CENG 384 - Signals and Systems for Computer Engineers
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Homework 2

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1. (a) $x(t) - 5y(t) = y'(t)$

   $\Longrightarrow y'(t) + 5y(t) = x(t)$

   (b) As we learned, $y(t) = y_h(t) + y_p(t)$ and now lets calculate firstly $y_h(t)$:

   To calculate $y_h(t)$, we will use characteristic equation which is:

   $r + 5 = 0 \Longrightarrow r = -5$

   Therefore,

   $y_h(t) = Ae^{rt} \Longrightarrow y_h(t) = Ae^{-5t}$

   Now lets calculate $y_p(t)$,

   $y_p(t) = Ce^{-t} + De^{-3t}$

   $y_p'(t) = -Ce^{-t} - 3De^{-3t}$

   Substitute this equations to $y'(t) + 5y(t) = x(t)$ (u(t)'s will cancel):

   $-Ce^{-t} - 3De^{-3t} + 5(Ce^{-t} + De^{-3t}) = e^{-t} + e^{-3t}$

   $\Longrightarrow 4Ce^{-t} + 2De^{-3t} = e^{-t} + e^{-3t} \Longrightarrow 4Ce^{-t} = e^{-t}$ and $2De^{-3t} = e^{-3t}$

   Therefore, $4C = 1 \Rightarrow C = \frac{1}{4}$ and $2D = 1 \Rightarrow D = \frac{1}{2}$

   As mentioned before $y(t) = y_h(t) + y_p(t)$, and

   $y(t) = Ae^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t}$

   Question mention that the system is initially at rest. Namely, y(0)=0.

   $y(0) = Ae^0 + \frac{1}{4}e^0 + \frac{1}{2}e^0 = 0$

   $\Longrightarrow A + \frac{1}{4} + \frac{1}{2} = 0 \Longrightarrow A = \frac{-3}{4}$

   In conculution,

   $y(t) = (\frac{-3}{4}e^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t})u(t)$

   (u(t) is taken into account now)

2. (a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$= x[-1]h[n+1] + x[0]h[n]$

$= h[n+1] + 2(h[n])$

$= (\delta[n] + 2\delta[n+2]) + 2(\delta[n-1] + 2\delta[n+1])$

$= 2\delta[n-1] + \delta[n] + 4\delta[n+1] + 2\delta[n+2]$

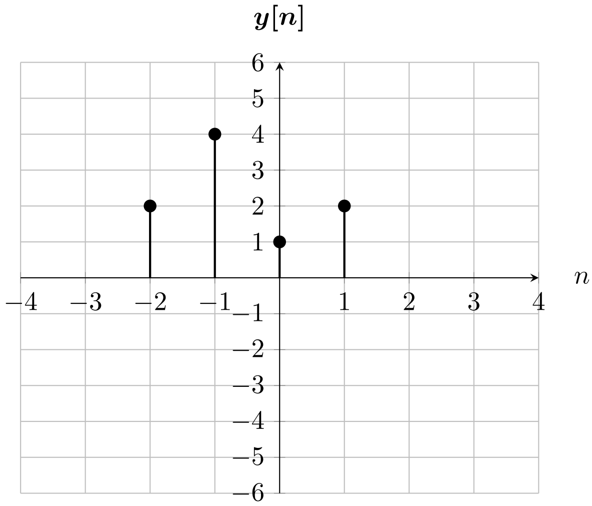

Figure 1: $y[n]$ - n

(b) $x(t) = u(t-1) + u(t+1)$

$\frac{dx(t)}{dt} = \delta[t-1] + \delta[t+1]$

$y(t) = (\delta(t-1) + \delta[t+1]) * h(t)$

$y(t) = (\delta(t-1) * h(t)) + (\delta[t+1] * h(t))$

$(\delta(t-1) * h(t)) = \int_{-\infty}^{\infty} \delta(\tau - 1)h(t-\tau)\, d\tau$

Substituting $\tau = t - \lambda$, we get: $\delta(t-1) * h(t) = \int_{-\infty}^{\infty} \delta(t-1-\lambda)h(\lambda)\, d\lambda$

Since $\delta(t-1-\lambda)$ is non-zero only when $\lambda = t-1$, the above expression simplifies to:

$\delta(t-1) * h(t) = h(t-1)$

Same calculation for $\delta[t+1] * h(t)$ and hence:

$\delta[t+1] * h(t) = h(t+1)$

Therefore;

$y(t) = h(t-1) + h(t+1)$

$= e^{1-t} sin(t-1)u(t-1) + e^{-1-t} sin(t+1)u(t+1)$

3. (a) $y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2t+2\tau}u(t-\tau)\, d\tau$

For $t < 0$ $x(\tau)$ and $h(t-\tau)$ don't overlap.

