CENG 424 - Logic for Computer Science
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Homework 3

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1. (a) $\forall x(C(x) \Rightarrow \exists y(D(y) \land Friends(x, y)))$

   This sentence means "For all x, if x is a cat then there exists and animal y such that y is a dog and, x and y are friends." So, it can be translated simply as follows:

   "Each cat has at least one dog friend."

   (b) $\exists x(C(x) \land \forall y(D(y) \Rightarrow Friends(x, y)))$

   This sentence means "There exists an x such that x is a cat and for all y, if y is a dog, then x and y are friends, so it can be translated simply as follows:

   "There exists at least one cat that is friends with every dog."

2. (a) $\exists x.(\forall y.p(x,y) \Rightarrow p(z,z))$

Take the implications out:

$\exists x.(\neg\forall y.p(x,y) \lor p(z,z))$

Take the negations in:

$\exists x.(\exists y.\neg p(x,y)) \lor p(z,z))$

Now, let's apply skolemization to x and y by replacing them with brand new constants a and b, respectively.

$\exists y.\neg p(a,y) \lor p(z,z)$

$\neg p(a,b) \lor p(z,z)$

Be careful that z is considered as a constant since it is not in the scope of any quantifier.

Our Herbrand Base is $\{\neg p(a,b), p(z,z)\}$ and Herbrand Interpretations and their results are as follows:

Herbrand Interpretations	Result
$\{\}$	0
$\{\neg p(a,b)\}$	1
$\{p(z,z)\}$	1
$\{\neg p(a,b), p(z,z)\}$	1

Table 1: Evaluations of Herbrand Interpretations

As you can see from the table, results contain both 0 and 1s, so this sentence is **CONTINGENT**.

(b) $(\forall x.(p(x) \vee q(x)) \Rightarrow (\exists y.p(y) \Rightarrow (p(x) \Rightarrow \forall y.p(y)))$

Take the implications out:

$(\forall x.(p(x) \vee q(x)) \Rightarrow (\exists y.p(y) \Rightarrow (\neg p(x) \vee \forall y.p(y)))$

$(\forall x.(p(x) \vee q(x)) \Rightarrow (\neg \exists y.p(y) \vee (\neg p(x) \vee \forall y.p(y)))$

$(\neg \forall x.(p(x) \vee q(x)) \vee (\neg \exists y.p(y) \vee (\neg p(x) \vee \forall y.p(y)))$

Take the negations in:

$(\exists x.(\neg p(x) \wedge \neg q(x)) \vee (\forall y.\neg p(y) \vee (\neg p(x) \vee \forall y.p(y)))$

Now let's apply skolemization to x belonging to $\exists x$:

$(\neg p(a) \wedge \neg q(a)) \vee (\forall y.\neg p(y) \vee (\neg p(x) \vee \forall y.p(y)))$

Observe that we didn't replace the remaining "x" with "a" since it is not in the scope of the quantifier.

Notice that "y" which is is at the end and "y" which is before it are not the same because they are in the scope of different quantifiers. Let's rename the last one to avoid any confusion:

$(\neg p(a) \wedge \neg q(a)) \vee (\forall y.\neg p(y) \vee (\neg p(x) \vee \forall z.p(z)))$

Note that remaining x is a constant since it is not in the scope of any quantifier, so there is no need to handle it.

So, our final expression S is as follows:
S = $(\neg p(a) \wedge \neg q(a)) \vee \forall y.\neg p(y) \vee \neg p(x) \vee \forall z.p(z)$
¬ S = $(p(a) \vee q(a)) \wedge (\exists y.p(y)) \wedge p(x) \wedge (\exists z.\neg p(z))$

Let's apply skolemization to $\exists y$ and $\exists x$ in ¬ S:
$(p(a) \vee q(a)) \wedge p(b) \wedge p(x) \wedge \neg p(c)$

Our Herbrand Base for $\neg S$ is $\{p(a), q(a), p(b), p(x), \neg p(c)\}$ and there are $2^5 = 32$ Herbrand Interpretations. For simplicity, I will just show two of them which evaluates to 0 and 1 for ¬ S respectively.

(a) $\{\}$ makes ¬ S 0, hence S 1
(b) $\{p(a), p(b), p(x), \neg p(c)\}$ makes ¬ S 1, hence S 0

Therefore, the sentence is **CONTINGENT** because some interpretations satisfies it and some interpretations falsifies it.

