CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 2 Solutions

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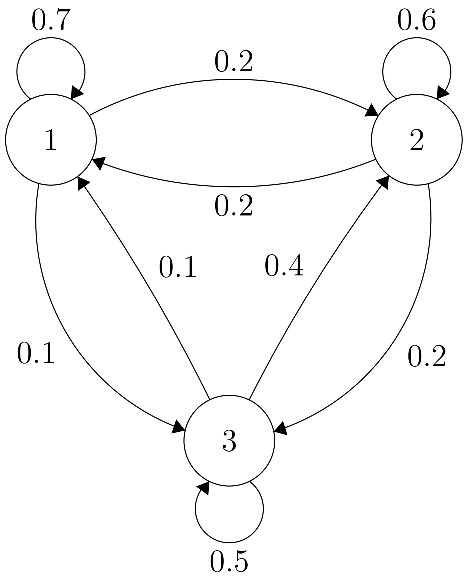
1. (a) Markov chain can be modeled with the following diagram where 1 represents professionals, 2 represents skilled laborers and 3 represents unskilled laborers.



The state transition matrix corresponding to this markov chain is found as:

$$A = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

(b) To find this probability, we need to find A^2

$$A^2 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.54 & 0.3 & 0.16 \\ 0.28 & 0.48 & 0.24 \\ 0.2 & 0.46 & 0.34 \end{bmatrix}$$

The entry a_{31} of this matrix, which is 0.2, gives the probability asked in the question.

Answer: 0.2

(c) Again, in order to find this probability, we need to find A^2.

$$A^2 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} . \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.54 & 0.3 & 0.16 \\ 0.28 & 0.48 & 0.24 \\ 0.2 & 0.46 & 0.34 \end{bmatrix}$$

The entry a_{11} of this matrix, which is 0.54, gives the probability asked in the question.

Answer: 0.54

(d) First we need to find the transpose of A:

$$A^T = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}$$

We know that $\lambda = 1$ is eigenvalue of stochastic matrices. We need to compute corresponding eigenvector such that $A^T.\ v = v$:

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.5 \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This gives the equations:

$$\begin{aligned} 0.7v_1 + 0.2v_2 + 0.1v_3 &= v_1 \\ 0.2v_1 + 0.6v_2 + 0.4v_3 &= v_2 \\ 0.1v_1 + 0.2v_2 + 0.5v_3 &= v_3 \\ v_1 + v_2 + v_3 &= 1 \end{aligned}$$

From there, we found $v_1 = \dfrac{6}{17}, v_2 = \dfrac{7}{17}, v_3 = \dfrac{4}{17}$.

So, we compute the eigenvector v as $v = \begin{bmatrix} 6/17 \\ 7/17 \\ 4/17 \end{bmatrix}$.

Hence, long term behavior of the matrix A, which is a regular Markov Chain representation, can be computed by replacing the rows of the matrix with transpose of the eigenvector, v^T.

$$A^{100} = A^{\infty} = \begin{bmatrix} 6/17 & 7/17 & 4/17 \\ 6/17 & 7/17 & 4/17 \\ 6/17 & 7/17 & 4/17 \end{bmatrix} = \begin{bmatrix} 0.352941 & 0.411765 & 0.235294 \\ 0.352941 & 0.411765 & 0.235294 \\ 0.352941 & 0.411765 & 0.235294 \end{bmatrix}$$

The behavior of the markov chain after 100^{th} generation can be summarized by the following:

100^{th} generation son of a person of any type is professional with the probability 0.352941, skilled laborer with the probability 0.411765 and unskilled laborer with the probability 0.235294.

That is,

$$p(100) \approx \lim_{t\to\infty} p(m) = \begin{bmatrix} 6/17 \\ 7/17 \\ 4/17 \end{bmatrix} = \begin{bmatrix} 0.352941 \\ 0.411765 \\ 0.235294 \end{bmatrix}$$

2. (a) In order to show that the system is controllable, we need to check controllability matrix M, if M has rank n (equal to number of rows and columns). If that's the case , the system is controllable.

