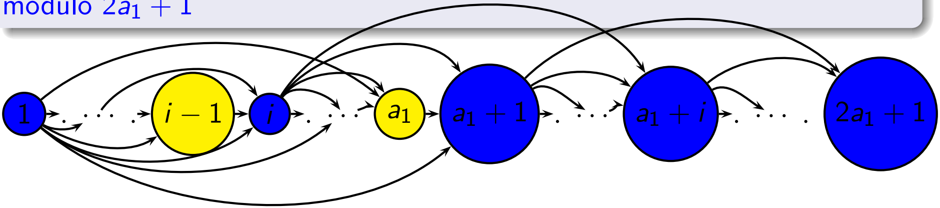

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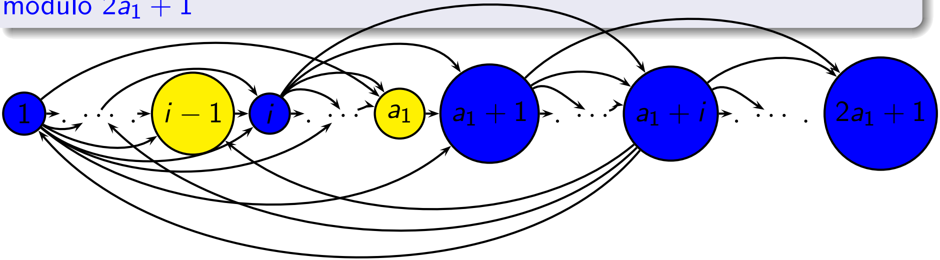

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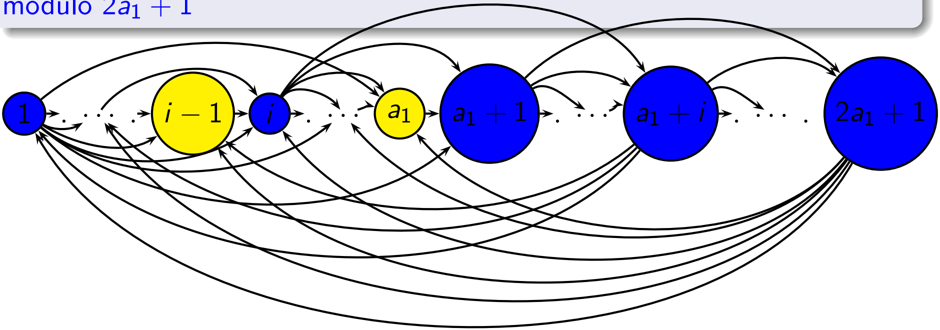

# Asymmetric Digraph $\wedge$-realization

Divide the vertices of $G_1$ in 3 components - $C_x, C_y, C_z$

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| $1\ 2 \cdots a_1$ | $(a_1+1)\ (a_1+2) \cdots 2a_1$ | $2a_1+1$ |
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Now add one component $C_1$ containing $(a_2 - a_1)$ isolated vertices and the edge set

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to $G_1$ to get $G_2$.

# Asymmetric Digraph $\wedge$-realization

<u>*Base Case*</u>(for $|S| = 2$): $G_2$ for $\{a_1 < a_2\}$ is constructed from $G_1$ as below

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$d^-(v) = d^+(v) = a_1$ $\quad$ $d^-(v) = d^+(v) = a_1$ $\quad$ $d^-(v) = d^+(v) = a_1$

$1\,2 \cdots (a_2 - a_1)$

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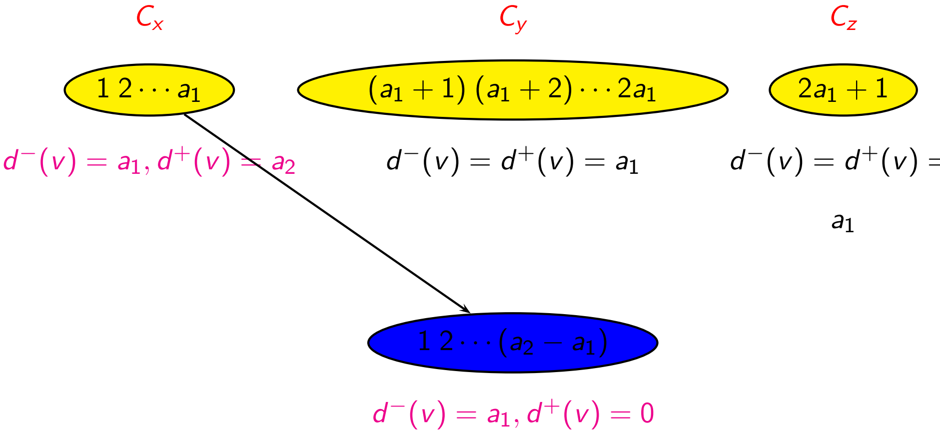

