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Question 1

I am asked to prove the following logical equivalence.

$$\neg(p \wedge q) \leftrightarrow (\neg q \to p) \equiv (p \vee q) \wedge (\neg p \vee \neg q)$$

Let's define the equivalence $p \to q \equiv \neg p \vee q$ as **identity1**.
Let's also define the equivalence $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ as **identity2**.

Steps and Justifications

1. $\neg(p \wedge q) \leftrightarrow (\neg q \to p)$ **LHS of the equivalence**
2. $\equiv (\neg p \vee \neg q) \leftrightarrow (\neg q \to p)$ By the first De Morgan law
3. $\equiv (\neg p \vee \neg q) \leftrightarrow (\neg(\neg q) \vee p)$ Using **identity1**
4. $\equiv (\neg p \vee \neg q) \leftrightarrow (q \vee p)$ By the double negation law
5. $\equiv [(\neg p \vee \neg q) \to (q \vee p)] \wedge [(q \vee p) \to (\neg p \vee \neg q)]$ Using **identity2**
6. $\equiv [\neg(\neg p \vee \neg q) \vee (q \vee p)] \wedge [(q \vee p) \to (\neg p \vee \neg q)]$ Using **identity1**
7. $\equiv [\neg(\neg p \vee \neg q) \vee (q \vee p)] \wedge [\neg(q \vee p) \vee (\neg p \vee \neg q)]$ Using **identity1**
8. $\equiv [(\neg(\neg p) \wedge \neg(\neg q)) \vee (q \vee p)] \wedge [(\neg q \wedge \neg p) \vee (\neg p \vee \neg q)]$ By the second De Morgan law
9. $\equiv [(p \wedge q) \vee (q \vee p)] \wedge [(\neg q \wedge \neg p) \vee (\neg p \vee \neg q)]$ By the double negation law
10. $\equiv [((p \wedge q) \vee q) \vee ((p \wedge q) \vee p)] \wedge [(\neg q \wedge \neg p) \vee (\neg p \vee \neg q)]$ Distributive laws
11. $\equiv [((p \wedge q) \vee q) \vee ((p \wedge q) \vee p)] \wedge [((\neg q \wedge \neg p) \vee \neg p) \vee ((\neg q \wedge \neg p) \vee \neg q)]$ Distributive laws
12. $\equiv [(q \vee (p \wedge q)) \vee (p \vee (p \wedge q))] \wedge [(\neg p \vee (\neg q \wedge \neg p)) \vee (\neg q \vee (\neg q \wedge \neg p))]$ Commutative laws
13. $\equiv [(q \vee (q \wedge p)) \vee (p \vee (p \wedge q))] \wedge [(\neg p \vee (\neg p \wedge \neg q)) \vee (\neg q \vee (\neg q \wedge \neg p))]$ Commutative laws
14. $\equiv [q \vee (p \vee (p \wedge q))] \wedge [(\neg p \vee (\neg p \wedge \neg q)) \vee (\neg q \vee (\neg q \wedge \neg p))]$ Absorption laws
15. $\equiv [q \vee p] \wedge [(\neg p \vee (\neg p \wedge \neg q)) \vee (\neg q \vee (\neg q \wedge \neg p))]$ Absorption laws
16. $\equiv [q \vee p] \wedge [\neg p \vee (\neg q \vee (\neg q \wedge \neg p))]$ Absorption laws
17. $\equiv [q \vee p] \wedge [\neg p \vee \neg q]$ Absorption laws
18. $\equiv (p \vee q) \wedge (\neg p \vee \neg q)$ Commutative laws

I have obtained RHS of the equivalence at the end. Hence, we have prove that $\neg(p \wedge q) \leftrightarrow (\neg q \to p)$ is logically equivalent to $(p \vee q) \wedge (\neg p \vee \neg q)$.

Question 2

I(x; y): x is an intern in faculty y.
E(x; y): x has employee id number y.
S(x; y): x is supervised by y.
A(x; y): x is admitted to job position y.
J(x; y): x is a job position in faculty y.

a. Two different interns in the same faculty cannot have the same employee id number.

