CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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1. Firstly lets start with Fourier representation of a continuous periodic function x(t) with period T, and Fourier series coefficients a_k. Given that $a_0 = 0$.

   $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

   The question asking from us $\int_{-\infty}^{t} x(s)\,ds$. Lets integrate both side of equation.

   $\int_{-\infty}^{t} x(s)\,ds = \int_{-\infty}^{t} (\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t})\,ds$

   Now swap the integral and summation since they are both linear operation and the function is continuous.

   $\int_{-\infty}^{t} x(s)\,ds = \sum_{k=-\infty}^{\infty} (\int_{-\infty}^{t} a_k e^{jk\omega_0 t})\,ds$

   $= \sum_{k=-\infty}^{\infty} (\frac{1}{jk\omega_0} a_k e^{jk\omega_0 s}\Big|_{-\infty}^{t})$

   $= \sum_{k=-\infty}^{\infty} (\frac{1}{jk\omega_0} a_k e^{jk\omega_0 t} - 0)$

   $\implies \int_{-\infty}^{t} x(s)\,ds = \sum_{k=-\infty}^{\infty} \frac{1}{jk\omega_0} a_k e^{jk\omega_0 t}$

   Lets define b_k as Fourier series coefficient of $\int_{-\infty}^{t} x(s)\,ds$. Then according to the result we found,

   $b_k = \frac{1}{jk\omega_0} a_k$ where $\omega_0 = \frac{2\pi}{T}$.

   Therefore

   $b_k = \frac{1}{jk\frac{2\pi}{T}} a_k$

2. (a) Note: $\longleftrightarrow$ represents $\underset{\longleftrightarrow}{F.S}$

   We will use Multiplication property of CT Fourier series.Property says that :

   Lets $x(t) \longleftrightarrow a_k$ and $y(t) \longleftrightarrow b_k$.

   Then;

   $x(t)y(t) \longleftrightarrow c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

   If we use this property on our question:

   $x(t)x(t) \longleftrightarrow c_k = a_k * a_k = \sum_{l=-\infty}^{\infty} a_l a_{k-l}$

   (b) Note: $\longleftrightarrow$ represents $\underset{\longleftrightarrow}{F.S}$

   Firstly lets find $Ev\{x(t)\}$.

   $Ev\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$

By time reversal property of CT Fourier Series:

if $x(t) \longleftrightarrow a_k$ then $x(-t) \longleftrightarrow a_{-k}$

Now to solve this question we will use another property which is linearity property of CT Fourier Series:

$Ev\{x(t)\} = \frac{1}{2}(x(t) + x(-t)) = \frac{1}{2}x(t) + \frac{1}{2}x(-t) \longleftrightarrow \frac{1}{2}a_k + \frac{1}{2}a_{-k}$

(c) Note: $\longleftrightarrow$ represents $\underleftrightarrow{F.S}$

In this question, we will use linearity and time shifting properties of CT Fourier Series:

Time-Shifting:

if $x(t) \longleftrightarrow a_k$ then $x(t - t_0) \longleftrightarrow e^{-jkw_0t_0}a_k$

Same as above:

if $x(t) \longleftrightarrow a_k$ then $x(t + t_0) \longleftrightarrow e^{jkw_0t_0}a_k$

Lastly, to solve this question we will use linearity property of CT Fourier Series:

$x(t - t_0) + x(t + t_0) \longleftrightarrow e^{-jkw_0t_0}a_k + e^{jkw_0t_0}a_k = 2cos(k\omega_0t_0)a_k$

3. Note: $\longleftrightarrow$ represents $\underleftrightarrow{F.S}$

$x(t) \longleftrightarrow a_k$, T=4

$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0t}\,dt$

$= \frac{1}{4}\int_0^4 x(t)e^{-jk\omega_0t}\,dt$

$= \frac{1}{4}\Big(\int_0^1 2e^{-jk\omega_0t}\,dt + \int_2^3(-2)e^{-jk\omega_0t}\,dt\Big)$

$= \frac{1}{2}\Big(\int_0^1 e^{-jk\omega_0t}\,dt - \int_2^3 e^{-jk\omega_0t}\,dt\Big)$

$= \frac{1}{2}\Big(-\frac{1}{jk\omega_0}e^{-jk\omega_0t}\Big|_0^1 - \big(-\frac{1}{jk\omega_0}\big)e^{-jk\omega_0t}\Big|_2^3\Big)$

$= -\frac{1}{2jk\omega_0}(e^{-jk\omega_0} - 1 - e^{-3jk\omega_0} + e^{-2jk\omega_0})$

Note that $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

So,

$a_k = -\frac{1}{2jk(\frac{\pi}{2})}(e^{-jk(\frac{\pi}{2})} - 1 - e^{-3jk(\frac{\pi}{2})} + e^{-2jk(\frac{\pi}{2})})$

$a_k = -\frac{1}{jk\pi}(e^{-jk\frac{\pi}{2}} - 1 - e^{-\frac{3}{2}jk\pi} + e^{-jk\pi})$

4. (a) Note: $\longleftrightarrow$ represents $\underleftrightarrow{F.S}$

$x(t) = 1 + sin(\omega_0t) + 2cos(\omega_0t) + cos(2\omega_0t + \frac{\pi}{4})$

Let;

$x_1(t) = 1 \longleftrightarrow b_k$

$x_2(t) = sin(\omega_0t) \longleftrightarrow c_k$

$x_3 = 2cos(\omega_0t) \longleftrightarrow d_k$

$x_4 = cos(2\omega_0 t + \frac{\pi}{4} \longleftrightarrow f_k$

Then;

$x_1(t) = 1 \longleftrightarrow b_0 = 1$, $b_k = 0$ for all other k's.

$x_2(t) = \dfrac{1}{2j}e^{j\omega_0 t} - \dfrac{1}{2j}e^{-j\omega_0 t} \longleftrightarrow c_k$ where

$c_1 = \dfrac{1}{2j} = \dfrac{-j}{2}$,

$c_{-1} = \dfrac{-1}{2j} = \dfrac{j}{2}$

$c_k = 0$ for all other k's.

$x_3(t) = e^{j\omega_0 t} + e^{-j\omega_0 t} \longleftrightarrow d_k$ where

$d_1 = d_{-1} = 1$,

$d_k = 0$ for all other k's.

$x_4(t) = \dfrac{1}{2}e^{j2\omega_0 t}e^{j\frac{\pi}{4}} + \dfrac{1}{2}e^{-j2\omega_0 t}e^{-j\frac{\pi}{4}} \longleftrightarrow f_k$ where

$f_2 = \dfrac{1}{2}e^{j\frac{\pi}{4}} = \dfrac{\sqrt{2}}{4}(1+j)$,

$f_{-2} = \dfrac{1}{2}e^{-j\frac{\pi}{4}} = \dfrac{\sqrt{2}}{4}(1-j)$,

$f_k = 0$ for all other k's.