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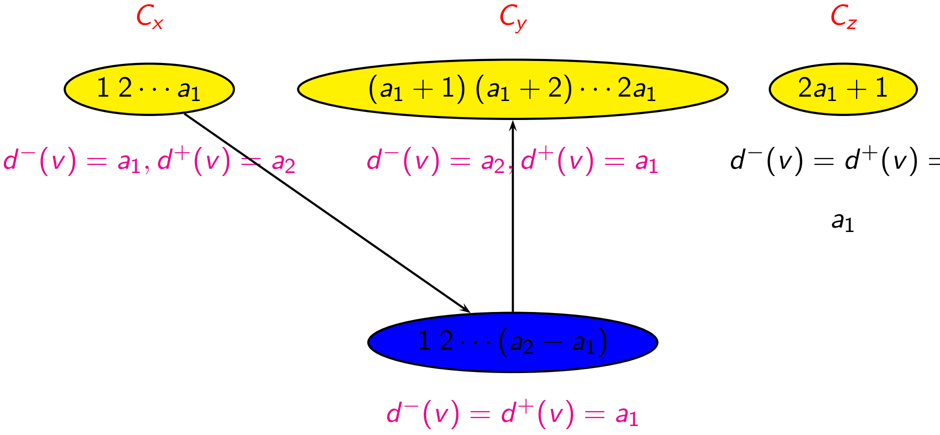

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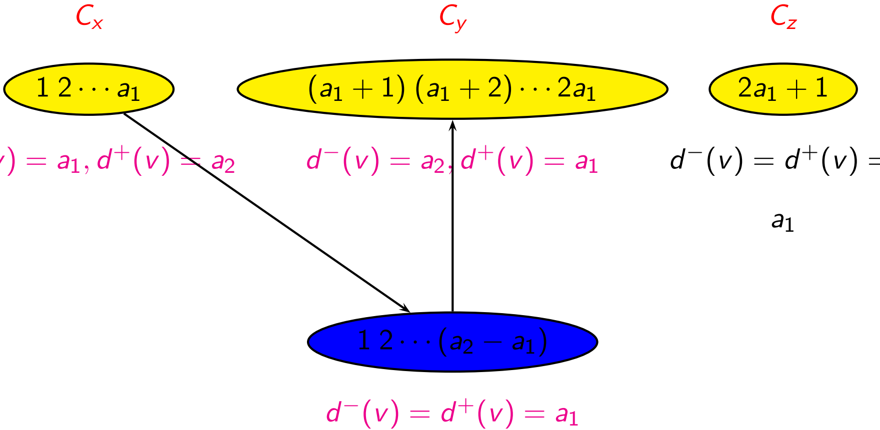


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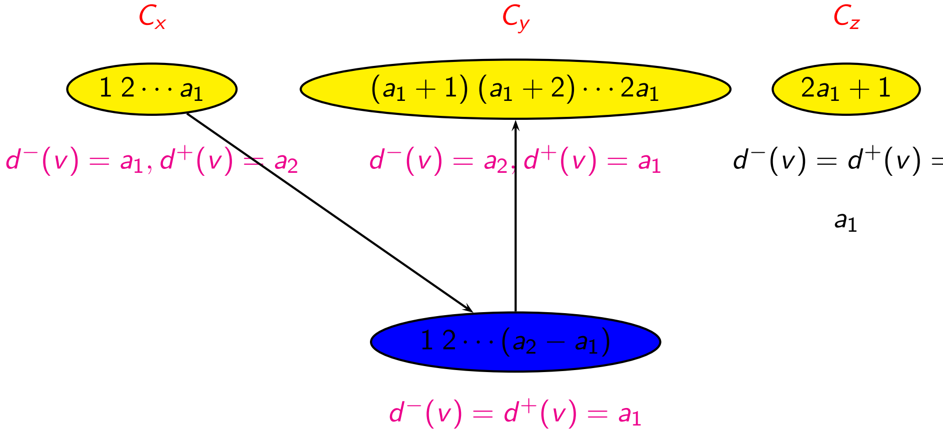


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*Hypothesis :* Consider there exists an asymmetric directed graph $G_{n_0}$ with degree set $\{a_1 < a_2 < \ldots < a_{n_0}\}, n_o < n$ with $a_{n_0} + a_{n_0 - 1} + 1$ vertices and $2n_0$ components.

