Student Information

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Answer 1

a)

For X and Y to be independent, they must satisfy the condition:

$$f_{(X,Y)}(x,y) = f_X(x).f_Y(y) \qquad \text{for all x} \in [-1,1], y \in [-1,1] \qquad (\star)$$

where f_X and f_Y are marginal densities of X and Y respectively.

Now, let's calculate marginal densities, namely f_X and f_Y.

for x $\in [-1,1]$;

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{(X,Y)}(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy = \frac{2\sqrt{1-x^2}}{\pi}$$

for y $\in [-1,1]$;

$$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{(X,Y)}(x,y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{2\sqrt{1-y^2}}{\pi}$$

Observe that for $x^2 + y^2 \leq 1$

$$f_{(X,Y)}(x,y) = \frac{1}{\pi} \neq \frac{4\sqrt{1-x^2}\sqrt{1-y^2}}{\pi^2} = f_X(x).f_Y(y)$$

So, they do not satisfy the condition $(\star)$.

Hence, X and Y are **NOT** independent.

b)

Marginal pdfs for X and Y denoted by $f_X(x)$ and $f_Y(y)$ respectively can be found as follows:

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{(X,Y)}(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy = \frac{1}{\pi}y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{2\sqrt{1-x^2}}{\pi}$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{(X,Y)}(x,y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{1}{\pi}x\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \frac{2\sqrt{1-y^2}}{\pi}$$

Therefore ;

marginal pdf of X is $f_X(x) = \dfrac{2\sqrt{1-x^2}}{\pi}$ **,** $\mathbf{x} \in [-1, 1]$

marginal pdf of Y is $f_Y(y) = \dfrac{2\sqrt{1-y^2}}{\pi}$ **,** $\mathbf{y} \in [-1, 1]$

c)

$$\mu_X = E(X) = \int_{-1}^{1} x f_X(x)dx = \int_{-1}^{1} \frac{x.2\sqrt{1-x^2}}{\pi} = \int_{-1}^{1} \frac{2x\sqrt{1-x^2}}{\pi}$$

applying the substitution:

$u = \sqrt{1-x^2}$
$du = -2xdx$

the integral becomes:

$$\int \frac{-\sqrt{u}}{\pi}du = \frac{-2}{3}\frac{u^{3/2}}{\pi} = \frac{-2}{3}\frac{(1-x^2)^{3/2}}{\pi}$$

putting boundaries;

$$\frac{-2}{3}\frac{(1-x^2)^{3/2}}{\pi}\Big|_{-1}^{1} = 0$$

Hence, expected value of X, E(X), is 0.

d)

$\sigma^2 = Var(X) = \int_{-1}^{1} x^2 f_X(x)dx - \mu^2 = \int_{-1}^{1} x^2 f_X(x)dx - 0^2 = \int_{-1}^{1} x^2 f_X(x)dx = \int_{-1}^{1} \frac{x^2.2\sqrt{1-x^2}}{\pi}dx$
$= \frac{2}{\pi}\int_{-1}^{1} x^2\sqrt{1-x^2}dx$

Now, let's compute the integral $\int x^2\sqrt{1-x^2}dx$, then put boundaries and then multiply it with $\frac{2}{\pi}$.

Apply the substitution on $\int x^2\sqrt{1-x^2}dx$:

$x = sin(u)$
$dx = cos(u)du$

$\int sin^2(u).\sqrt{1-sin^2(u)}.cos(u).du = \int sin^2(u).cos(u).cos(u).du = \int sin^2(u).cos^2(u).du = \frac{1}{4}\int sin^2(2u)du$
$= \frac{1}{4}\int \frac{1}{2}(1-cos(4u))du = \frac{1}{8}\int(1-cos(4u))du = \frac{1}{8}(u - \frac{sin(4u)}{4}) = \frac{1}{8}u - \frac{1}{32}sin(4u)$

Now back-substitute:

$\int x^2\sqrt{1-x^2}dx = \frac{1}{8}.arcsin(x) - \frac{1}{32}.sin(4.arcsin(x)) + C$

By putting boundaries;

$\int_{-1}^{1} x^2\sqrt{1-x^2}dx = [\frac{1}{8}arcsin(x) - \frac{1}{32}sin(4.arcsin(x)) + C]\Big|_{-1}^{1} = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}$

Multiply by $\frac{2}{\pi}$:

$$Var(X) = \frac{2}{\pi}\int_{-1}^{1} x^2\sqrt{1-x^2} = \frac{2}{\pi}.\frac{\pi}{8} = \frac{1}{4} = 0.25$$

Hence, variance of X, $Var(X)$ is $\frac{1}{4} = 0.25$.

