Student Information

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Question 1

Say $\sum\limits_{n=0}^{\infty} a_n.x^n = A(x)$

Summing both side of the recurrence relation from n=1 to $n = \infty$, we obtain

$$\sum_{n=1}^{\infty} a_n.x^n = \sum_{n=1}^{\infty} (a_{n-1} + 2^n).x^n$$

$$A(x) - a_0.x^0 = \sum_{n=1}^{\infty} a_{n-1}.x^n + \sum_{n=1}^{\infty} 2^n.x^n$$

$$A(x) - 1 = x.\sum_{n=1}^{\infty} a_{n-1}.x^{n-1} + \sum_{n=1}^{\infty} 2^n.x^n$$

$$A(x) - 1 = x.\sum_{n=0}^{\infty} a_n.x^n + \sum_{n=1}^{\infty} 2^n.x^n$$

$$A(x) - 1 = x.A(x) + ((\sum_{n=0}^{\infty} 2^n.x^n) - 2^0.x^0)$$

$$A(x) - 1 = x.A(x) + ((\sum_{n=0}^{\infty} 2^n.x^n) - 1)$$

$$A(x) - 1 = x.A(x) + (\frac{1}{1-2x} - 1)$$

$$A(x) - x.A(x) = \frac{1}{1-2x} \quad \rightarrow \quad A(x)(1-x) = \frac{1}{1-2x} \quad \rightarrow \quad A(x) = \frac{1}{(1-x).(1-2x)}$$

$$A(x) = (-1).\frac{1}{1-x} + (2).\frac{1}{1-2x} \qquad \textbf{(By partial fractions)}$$

$$(1^0, 1^1, 1^2, ..., 1^n, ...) = \frac{1}{1-x} \implies ((-1).1^0, (-1).1^1, (-1).{}^2, ..., (-1).1^n, ...) = (-1).\frac{1}{1-x}$$

$$(2^0, 2^1, 2^2, ..., 2^n, ...) = \frac{1}{1-2x} \implies (2.2^0, 2.2^1, 2.{}^2, ..., 2.2^n, ...) = (2).\frac{1}{1-2x}$$

$$A(x) = (-1).\frac{1}{1-x} + (2).\frac{1}{1-2x} = ((-1).1^0, (-1).1^1, (-1).{}^2, ..., (-1).1^n, ...) + (2.2^0, 2.2^1, 2.{}^2, ..., 2.2^n, ...)$$

$$A(x) = \sum_{n=0}^{\infty} a_n.x^n = (2.2^0 - 1^0, 2.2^1 - 1^1, 2.2^2 - 1^2, ..., 2.2^n - 1^n, ...)$$

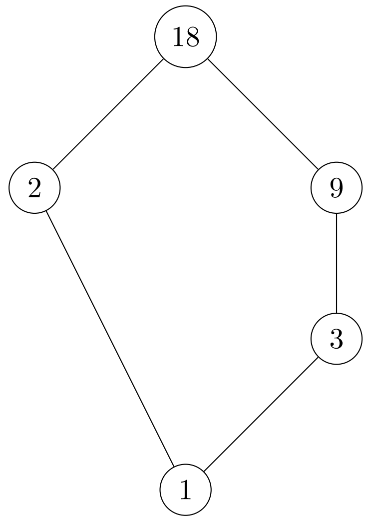
Hence, $a_n = 2.2^n - 1^n = 2^{n+1} - 1$.

The recurrence relation has been solved as $a_n = 2^{n+1} - 1$.

Question 2

$R = \{(1,1),(1,2),(1,3),(1,9),(1,18),(2,2),(2,18),(3,3),(3,9),(3,18),(9,9),(9,18),(18,18)\}$

a) Hasse Diagram of R



b) Matrix Representation of R

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Above matrix is representation of the relation aRb where a is represented by rows and b is represented by columns. Also, note that the columns are for {1,2,3,9,18} from left to right and the rows are for {1,2,3,9,18}.

c)

Yes, (A, R) is a lattice.

Explanation:

We know that a partial order relation is a **lattice** if for every pair of elements there is a unique Least Upper Bound (LUB) and unique Greatest Lower Bound (GLB).

$LUB(1, 2) = 2$ $\quad GLB(1, 2) = 1$

$LUB(1, 3) = 3$ $\quad GLB(1, 3) = 1$

$LUB(1, 9) = 9$ $\quad GLB(1, 9) = 1$

$LUB(1, 18) = 18$ $\quad GLB(1, 18) = 1$

$LUB(2, 3) = 18$ $\quad GLB(2, 3) = 1$

$LUB(2, 9) = 18$ $\quad GLB(2, 9) = 1$

$LUB(2, 18) = 18$ $\quad GLB(2, 18) = 2$

$LUB(3, 9) = 9$ $\quad GLB(3, 9) = 3$

$LUB(3, 18) = 18$ $\quad GLB(3, 18) = 3$

$LUB(9, 18) = 18$ $\quad GLB(9, 18) = 9$

So, for every pair of elements there is a unique Least Upper Bound (LUB) and unique Greatest Lower Bound (GLB).