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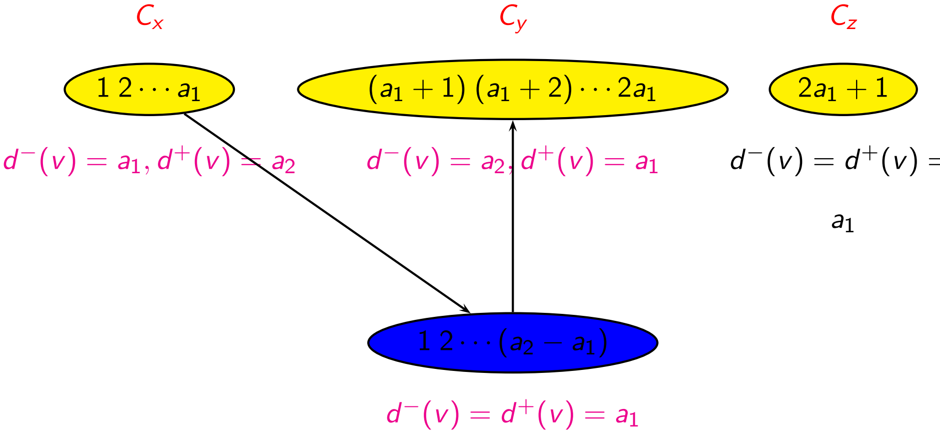


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*Need to construct $G_{n_o+1}$ from $G_{n_o}$.*

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- $C_{n_0-1}$ : $|V| = a_{n_0} - a_{n_0-1}$, $d^-(v) = d^+(v) = a_1$
- $C_i(\forall 1 \leq i \leq n_0 - 2)$ : $|V_i| = a_{i+1} - a_i$, $d^-(v) = a_{n_0-1-i}, d^+(v) = a_1$
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- Components from base graph $G_1$
  - $C_x$ : $|V| = a_1$, $d^-(v) = a_{n_0-1}, d^+(v) = a_{n_0}$
  - $C_y$ : $|V| = a_1$, $d^-(v) = a_{n_0}, d^+(v) = a_{n_0-1}$
  - $C_z$ : $|V| = 1$, $d^-(v) = d^+(v) = a_1$

$C_{n_o-1}$

$|V| = a_{n_0} - a_{n_o-1}$

$d^-(v) = d^+(v) = a_1$

$C_1 \dots C_i \dots C_{n_o-2}$

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$C_y$

$|V| = a_1$

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# Asymmetric Digraph $\wedge$-realization

To obtain $G_{n_0+1}$ from $G_{n_0}$, we add two new components - $C_{n_0}(|V| = a_{n_0+1} - a_{n_0}), C'_{n_0-1}(|V| = a_{n_0} - a_{n_0-1})$