Answer 2

a)

T_A and T_B are uniformly distributed on [0,100], so their marginal density functions are as follows:

$$f_A(t_A) = \frac{1}{100-0} = \frac{1}{100}$$
$$\mathbf{f}_B(t_B) = \frac{1}{100-0} = \frac{1}{100}$$

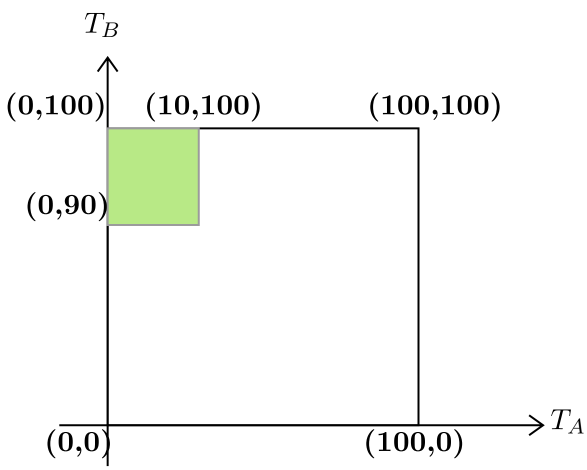
Because they are independent, their joint density function can be found as follows:

$$f(t_A, t_B) = f_A(t_A).f_B(t_B) = \frac{1}{100}.\frac{1}{100} = \frac{1}{10000}$$

Their joint cumulative distribution function (cdf) can be found as follows:

$$F(t_A, t_B) = \int\int f(t_A, t_B).dt_A.dt_B = \int\int \frac{1}{10000}.dt_A.dt_B = \frac{t_A.t_B}{10000}$$
$$F(t_A, t_B) = \frac{t_A.t_B}{10000}$$

b)



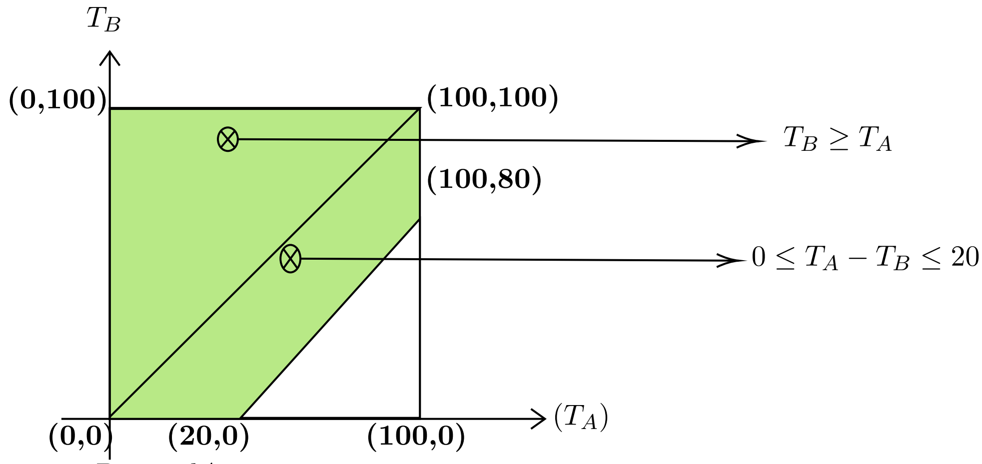
Since T_A and T_B are uniformly distributed, we can compute the probability as follows:

$$\mathbf{P} = \frac{DesiredArea}{TotalArea} = \frac{10.10}{100.100} = \frac{1}{100}$$

c)

The event is E = $\{T_A - T_B \leq 20\}$.

Since T_A **and** T_B **are uniformly distributed, the probability can be found as follows:**

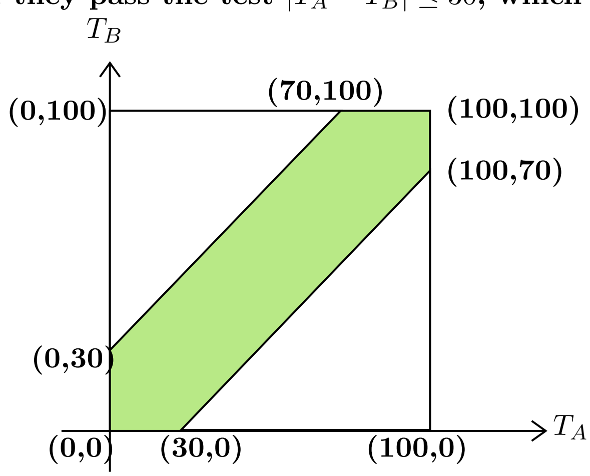


P(E) = $\dfrac{Desired Area}{Total Area} = \dfrac{6800}{1000} = \dfrac{68}{100}$

d)

They fail if $|T_A - T_B| > 30$ **.**

So, they pass the test $|T_A - T_B| \leq 30$**, which also means they pass if** $-30 \leq T_A - T_b \leq 30$**.**



Since T_A **and** T_B **are independent, the probability can be found as follows:**

P(pass) = $\dfrac{Desired Area}{Total Area} = \dfrac{5100}{10000} = \dfrac{51}{100}$

Answer 3

a)

Note that cdf of an exponential random variable X with λ is. F(t) = P(X $\leq t) = 1-$e$^{-\lambda t}$ So, complement of it is P(X > t) = $e^{-\lambda t}$.