Hence, (A, R) is a lattice.

d)

$R = \{(1,1),(1,2),(1,3),(1,9),(1,18),(2,2),(2,18),(3,3),(3,9),(3,18),(9,9),(9,18),(18,18)\}$

$S = \{(1,1),(2,1),(3,1),(9,1),(18,1),(2,2),(18,2),(3,3),(9,3),(18,3),(9,9),(18,9),(18,18)\}$

$R_s = R \cup S = \{(1,1),(1,2),(1,3),(1,9),(1,18),(2,2),(2,18),(2,1),(3,3),(3,9),(3,18),(3,1),$
$(9,3),(9,9),(9,18),(9,1),(18,18),(18,3),(18,2),(18,9),(18,1)\}$

Matrix representation of symmetric closure R_s of R

$$M_{Rs} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Above matrix is representation of the symmetric closure aR_sb where a is represented by rows and b is represented by columns. Also, note that the columns are for {1,2,3,9,18} from left to right and the rows are for {1,2,3,9,18}.

e)

i) 2 and 9 are not comparable because 2 $\not R$ 9 and 9 $\not R$ 2 .

ii) 3 and 18 are comparable because 18 $\not R$ 3 but 3 R 18 .

Question 3

a)

For a relation R on A to be anti-symmetric, $\forall x, y \in A\ (xRy \land yRx \to x = y)$ must be satisfied (i.e if xRy and x and y are distinct, then $y \not R x$, for any (x,y) ordered pair).

For the entries corresponding to ordered pairs that are located on the diagonal of the matrix representation of relation R on A (i.e (x,y) pairs such that x = y)

There are two possible choices for each entry (pair): i) xRx , ii) $x \not R x$
And, there are n such entries

So,we have 2^n different choices for these entries (pairs).

For the entries corresponding to ordered pairs which are **not** located on the diagonal of the matrix representation of relation R on A (i.e (x,y) pairs such that $x \neq y$) .

There are three possible choices for each entry (pair) :

i) xRy and yRx
ii) $x \not R\, y$ and yRx
iii) $x \not R\, y$ and $y \not R\, x$

And, there are $\binom{n}{2} = \dfrac{n.(n-1)}{2}$ such pairs

So, we have $3^{\binom{n}{2}} = 3^{\dfrac{n.(n-1)}{2}}$ different choices for these entries (pairs) .

Hence, there are $2^n.3^{\binom{n}{2}} = 2^n.3^{\dfrac{n.(n-1)}{2}}$ different anti-symmetric binary relations R on A.

Answer $= 2^n.3^{\binom{n}{2}} = 2^n.3^{\dfrac{n.(n-1)}{2}}$

b)

For a relation R on A to be reflexive, $\forall x \in A$ xRx must be satisfied.

For a relation R on A to be anti-symmetric, $\forall x, y \in A$ $(xRy \wedge yRx \rightarrow x = y)$ must be satisfied (i.e if xRy and x and y are distinct, then $y \not R x$, for any x,y ordered pair).

For the entries corresponding to ordered pairs which are located on the diagonal of the matrix representation of relation R (i.e (x,y) ordered pairs such that x = y)

We want R to be anti-symmetric, then there are two possible choices: i) xRx , ii) $x \not R x$

But, we want also R to be reflexive, so **only 1** choice would be: xRx

So, we have 1 choice for each pair. There are n such pairs. Hence there are $1^n = 1$ choices for these pairs.

For the entries corresponding to ordered pairs which are **not** located on the diagonal of the matrix representation of relation R on A (i.e (x,y) pairs such that $x \neq y$)

There are three possible choices for each entry (pair) :

i) xRy and yRx
ii) $x \not R y$ and yRx
iii) $x \not R y$ and $y \not R x$

And, there are $\binom{n}{2} = \dfrac{n.(n-1)}{2}$ such pairs

So, we have $3^{\binom{n}{2}} = 3^{\dfrac{n.(n-1)}{2}}$ different choices for these entries (pairs) .

Hence, there are $1 \ . \ 3^{\binom{n}{2}} = 3^{\binom{n}{2}} = 3^{\dfrac{n.(n-1)}{2}}$ relations that are both reflexive and anti-symmetric on A.

Answer $= 3^{\binom{n}{2}} = 3^{\binom{n}{2}} = 3^{\dfrac{n.(n-1)}{2}}$.