For $t \geq 0$, $x(\tau)$ and $h(t-\tau)$ have same overlapping between 0 and t. So;

$= \int_0^t e^{-\tau}e^{-2t+2\tau}\, d\tau$

$= e^{-2t}\int_0^t e^{\tau}\, d\tau = e^{-2t}(e^t - 1)u(t) = (e^{-t} - e^{-2t})u(t)$

(b) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

$= \int_{-\infty}^{\infty} e^{3(t-\tau)}u(t-\tau)(u(\tau)-u(\tau-1))\,d\tau$

If we write this equation piecewise:

$y(t) = \int_{-\infty}^{\infty} e^{3(t-\tau)}u(t-\tau)u(\tau)\,d\tau - \int_{-\infty}^{\infty} e^{3(t-\tau)}u(t-\tau)u(\tau-1)\,d\tau$

$= \int_{\tau=0}^{t} e^{3(t-\tau)}\,d\tau - \int_{\tau=1}^{t} e^{3(t-\tau)}\,d\tau$

If $t < 1, then:$

$y(t) = e^{3t}\int_{\tau=0}^{t} e^{-3\tau}\,d\tau = \frac{-1}{3}e^{3t}(e^{-3t}-1)$

$= \frac{1}{3}(e^{3t}-1)(u(t)-u(t-1))$

If $t > 1$:

$y(t) = \int_{\tau=0}^{t} e^{3(t-\tau)}\,d\tau - \int_{\tau=1}^{t} e^{3(t-\tau)}\,d\tau$

$= \int_{\tau=0}^{1} e^{3(t-\tau)}\,d\tau + \int_{\tau=1}^{t} e^{3(t-\tau)}\,d\tau - \int_{\tau=1}^{t} e^{3(t-\tau)}\,d\tau$

$= \int_{\tau=0}^{1} e^{3(t-\tau)}\,d\tau = \frac{-1}{3}e^{3t}(e^{-3}-1)$

$= \frac{1}{3}(e^{3t}-e^{3t-3})u(t-1)$

4. (a) We will solve this question with characteristic equation:

$r^2 - r - 1 = 0$

If we solve this equation, we will find $\Delta = 1 + 2^2 = 5$ and therefore;

$r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.

$y[n] = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$

Now we will use y[0] = 1 and y[1] = 1.

$y[0] = c_1 + c_2 = 1$, and

$y[1] = c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2})$

$= \frac{1}{2}(c_1+c_2) + \frac{\sqrt{5}}{2}(c_1-c_2) = 1$

if we subtitute $c_1 + c_2$ to $\frac{1}{2}(c_1+c_2) + \frac{\sqrt{5}}{2}(c_1-c_2)$,we will obtain:

$\frac{1}{2} + \frac{\sqrt{5}}{2}(c_1-c_2) = 1$.
Then,

$c_1 - c_2 = \frac{1}{\sqrt{5}}$

We know $c_1 + c_2 = 1$ and $c_1 - c_2 = \frac{1}{\sqrt{5}}$. Then

$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$ and $c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$

Therefore;

$y(n) = \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

(b) Again we will solve this question with characteristic equation: $r^3 - 6r^2 + 13r - 10 = 0$

If we try to factor this expression: $(r-2)(r^2-4r+5) = 0$.Now we will find roots of this equation.First root $r_1 = 2$.To find other roots we need to find roots of $(r^2 - 4r + 5) = 0$.