We know that $x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$

Let $x(t) \longleftrightarrow a_k$

So,

$a_k = b_k + c_k + d_k + f_k$

$a_{-2} = b_{-2} + c_{-2} + d_{-2} + f_{-2} = 0 + 0 + 0 + \dfrac{\sqrt{2}}{4}(1-j) = \dfrac{\sqrt{2}}{4}(1-j)$

$a_{-1} = b_{-1} + c_{-1} + d_{-1} + f_{-1} = 0 + \dfrac{j}{2} + 1 + 0 = 1 + \dfrac{j}{2}$

$a_0 = b_0 + c_0 + d_0 + f_0 = 1 + 0 + 0 + 0 = 1$

$a_1 = b_1 + c_1 + d_1 + f_1 = 0 + \dfrac{-j}{2} + 1 + 0 = 1 - \dfrac{j}{2}$

$a_2 = b_2 + c_2 + d_2 + f_2 = 0 + 0 + 0 + \dfrac{\sqrt{2}}{4}(1+j) = \dfrac{\sqrt{2}}{4}(1+j)$

and,

$a_k = 0$ for all $k \notin \{-2, -1, 0, 1, 2\}$

Now lets find magnitude of a_k:

$|a_{-2}| = \dfrac{1}{2}, |a_{-1}| = \dfrac{\sqrt{5}}{2}, |a_0| = 1, |a_1| = \dfrac{\sqrt{5}}{2}, |a_2| = \dfrac{1}{2}$, and $|a_k| = 0$ for all $k \notin \{-2, -1, 0, 1, 2\}$

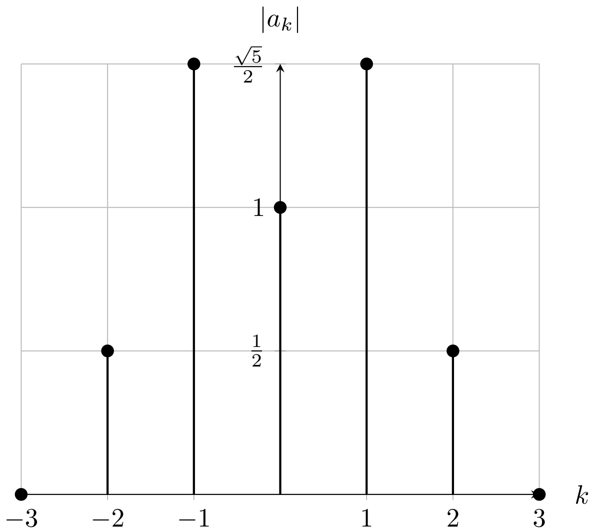


Figure 1: k vs. $|a_k|$.

Now lets find phase of a_k:

$\angle a_{-2} = arctan(-1) = -\dfrac{\pi}{4}$, $\angle a_{-1} = arctan(\frac{1}{2}) = 0.1476\pi$, $\angle a_0 = 0$, $\angle a_1 = -arctan(\frac{1}{2}) = -0.1476\pi$, $\angle a_2 = arctan(1) = \dfrac{\pi}{4}$

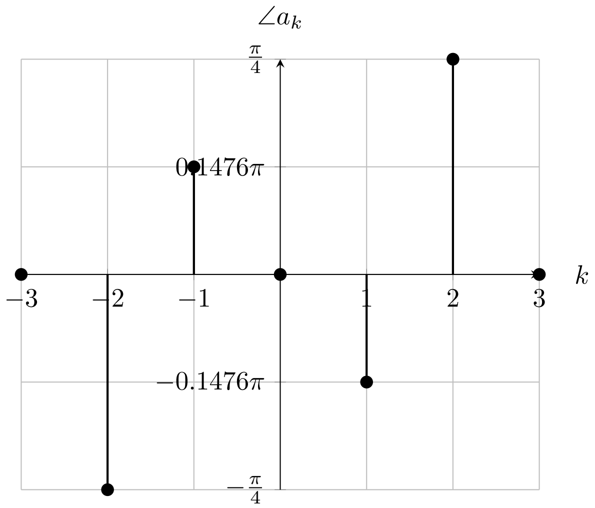


Figure 2: k vs. $\angle a_k$.

(b) Transfer Function of the system = Eigenvalue of the system = Frequency response of the system

Lets find frequency response of the system.

$$\frac{dy(t)}{dt} + y(t) = x(t)$$

Since this is an LTI system , $y(t) = H(j\omega)e^{j\omega t}$ when $x(t) = e^{j\omega t}$

$$(j\omega)H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$(j\omega + 1)H(j\omega) = 1$$

$$H(j\omega) = \frac{1}{1 + j\omega}$$

So transfer function of the system is $\dfrac{1}{1 + j\omega}$, and eigenvalue of the system is $\dfrac{1}{1 + j\omega}$.

(c) We found that $H(j\omega) = \dfrac{1}{1+j\omega} \longrightarrow H(jk\omega_0) = \dfrac{1}{1+jk\omega_0}$ in part b.

There is a relation between a_k and b_k such that

$$b_k = H(jk\omega_0)a_k$$

$$b_k = \frac{1}{1+jk\omega_0}a_k$$

Remember that we found the followings in part a:

$$a_{-2} = \frac{\sqrt{2}}{4}(1-j),\ a_{-1} = 1+\frac{j}{2},\ a_0 = 1,\ a_1 = 1-\frac{j}{2},\ a_2 = \frac{\sqrt{2}}{4}(1+j)$$

Then;

$$b_0 = \frac{1}{1+0}a_0 = 1$$

$$b_1 = \frac{1}{1+j\omega_0}a_1 = \frac{1}{1+j\omega_0}\frac{2-j}{2} = \frac{2-j}{2+2j\omega_0} = \frac{2-j}{2+4\pi j} = \frac{(4-4\pi)-(8\pi+2)j}{4+16\pi^2}$$

$$b_{-1} = \frac{1}{1-j\omega_0}a_{-1} = \frac{1}{1-j\omega_0}\frac{2+j}{2} = \frac{2+j}{2-2j\omega_0} = \frac{2+j}{2-4\pi j} = \frac{(4-4\pi)+(8\pi+2)j}{4+16\pi^2}$$

$$b_2 = \frac{1}{1+2j\omega_0}a_2 = \frac{1}{1+2j\omega_0}\frac{\sqrt{2}(1+j)}{4} = \frac{\sqrt{2}(1+j)}{4(1+2j\omega_0)} = \frac{\sqrt{2}(1+j)}{4(1+4\pi j)} = \frac{\sqrt{2}}{4}\Big(\frac{(1+4\pi)-(4\pi-1)j}{1+16\pi^2}\Big)$$

$$b_{-2} = \frac{1}{1-2j\omega_0}a_{-2} = \frac{1}{1-2j\omega_0}\frac{\sqrt{2}(1-j)}{4} = \frac{\sqrt{2}(1-j)}{4(1-2j\omega_0)} = \frac{\sqrt{2}(1-j)}{4(1-4\pi j)} = \frac{\sqrt{2}}{4}\Big(\frac{(1+4\pi)+(4\pi-1)j}{1+16\pi^2}\Big)$$

$b_k = 0$ for all other k's.

To simplify equations take $\pi = 3.14$, then:

$$b_0 = 1$$

$$b_1 = -0.053 - 0.168j$$

$$b_{-1} = -0.053 + 0.168j$$

$$b_2 = \frac{\sqrt{2}}{4}(0.085 - 0.073j) = 0.03 - 0.026j$$

$$b_{-2} = 0.03 + 0.026j$$

Now let determine magnitude of this values:

$$|b_0| = 1,$$

$$|b_1| = \sqrt{\frac{5}{4+16\pi^2}} = 0.176,$$

$$|b_{-1}| = \sqrt{\frac{5}{4+16\pi^2}} = 0.176,$$

$$|b_2| = \frac{1}{2}\sqrt{\frac{1}{1+16\pi^2}} = 0.0397,$$

$$|b_{-2}| = \frac{1}{2}\sqrt{\frac{1}{1+16\pi^2}} = 0.0397$$

for all other b_k's, $|b_k| = 0$

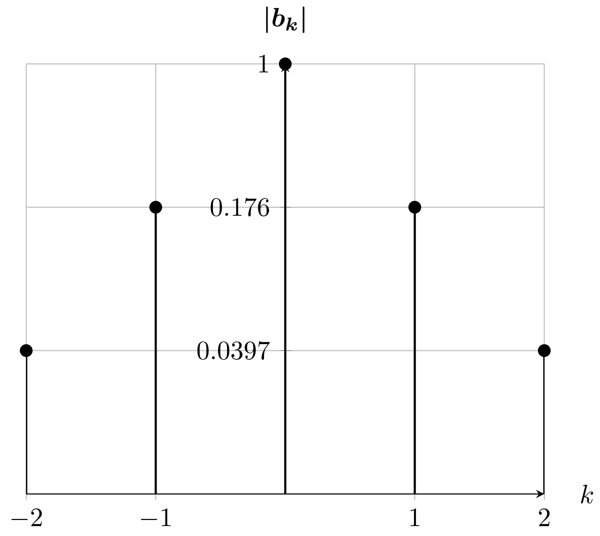


Figure 3: k vs. $|b_k|$

Now find phases:

