Student Information

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Question 1

I am asked to prove the following logical equivalence.

$$\neg (p \land q) \leftrightarrow (\neg q \to p) \equiv (p \lor q) \land (\neg p \lor \neg q)$$

Let's define the equivalence $p \to q \equiv \neg p \lor q$ as **identity1**.

Let's also define the equivalence $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$ as **identity2**.

Steps and Justifications

- 1. $\neg (p \land q) \leftrightarrow (\neg q \rightarrow p)$ LHS of the equivalence
- 2. $\equiv (\neg p \lor \neg q) \leftrightarrow (\neg q \to p)$ By the first De Morgan law
- 3. $\equiv (\neg p \lor \neg q) \leftrightarrow (\neg (\neg q) \lor p)$ Using **identity1**
- 4. $\equiv (\neg p \lor \neg q) \leftrightarrow (q \lor p)$ By the double negation law
- 5. $\equiv [(\neg p \lor \neg q) \to (q \lor p)] \land [(q \lor p) \to (\neg p \lor \neg q)]$ Using **identity2**
- 6. $\equiv [\neg(\neg p \lor \neg q) \lor (q \lor p)] \land [(q \lor p) \rightarrow (\neg p \lor \neg q)]$ Using **identity1**
- 7. $\equiv [\neg(\neg p \lor \neg q) \lor (q \lor p)] \land [\neg(q \lor p) \lor (\neg p \lor \neg q)]$ Using **identity1**
- 8. $\equiv [(\neg(\neg p) \land \neg(\neg q)) \lor (q \lor p)] \land [(\neg q \land \neg p) \lor (\neg p \lor \neg q)]$ By the second De Morgan law
- 9. $\equiv [(p \land q) \lor (q \lor p)] \land [(\neg q \land \neg p) \lor (\neg p \lor \neg q)]$ By the double negation law
- 10. $\equiv [((p \land q) \lor q) \lor ((p \land q) \lor p)] \land [(\neg q \land \neg p) \lor (\neg p \lor \neg q)]$ Distributive laws
- 11. $\equiv [((p \land q) \lor q) \lor ((p \land q) \lor p)] \land [((\neg q \land \neg p) \lor \neg p) \lor ((\neg q \land \neg p) \lor \neg q)]$ Distributive laws
- 12. $\equiv [(q \lor (p \land q)) \lor (p \lor (p \land q))] \land [(\neg p \lor (\neg q \land \neg p)) \lor (\neg q \lor (\neg q \land \neg p))]$ Commutative laws
- 13. $\equiv [(q \lor (q \land p)) \lor (p \lor (p \land q))] \land [(\neg p \lor (\neg p \land \neg q)) \lor (\neg q \lor (\neg q \land \neg p))]$ Commutative laws
- 14. $\equiv [q \lor (p \lor (p \land q))] \land [(\neg p \lor (\neg p \land \neg q)) \lor (\neg q \lor (\neg q \land \neg p))]$ Absorption laws
- 15. $\equiv [q \lor p] \land [(\neg p \lor (\neg p \land \neg q)) \lor (\neg q \lor (\neg q \land \neg p))]$ Absorption laws
- 16. $\equiv [q \lor p] \land [\neg p \lor (\neg q \lor (\neg q \land \neg p))]$ Absorption laws
- 17. $\equiv [q \vee p] \wedge [\neg p \vee \neg q]$ Absorption laws
- 18. $\equiv (p \vee q) \wedge (\neg p \vee \neg q)$ Commutative laws

I have obtained RHS of the equivalence at the end. Hence, we have prove that $\neg(p \land q) \leftrightarrow (\neg q \rightarrow p)$ is logically equivalent to $(p \lor q) \land (\neg p \lor \neg q)$.

Question 2

I(x; y): x is an intern in faculty y.

E(x; y): x has employee id number y.

S(x; y): x is supervised by y.

A(x; y): x is admitted to job position y.

J(x; y): x is a job position in faculty y.

a. Two different interns in the same faculty cannot have the same employee id number.

Answer a:

$$\forall x \forall y \forall z \forall w ([(x \neq y) \land I(x,z) \land I(y,z) \land E(x,w)] \rightarrow \neg E(y,w))$$

where the domain of discourse for x,y is all people, the domain of discourse for z is all faculties and, the domain of discourse for w is all id numbers.

b. There are some interns in all faculties who are supervised by no one but themselves.

Answer b:

$$\forall x \exists y \forall z [I(y,x) \land S(y,y) \land (S(y,z) \rightarrow (z=y))]$$

where the domain of discourse for x is all faculties, the domain of discourse for y and z is all people.

c. At most two interns can be admitted to each job position in the medicine faculty.

Answer c:

$$\forall x \forall y \forall z \forall w [((x \neq y) \land (x \neq z) \land (y \neq z) \land (x, medicine) \land I(y, medicine) \land I(x, medicine) \land A(x, w) \land A(y, w)) \rightarrow \neg A(z, w)]$$

where the domain of discourse for x,y and z is all people, the domain of discourse for w is all job positions.

Lemmas

In the following questions, I am going to use use several lemmas which I will prove first here.