Firstly we need to find delta of $(r^2 - 4r + 5) = 0$:

$\Delta = 16 - 20 = -4$ and roots of it are: $r_2 = \frac{4+\sqrt{-4}}{2} = 2 + i$ and $r_3 = \frac{4-\sqrt{-4}}{2} = 2 - i$.Then,

$y(t) = c_1 e^{2t} + c_2 e^{(2+i)t} + c_3 e^{(2-i)t}$

To find c_1, c_2, c_3, we will use $y''(0) = 3, y'(0) = \frac{3}{2}, y(0) = 1$

$y'(t) = 2c_1 e^{2t} + (2+i)c_2 e^{(2+i)t} + (2-i)c_3 e^{(2-i)t}$, and

$y''(t) = 4c_1 e^{2t} + (3+4i)c_2 e^{(2+i)t} + (3-4i)c_3 e^{(2-i)t}$

$y(0) = c_1 + c_2 + c_3 = 1$

$y'(0) = 2c_1 + (2+i)c_2 + (2-i)c_3 = \frac{3}{2}$

$y''(0) = 4c_1 + (3+4i)c_2 + (3-4i)c_3 = 3$

Now lets do some calculation,

$2c_1 + (2+i)c_2 + (2-i)c_3 = 2(c_1 + c_2 + c_3) + i(c_2 - c_3) = 2 + i(c_2 - c_3) = \frac{3}{2}$

$\Longrightarrow i(c_2 - c_3) = \frac{-1}{2}$

$4c_1 + (3+4i)c_2 + (3-4i)c_3 = 3(c_1 + c_2 + c_3) + c_1 + 4i(c_2 - c_3) = 3 + c_1 + 4i(c_2 - c_3) = 3$

$\Longrightarrow c_1 + 4i(c_2 - c_3) = 0 \Longrightarrow c_1 - 2 = 0$

$\Longrightarrow c_1 = 2$

$c_1 + c_2 + c_3 = 1 \Longrightarrow c_2 + c_3 = -1$

If we sum $c_2 + c_3 = -1$ and $c_2 - c_3 = \frac{i}{2}$

$2c_2 = \frac{i}{2} - 1 \Longrightarrow c_2 = \frac{i-2}{4}$

$\frac{i-2}{4} + c_3 = -1 \Longrightarrow c_3 = \frac{-i-2}{4}$.

Therefore;

$y(t) = 2e^{2t} + (\frac{i-2}{4})e^{(2+i)t} + (\frac{-i-2}{4})e^{(2-i)t} = 2e^{2t} - e^{2t}cos(t) - \frac{1}{2}e^{2t}sin(t)$

5. (a) $y_p(t) = Acos(5t) + Bsin(5t)$

$y_p'(t) = -5Asin(5t) + 5Bcos(5t)$

$y_p''(t) = -25Acos(5t) - 25Bsin(5t)$

$y_p''(t) + 5y_p'(t) + 6y_p(t) = x(t)$

$= -25Acos(5t) - 25Bsin(5t) - 25Asin(5t) + 25Bcos(5t) + 6Acos(5t) + 6Bsin(5t)$

$= (25B - 19A)cos(5t) - (25A + 19B)sin(5t) = cos(5t)$

$25B - 19A = 1$ and $25A + 19B = 0$

$\Longrightarrow A = \frac{-19}{986}$ and $B = \frac{25}{986}$

Therefore,

$y_p(t) = \frac{-19}{986} cos(5t) + \frac{25}{986} sin(5t)$

(b) To find the homogeneous solution, we will use characteristic equation:

$r^2 + 5r + 6 = 0$ and roots of this equation are:

$r_1 = -3$ and $r_2 = -2$. Therefore,

$y_h(t) = Me^{-3t} + Ke^{-2t}$

(c) $y(t) = y_p(t) + y_h(t) = \frac{-19}{986} cos(5t) + \frac{25}{986} sin(5t) + Me^{-3t} + Ke^{-2t}$

The system is initially at rest, so $y(0) = y'(0) = 0$.

$y'(t) = \frac{95}{986} sin(5t) + \frac{125}{986} cos(5t) - 3Me^{-3t} - 2Ke^{-2t}$

$y(0) = \frac{-19}{986} + K + M = 0 \Longrightarrow K + M = \frac{19}{986}$

$y'(0) = \frac{125}{986} - 2K - 3M = 0 \Longrightarrow 2K + 3M = \frac{125}{986}$

If we do some calculations,

$\Longrightarrow K = \frac{-68}{986}$ and $M = \frac{87}{986}$

Hence,

$y(t) = \frac{-19}{986} cos(5t) + \frac{25}{986} sin(5t) + \frac{87}{986} e^{-3t} + \frac{-68}{986} e^{-2t}$

6. (a) The system consists of two parts,the first part gets input x[n] and gives output w[n],the second part takes input w[n] and gives output y[n].