$\angle b_0 = 0$, $\angle b_1 = -0.6\pi$, $\angle b_{-1} = 0.6\pi$, $\angle b_2 = -0.23\pi$, $\angle b_{-2} = 0.23\pi$

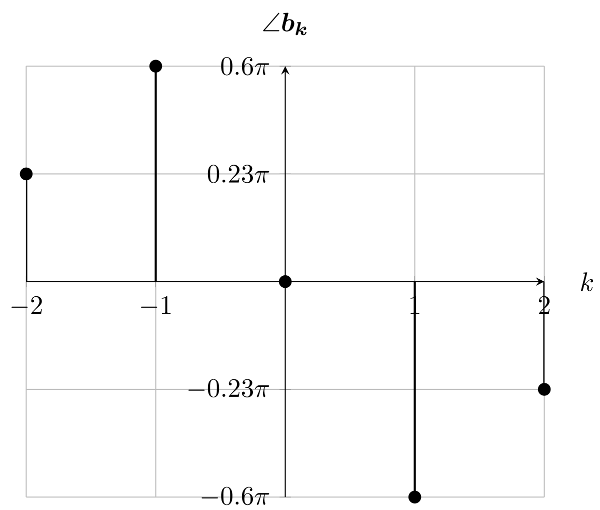


Figure 4: k vs. $\angle b_k$

(d) To solve this question, we will use Synthesis equation which is:

$y(t) = \sum_{-\infty}^{\infty} b_k e^{jk\omega_0 t}$

$= \sum_{-2}^{2} b_k e^{jk\omega_0 t}$ Since $b_k = 0$ when $k \notin \{-2, -1, 0, 1, 2\}$

$y(t) = b_{-2}e^{-2j\omega_0 t} + b_{-1}e^{-j\omega_0 t} + b_0 e^0 + b_1 e^{j\omega_0 t} + b_2 e^{2j\omega_0 t}$

$= (0.03 + 0.026j)e^{-2j\omega_0 t} + (-0.053 + 0.168j)e^{-j\omega_0 t} + 1 + (-0.053 - 0.168j)e^{j\omega_0 t} + (0.03 - 0.026j)e^{2j\omega_0 t}$

$= (0.03 + 0.026j)e^{-4\pi jt} + (-0.053 + 0.168j)e^{-2\pi jt} + 1 + (-0.053 - 0.168j)e^{2\pi jt} + (0.03 - 0.026j)e^{4\pi jt}$

5. Note: $\longleftrightarrow$ represents $\underset{\longleftrightarrow}{F.S}$

Let's first name the coefficients for each signal.

$x[n] \longleftrightarrow a_k$

$y[n] \longleftrightarrow b_k$

$x[n]y[n] \longleftrightarrow d_k$

(a) Lets first calculate period of x[n]:

$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{\pi}{2}} = 4$. Then

Now we will use euler equation to convert $sin\frac{\pi}{2}n$ to as e base.

$x[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$, $\omega_0 = \frac{\pi}{2}$

$a_{-2} = 0$

$a_{-1} = -\frac{1}{2j} = \frac{j}{2}$

$a_0 = 0$

$a_1 = \frac{1}{2j} = -\frac{j}{2}$

Other a_k's repeat periodically with N=4. That is:

$a_k = a_{k+N}$, where $N = 4$.

(b) Lets first calculate period of x[n]:

$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{\pi}{2}} = 4$. Then

Now we will use euler equation to convert $1 + cos\frac{\pi}{2}n$ to as e base.

$y[n] = 1 + \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})$, $\omega_0 = \frac{\pi}{2}$

$b_{-2} = 0$

$b_{-1} = \frac{1}{2}$

$b_0 = 1$

$b_1 = \frac{1}{2}$

Other b_k's repeat periodically with N=4. That is:

$b_k = b_{k+N}$, where $N = 4$.

(c) In this question we will use directly multiplication property which is:

$x[n]y[n] \longleftrightarrow \sum_{l=<N>} a_l b_{k-l}$

$d_k = \sum_{l=0}^{3} a_l b_{k-l}$ since N=4

$= a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$ where $a_0 = a_2 = 0$

$= a_1 b_{k-1} + a_3 b_{k-3}$ where $a_3 = a_{-1}$

$= a_1 b_{k-1} + a_{-1} b_{k-3}$

Now We'll give k the values {-2,-1,0,1,2} and then we will find coefficient of $x[n]$x$y[n]$

$d_0 = a_1 b_{-1} + a_{-1} b_{-3} = a_1 b_{-1} + a_{-1} b_1 = \frac{1}{4j} - \frac{1}{4j} = 0$

$d_2 = d_{-2} = a_1 b_1 + a_{-1} b_{-1} = \frac{1}{4j} - \frac{1}{4j} = 0$

$d_1 = a_1 b_0 + a_{-1} b_{-2}$ where $b_{-2} = 0$

$d_1 = \frac{1}{2j} = -\frac{j}{2}$

$d_{-1} = a_1 b_{-2} + a_{-1} b_{-4}$ where $b_{-2} = 0$ and $b_{-4} = b_0$

$d_{-1} = a_{-1} b_0 = -\frac{1}{2j} = \frac{j}{2}$

Since the period of two signals is 4, the period of their product is also 4.Namely $N = 4$

$d_{-2} = 0$

$d_{-1} = \frac{j}{2}$

$d_0 = 0$

$d_1 = -\frac{j}{2}$

Other d_k's repeat periodically with N=4. That is:

$d_k = d_{k+N}$, where $N = 4$.

(d) Now to solve this question we will multiply x[n] and y[n] directly.Then we will find coefficient of $x[n]$x$y[n]$.

$x[n]y[n] = (sin\frac{\pi}{2}n)(1 + cos\frac{\pi}{2}n)$

$= sin\frac{\pi}{2} + \frac{1}{2}sin\pi n$ where $sin\pi n$ is always equal to 0

$= sin\frac{\pi}{2}n$

Now we will use euler equation to convert $sin\frac{\pi}{2}n$ to as e base.Then,

$x[n]y[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$

$d_1 = \frac{1}{2j} = -\frac{j}{2}$, $d_{-1} = -\frac{1}{2j} = \frac{j}{2}$.

Since the period of two signals is 4, the period of their product is also 4.Namely $N = 4$

$d_{-2} = 0$

$d_{-1} = \frac{j}{2}$

$d_0 = 0$

$d_1 = -\frac{j}{2}$

Other d_k's repeat periodically with N=4. That is:

$d_k = d_{k+N}$, where $N = 4$.
When we compare the results in part c and part d, we see that the results are the same.

6. (a) As you can see on graph of part a $N = 4$ and therefore we can find $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$.

