CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 1 Solutions

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November 14, 2022

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1. (a) $y(k+3) + 2y(k+1) - y(k) = 5k + 8$

   The difference equation of order $n = 3$ is of the form:

   $a_n(k)y(k+n) + a_{n-1}(k)y(k+n-1) + ... + a_1(k)y(k+1) + a_0(k)y(k) = g(k)$

   Thus, the system is **"linear"**.

   Since all of the coefficients a_i does not depend on k, i.e. they are all constants, the system is **"time-invariant"**.

   It is **"forced"**, as the forcing term, $g(k)$, is not equal to zero, i.e. $(g(k) \neq 0)$.

   Answer = **linear, time-invariant, forced**

   (b) $\ddot{y}(t) - (t+1)^2\dot{y}(t) - y(t) = 0$

   The differential equation of order $n = 2$ is of the form:

   $a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_1(t)\dot{y}(t) + a_0(t)y(t) = g(t)$

   Hence, the system is **"linear"**.

   Since there is a coefficient depending on t, which is $-(t+1)^2$, it is **"time-varying"**.

   Because the forcing term, $g(t)$, is equal to zero, i.e. $(g(t) = 0)$, it is **"unforced"**

   Answer = **linear, time-varying, unforced**

(c) $\ddot{y}(t) - 5\dot{y}(t) + 6y(t) = y^2(t) + 3$

The linear continuous systems can be written in the form of

$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_1(t)\dot{y}(t) + a_0(t)y(t) = g(t).$

However, this system cannot be written in that form because of the term $y^2(t)$, so it is **"non-linear"**.

Since all coefficients does not depend on t, it is **"time-invariant"**.

It is **"forced"**, as the forcing term, g(t), is not equal to zero.

Answer = **non-linear, time-invariant, forced**.

2. (a) Find eigenvalues:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -2 \\ -5 & \lambda + 1 \end{vmatrix} = \lambda^2 - \lambda - 12 = 0 \longrightarrow \lambda_1 = 4,\ \lambda_2 = -3$$

Find corresponding eigenvectors:
for $\lambda_1 = 4$:

$$As_1 = \lambda_1 s_1 \longrightarrow \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} \longrightarrow x_1 = x_2 \longrightarrow s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -3$:

$$As_2 = \lambda_2 s_2 \longrightarrow \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 \\ -3x_2 \end{bmatrix} \longrightarrow 5x_1 = -2x_2 \longrightarrow s_2 = \begin{bmatrix} 1 \\ \frac{-5}{2} \end{bmatrix}$$

For $A = S\Lambda S^{-1}$:

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-5}{2} \end{bmatrix}, S^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{-2}{7} \end{bmatrix}$$

The system can be written as:

$$x' = (S\Lambda S^{-1})x + b$$

Multiply on the left by S^{-1}:

$$S^{-1}x' = \Lambda S^{-1}x + S^{-1}b$$
$$(S^{-1}x)' = \Lambda S^{-1}x + S^{-1}b$$

Let $u = S^{-1}x$ and c = $\mathrm{S}^{-1}b$, then the system becomes:

$$u' = \Lambda u + c$$

where $c = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{-2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ \frac{-2}{7} \end{bmatrix}$.

So, the system looks like

$$u' = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} u + \begin{bmatrix} \frac{9}{7} \\ \frac{-2}{7}, \end{bmatrix}$$

with $u_0 = S^{-1}x_0 = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{-2}{7} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{7} \\ \frac{-6}{7} \end{bmatrix}$

So, the matrix equation can be written simply as:

$$u_1'(t) = 4u_1(t) + \frac{9}{7},\ u_1(0) = \frac{-1}{7}$$
$$u_2'(t) = -3u_2(t) + \frac{-2}{7},\ u_2(0) = \frac{-6}{7}$$

We can solve them seperately. Recall that the solution to the one-dimensional equation $x' = ax + b$, $x(0) = x_0$ is $x(t) = e^{at}(x_0 + \frac{b}{a}) - \frac{b}{a}$. Therefore $u_1(t)$ and $u_2(t)$ is found as follows:

$$u_1(t) = e^{4t}(\frac{-1}{7} + \frac{9}{28}) - \frac{9}{28} = \frac{5}{28}e^{4t} - \frac{9}{28}$$

$$u_2(t) = e^{-3t}(\frac{-6}{7} + \frac{2}{21}) - \frac{2}{21} = \frac{-16}{21}e^{-3t} - \frac{2}{21}$$

As a result:

$$u(t) = \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{9}{28} \\ \\ \frac{-16}{21}e^{-3t} - \frac{2}{21} \end{bmatrix}$$

by back substitution $x = Su$:

$$x(t) = \begin{bmatrix} 1 & 1 \\ \\ 1 & \frac{-5}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{9}{28} \\ \\ \frac{-16}{21}e^{-3t} - \frac{2}{21} \end{bmatrix} = \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{16}{21}e^{-3t} - \frac{245}{588} \\ \\ \frac{5}{28}e^{4t} + \frac{40}{21}e^{-3t} - \frac{49}{588} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{16}{21}e^{-3t} - \frac{245}{588} \\ \\ \frac{5}{28}e^{4t} + \frac{40}{21}e^{-3t} - \frac{49}{588} \end{bmatrix}$$

(b)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{16}{21}e^{-3t} - \frac{245}{588} \\ \\ \frac{5}{28}e^{4t} + \frac{40}{21}e^{-3t} - \frac{49}{588} \end{bmatrix} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

The term e^{-3t} vanishes and the term e^{4t} dominates by exploading as $t \longrightarrow \infty$.