If we give x[n]=$\delta[n]$ as an input to the first part of the system ,we get $h_0[n]$ as output which is impulse response of this part.

So, $w[n] = h_0[n]$

Also,we know $w[n] - \frac{1}{2}w[n-1] = x[n]$.Hence;

$h_0[n] - \frac{1}{2}h_0[n-1] = \delta[n]$

$h_0[n] = \frac{1}{2}h_0[n-1] + \delta[n]$

Since the system is initially at rest,$h_0[n] = 0$ for $n < 0$.

$h_0[0] = \frac{1}{2}h_0[-1] + \delta[0] = 1$

$h_0[1] = \frac{1}{2}h_0[0] + \delta[1] = \frac{1}{2}$

$h_0[2] = \frac{1}{2}h_0[1] + \delta[2] = (\frac{1}{2})^2$

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$h_0[n] = (\frac{1}{2})^n$ for $n \geq 0$

Thus,

$h_0[n] = (\frac{1}{2})^n u(n)$

(b) To find the overall impulse response h[n],we need to calculate

$h[n] = h_0[n] * h_0[n] = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^k u[k](\frac{1}{2})^{n-k}u[n-k]$

For $k < 0$, u[k]=0 and for $k \geq n$,u[n-k]=0

Therefore,the sum is reduced to

$= \sum_{k=0}^{n}(\frac{1}{2})^k(\frac{1}{2})^{n-k} = \sum_{k=0}^{n}(\frac{1}{2})^n = n(\frac{1}{2})^n$

For $n \geq 0$, $h[n] = n(\frac{1}{2})^n$ otherwise $h[n] = 0$

Thus,

$\implies h[n] = n(\frac{1}{2})^n u[n]$

(c) First part and second part of the system are actually identical since them impulse response are same. So,

$w[n] - \frac{1}{2}w[n-1] = x[n] \rightarrow i) y[n] - \frac{1}{2}y[n-1] = w[n]$

Since the system is time-invariant

$ii) y[n-1] - \frac{1}{2}y[n-2] = w[n-1]$

Now take the difference of $i)$ and $ii)$ by multiplying $ii)$ by $\dfrac{1}{2}$

$y[n] - \frac{1}{2}y[n-1] - (\frac{1}{2}y[n-1] - \frac{1}{4}y[n-2]) = w[n] - \frac{1}{2}w[n-1] = x[n]$

Thus,

$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$

represents the relationship between input and output.

7. (a)

```python
import matplotlib.pyplot as plt
import numpy as np
def convolution(signal_x,starting_index_of_x,signal_h,starting_index_of_h):
    y = []
    s_yi = starting_index_of_x + starting_index_of_h
    time=list(range(int(s_yi),int(s_yi)+len(signal_x)+len(signal_h)-1))
    for n in range(len(signal_x)+len(signal_h)-1):
        y.append(0)
        for k in range(len(signal_x)):
            if n - k >= 0 and n - k < len(signal_h):
                y[n] += signal_x[k] * signal_h[n - k]
    return (time, y)


def load_data(filepath):
    file=open(filepath)
    data=list(map(lambda x:float(x),file.readline().split(",")))
    starting_index=data[0]
    signal=data[1:]
    return signal,starting_index

def createdelta():
    x=[]
    x.append(1)
    starting_index=5
    return x,starting_index




signal_x,starting_index_of_x=load_data("./hw2_signal.csv")
signal_h,starting_index_of_h=createdelta()
r=convolution(signal_x,starting_index_of_x,signal_h,starting_index_of_h)
print(r)
plt.stem(r[0],r[1])
plt.xlabel("n",fontsize=20)
plt.ylabel("y[n]",fontsize=20)
plt.xlim(r[0][0]-10,r[0][-1]+10)
plt.xticks(np.arange(r[0][0]-10,r[0][-1]+10,10))
plt.savefig("Question 7 part_a.pdf",format="pdf",bbox_inches="tight")
plt.cla()
```

