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Question 1

I am asked to prove the following set equivalence using membership notation and logical equivalences.

$$(A \cup B) \setminus (A \cap B) \equiv (A \setminus B) \cup (B \setminus A)$$

Steps and Justifications

1. $(A \cup B) \setminus (A \cap B)$ LHS of the equivalence
2. $\equiv \{x | x \in (A \cup B) \wedge x \notin (A \cap B)\}$ Defn. of set difference
3. $\equiv \{x | x \in (A \cup B) \wedge \neg(x \in (A \cap B))\}$ Defn. of $\notin$
4. $\equiv \{x | (x \in A \vee x \in B) \wedge \neg(x \in (A \cap B))\}$ Defn. of union
5. $\equiv \{x | (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$ Defn. of intersection
6. $\equiv \{x | (x \in A \vee x \in B) \wedge (\neg(x \in A) \vee \neg(x \in B))\}$ De Morgan's Law for Propositional Logic
7. $\equiv \{x | (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B)\}$ Defn. of $\notin$
8. $\equiv \{x | ((x \in A \vee x \in B) \wedge x \notin A) \vee ((x \in A \vee x \in B) \wedge x \notin B)\}$ Distributive laws
9. $\equiv \{x | (x \notin A \wedge (x \in A \vee x \in B)) \vee (x \notin B \wedge (x \in A \vee x \in B))\}$ Commutative laws
10. $\equiv \{x | ((x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)) \vee ((x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B))\}$ Distributive laws
11. $\equiv \{x | ((\neg(x \in A) \wedge x \in A) \vee (x \notin A \wedge x \in B)) \vee ((x \notin B \wedge x \in A) \vee (\neg(x \in B) \wedge x \in B))\}$ Defn. of $\notin$
12. $\equiv \{x | (F \vee (x \notin A \wedge x \in B)) \vee ((x \notin B \wedge x \in A) \vee F)\}$ Negation laws
13. $\equiv \{x | (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A)\}$ Identity laws
14. $\equiv \{x | (x \in B \wedge x \notin A) \vee (x \in A \wedge x \notin B)\}$ Commutative laws
15. $\equiv \{x | (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$ Commutative laws
16. $\equiv \{x | (x \in (A \setminus B)) \vee (x \in (B \setminus A))\}$ Defn. of set difference
17. $\equiv \{x | x \in ((A \setminus B) \cup (B \setminus A))\}$ Defn. of union
18. $\equiv (A \setminus B) \cup (B \setminus A)$

By starting with LHS, I have obtained RHS, so the set equivalence has been proved.

A Lemma

In this section I am going to prove a lemma in order to use it in the 2nd question.

Claim: If A is an uncountably infinite set and B is a countably infinite set. Then A $\setminus$ B is an uncountably infinite set.

Let A be an uncountably set and B be a countably set.

Assume that A $\setminus$ B is a countably infinite set. (We will contradict.) **(1)**

B is a countably infinite set, so it would mean that $(A \setminus B) \cup B$ is countably infinite set **(finite union of countable sets is clearly countable)**. But then A $\subseteq (A \setminus B) \cup B$, so A is contained in a countable set, then A must be countable **(2)**

We have obtained a contradiction by **(1)**,**(2)**

Therefore, our assumption has been contradicted. Hence, A $\setminus$ B is an uncountably infinite set.

Lemma: If A is an uncountably infinite set and B is a countably infinite set, then A $\setminus$ B is an uncountably infinite set.

Question 2

Solution

Let A $= \{f \mid f \subseteq \mathbb{N} \times \{0,1\}\}$

I will represent the mappings from $\mathbb{N}$ to $\{0,1\}$

$f(1) = a_1$
$f(2) = a_2$
$f(3) = a_3$ as n-tuples of $(a_1, a_2, a_3, ...)$ where $a_i = 0$ or $a_i = 1$ for i = 1,2,3,.........
.
.
.

Assume that A, the set of all mappings defined from $\mathbb{N}$ to $\{0,1\}$ (plus empty set) is countable. Then there exists a 1-to-1 correspondence between $\mathbb{N}$ and the set A.

Suppose we have it.

$1 \to (a_1, a_2, a_3, ...) = f_1$
$2 \to (b_1, b_2, b_3, ...) = f_2$
$3 \to (c_1, c_2, c_3, ...) = f_3$
.
.
.

Now construct a mapping from $\mathbb{N}$ to $\{0,1\}$ that is missed by the enumeration.