Answer a:

$$\forall x \forall y \forall z \forall w([(x \neq y) \wedge I(x,z) \wedge I(y,z) \wedge E(x,w)] \rightarrow \neg E(y,w))$$

where the domain of discourse for x,y is all people,the domain of discourse for z is all faculties and, the domain of discourse for w is all id numbers.

b. There are some interns in all faculties who are supervised by no one but themselves.

Answer b:

$$\forall x \exists y \forall z[I(y,x) \wedge S(y,y) \wedge (S(y,z) \rightarrow (z = y))]$$

where the domain of discourse for x is all faculties, the domain of discourse for y and z is all people.

c. At most two interns can be admitted to each job position in the medicine faculty.

Answer c:

$$\forall x \forall y \forall z \forall w[((x \neq y) \wedge (x \neq z) \wedge (y \neq z) \wedge I(x, medicine) \wedge I(y, medicine) \wedge I(z, medicine) \wedge J(w, medicine) \wedge A(x,w) \wedge A(y,w)) \rightarrow \neg A(z,w)]$$

where the domain of discourse for x,y and z is all people, the domain of discourse for w is all job positions.

Lemmas

In the following questions, I am going to use use several lemmas which I will prove first here.

1.	$\neg p \land \neg q$	*premise*
2.	$\neg p$	$\land e, 1$
3.	$\neg q$	$\land e, 1$
4.	$p \lor q$	assumption
5.	p	assumption
6.	$\bot$	$\neg e, 2, 5$
7.	$p \to \bot$	$\to i, 5-6$
8.	q	assumption
9.	$\bot$	$\neg e, 3, 8$
10.	$q \to \bot$	$\to i, 8-9$
11.	$\bot$	$\lor e, 4, 7, 10$
12.	$\neg(p \lor q)$	$\neg i, 4-11$

Table 1:De Morgan's Law: $\neg p \land \neg q \vdash \neg(p \lor q)$. This law will be refferred as **DM**

1.	$p \lor q$	*premise*
2.	$\neg q$	*premise*
3.	$\neg p$	*assumption*
4.	$\neg p \land \neg q$	$\land i, 2, 3$
5.	$\neg(p \lor q)$	**DM**,4
6.	$\bot$	$\neg e, 1, 5$
7.	$\neg\neg p$	$\neg i, 3-6$
8.	p	$\neg\neg e, 7$

Table 2: Lemma: $p \lor q, \neg q \vdash p$. I will refer this lemma as **Göçer's Rule**.

1. $\neg(p \to q)$	premise
2. $\neg(p \wedge \neg q)$	assumption
3. p	assumption
4. $\neg q$	assumption
5. $p \wedge \neg q$	$\wedge i, 3, 4$
6. $\perp$	$\neg e, 2, 5$
7. $\neg\neg q$	$\neg i, 4-6$
8. q	$\neg\neg e, 7$
9. $p \to q$	$\to i, 3-8$
10. $\perp$	$\neg e 1, 9$
11. $\neg\neg(p \wedge q)$	$\to i, 2-10$
12. $p \wedge \neg q$	$\neg\neg e, 11$

Table 3: Lemma:$\neg(p \to q) \vdash p \wedge \neg q$. This lemma will be referred as **Beautiful Rule**.

1. $\exists x \neg P(x)$	premise
2. $\forall x P(x)$	assumption
3. $\neg P(a)$	assumption
4. P(a)	$\forall e, 2$
5. $\perp$	$\neg e, 3, 4$
6. $\perp$	$\exists e, 1, 3-5$
7. $\neg \forall x P(x)$	$\neg i, 2-6$

Table 4: Lemma: $\exists x \neg P(x) \vdash \neg \forall x P(x)$. This lemma will be referred as **Ankara's Rule**.