To find coefficients of the Fourier series representation of x[n] we need to use formula which is:

$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$

So

$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\omega_0 n}$

$a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^0 = \frac{1}{4}(0+1+2+1) = 1$

$a_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = -\frac{1}{2}$

$a_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n} = 0$

$a_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{3\pi}{2}n} = -\frac{1}{2}$

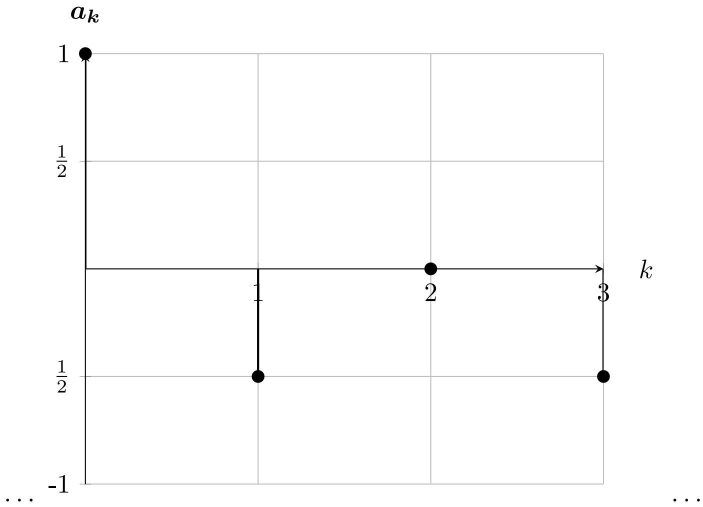


Figure 5: k vs. a_k

$a_k = a_{k+N}$,a_k will repeat with $N = 4$.

To plot magnitude of spectral coefficient, we need to find them:

$|a_0| = 1$, $|a_1| = \frac{1}{2}$,$|a_2| = 0$,$|a_3| = \frac{1}{2}$, $|a_k| = |a_{k+N}|$,$|a_k|$ will repeat with $N = 4$

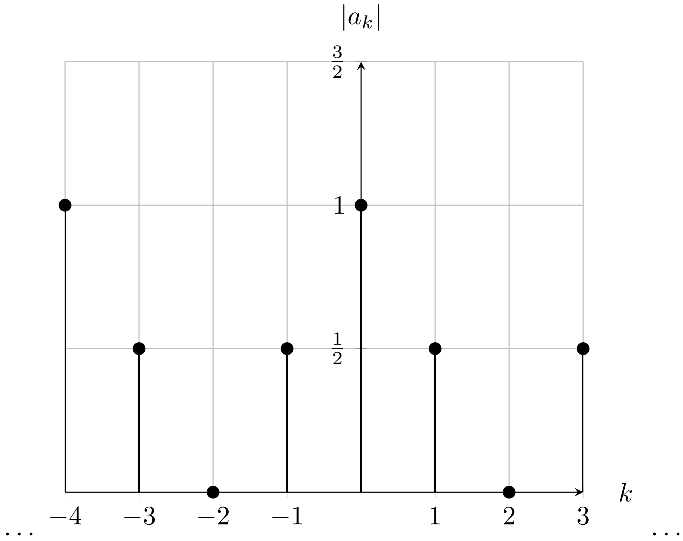


Figure 6: k vs. $|a_k|$.

So the phase spectrum of the coefficients:

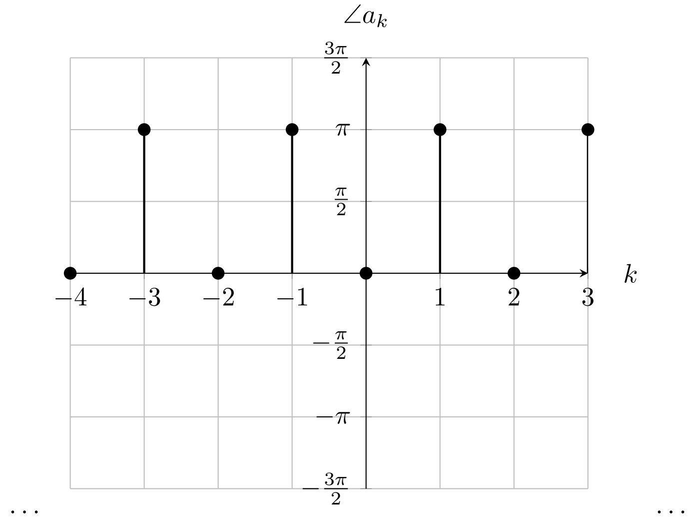


Figure 7: k vs. $\angle a_k$.

(b) **i.** We can write y[n] in terms of x[n] as:

$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - 3 + Nk]$ where $k \in Z$, $N = 4$.

ii. To find coefficients of the Fourier series representation of y[n] we need to use following formula:

$a_k = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-jk\frac{\pi}{2}n}$

Now lets calculate coefficients of the Fourier series representation of y[n]:

$a_0 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{0} = \frac{1}{4}(0 + 1 + 2 + 0) = \frac{3}{4}$

$a_1 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{\pi}{2}n} = -\frac{j}{4} - \frac{1}{2}$

$a_2 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\pi n} = \frac{1}{4}$

$a_3 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{3\pi}{2}n} = \frac{j}{4} - \frac{1}{2}$

From periodicity we can simply say: $a_n = a_{n+4} = a_{n-4}$, so other coefficients can be found via this fact.

So the magnitude spectrum of the coefficients:

$|a_0| = \frac{3}{4}$, $|a_1| = |-\frac{j}{4} - \frac{1}{2}| = \sqrt{\frac{5}{16}}$, $|a_2| = |\frac{1}{4}| = \frac{1}{4}$, $|a_3| = |\frac{j}{4} - \frac{1}{2}| = \sqrt{\frac{5}{16}}$

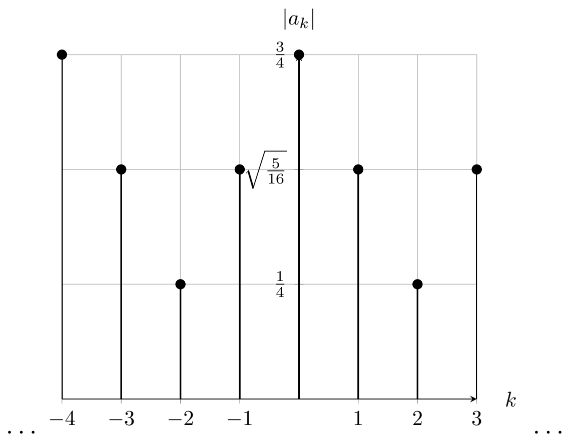


Figure 8: k vs. $|a_k|$.

So the phase spectrum of the coefficients:

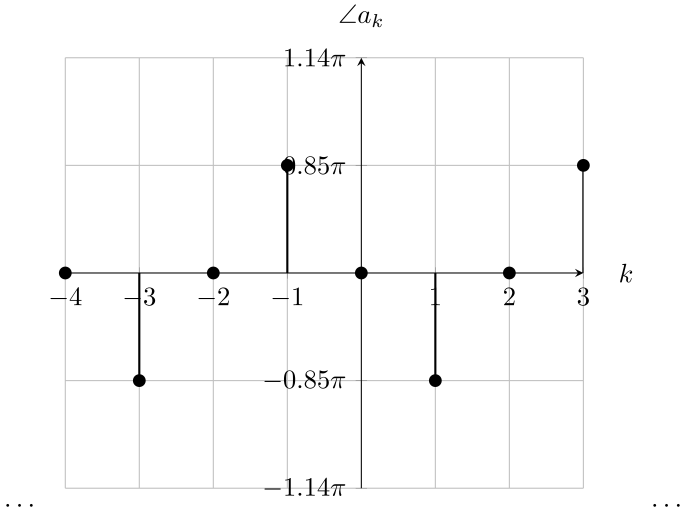


Figure 9: k vs. $\angle a_k$.