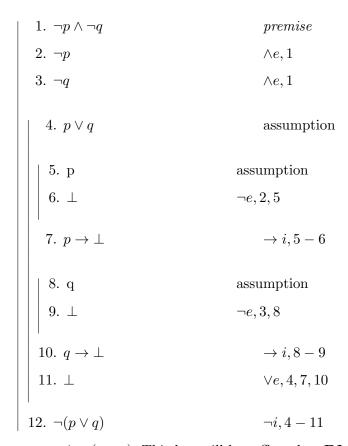


Table 1:De Morgan's Law: $\neg p \land \neg q \vdash \neg (p \lor q)$. This law will be refferred as **DM**

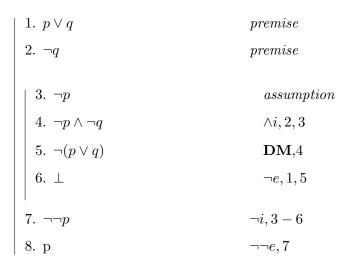


Table 2: Lemma: $p \lor q, \neg q \vdash p$. I will refer this lemma as **Göçer's Rule**.

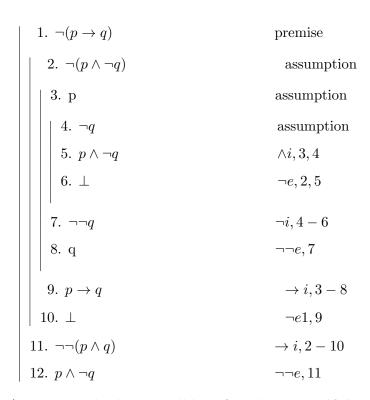


Table 3: Lemma: $\neg(p \to q) \vdash p \land \neg q$. This lemma will be referred as **Beautiful Rule**.

1. $\exists x \neg P(x)$	premise
$ 2. \forall x P(x)$	assumption
$\begin{vmatrix} 2. & \forall x \Gamma(x) \\ & 3. & \neg P(a) \\ & 4. & P(a) \\ & 5. & \bot \\ & 6. & \bot \end{vmatrix}$	assumption
4. P(a)	$\forall e, 2$
5. ⊥	$\neg e, 3, 4$
6. ⊥	$\exists e, 1, 3-5$
7. $\neg \forall x P(x)$	$\neg i, 2-6$

Table 4: Lemma: $\exists x \neg P(x) \vdash \neg \forall x P(x)$. This lemma will be referred as **Ankara's Rule**.

$1. \ \forall x \neg P(x)$	premise
$ 2. \exists x P(x)$	assumption
3. P(a)	assumption
3. $P(a)$ 4. $\neg P(a)$ 5. \bot	$\forall e,2$
	$\neg e, 3, 4$
6. ⊥	$\exists e, 2, 3-5$
7. $\neg \exists x P(x)$	$\neg i, 2-6$

Table 5: Lemma: $\forall x \neg P(x) \vdash \neg \exists x P(x)$. This lemma will be refferred as **METU's Rule**

1. $p \rightarrow q$	Premise
2. ¬q	Premise
3. p	Assumption
3. p 4. q 5. ⊥	$\rightarrow e, 1, 3$
5. ⊥	$\neg e, 2, 4$
$ 6. \neg p $	$\neg i, 3-5$

Table6: Lemma: $p \to q, \neg q \vdash \neg p$. This lemma will be refferred as **Modus Tollens**.

Question 3

a. $p \lor \neg q, p \lor r \vdash (r \to q) \to p$.

b.
$$\vdash ((q \rightarrow p) \rightarrow q) \rightarrow q$$
.

$$\begin{vmatrix} 1. & ((q \rightarrow p) \rightarrow q) & \text{assumption} \\ 2. & \neg q & \text{assumption} \\ 3. & \neg (q \rightarrow p) & \textbf{Modus Tollens}, 1, 2 \\ 4. & q \land \neg p & \textbf{Beautiful Rule}, 3 \\ 5. & q & \land e, 4 \\ 6. & \bot & \neg e, 2, 5 \\ 7. & \neg \neg q & \neg i, 2, 6 \\ 8. & q & \neg \neg e, 7 \\ 9. & ((q \rightarrow p) \rightarrow q) \rightarrow q & \rightarrow i, 1 - 8 \end{aligned}$$

Question 4

a.
$$\neg \forall x (P(x) \to Q(x)) \vdash \exists x (P(x) \land \neg Q(x)).$$

b.
$$\forall x \forall y (P(x,y) \rightarrow \neg P(y,x)), \forall x \exists y P(x,y) \vdash \neg \exists v \forall z P(z,v).$$

$$\forall x \forall y (P(x,y) \rightarrow \neg P(y,x)), \forall x \exists y P(x,y) \vdash \neg \exists v \forall z P(z,v).$$

$$1. \ \forall x \forall y (P(x,y) \rightarrow \neg P(y,x))) \qquad \text{premise}$$

$$2. \ \forall x \exists y P(x,y) \qquad \qquad \forall e, 2$$

$$3. \ a$$

$$4. \ \exists y P(a,y) \qquad \qquad \forall e, 2$$

$$5. \ b, P(a,b) \qquad \text{assumption}$$

$$6. \ P(a,b) \rightarrow \neg P(b,a) \qquad \qquad \forall \forall e, 1$$

$$7. \ \neg P(b,a) \qquad \qquad \forall e, 5, 6$$

$$8. \ \exists y \neg P(y,a) \qquad \qquad \exists i, 6$$

$$9. \ \forall x \exists y \neg P(y,x) \qquad \qquad \forall xi, 3, 4-7$$

$$10. \ \forall v \exists z \neg P(z,v) \qquad \qquad \text{the same as 9}$$

$$11. \ \forall v \neg \forall z P(z,v) \qquad \qquad \text{Ankara's Rule,10}$$

$$12. \ \neg \exists v \forall z P(z,v) \qquad \qquad \text{METU's Rule,11}$$