1. $\forall x \neg P(x)$	premise
2. $\exists x P(x)$	assumption
3. P(a)	assumption
4. $\neg P(a)$	$\forall e, 2$
5. $\bot$	$\neg e, 3, 4$
6. $\bot$	$\exists e, 2, 3-5$
7. $\neg \exists x P(x)$	$\neg i, 2-6$

Table 5: Lemma: $\forall x \neg P(x) \vdash \neg \exists x P(x)$. This lemma will be refferred as **METU's Rule**

1. $p \rightarrow q$	Premise
2. $\neg q$	Premise
3. p	Assumption
4. q	$\rightarrow e, 1, 3$
5. $\bot$	$\neg e, 2, 4$
6. $\neg p$	$\neg i, 3-5$

Table6: Lemma: $p \rightarrow q, \neg q \vdash \neg p$. This lemma will be refferred as **Modus Tollens**.

Question 3

a. $p \vee \neg q, p \vee r \vdash (r \to q) \to p$.

1. $p \vee \neg q$	premise
2. $p \vee r$	premise
3. $r \to q$	assumption
4. $\neg p$	assumption
5. $\neg q$	**Göçer's Rule** 1,4
6. r	**Göçer's Rule** 2,4
7. q	$\to e, 3, 6$
8. $\perp$	$\neg e, 5, 7$
9. $\neg\neg p$	$\neg i, 4-8$
10. p	$\neg\neg e, 9$
11. $(r \to q) \to p$	$\to i, 3-10$

b. $\vdash ((q \to p) \to q) \to q$.

1. $((q \to p) \to q)$	assumption
2. $\neg q$	assumption
3. $\neg(q \to p)$	**Modus Tollens**,1,2
4. $q \wedge \neg p$	**Beautiful Rule**,3
5. q	$\wedge e, 4$
6. $\perp$	$\neg e, 2, 5$
7. $\neg\neg q$	$\neg i, 2, 6$
8. q	$\neg\neg e, 7$
9. $((q \to p) \to q) \to q$	$\to i, 1-8$

Question 4

a. $\neg\forall x(P(x) \to Q(x)) \vdash \exists x(P(x) \land \neg Q(x))$.

1. $\neg\forall x(P(x) \to Q(x))$	*premise*
2. $\neg(P(x_0) \to Q(x_0))$	$\forall x e, 1$
3. $\neg(P(x_0) \land \neg Q(x_0))$	*assumption*
4. $P(x_0)$	*assumption*
5. $\neg Q(x_0)$	*assumption*
6. $P(x_0) \land \neg Q(x_0)$	$\land i, 4, 5$
7. $\perp$	$\neg e, 3, 6$
8. $\neg\neg Q(x_0)$	$\neg i, 5-7$
9. $Q(x_0)$	$\neg\neg e, 8$
10. $P(x_0) \to Q(x_0)$	$\to i, 4-9$
11. $\perp$	$\neg e, 2, 10$
12. $\neg\neg(P(x_0) \land \neg Q(x_0))$	$\neg i, 3-11$
13. $P(x_0) \land \neg Q(x_0)$	$\neg\neg e, 12$
14. $\exists x(P(x) \land \neg Q(x))$	$\exists x i, 13$

b. $\forall x\forall y(P(x,y) \to \neg P(y,x)), \forall x\exists y P(x,y) \vdash \neg\exists v\forall z P(z,v)$.

1. $\forall x\forall y(P(x,y) \to \neg P(y,x)))$	premise
2. $\forall x\exists y P(x,y)$	premise
3. a	
4. $\exists y P(a,y)$	$\forall e, 2$
5. b,$P(a,b)$	assumption
6. $P(a,b) \to \neg P(b,a)$	$\forall\forall e, 1$
7. $\neg P(b,a)$	$\to e, 5, 6$
8. $\exists y \neg P(y,a)$	$\exists i, 6$
9. $\forall x\exists y \neg P(y,x)$	$\forall x i, 3, 4-7$
10. $\forall v\exists z \neg P(z,v)$	the same as 9
11. $\forall v \neg\forall z P(z,v)$	**Ankara's Rule,10**
12. $\neg\exists v\forall z P(z,v)$	**METU's Rule,11**