Now, we will examine T = min$(X_1, X_2, ..., X_n)$ more closely.

If we want P(T > t) for some t > 0, we just want

P(T > t) = P(min$(X_1, X_2, ..., X_N) >$ t) = P($X_1 > t$, $X_2 > t$,....,$X_n > t$)

= P($X_1 >$ t) . P($X_2 >$ t) . ... .P($X_n >$ t)

= $e^{-\lambda_1 t}$. $e^{-\lambda_2 t}$. .... $e^{-\lambda_n t}$

= $e^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$

So, we found P(T > t) = $e^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$. And now we can find $F_T(t)$, cdf of T.

$F_T(t)$ **= P (T** $\leq t) = 1 - P(T > t) = 1-$**e**$^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$

As you can see above, T itself is exponential with $\lambda = \lambda_1 + \lambda_2 + ...\lambda_n$ and cdf $F_T(t)$

$$F_T(t) = \mathbf{1} \text{ - } e^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$$

b)

Let $X_1, X_2, ...X_n$ **be independent exponential random variables with** λ_i **for i = 1,2,....n, and T =** $\mathbf{min}(X_1, X_2, ..., X_n)$**.**

Now, let's investigate complement of cdf of T.

P(T $>$ **t) = P(min**$(X_1, X_2, ..., X_N) >$ **t) = P(**$X_1 > t$**,** $X_2 > t$**,....,**$X_n > t$**)**

= P($X_1 >$ **t) . P(**$X_2 >$ **t) . ... P(**$X_n >$ **t)**

$= e^{-\lambda_1 t}$ **.** $e^{-\lambda_2 t}$**. ....** $e^{-\lambda_n t}$

$= e^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$

So, cdf of T, $F_T(t) =$ **1 -** $e^{-(\lambda_1+\lambda_2+.....+\lambda_n)t}$

This shows that T itself is also an exponential random variable.
Hence, we get the below fact:

Fact: Let $X_1, X_2, ...X_n$ **be independent exponential random variables with** λ_i **for i = 1,2,....n. Then, T = min**$(X_1, X_2, ..., X_n)$ **is also an exponential random variable with** $\lambda = \lambda_1 + \lambda_2 + ...\lambda_n$ **.**

The question is asking for E(T = min$(C_1, C_2, ..., C_{10})$**) .**

Since E(C_n**)** $= \dfrac{10}{n}$ **for** $n = 1, 2, ...10$**, we get** $\lambda_n = \dfrac{n}{10} for$ **n = 1,2,...,10**

Then, $\lambda_T = \lambda_1 + \lambda_2 + ... + \lambda_{10} = \dfrac{1+2+3+4+5+6+7+8+9+10}{10} = \dfrac{55}{10}$

Therefore, the expected time before one of the computer fails, E(T)

$$\mathbf{E(T)} = \frac{1}{\lambda_T} = \frac{10}{55} = \frac{2}{11}$$

Answer 4

a)

The number X of undergraduate students has binomial distribution with

$\mathbf{n = 100}$
$\mathbf{p = 0.74}$
$\mu = \mathbf{np = 74}$
$\sigma = \sqrt{np(1-p)} = 4.3863$

Applying the Central Limit Theorem with the continuity correction,

$$\mathbf{P(X \geq} \frac{70}{100}.100) = P(X \geq 70) = 1 - P(X < 70)$$
$$= 1 - P(X < 69.5) = 1 - P(\frac{X-74}{4.3863} \leq \frac{69.5-74}{4.3863})$$
$$= 1 - \phi(-1.0259) \mathbf{= 1 -} \phi(-1.03) \mathbf{= 1 - 0.1515 = 0.8485.}$$

Answer = 0.8485

b)

The number X of participants pursuing a doctoral degree has binomial distribution with

$\mathbf{n = 100}$
$\mathbf{p = 0.10}$
$\mu = \mathbf{np = 10}$
$\sigma = \sqrt{np(1-p)} = 3$

Applying the Central Limit Theorem with the continuity correction,

$$\mathbf{P(X \leq} \frac{5}{100}.100) = P(X \leq 5) = P(X \leq 5.5) =$$
$$= P(\frac{x-10}{3} < \frac{5.5-10}{3}) = \phi(-1.5) \mathbf{= 0.0668}$$

Answer = 0.0668

(Values of standart normal distribution cdf, ϕ, was taken from TABLE A4 in textbook.)