Let's calculate M;

$$\mathrm{M} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

We know that;

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

So, we find M as;

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

M is already in echelon form and each column has a pivot. We see that M has 3 linearly independent rows, so it has rank $n = 3$.

This means that the sytem is **controllable**.

(b) We have $x(0) = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $x(3) = \begin{bmatrix}4\\4\\4\end{bmatrix}$.

$$x(1) = Ax(0) + Bu(0) = \begin{bmatrix}0 & 0 & 1\\-2 & 0 & -1\\0 & 1 & 0\end{bmatrix} . \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}u(0)\\0\\0\end{bmatrix} = \begin{bmatrix}u(0)+1\\-3\\1\end{bmatrix}$$

$$x(2) = Ax(1) + Bu(1) = \begin{bmatrix}0 & 0 & 1\\-2 & 0 & -1\\0 & 1 & 0\end{bmatrix} . \begin{bmatrix}u(0)+1\\-3\\1\end{bmatrix} + \begin{bmatrix}u(1)\\0\\0\end{bmatrix} = \begin{bmatrix}u(1)+1\\-2u(0)-3\\-3\end{bmatrix}$$

$$x(3) = Ax(2) + Bu(2) = \begin{bmatrix}0 & 0 & 1\\-2 & 0 & -1\\0 & 1 & 0\end{bmatrix} . \begin{bmatrix}u(1)+1\\-2u(0)-3\\-3\end{bmatrix} + \begin{bmatrix}u(2)\\0\\0\end{bmatrix} = \begin{bmatrix}u(2)-3\\-2u(1)+1\\-2u(0)-3\end{bmatrix}$$

It follows as:

$$x(3) = \begin{bmatrix}4\\4\\4\end{bmatrix} = \begin{bmatrix}u(2)-3\\-2u(1)+1\\-2u(0)-3\end{bmatrix}$$

From there, we find $u(0) = -7/2, u(1) = -3/2$ and $u(2) = 7$.

This sequence of inputs leads to system to the given point $\begin{bmatrix}4\\4\\4\end{bmatrix}$ in exactly 3 steps.

$$\begin{bmatrix}u(0)\\u(1)\\u(2)\end{bmatrix} = \begin{bmatrix}\frac{-7}{2}\\ \\ \frac{-3}{2}\\ \\ 7\end{bmatrix}$$

3. (a) By using the fact $y(k) = \begin{bmatrix} 0 & -2 & 4 \end{bmatrix} x(k)$, we obtained the following equations:

$$\begin{aligned} y(0) &= -2x_2(0) - 4x_3(0) \\ y(1) &= -2x_2(1) - 4x_3(1) \\ y(2) &= -2x_2(2) - 4x_3(2) \end{aligned} \tag{1}$$

Then using the system equation $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$, we get:

$$\begin{aligned} x_1(1) &= x_3(0) \\ x_2(1) &= -2x_1(0) - x_3(0) \\ x_3(1) &= x_2(0) \end{aligned} \tag{2}$$

$$\begin{aligned} x_1(2) &= x_3(1) \\ x_2(2) &= -2x_1(1) - x_3(1) \\ x_3(2) &= x_2(1) \end{aligned} \tag{3}$$

We replace the equations (2) into the equations (3), and we obtained:

$$\begin{aligned} x_1(2) &= x_2(0) \\ x_2(2) &= -2x_3(0) - x_2(0) \\ x_3(2) &= -2x_1(0) - x_3(0) \end{aligned} \tag{4}$$

Now, we replace what we found in the equations (2) and (4) into the equations (1). As a result, we get:

$$\begin{aligned} y(0) &= -2x_2(0) - 4x_3(0) \\ y(1) &= 4x_1(0) - 4x_2(0) + 2x_3(0) \\ y(2) &= 8x_1(0) + 2x_2(0) + 8x_3(0) \end{aligned} \tag{5}$$

