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Question 1

Lemma: $((k \in Z^+) \wedge (n \in Z^+)) \rightarrow (k^n \in Z^+)$. **(The lemma can be used without giving a proof.)**

Assume that 1 is not the smallest positive integer, since 1 is positive, by the Well-Ordering Principle, there is a smallest positive integer, say k, and $k < 1$. If we multiply the inequality

$$0 < k < 1$$

by k,then we get

$$0 < k^2 < k \quad .$$

By the **Lemma**, $k^2 \in Z^+$, implying that there is an element k^2 which is smaller than k and k is not the smallest integer. Therefore, our assumption has been contradicted.

Hence, 1 is the smallest positive integer.

Question 2

First part of the proof:

For $S(1,n)$:

Basis step: $S(1,1): x_1 = 1$, there is 1 solution which is $x_1 = 1$
Also,$f(1,1) = \dfrac{(1+1-1)!}{1!.(1-1))} = \dfrac{1!}{1!.0!} = 1$. So,$S(1,1)$ is correct.

Inductive step: Assume $S(1,n)$, i.e $x_1 = n$ has $f(1,n) = \dfrac{(n+1-1)!}{n!.(1-1)!} = 1$ solutions.
Then, for $S(1,n+1)$, i.e $x_1 = n+1$
By assumption, we should have 1 solution since we are just adding 1 on RHS.
Also, $f(1,n+1) = \dfrac{(n+1+1-1)!}{(n+1)!(1-1)!} = \dfrac{(n+1)!}{(n+1)!} = 1$.
$\therefore$ Hence , $S(1,n)$ is true.

For $S(m,1)$:

Basis step: $S(1,1): x_1 = 1$, there is 1 solution which is $x_1 = 1$
Also,$f(1,1) = \dfrac{(1+1-1)!}{1!.(1-1))} = \dfrac{1!}{1!.0!} = 1$. So,$S(1,1)$ is correct.

Inductive step: Assume $S(m,1)$, i.e $x_1 + x_2 + ... + x_m = 1$ has $f(m,1) = \dfrac{(1+m-1)!}{1!.(m-1)!} = \dfrac{(m!)}{(m-1)!} = m$ solutions.
Then, for $S(m+1,1)$ i.e ,$x_1 + x_2 + ... + x_m + x_{m+1} = 1$:
There are two cases: $x_{m+1} = 0$ and $x_{m+1} = 1$

For $x_{m+1} = 0$: the equation becomes $x_1 + x_2 + ... + x_m = 1$ and by assumption there are m solutions.
For $x_{m+1} = 1$: There is only 1 solution represented by (m+1)-tuples as (0,0,0,.....,1).

Therefore, total number of solutions is m + 1.
Also, $f(m+1,1) = \dfrac{(1+m+1-1)!}{1!.(m+1-1)!} = \dfrac{(m+1)!}{m!} = m+1$
$\therefore$ Hence , $S(m,1)$ is true.

Second part of the proof:

Assume $S(m,n+1)$ and $S(m+1,n)$ are true, i.e the numbers of solutions for

$$x_1 + x_2 + ... + x_m = n+1 \qquad (1)$$

$$x_1 + x_2 + ... + x_m + x_{m+1} = n \qquad (2)$$

are $f(m, n+1) = \dfrac{(n+m)!}{(n+1)!.(m-1)!}$ and $f(m+1, n) = \dfrac{(n+m)!}{n!.m!}$, respectively.

For $S(m+1, n+1)$:

the solutions of the equation

$x_1 + x_2 + ... + x_m + x_{m+1} = n+1 \qquad (3)$

can be divided into two parts : $x_{m+1} = 0$ and $x_{m+1} > 0$.

i) For $x_m + 1 = 0$, equation (3) becomes equation (1), hence number of solutions is

$$f(m, n+1) = \frac{(n+m)!}{(n+1)!.(m-1)!}$$

ii) For $x_m + 1 > 0$, x_{m+1} can be replaced by $x'_{m+1} + 1$. It is guaranteed that $x'_{m+1} \geqslant 0$ since $x_{m+1} > 0$. Therefore, we will not have any problem with restrictions in the question by doing this replacing.

