CEng 382 - Analysis of Dynamic Systems
20221
Take Home Exam 3 Solutions

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1. (a) First, calculate the Jacobian matrix:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 2x^2 \\ 6x_1 & -2 \end{bmatrix}$$

for $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

$$Df(\tilde{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Now, calculate eigenvalues:

$$\begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2) = 0 \quad \longrightarrow \quad \lambda_1 = -1, \lambda_2 = -2$$

This is a continuous time system and all eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = -2$, have negative real parts, so the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is **stable**.

(b) (i) $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{4}$ is a polynomial function, so V is continuous on R^2. Also, V has continuous first partial derivatives, since it is a polynomial. In particular, $\frac{\partial V}{\partial x_1} = x_1$ and $\frac{\partial V}{\partial x_2} = \frac{x_2}{2}$ are the first partial derivatives of V. Since they are also polynomial, they are continuous.

(ii) $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{4}$ contains the terms x_1^2, x_2^2. These will be minimum if $x_1 = 0$, $x_2 = 0$. Thus, V gets its unique minimum at the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(iii) Finally, we check the difference function $V' = \nabla V(x).f(x)$ and examine if it satisfies $V' \leq 0$ for each x.

$$
\begin{aligned}
V' &= \nabla V(x).f(x) \\
&= \frac{\partial V}{\partial x_1} x_1' + \frac{\partial V}{\partial x_2} x_2' \\
&= x_1(-x_1 + x_2^2) + \frac{x_2}{2}(-2x_2 + 3x_1^2) \\
&= -x_1^2 + x_1 x_2^2 - x_2^2 + \frac{3}{2} x_2 x_1^2 \\
&= -x_1^2(1 - \frac{3}{2} x_2) - x_2^2(1 - x_1)
\end{aligned}
$$

V' is not less than or equal to 0 for all $(x_1, x_2) \in R^2$. However, we still have a chance to show stability.

Notice that $V' < 0$ for all (x_1, x_2) such that $x_1 < 1$ and $x_2 < \frac{2}{3}$. We can find such a region including the fixed point $\tilde{x}$:

For example;

Observe that level curves of the lyapunov function V are the ellipses centered at origin with the axes $2\sqrt{\alpha}$ and $\sqrt{2\alpha}$. We can use these ellipses by satisfying the following conditions:

$2\sqrt{\alpha} < \frac{2}{3} \longrightarrow \alpha < \frac{1}{9}$

$\sqrt{2\alpha} < 1 \longrightarrow \alpha < \frac{1}{2}$

These two inequalities gives the fact that $\alpha < \frac{1}{9}$.

So, we found the region as an ellipse:

$$\frac{x_1^2}{2} + \frac{x_2^2}{4} = \frac{1}{9}$$

This region includes the fixed point and satisfies the conditions $x_1 < 1$ and $x_2 < \frac{2}{3}$ meaning $V' < 0$.

We can take the largest circle inside the ellipse as the attraction of region by setting the radius to the half of the minor axis of the ellipse:

$$x_1^2 + x_2^2 = \frac{2}{9}$$

Thus, if the system starts near the the fixed point (origin) in this circle, it must tend to go to the fixed point.

Thus, the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is **stable**.

2. I will use the function $V(x_1, x_2, x_3) = 4(x_1^2 + x_2^2 + x_3^3)$.

(i) V is a polynomial function, so it is continuous.

(ii) V contains terms x_1^2, x_2^2 and x_3^2. These will be minimum if $x_i = 0$ for $i = 1, 2, 3$. Thus, V has a unique minimum at the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(iii) Now, we will compare $V(x_1(k+1), x_2(k+1), x_3(k+1))$ and $V(x_1(k), x_2(k), x_3(k))$. In other words, we will calculate the difference function

$$\Delta V = V(f(x)) - V(x) = V(x_1(k+1), x_2(k+1), x_3(k+1)) - V(x_1(k), x_2(k), x_3(k))$$

and check if it satisfies $\Delta V \leq 0$ for all x.

$$V(x_1(k), x_2(k), x_3(k)) = 4x_1^2(k) + 4x_2^2(k) + 4x_3^2(k)$$

$$\begin{aligned} V(x_1(k+1), x_2(k+1), x_3(k+1)) &= 4(\frac{1}{2}x_1(k) + \frac{1}{2}x_2(k))^2 + 4(\frac{1}{2}x_3(k))^2 + 4(\frac{1}{2}x_1(k) - \frac{1}{2}x_2(k))^2 \\ &= x_1^2(k) + x_1(k)x_2(k) + x_3^2(k) + x_1^2(k) - x_1(k)x_2(k) + x_3^2(k) \\ &= 2x_1^2(k) + 2x_2^2(k) + x_3^2(k) \end{aligned}$$

