MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF COMPUTER ENGINEERING

CENG 382

Fall 2022
Take Home Exam 3

Regulations:

- **Submission:** We provide a latex template for your solutions. Use that template and submit a "the3.tex" file that creates a "the3.pdf" file with the following command:

  pdflatex the3.tex

  Not using the given template, or failing to generate a valid pdf with the above command will result in you receiving a zero grade. This command will be run on inek machines where you can try generating the pdf file yourselves.
- **Deadline:** 23:55, 8 January, 2022 (Sunday).
- **Late Submission:** The solutions will be available after the deadline. Therefore, late submissions will not be allowed.

---

1. (25 pts, 10 + 15 pts) Consider the following continuous time system.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-x_1 + x_2^2\\ -2x_2 + 3x_1^2\end{bmatrix}$$

   Determine the stability of the fixed point $\tilde{x} = \begin{bmatrix}0\\0\end{bmatrix}$,

   (a) via linearization.

   (b) using the Lyapunov theorem. You can use the Lyapunov function $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{4}$. First, show that the Lyapunov theorem is applicable in this case.

2. (25 pts) Consider the following discrete time system. Determine the stability of the fixed point $\tilde{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ using the Lyapunov theorem. First, find a suitable Lyapunov function and show that the Lyapunov theorem is applicable in this case.

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\\ x_3(k+1)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}x_1 + \frac{1}{2}x_2\\ \frac{1}{2}x_3\\ \frac{1}{2}x_1 - \frac{1}{2}x_2\end{bmatrix}$$

3. (25 pts) Consider the following continuous time system.

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}x_1 + x_2 - 4x_1(x_1^2 + x_2^2)\\ -x_1 + x_2 - 4x_2(x_1^2 + x_2^2)\end{bmatrix}$$

   Show that the system has a periodic limit cycle by using the Poincare-Bendixson theorem. You can use the Lyapunov function $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$. (*Hint:* Show that the origin is the only fixed point.)

4. (25 pts, 5 + 15 + 5 pts) Consider the following discrete time system.

$$x(k+1) = 3 - x(k)^2$$

   (a) Find the fixed points of this system.

   (b) Find the periodic points of prime period 2. Comment on the relation between the fixed points and the periodic points of prime period 2 in terms of the equations that you use to calculate them.

   (c) For each periodic point, determine its stability.