By solving the system of equations in (5), we reach:

$$\begin{aligned} x_1(0) &= \frac{9}{32}y(0) - \frac{1}{16}y(1) + \frac{5}{32}y(2) \\ x_2(0) &= \frac{1}{8}y(0) - \frac{1}{4}y(1) + \frac{1}{8}y(2) \\ x_1(0) &= -\frac{5}{16}y(0) + \frac{1}{8}y(1) - \frac{1}{16}y(2) \end{aligned} \tag{6}$$

Therefore, we obtained $x(0)$ in terms of $y(0), y(1)$ and $y(2)$. That is:

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} \frac{9}{32}y(0) - \frac{1}{16}y(1) + \frac{5}{32}y(2) \\ \\ \frac{1}{8}y(0) - \frac{1}{4}y(1) + \frac{1}{8}y(2) \\ \\ -\frac{5}{16}y(0) + \frac{1}{8}y(1) - \frac{1}{16}y(2) \end{bmatrix}$$

Consequently, we could observe $x(0)$. Hence, the system is **observable**.

(b) Let's call observability matrix as M:

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}, \text{ where } C = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \text{ and, } A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Calculate CA and CA^2 as:

$$CA = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix} \text{ and } CA^2 = \begin{bmatrix} 8 & 2 & 8 \end{bmatrix}$$

We found observability matrix M as:

$$M = \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix}$$

Note that;

$$\alpha_1 \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 & -4 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 8 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

is satisfied only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, rows of M are linearly independent. We have 3 linearly independent rows, which indicates that observability matrix M has rank 3. This shows that the system is **observable**, which is consistent with the result we found in part a.

Additionally, notice that the matrix M gives the coefficients of $x_1(0), x_2(0)$ and $x_3(0)$ in the system of equations (5) in part (a). This also shows that the result I found in part (a) is accurate.

4. (a) $\dot{x}(t) = 3x^2 - 3x^3 = f(x)$

For this system, since it is a continuous case, we will use $\dot{x} = 0$:

$$3x^2 - 3x^3 = 0 \longrightarrow 3x^2(1-x) = 0 \longrightarrow x_{1,2} = 0\ ,\ \ x_3 = 1$$

We have 2 fixed points which are $\tilde{x_1} = 0$ and $\tilde{x_2} = 1$.

For the fixed point $\tilde{x} = 0$;

We use linearization formular which is

$$f(x) \approx \frac{df(x=\tilde{x})}{dx}(x - \tilde{x}) + f(\tilde{x})$$

We will check for the coefficient of x. If it is less than 0, this means $\tilde{x} = 0$ is stable. If it is greater than 0, this means $\tilde{x} = 0$ is unstable. If it is equal to 0, this means linearization test fails.

$$\frac{df(x)}{dx} = 6x - 9x^2 \qquad \longrightarrow \qquad \frac{df(x=0)}{dx} = 0$$

So, the test with linearization fails. We need to analyze the system around the fixed points.

For values $x < 0$ but close to 0, derivative $\dot{x}(t)$ is positive, this means x tends to go back to 0. However, for values $x > 0$ but close to 0, derivative $\dot{x}(t)$ is positive, this means $x(t)$ is increasing and moving farther and farther away from 0.

Thus, the fixed point $\tilde{x} = 0$ is **unstable**.

For the fixed point $\tilde{x} = 1$;

We again apply linearization formula;

$$f(x) \approx \frac{df(x=\tilde{x})}{dx}(x - \tilde{x}) + f(\tilde{x})$$

and check for the coefficient of x.

$$\frac{df(x=\tilde{x})}{dx} = 6x - 9x^2 \qquad \longrightarrow \qquad \frac{df(x=1)}{dx} = -3$$

Since $-3 < 0$, the fixed point $\tilde{x} = 1$ is **stable**.

(b)