7. (a) Note: $\longleftrightarrow$ represents $\underset{\longleftrightarrow}{F.S}$

We know that

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Also we know that y(t)=x(t), so

$y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)e^{jk\omega_0 t} = x(t) \Longrightarrow a_k = H(jk\omega_0)a_k$ where $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{\frac{\pi}{K}} = 2K$

Since $H(j\omega)$ is zero for $|\omega| > 80$, the largest value of $|k|$ for which a_k is non-zero must be as follows:

$|k|\omega_0 \leq 80$

$|k|2K \leq 80$

$|k| \leq \dfrac{40}{K}$ (K is assumed to be positive because $\dfrac{\pi}{K}$ represents a period)

Therefore,for $|k| > \lfloor\dfrac{40}{K}\rfloor$, a_k is guaranteed to be 0.

Hence what we can conclude about a_k is that:

For $|k| > \lfloor\dfrac{40}{K}\rfloor$, a_k is guaranteed to be 0.

For $|k| \leq \lfloor\dfrac{40}{K}\rfloor$, a_k may or may not be 0.

(b) $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)a_k e^{jk\omega_0 t}$, where $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{\frac{\pi}{K}} = 2K$

Lets call $H(jk\omega_0)a_k$ as b_k, i.e. $b_k = H(jk\omega_0)a_k$

So,$y(t) \neq x(t) \Longrightarrow a_k \neq b_k$ for at least one k.

For $|k|2K \leq 80$, i.e $|k| \leq \dfrac{40}{K}$, transfer function $H(j\omega)$ returns 1 which causes $b_k = a_k$

For $|k| > \lfloor\dfrac{40}{K}\rfloor$, transfer function $H(j\omega)$ returns zero, which causes b_k to be 0. Therefore, for $|k| > \lfloor\dfrac{40}{K}\rfloor$, a_k is non-zero for at least one value of k.

Hence we conclude that

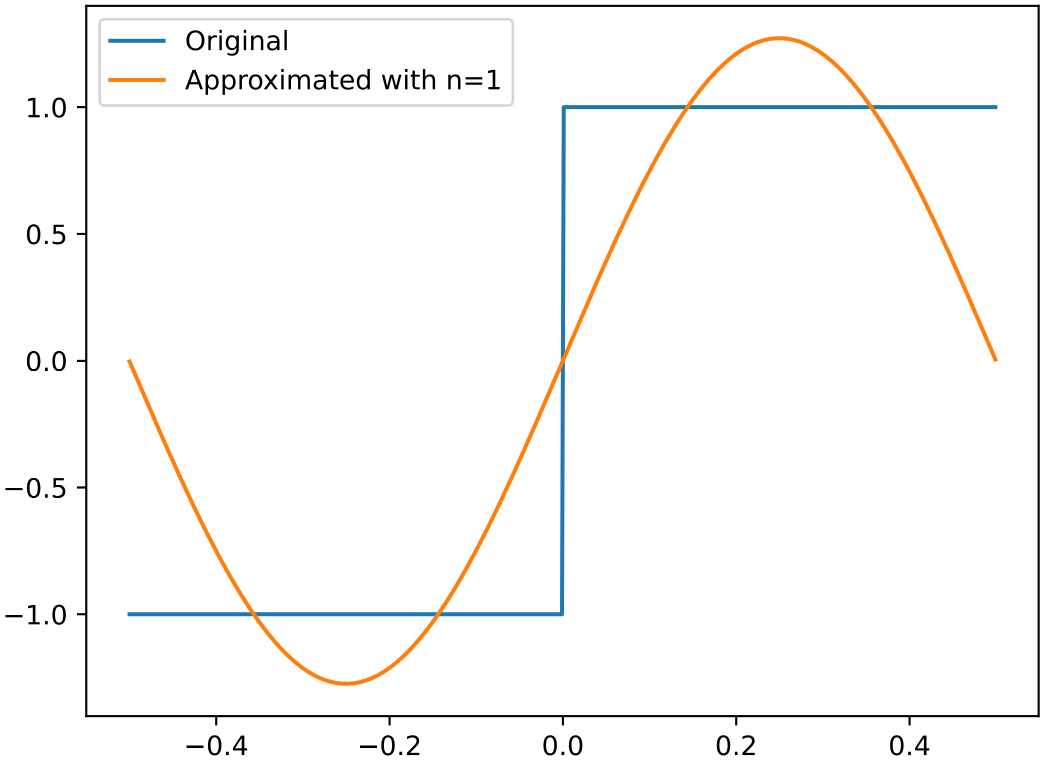
For $|k| > \lfloor \frac{40}{K} \rfloor$, a_k is non-zero for at least one value of k.

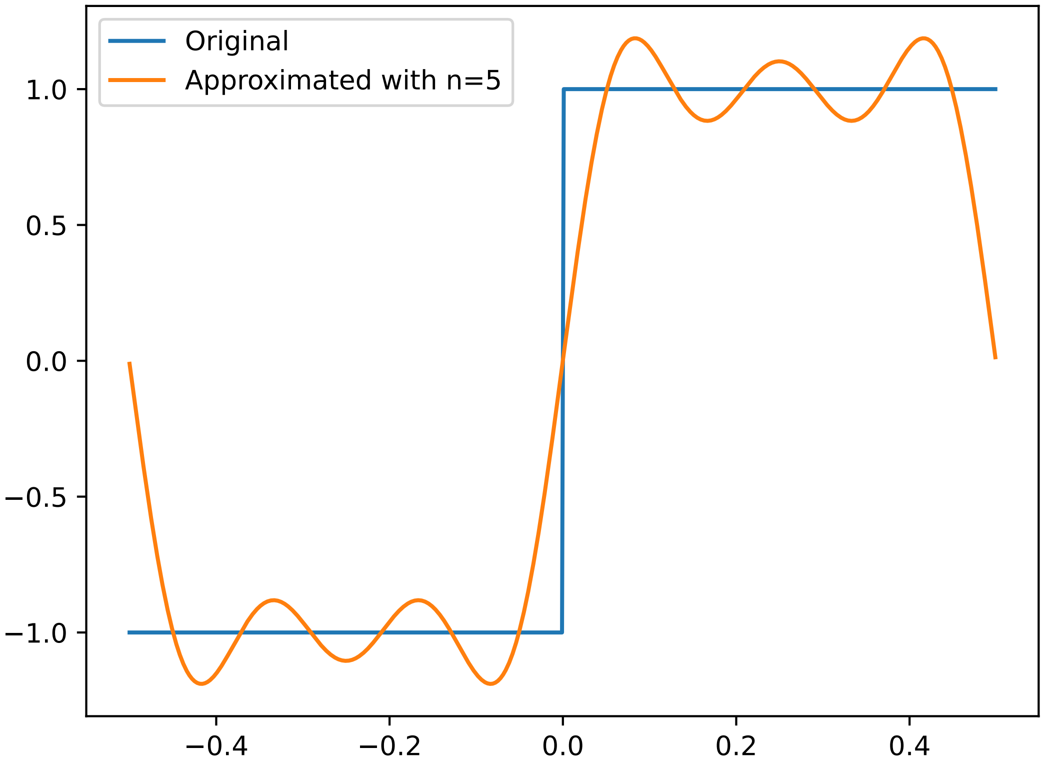
8.