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$|V| = a_{n_0} - a_{n_o-1}$

$d^-(v) = d^+(v) = a_1$

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$|V| = a_{n_o+1} - a_{n_o}$

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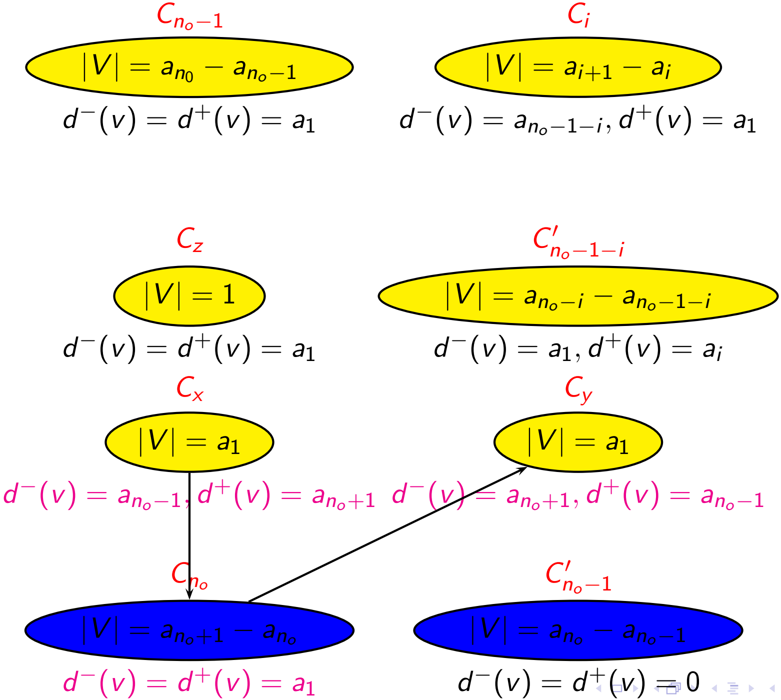
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$C'_{n_o-1-i}$
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$C_x$
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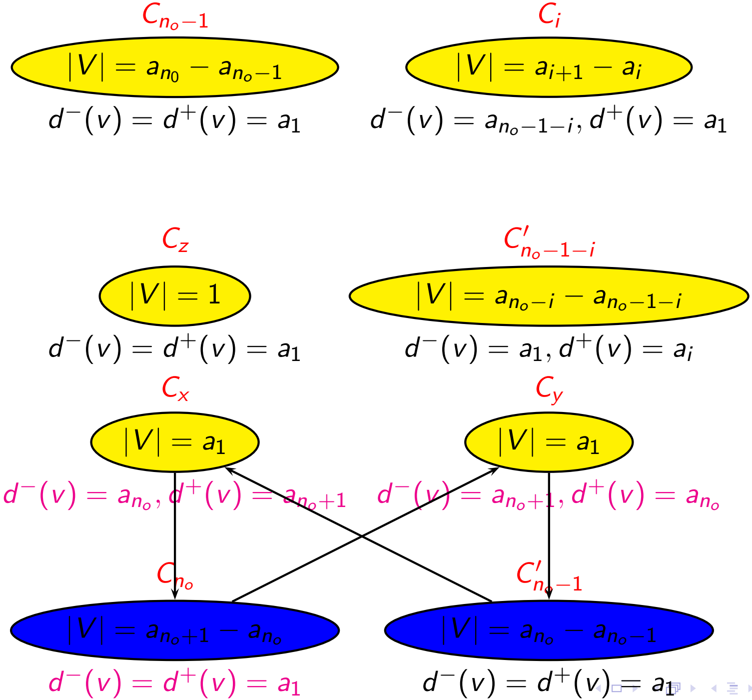
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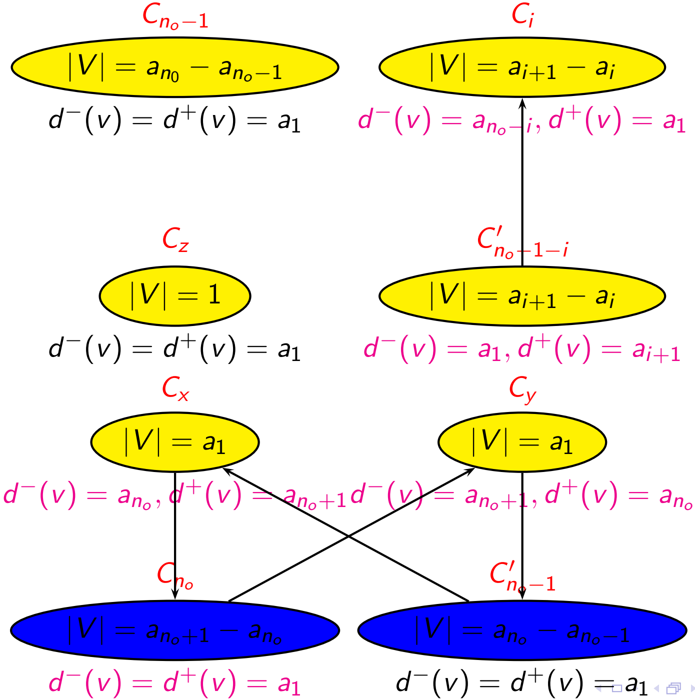
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Hence, $\mu_A(S) \leq a_n + a_{n-1} + 1$

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Hence, $\mu_A(S) \leq a_n + a_{n-1} + 1$

*Extension* : Always possible if $r \geq a_n + a_{n-1} + 1$, by costructing $G_1$ for $\{a_1\}$ with $(2a_1 + 1) + r - (a_n + a_{n-1} + 1)$ vertices and then construct the graph using same inductive approach.