(c) $\exists y.(p(y) \Rightarrow \exists x.q(x,y)) \Rightarrow \neg\exists x.q(y,x)$

Take the implications out:

$\neg\exists y.(p(y) \Rightarrow \exists x.q(x,y)) \vee \neg\exists x.q(y,x)$

$\neg\exists y.(\neg p(y) \vee \exists x.q(x,y)) \vee \neg\exists x.q(y,x)$

Take the negations in:

$\forall y.(p(y) \wedge \forall x.\neg q(x,y)) \vee \forall x.\neg q(y,x)$

Now, observe that "y" in the last universal quantifier is a constant because it is not in the scope of any quantifier. In order to avoid any confusion, let's rename it as "d".

$\forall y.(p(y) \wedge \forall x.\neg q(x,y)) \vee \forall x.\neg q(d,x)$

I obtained the above representation for the formula F. I just want to find ¬ F, because it will be easy for me (no specific reason).

¬ F $= \neg[\forall y.(p(y) \wedge \forall x.\neg q(x,y)) \vee \forall x.\neg q(d,x)]$

$\neg\forall y.(p(y) \wedge \forall x.\neg q(x,y)) \wedge \neg\forall x.\neg q(d,x)$

$\exists y.(\neg p(y) \vee \neg\forall x.\neg q(x,y)) \wedge \exists x.q(d,x)$

$\exists y.(\neg p(y) \vee \exists x.q(x,y)) \wedge \exists x.q(d,x)$

Now, let's apply skolemization to ¬ F:

$\exists y.(\neg p(y) \vee q(a,y)) \wedge \exists x.q(d,x)$

$(\neg p(b) \vee q(a,b)) \wedge \exists x.q(d,x)$

$(\neg p(b) \vee q(a,b)) \wedge q(d,c)$

Herbrand Base for ¬ F: $\{\neg p(b), q(a,b), q(d,c)\}$

Herbrand Interpretations	¬ F	F
$\{\}$	0	1
$\{\neg p(b)\}$	0	1
$\{q(a,b)\}$	0	1
$\{q(d,c)\}$	0	1
$\{\neg p(b), q(a,b)\}$	0	1
$\{\neg p(b), q(d,c)\}$	1	0
$\{q(a,b), q(d,c)\}$	1	0
$\{\neg p(b), q(a,b), q(d,c)\}$	1	0

Table 2: Evaluations of Herbrand Interpretations

3.

1. $\forall x(p(x) \Rightarrow q(x))$	Premise
2. $\neg\exists z r(z)$	Premise
3. $\exists y p(y) \vee r(a)$	Premise
4. $p(a) \Rightarrow q(a))$	UI: 1
5. $\neg r(a)$	EI: 2
6. $p(a) \vee r(a)$	EI: 3
7. $(p(a) \vee r(a)) \Leftrightarrow (\neg p(a) \Rightarrow r(a))$	OQ
8. $((p(a) \vee r(a)) \Leftrightarrow (\neg p(a) \Rightarrow r(a))) \Rightarrow ((p(a) \vee r(a)) \Rightarrow (\neg p(a) \Rightarrow r(a)))$	EQ
9. $(p(a) \vee r(a)) \Rightarrow (\neg p(a) \Rightarrow r(a))$	MP: 7, 8
10. $\neg p(a) \Rightarrow r(a))$	MP: 6,9
11. $(\neg p(a) \Rightarrow r(a)) \Rightarrow ((\neg p(a) \Rightarrow \neg r(a)) \Rightarrow p(a))$	CR
12. $(\neg p(a) \Rightarrow \neg r(a)) \Rightarrow p(a)$	MP: 10, 11
13. $\neg r(a) \Rightarrow (\neg p(a) \Rightarrow \neg r(a))$	II
14. $\neg p(a) \Rightarrow \neg r(a)$	MP: 5, 13
15. $p(a)$	MP: 12, 14
16. $q(a)$	MP: 4, 15
17. $\exists z q(z)$	EG: 16

4. Premise 1: $\forall y A(a, y) = \{A(a, y)\}$

   Premise 2: $\forall x \forall y(A(x, y) \Rightarrow A(B(x), B(y))) = \forall x \forall y(\neg A(x, y) \vee A(B(x), B(y)))$
   $= \{\neg A(x, y), A(B(x), B(y))\}$

   Negated Goal: $\neg \exists z(A(a, z) \wedge A(z, B(B(a))) = \forall(\neg A(a, z) \vee \neg A(z, B(B(a))) = \{\neg A(x, y), A(B(x), B(y))\}$

1. $\{A(a, y)\}$	Premise
2. $\{\neg A(x, y), A(B(x), B(y))\}$	Premise
3. $\{\neg A(a, z), \neg A(z, B(B(a)))\}$	Negated Goal
4. $\{A(B(a), B(y))\}$	1,2: $\{x \leftarrow a\}$
5. $\{\neg A(z, B(B(a))\}$	1,3: $\{y \leftarrow z\}$
6. $\{\}$	4,5: $\{z \leftarrow B(a), y \leftarrow B(a)\}$