$f(1) = x_1$
$f(2) = x_2$
$f(3) = x_3$
.
.
.
represented as $f_x = (x_1, x_2, x_3, ...)$

such that
$x_1 \neq a_1 \to f_1 \neq f_x$
$x_2 \neq b_2 \to f_2 \neq f_x$
$x_3 \neq c_3 \to f_3 \neq f_x$
.
.
.

By design $f_x \in A$ and it is missed by the enumeration. So, there does not exist an enumeration listing each mapping in A. Hence A is an uncountably infinite set.

Let B = $\{f \mid f : \{0,1\} \rightarrow \mathbb{N}, f$ is a function $\}$

I will represent the functions defined from $\{0,1\}$ to $\mathbb{N}$ as such 2-tuples.

$f_a(0) = a_0$
$f_a(1) = a_1 \quad \rightarrow(a_0, a_1), a_0 \in \mathbb{N}, a_1 \in \mathbb{N}$

By the following enumeration.

(1,1),	$\rightarrow$	$a_0 + a_1 = 2$	$\rightarrow$	correspondence to (2-1) 1
(1,2),(2,1),	$\rightarrow$	$a_0 + a_1 = 3$	$\rightarrow$	correspondence to (3-1) 2
(1,3),(2,2),(2,1),	$\rightarrow$	$a_0 + a_1 = 4$	$\rightarrow$	correspondence to (4-1) 3
(1,4),(2,3),(3,2),(4,1),	$\rightarrow$	$a_0 + a_1 = 5$	$\rightarrow$	correspondence to (5-1) 4

.
.
.

There is an one-to-one correspondence between B and $\mathbb{N}$, we have an enumeration method listing all elements of B. Hence B is a countably infinite set.

By the **Lemma**, since A is an uncountably infinite set and B is countably infinite set. The given set $\{f \mid f \subseteq \mathbb{N} \times \{0,1\}\} \setminus \{f \mid f : \{0,1\} \rightarrow \mathbb{N}, f$ is a function $\}$, which is equal to $A \setminus B$,would be uncountably infinite set.

Question 3

Prove that the function $f(n) = 4^n + 5n^2 logn$ is not $O(2^n)$.

Solution

Assume that $f(n) = 4^n + 5n^2 logn$ is $O(2^n)$. (We will contradict.)

Then, there exists c and k constants such that

$$4^n + 5n^2 logn < c.2^n \ , \ \text{for all } n \geq k$$

$$4^n/2^n + (5n^2 logn)/2^n < c \ , \ \text{for all } n \geq k$$

$$2^n + (5n^2 logn)/2^n < c \ , \ \text{for all } n \geq k$$

$$2^n < c, \text{for all } n \geq k$$

This cannot hold for all $n \geq k$, because as n goes to infinity LHS of the inequality goes to infinity while RHS of the inequality remains constant, we get a contradiction.

The assumption has been contradicted.

Hence,$f(n) = 4^n + 5n^2 logn$ is not $O(2^n)$.

Question 4

Given two positive integers x and n such that $x > 2$ and $n > 2$, and the congruence relation $(2x-1)^n - x^2 \equiv -x-1(mod(x-1))$. I am required to determine the value of x.

Solution

$(2x-1)^n - x^2 \equiv -x-1(mod(x-1))$

$(2x-1)^n - x^2 + x + 1 \equiv 0(mod(x-1))$

$(2x-1)^n(mod(x-1)) + (-x^2+x+1)(mod(x-1)) \equiv 0(mod(x-1))$

$[(2x-1)(mod(x-1))]^n(mod(x-1)) + (-x^2+x+1)(mod(x-1)) \equiv 0(mod(x-1))$

$2x-1 = 2(x-1)+1 \rightarrow remainder = 1 \rightarrow 2x-1 \equiv 1(mod(x-1))$

$-x^2+x+1 = -x.(x+1)+1 \rightarrow remainder = 1 \rightarrow -x^2+x+1 \equiv 1(mod(x-1))$

$1^n(mod(x-1)) + 1(mod(x-1)) \equiv 0(mod(x-1))$

$1(mod(x-1)) + 1(mod(x-1)) \equiv 0(mod(x-1))$

$2(mod(x-1)) \equiv 0(mod(x-1))$

$x-1 \mid 2, x > 2$

$x-1 = 2$

$x = 3$

The value of x is 3.