MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF COMPUTER ENGINEERING

CENG 382

Fall 2022
Take Home Exam 2

Regulations:

- **Submission:** You need to submit a pdf file named "the2.pdf" to the odtuclass page of the course. You need to use the given template "the2.tex" to generate your pdf files. Otherwise you will receive zero.
- **Deadline:** 23:55, 12 December, 2022 (Monday).
- **Late Submission:** The solutions will be available after the deadline. Therefore, late submissions will not be allowed.

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1. (30 pts, 10 + 5 + 5 + 10 pts) Sociologists often assume that the social classes of successive generations in a family can be regarded as a Markov chain. Thus, the occupation of a son is assumed to depend only on his father's occupation and not on his grandfather's. For the sake of simplicity, only males are considered in this model. We can classify each man's profession as either professional, skilled laborer, or unskilled laborer. 70 percent of the sons of professional men are professional, 20 percent are skilled laborers, and 10 percent are unskilled laborers. 60 percent of sons of skilled laborers are skilled laborers, 20 percent are professional, and 20 percent are unskilled laborers. Finally, 50 percent of sons of unskilled laborers are unskilled laborers, 40 percent are skilled laborers and 10 percent are professionals. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations.

   (a) Find the state transition matrix for the described Markov chain. Model professionals to be the first state and unskilled laborers to be the third state.

   (b) Find the probability that a randomly chosen grandson of an unskilled laborer is a professional man.

   (c) Find the probability that a randomly chosen grandson of a professional man is a professional man.

   (d) What is the behavior of this Markov chain after 100 generations? You should show your calculations indicating how you reached that result. Answers without any justification will receive zero grade.

2. (20 pts, 10 pts each) Consider the following system.

$$x(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$

   (a) Show that the system is controllable. Show each step to receive full grade.

   (b) Starting from the initial state $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find a sequence of inputs that lead the system to $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ in exactly 3 steps.

3. (20 pts, 10 pts each) Consider the following system.

$$x(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x(k), \qquad y(k) = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} x(k)$$

   (a) Determine whether this system is observable by using the definition observability. That is to say, you should **not** calculate the observability matrix here, rather try to show you are able to obtain the initial state by following the output sequence.

   (b) Now, calculate the observability matrix and show that the result you found in part (a) is accurate. Show each step to receive full grade.

4. (30 pts, 10 + 20 pts) For each of the following systems, find all fixed points and determine their stability via linearization.

   (a) $\dot{x}(t) = 3x^2 - 3x^3$

   (b) $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1 - 2x_1x_2 - x_1^2 \\ x_2 - x_1 \end{bmatrix}$

Ungraded Example Questions

1. A person who is in jail has 3 dollars. The person can get out on bail if they have 8 dollars. The person makes a series of bets with a guard. If the person bets x dollars, s/he wins x dollars with probability 0.4 and loses x dollars with probability 0.6. Find the probability that the person wins 8 dollars before losing all of their money if

   (a) s/he bets 1 dollar each time (timid strategy).

   (b) s/he bets as much as possible but not more than necessary to bring her/his fortune up to 8 dollars each time (bold strategy).

   (c) Which strategy gives the person the better chance of getting out of jail?

2. Consider the following system.

$$x(k+1) = \begin{bmatrix} a & 0 \\ -2 & b \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

   Find conditions on a, b such that the system is controllable.

3. Consider the following system.

$$x(k+1) = \begin{bmatrix} a & 0 \\ -2 & b \end{bmatrix} x(k) \qquad y(k) = \begin{bmatrix} 1 & 3 \end{bmatrix} x(k)$$

   Find conditions on a, b such that the system is observable.

4. For the following system, find all fixed points and determine its stability via linearization.

$$\dot{x}(t) = 1 - e^{x^2}$$