That is $x(t)$ is exploading in the direction of the eigenvector corresponding the eigenvalue $\lambda = 4$.

3. $\dot{x}(t) = -7x(t) + 5$

Let's say $\tilde{x}$ is the fixed point of the system. Then,

$-7\tilde{x} + 5 = 0 \longrightarrow \tilde{x} = \dfrac{5}{7}$, so $\tilde{x} = \dfrac{5}{7}$ is the fixed point of the system.

Recall that solution to the equation $\dot{x}(t) = ax(t) + b, x(0) = x_0$ is $e^{at}(x_0 + \dfrac{b}{a}) - \dfrac{b}{a}$.
So, $x(t) = e^{-7t}(x_0 - \dfrac{5}{7}) + \dfrac{5}{7}$.

$$\lim_{t\to\infty} x(t) = \lim_{x\to\infty} e^{-7t}(x_0 - \frac{5}{7}) + \frac{5}{7} \;= \frac{5}{7}$$

Hence, $x(t)$ converges to the fixed point $\tilde{x} = \dfrac{5}{7}$ as $t \longrightarrow \infty$.

Because $x(t) \longrightarrow \tilde{x} = \dfrac{5}{7}$ as $t \longrightarrow \infty$, fixed point $\tilde{x}$ is **"stable"** fixed point. This means the system gravitates toward the fixed point.

4. $\dddot{x}(t) + t^3\ddot{x}(t) + (t+1)\dot{x}(t) - x(t) = 2t + 1$

Let

$$\begin{aligned} x(t) &= y_1(t) \\ \dot{x}(t) &= y_2(t) = \dot{y}_1(t) \\ \ddot{x}(t) &= y_3(t) = \dot{y}_2(t) \\ \dddot{x}(t) &= -t^3 y_3(t) - (t+1)y_2(t) + y_1(t) + 2t + 1 = \dot{y}_3(t) \end{aligned}$$

Then, the system can be represented as follows:

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

$$\dot{Y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} Y + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

5. (a) For $x^1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$x(1) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} x(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x(2) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x(1) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$x(3) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} x(2) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$x(4) = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} x(3) = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \end{bmatrix}$

$x(5) = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} x(4) = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$

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It can be inferred that $x(k) = \begin{bmatrix} 1 \\ \frac{(k+1)(k+2)}{2} - 1 \end{bmatrix}$.

For $x^1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

$x(1) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} x(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x(2) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x(1) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x(3) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} x(2) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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It can be concluded that $x(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$a \begin{bmatrix} 1 \\ \frac{(k+1)(k+2)}{2} - 1 \end{bmatrix} + \text{b} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ is satisfied only if $a = b = 0$. So, the solutions are linearly independent.

We have two linearly independent solutions. Hence, the fundamental set of solutions is $\{ \begin{bmatrix} 1 \\ \frac{(k+1)(k+2)}{2} - 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$.

It can be written in the matrix form as:

$$X(k) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix}$$

(b) We know $\Phi(k,0) = X(k)X^{-1}(0)$.

$$X^{-1}(k) = \begin{bmatrix} 1 & 0 \\ -\dfrac{(k+1)(k+2)}{2} + 1 & 1 \end{bmatrix} \longrightarrow X^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi(k,0) = \begin{bmatrix} 1 & 0 \\ \dfrac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix}$$

$$\Phi(k,0) = \begin{bmatrix} 1 & 0 \\ \dfrac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix}$$

(c) Say $\begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \end{bmatrix}$ is the fixed point of the system. Then,

$$\begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k+2 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \end{bmatrix}$$

$\longrightarrow \tilde{x_1} = \tilde{x_1}$
$\longrightarrow (k+2)\tilde{x_1} + \tilde{x_2} = \tilde{x_2}$

For the second equation to satisfy, either (k+2) or $\tilde{x_1}$ should be 0. since k (time) cannot be negative, (k+2) cannot be 0, so $\tilde{x_1} = 0$. Then, $\tilde{x_2}$ can be any constant, i.e $\tilde{x_2} = c, c \in R$.

Hence, there are infinitely many fixed points $\tilde{X}$ of the form $\begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \end{bmatrix}$, $\tilde{x_1} = 0$, $\tilde{x_2} = c$, $c \in R$.

Say $x(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then :

$$x(t) = \begin{bmatrix} 1 & 0 \\ \dfrac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ [\dfrac{(k+1)(k+2)}{2} - 1]c_1 + c_2 \end{bmatrix}$$

If $c_1 = 0$, then $x(t)$ converges to $\begin{bmatrix} 0 \\ c_2 \end{bmatrix}$ as $k \longrightarrow \infty$. In other words, if the system starts at one of the fixed points, then system remains there.

If $c_1 \neq 0$, then $x(t)$ diverges as $k \longrightarrow \infty$. In other words, if the system does not start at one of the fixed points, then it diverges.