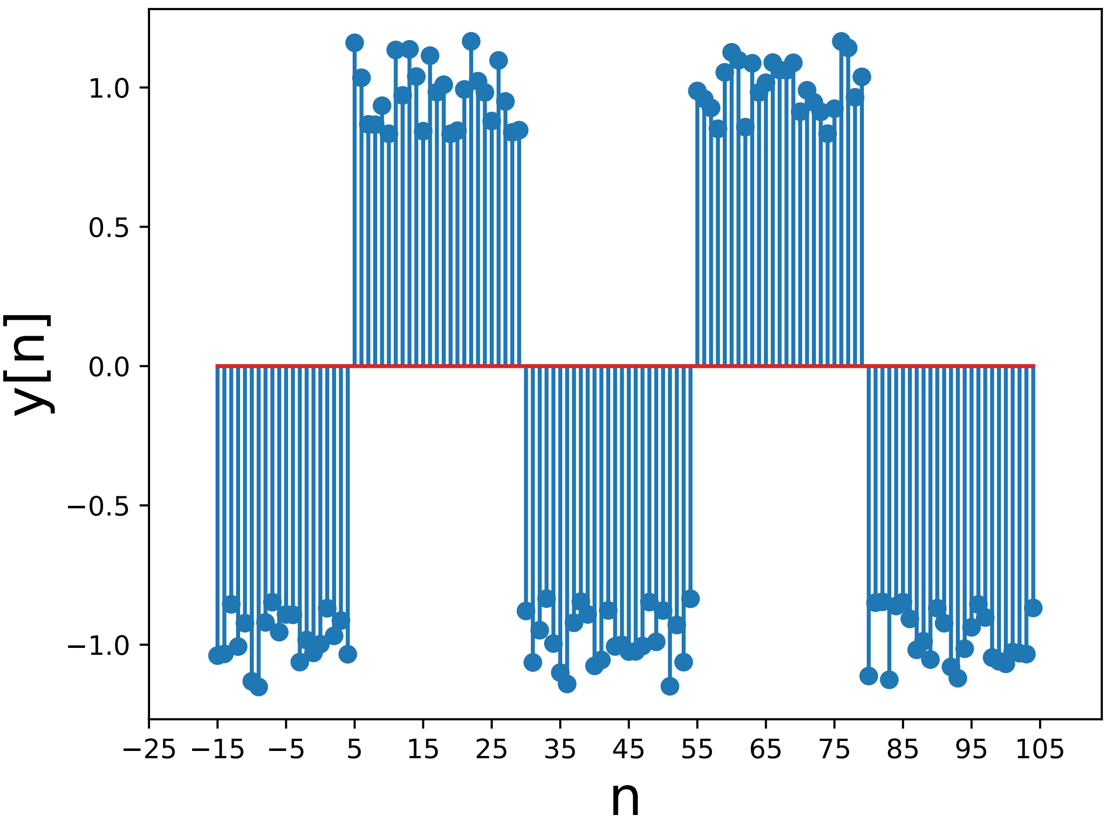

Figure 2: Question 7 $part_a$

It is shifted to right 5 units.

(b)
```
import matplotlib.pyplot as plt
import numpy as np
def convolution(signal_x,starting_index_of_x,signal_h,starting_index_of_h):
    y = []
    s_yi = starting_index_of_x + starting_index_of_h
    time=list(range(int(s_yi),int(s_yi)+len(signal_x)+len(signal_h)-1))
    for n in range(len(signal_x)+len(signal_h)-1):
        y.append(0)
        for k in range(len(signal_x)):
            if n - k >= 0 and n - k < len(signal_h):
                y[n] += signal_x[k] * signal_h[n - k]
    return (time, y)


def load_data(filepath):
    file=open(filepath)
    data=list(map(lambda x:float(x),file.readline().split(",")))
    starting_index=data[0]
    signal=data[1:]
    return signal,starting_index

def createNpoint(n):
    starting_index=0
    x=[]
    for i in range(n):
        x.append(1/n)
    return x,starting_index

N=[3,5,10,20]

signal_x,starting_index_of_x=load_data("./hw2_signal.csv")
for i in N:
    signal_h,starting_index_of_h=createNpoint(i)
    r=convolution(signal_x,starting_index_of_x,signal_h,starting_index_of_h)
    plt.stem(r[0],r[1])
    plt.xlabel("n",fontsize=20)
    plt.ylabel("y[n]",fontsize=20)
    plt.xlim(r[0][0]-10,r[0][-1]+10)
    plt.xticks(np.arange(r[0][0]-10,r[0][-1]+10,10))
    plt.savefig(f"Question 7 part_b N={i}.pdf",format="pdf",bbox_inches="tight")
    plt.cla()
```