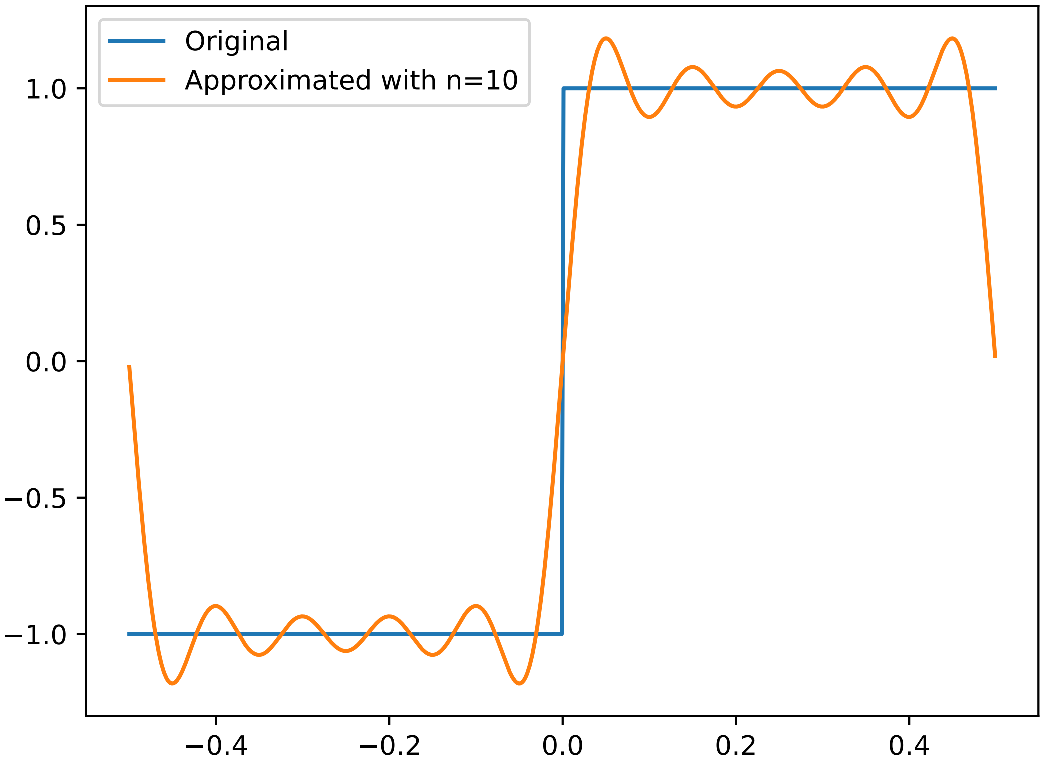
```python
import numpy as np
import matplotlib.pyplot as plt

# Part (a)
def fourier_series_coeffs(signal, T, n):
    N = len(signal)
    t = np.linspace(-T/2, T/2, N, endpoint=False)
    coefficients = []
    a0 = 2/N * np.sum(signal)
    coefficients.append(a0)

    for i in range(1, n+1):
        an = 2/N * np.sum(signal * np.cos(2 * np.pi * i * t / T))
        bn = 2/N * np.sum(signal * np.sin(2 * np.pi * i * t / T))
        coefficients.append((an, bn))
    return coefficients

# Part (b)
def fourier_series(coefficients, T, t):
    a0 = coefficients[0]
    result = a0 / 2
    n = len(coefficients) - 1
    for i in range(1, n+1):
        an, bn = coefficients[i]
        result += an * np.cos(2 * np.pi * i * t / T) + bn * np.sin(2 * np.pi * i * t / T)
    return result

# Part (c)
T = 1.0
t = np.linspace(-0.5, 0.5, 1000, endpoint=False)

signal_square = np.where(t < 0, -1, 1)
for n in [1, 5, 10, 50, 100]:
    coeffs = fourier_series_coeffs(signal_square, T, n)
    approx_signal = fourier_series(coeffs, T, t)

    plt.figure()
    plt.plot(t, signal_square, label='Original')
    plt.plot(t, approx_signal, label='Approximated with n={}'.format(n))
    plt.legend()
plt.show()

# Part (d)
signal_sawtooth = np.where(t < 0, 1 + 2*t, -1 + 2*t)
for n in [1, 5, 10, 50, 100]:
    coeffs = fourier_series_coeffs(signal_sawtooth, T, n)
    approx_signal = fourier_series(coeffs, T, t)

    plt.figure()
    plt.plot(t, signal_sawtooth, label='Original')
    plt.plot(t, approx_signal, label='Approximated with n={}'.format(n))
    plt.legend()
plt.show()
```

The larger the value of n, the closer our estimation is to the true value.