Now, equation (3) becomes

$x_1 + x_2 + ... + x_m + x'_{m+1} = n$, which is in the form consistent with (2)

so, the number of solution is

$$f(m+1, n) = \frac{(n+m)!}{n!.m!}$$

Hence,

$\therefore$ The total number of solution is

$$\frac{(n+m)!}{(n+1)!.(m-1)!} + \frac{(n+m)!}{n!.m!} = \frac{(n+m)!.[(n+1)+m]}{(n+1)!.m!}$$

$$= \frac{[(n+1)+(m+1)-1]!}{(n+1)!.[(m+1)-1]!}$$

Also $f(m+1, n+1) = \dfrac{[(n+1)+(m+1)-1]!}{(n+1)!.[(m+1)-1]!}$
$\therefore$ $S(m+1, n+1)$ is also true.

Hence, we have proven $S(m, n)$ is true.

Question 3

a.

Count the number of 1 x 1 squares in the figure = 21 .

Each square can contain 4 triangles in the desired orientation and size: 21 x 4 = 84 .

On the diagonal of the half square we can have another 7 triangles.

Total number of triangles is 84 + 7 = 91 .

Hence, there are **91** triangles congruent to the one drawn in the figure, with the same size and of any orientation.

b.

By **the Principle of Inclusion - Exclusion**, the number of functions from a set with 6 elements to a set with 4 elements is

$$4^6 - \binom{4}{1}.3^6 + \binom{4}{2}.2^6 - \binom{4}{3}.1^6 = 4916 - 2916 + 384 - 4 = 1560$$

Question 4

a.

Let a_n be the number of strings over the alphabet $\Sigma = \{0, 1, 2\}$ of length n that contain two consecutive symbols that are the same.
Also, say **valid** string means a string over the alphabet $\Sigma = \{0, 1, 2\}$ of length n that contain two consecutive symbols that are the same.

There a_{n-1} valid strings of length (n - 1). We can produce $3.a_{n-1}$ valid strings of length n by placing any of $\{0, 1, 2\}$ at the end of each valid string of length (n - 1).

There are 3^{n-1} - a_{n-1} non-valid strings of length (n - 1). If we put the $(n-1)th$ element again as the nth element at the end of each non-valid strings of length (n - 1), we can produce a valid string of length n. So, we can produce 3^{n-1} - a_{n-1} valid strings of length n.

Therefore,

$$a_n = 3.a_{n-1} + 3^{n-1} - a_{n-1}$$

$$a_n = 2.a_{n-1} + 3^{n-1}$$

Hence, we obtained the above recurrence relation for the number of strings over the alphabet $= 0, 1, 2$ of length n that contain two consecutive symbols that are the same.

b.

Initial conditions for the recurrence relation are $a_1 = 0$, $a_2 = 3$.

c.

$$a_n = 2.a_{n-1} + 3^{n-1}$$

Find homogeneous solution a_n^h :

Characteristic equation for the recurrence relation: $\alpha - 2$. So, $\alpha = 2$ is the characteristic root. Therefore, homogeneous solution a_n^h is in the form of A . 2^n where A is a constant.

$$a_n^h = A \ . \ 2^n$$

Find particular solution a_n^p :

Non-homogeneity factor is 3^{n-1} , so particular solution a_n^p is in the form of B . 3^n where B is a constant.

$$a_n^p = B \ . \ 3^n$$

Now, find the constants A and B using initial conditions.

$$\begin{cases} \text{a}_1 = A \ . \ 2^1 \ + B \ . \ 3^1 = 2A + 3B = 0 \\ \text{a}_2 = A \ . \ 2^2 \ + B \ . \ 3^2 = 4A + 9B = 3 \end{cases} \quad \rightarrow \quad \text{A} = \ \frac{-3}{2} \ , B = 1 \ .$$

We have $a_n = a_n^h + a_n^p$, $a_n^h = \dfrac{-3}{2} \ . \ 2^n$, $a_n^p = 1 \ . \ 3^n$

Thus,

$$a_n = \frac{-3}{2} \ . \ 2^n \ + 1 \ . \ 3^n = -3 \ . \ 2^{n-1} + 3^n$$

Hence, by solving the recurrence relation we get

$$a_n = -3 \ . \ 2^{n-1} + 3^n$$