Then,

$$\begin{aligned} \Delta V &= (2x_1^2(k) + 2x_2^2(k) + x_3^2(k)) - (4x_1^2(k) + 4x_2^2(k) + 4x_3^2(k)) \\ &= -2x_1^2(k) - 2x_2^2(k) - 3x_3^2(k) \end{aligned}$$

Since all square terms have negative sign, $\Delta V \leq 0$ for all $x \in R^3$.

This means that V is not increasing along trajectories.

Hence, the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is **stable** by Lyapunov Theorem.

3. Find the fixed points of the system:

$$x_1'(t) = x_1 + x_2 - 4x_1(x_1^2 + x_2^2) = 0$$
$$x_2'(t) = -x_1 + x_2 - 4x_2(x_1^2 + x_2^2) = 0$$

Multiply the first equation by x_2 and multiply the second equation by x_1:

$$x_1x_2 + x_2^2 - 4x_1x_2(x_1^2 + x_2^2) = 0$$
$$-x_1^2 + x_1x_2 - 4x_1x_2(x_1^2 + x_2^2) = 0$$

Subtract the second equation from the first equation:

$$x_1^2 + x_2^2 = 0$$

This gives us that $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is **the only fixed point** of the system.

Now, apply linearization around the fixed point $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:
Calculate the Jacobian matrix:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 12x_1^2 - 4x_2^2 & 1 - 8x_1x_2 \\ \\ -1 - 8x_1x_2 & 1 - 4x_1^2 - 12x_2^2 \end{bmatrix}$$
$$Df(\tilde{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Calculate the eigenvalues:

$$\begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = 0 \longrightarrow (\lambda - 1)^2 + 1 = 0 \Longrightarrow \lambda_1 = 1 + i, \lambda_2 = 1 - i$$

All eigenvalues have positive real part. It is enough to have one eigenvalue with positive real part in order to say that a fixed point is unstable. So, $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is **unstable**.

This means that **the system does not converge to a fixed point**.

We know that the fixed point is unstable, but let's try $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_1^2}{2}$ (it cannot be a Lyapunov function) .

i) V is a polynomial function, so it is continuous. Its first partial derivatives are also polynomial. So, it has continuous first partial derivatives.

ii) It contains square terms x_1^2, x_2^2 which get their minimum at 0. Therefore, V gets its unique minimum at $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

iii) Now check V':

$$
\begin{aligned}
V' &= \frac{\partial V}{\partial x_1}x_1' + \frac{\partial V}{\partial x_2}x_2' \\
&= x_1(x_1 + x_2 + 4x_1^3 - 4x_1x_2^2) + x_2(-x_1 + x_2 - 4x_2x_1^2 - 4x_2^3) \\
&= x_1^2 + x_1x_2 - 4x_1^4 - 4x_1^2x_2^2 - x_1x_2 + x_2^2 - 4x_2^2x_1^2 - 4x_2^4 \\
&= x_1^2 + x_2^2 - 4x_1^4 - 4x_2^4 - 8x_1x_2 \\
&= x_1^2 + x_2^2 - 4(x_1^4 + 2x_1^2x_2^2 + x_2^4) \\
&= x_1^2 + x_2^2 - 4(x_1^2 + x_2^2)^2 \\
&= (x_1^2 + x_2^2)[1 - 4(x_1^2 + x_2^2)]
\end{aligned}
$$

We want $V' < 0$. However, we see that when $x_1^2 + x_2^2 < \frac{1}{4} = (\frac{1}{2})^2$, then V' is actually positive. On the other hand, if $x_1^2 + x_2^2 > \frac{1}{4} = (\frac{1}{2})^2$, then $V' < 0$. In other words, if x_1 and x_2 are large enough, then the system is heading back toward origin by getting closer to the region bounded by the circle $x_1^2 + x_2^2 = (\frac{1}{2})^2$. As a result, we see that **the system does not diverge to infinity**.