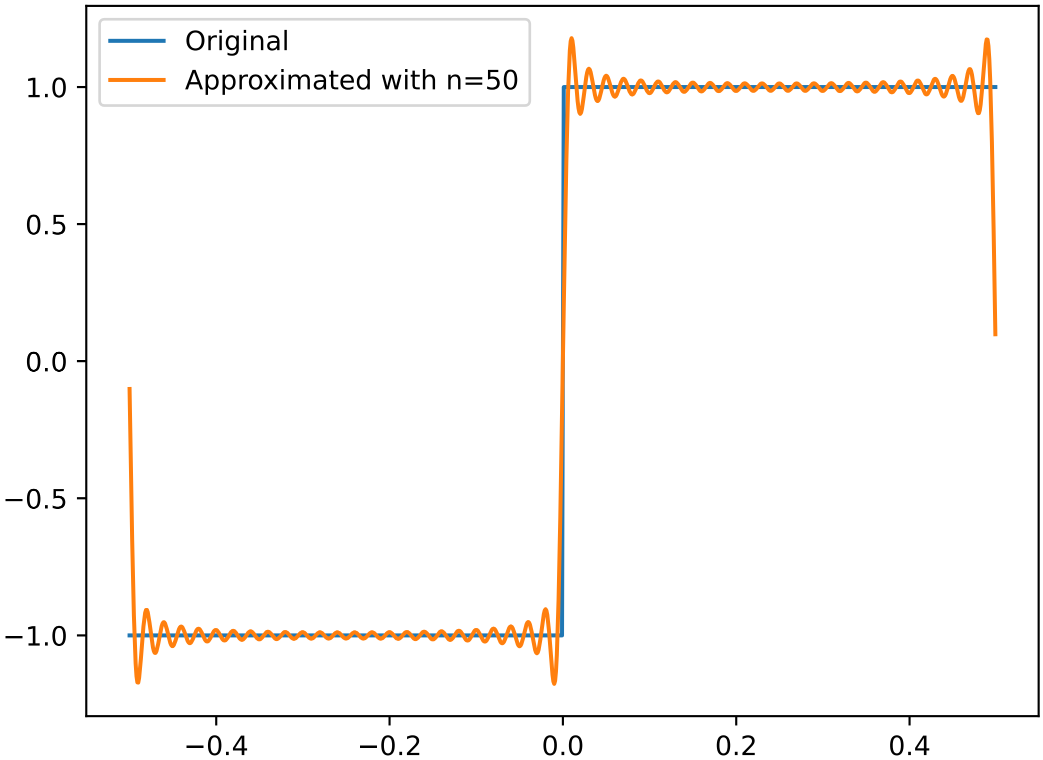
Graphs of square wave function



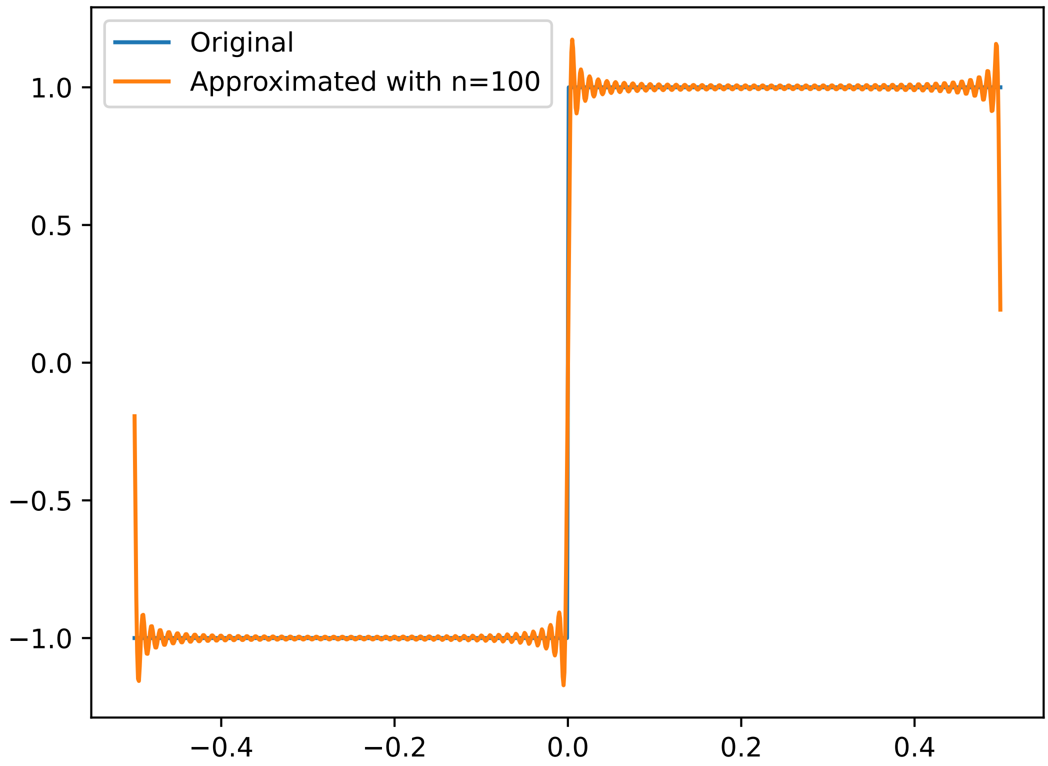




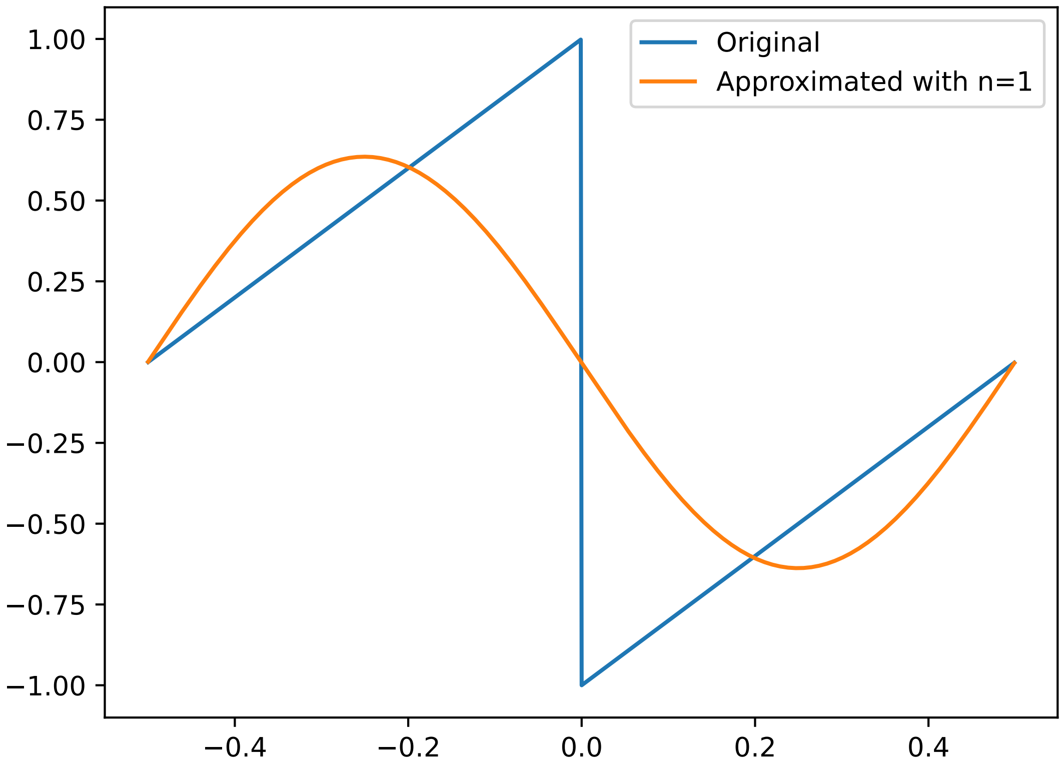
Original
Approximated with n=10
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