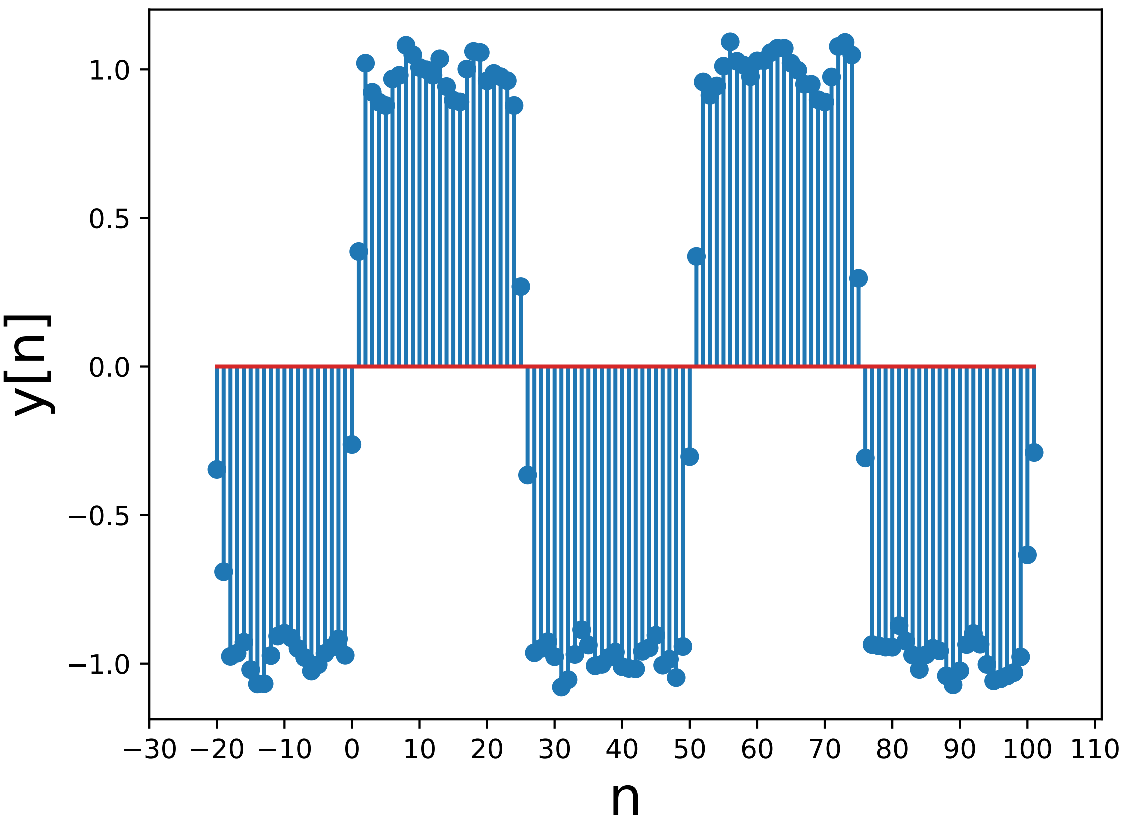

Figure 3: Question 7 part$_b N = 3$

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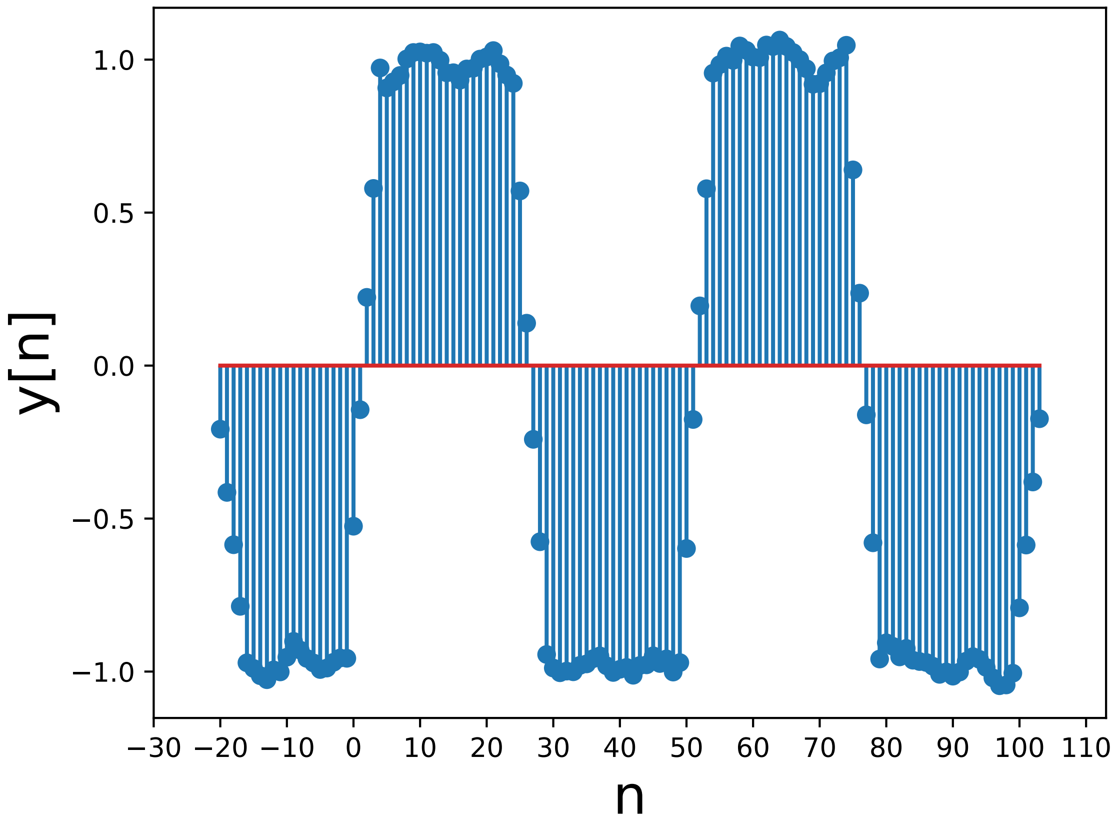