We found that the only fixed point, origin $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable, and divergent behavior is impossible. What's left ? The Poincare-Bendixson theorem leaves us only one possible behavior: As $t \longrightarrow \infty$, we must have $x(t)$ **tending to a periodic orbit.** And **this periodic orbit is the limit cycle we found above**.

Hence, **the system has a periodic limit cycle** $x_1^2 + x_2^2 = (\frac{1}{2})^2$.

4. (a) To find fixed points of the system, we need to solve $\tilde{x} = f(\tilde{x})$:

$$\tilde{x} = 3 - \tilde{x}^2 \longrightarrow \tilde{x}^2 + \tilde{x} - 3 = 0 \longrightarrow \tilde{x_1} = \frac{-1+\sqrt{13}}{2}, \tilde{x_2} = \frac{-1-\sqrt{13}}{2}$$

The fixed points are $\tilde{x_1} = \dfrac{-1+\sqrt{13}}{2}$ and $\tilde{x_2} = \dfrac{-1-\sqrt{13}}{2}$.

(b) We need to solve $f^2(x) = f(f(x)) = x$ where $f(x) = 3 - x^2$:

$$\begin{aligned} 3 - (3 - x^2)^2 = x &\longrightarrow 3 - x^4 - 9 + 6x^2 = x \\ &\longrightarrow x^4 - 6x^2 + x + 6 = 0 \\ &\longrightarrow (x+1)(x-2)(x^2 + x - 3) = 0 \end{aligned}$$

I have found the roots of the equation as $x_1 = \dfrac{-1+\sqrt{13}}{2}$, $x_2 = \dfrac{-1-\sqrt{13}}{2}$, $x_3 = -1$ and $x_4 = 2$.

Now, we know the periodic points. Let's have a closer look at them:

If we plug $x_1 = \dfrac{-1+\sqrt{13}}{2}$ or $x_2 = \dfrac{-1-\sqrt{13}}{2}$ into the system, we always get the same state as the previous step. Actually, this happens because x_1 and x_2 are fixed points of the system. This means that x_1 and x_2 have prime periods of 1.

If we plug $x_3 = -1$ into the system, we observe:

$$\begin{aligned} 3 - x_3^2 &= 3 - (-1)^2 = 2 = x_4 \\ 3 - x_4^2 &= 3 - 2^2 = -1 = x_3 \\ 3 - x_3^2 &= 3 - (-1)^2 = 2 = x_4 \\ &\vdots \end{aligned}$$

We see that, when we plug x_3 or x_4 into the system, state will oscillate between x_3 and x_4. This means that $x_3 = -1$ and $x_4 = 2$ are **periodic points of prime period 2.**

Remember that we solved the equation $x = f(x)$ to find fixed points which are nothing but periodic points with prime period 1. Moreover, remember that we solved the equation $f^2(x) = f(f(x)) = x$ to find periodic points with period up to 2. And from these, I concluded that the relation between the fixed points and the periodic points of prime period 2 in terms of the equations that I use to calculate them is the following:

$\{$ periodic points with prime period 2$\}$ = $\{$roots of $f^2(x) = f(f(x)) = x\}$ - $\{$roots of $x = f(x)\}$

In particular, if we apply polynomial division and solve the equation

$$\frac{f^2(x) - x}{f(x) - x} = 0$$

we can find the periodic points with prime period 2.

(c) Periodic points with prime period 2 are $x_3 = -1$ and $x_4 = 2$ which we found in part b. We should compute $(f^2)'(x_3)$ and $(f^2)'(x_4)$ to determine stability of them.

By chain rule:

$$(f^2)'(x) = \frac{df f^2(x)}{dx} = f'(f(x)).f'(x)$$

where $f(x) = 3 - x^2$ and $f'(x) = -2x$.

So,

$$\begin{aligned}(f^2)'(x) &= -2(3 - x^2)(-2x) \\ &= -4x^3 + 12x\end{aligned}$$

By plugging $x_3 = -1$ and $x_4 = 2$:

$$(f^2)'(x_3 = -1) = -4.(-1)^3 + 12.(-1) = -8$$
$$(f^2)'(x_4 = 2) = -4.(2)^3 + 12.(2) = -8$$

We see that both $(f^2)'(x_3)$ and $(f^2)'(x_4)$ have absolute value greater than 1 i.e. $|(f^2)'(x_3)| > 1$ and $|(f^2)'(x_4)| > 1$. So, the system do not tend to these points.

Therefore, both $x_3 = -1$ and $x_4 = 2$ are **unstable**.