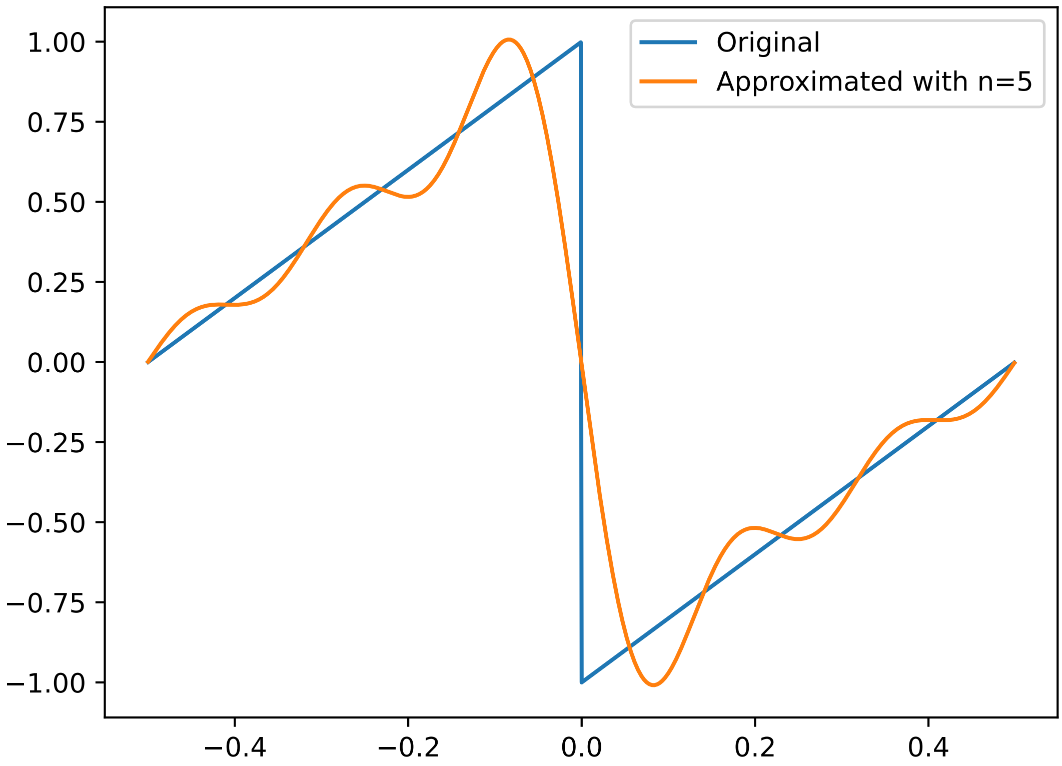


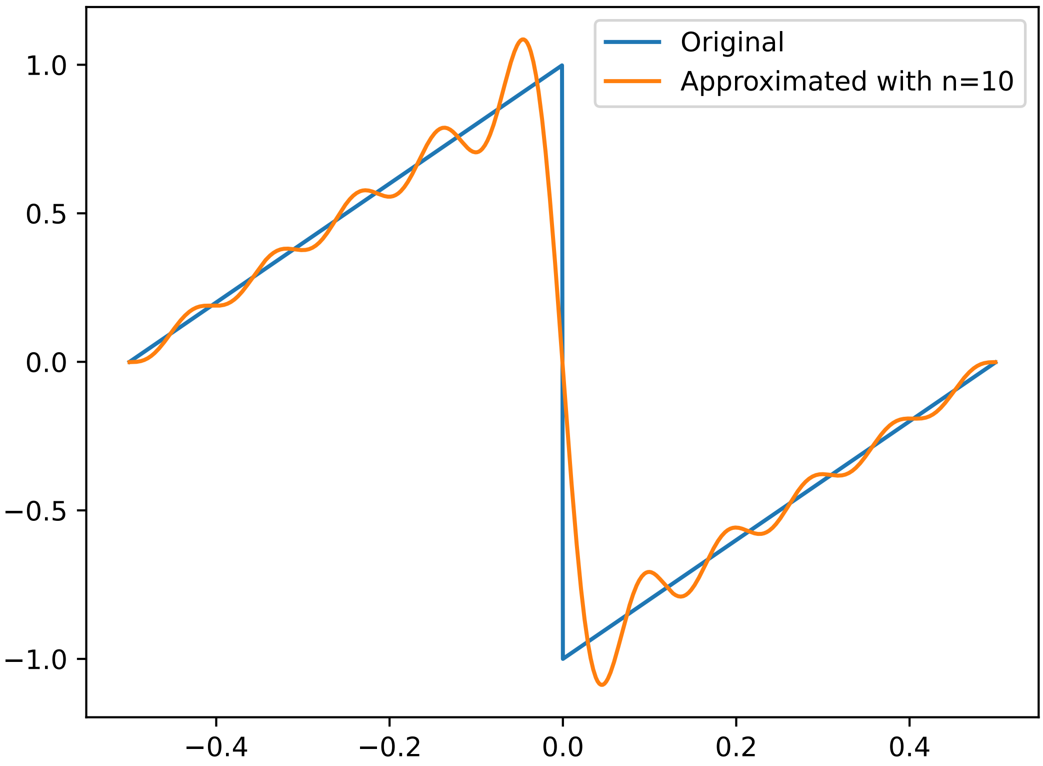
Original
Approximated with n=50
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

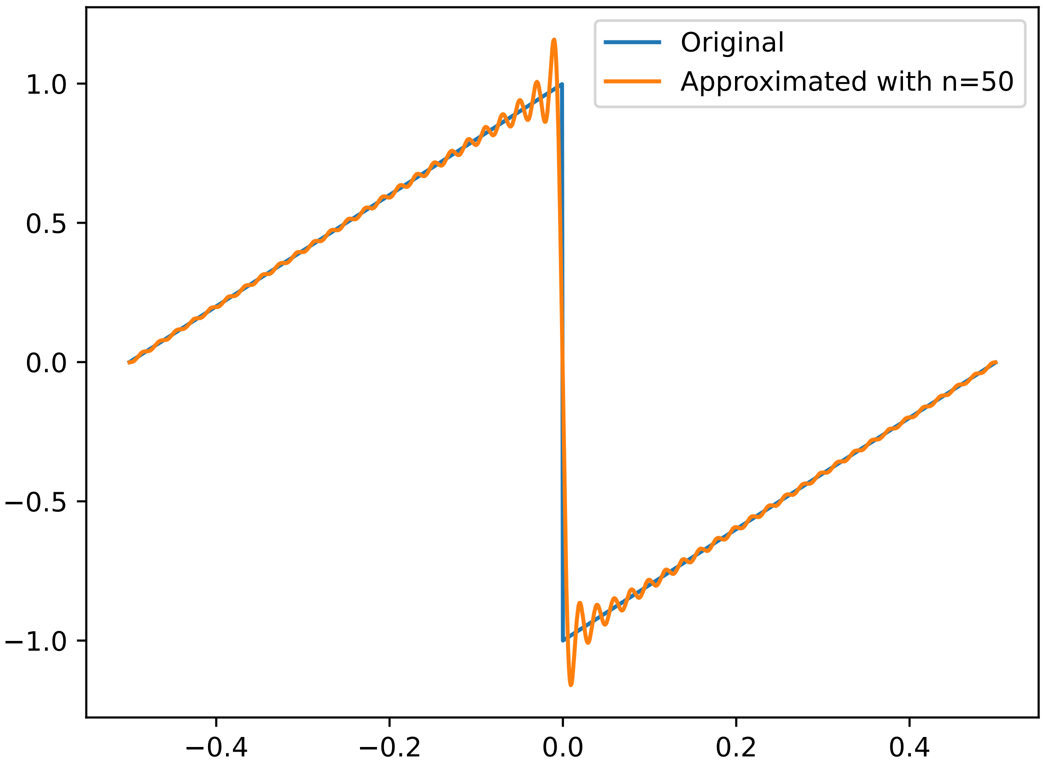


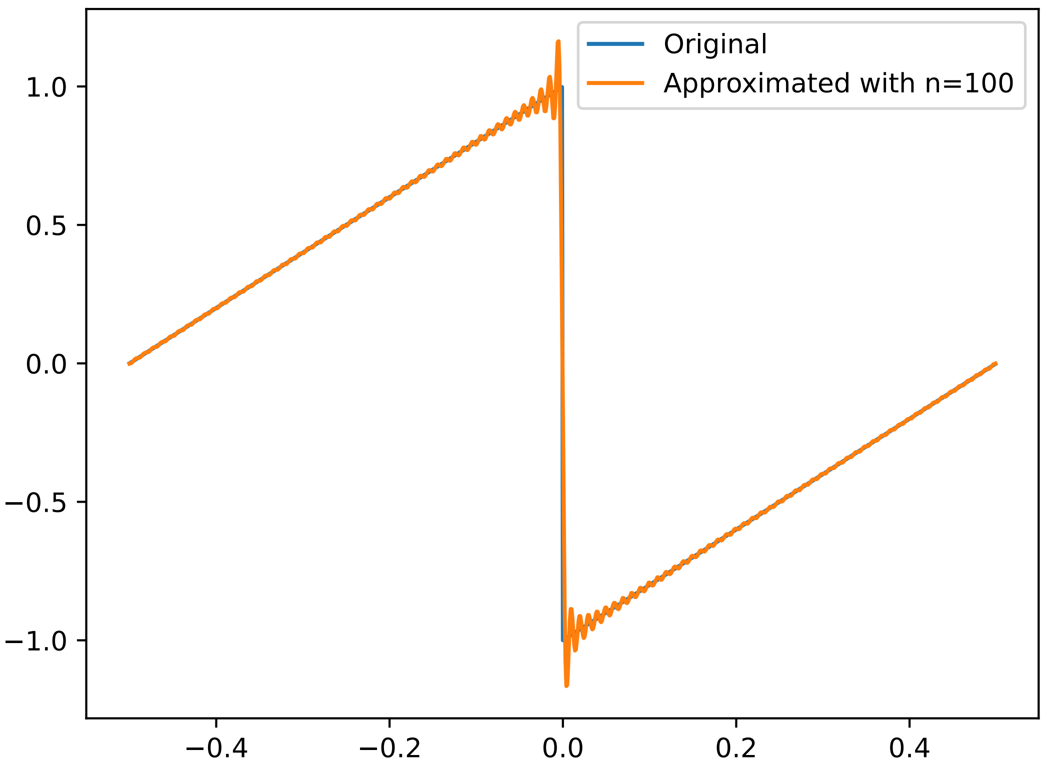
Graphs of sawtooth function




Original
Approximated with n=5
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−0.4
−0.2
0.0
0.2
0.4


Original
Approximated with n=10
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4


Original
Approximated with n=50
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4


Original
Approximated with n=100
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4