Figure 4: Question 7 part$_b N = 5$

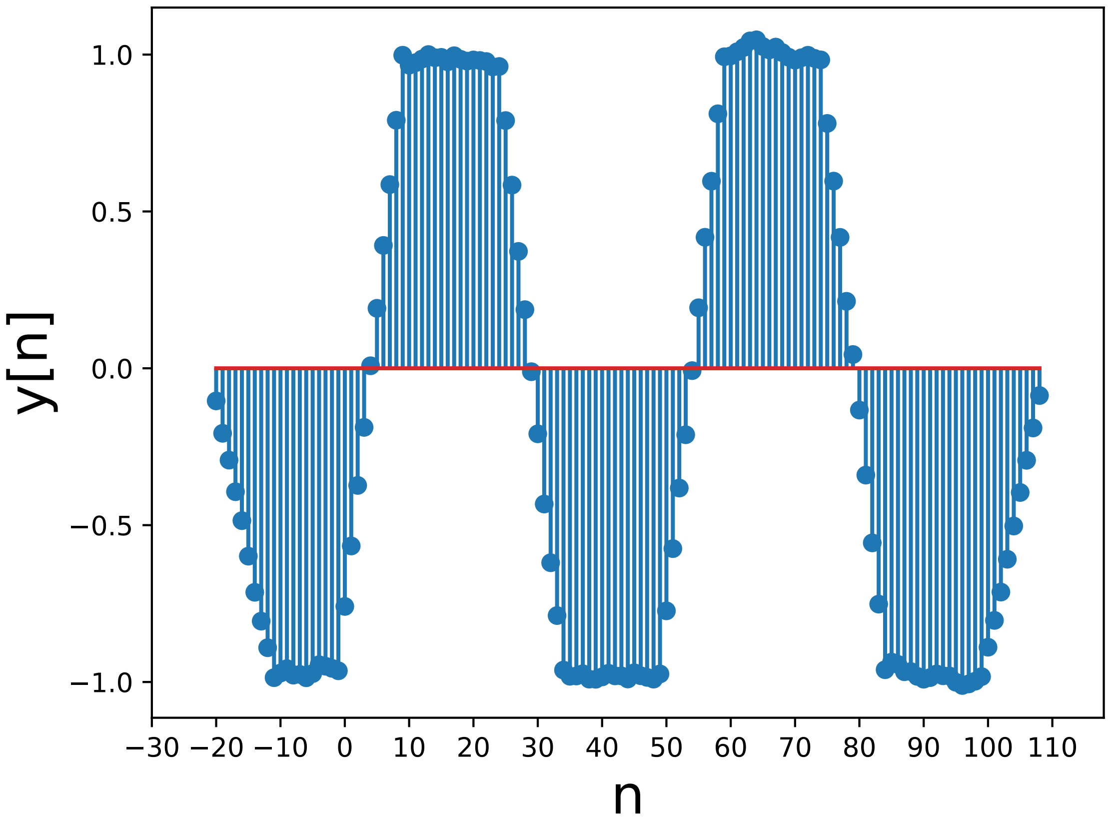

Figure 5: Question 7 part$_b N = 10$

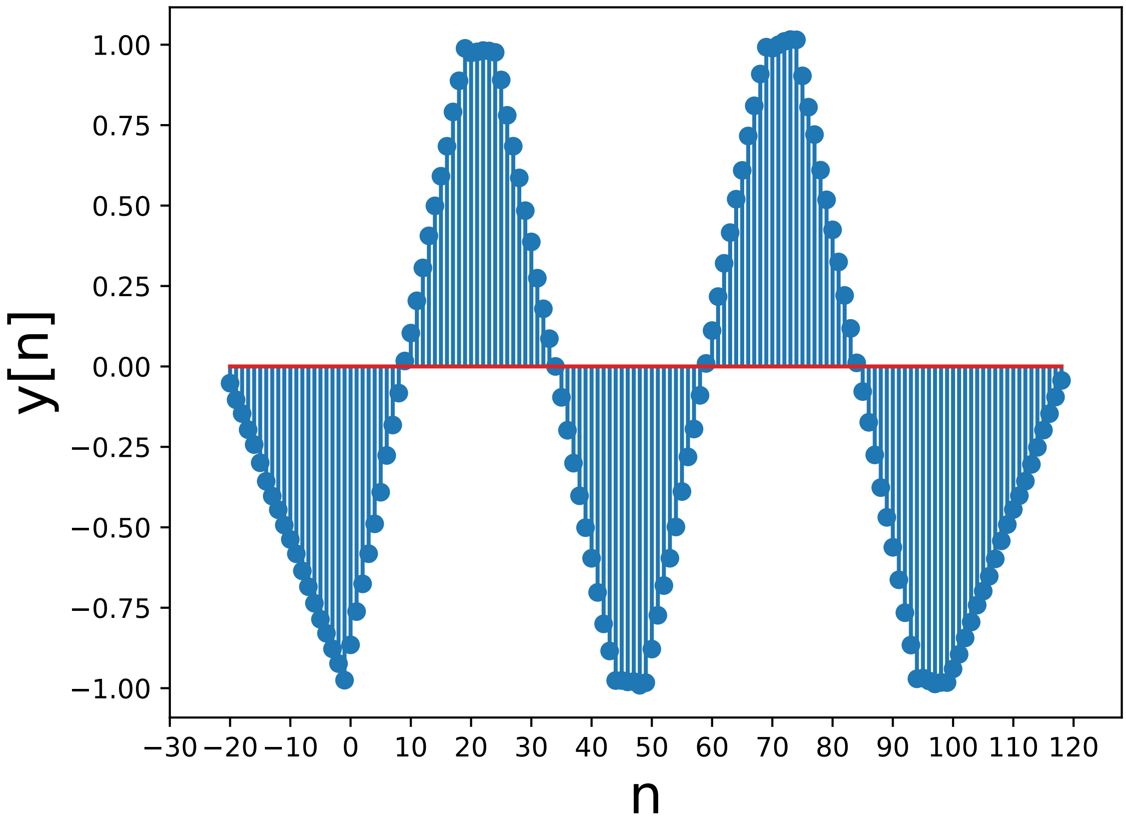

Figure 6: Question 7 part$_b N = 20$

This equation shows that the output at each time index n is obtained by taking a weighted sum of the current and past N-1 input samples, with the weights given by the coefficients of the filter h[n].

Intuitively, the effect of the filter is to smooth the input signal, reducing the high-frequency components and emphasizing the low-frequency components. The degree of smoothing is controlled by the length of N: a bigger N will result in a smoother output, while a smaller N will preserve more of the high-frequency details of the input.

In summary, the effect of h[n] on x[n] is to average the input signal over value of N, resulting in a smoothed output signal.