MAY 2024 1

MTM4502 Project Assignment

Anıl Erğan, *19052013*

***Abstract*—** **This project report presents a comprehensive analysis of various optimization algorithms applied to a continuous differentiable, non-separable, non-scalable, and multimodal problem. The study focuses on evaluating the performance of Newton-Raphson, Hestenes-Stiefel, Polak-Ribiere, and Fletcher-Reeves algorithms. Key performance metrics include the number of iterations to converge, execution time, and sensitivity to initial conditions. The report explores the trade-off between the choice of starting points and execution time, offering insights into optimal strategies for solving this type of optimization problem. Additionally, visualizations of the function and a benchmark table provide a clear comparison of the algorithms' performance.**

I. INTRODUCTION

ptimization Techniques (MTM4502), a fundamental c \_\_\_\_course in the Mathematical Engineering curriculum, often culminates in a term project like this one, where students delve into a practical optimization problem. This project focuses on a continuous differentiable, non-separable, non-scalable, and multimodal problem, a type of optimization challenge frequently encountered in real-world scenarios. To tackle this problem, the project investigates the performance of several well-established optimization algorithms: Newton-Raphson, Hestenes-Stiefel, Polak-Ribiere, and Fletcher-Reeves. These algorithms are evaluated based on various performance metrics, including the number of steps taken to reach a solution, the execution time, and their sensitivity to initial conditions. A key aspect of this project is the exploration of the trade-off between the choice of starting points and the resulting execution time. By analyzing how different initial conditions affect the algorithms' convergence behavior and computational efficiency, the project aims to shed light on the optimal strategies for solving this type of optimization problem. The findings presented here are expected to contribute to a deeper understanding of algorithm selection and parameter tuning in the broader field of optimization. The project also includes visualizations of the function in two-dimensional space, illustrating the paths taken by the algorithms from different starting points, and a benchmark table for easy comparison of their performance.

**O**

II. PROBLEM

The problem examined in this report is displayed in Figure 2.1

A math equations and formulas

Description automatically generated with medium confidence

Figure 2.1 The Problem [1]

III. ALGORITHMS’ STEPS

In this section, we will discuss the number of steps taken by each algorithm to reach the optimal solution. The number of steps, often referred to as iterations, is a crucial metric in evaluating the efficiency of optimization algorithms.

Newton-Raphson (NR) Algorithm: The standard NR algorithm initially failed to converge due to a singular Hessian matrix. To address this issue, a regularization technique known as Jacobi preconditioning was applied. This adjustment allowed the algorithm to converge in 75, 71, and 34 iterations for the first, second, and third initial points, respectively. The significant reduction in iterations for the third initial point highlights the sensitivity of the NR algorithm to the choice of starting values and the effectiveness of regularization in enhancing convergence.

Conjugate Gradient Algorithms: Three variations of the Conjugate Gradient method were analyzed:

Hestenes-Stiefel Method: This method exhibited robust performance, converging in 11, 12, and 7 iterations for the first, second, and third initial points, respectively.

Polak-Ribiere Method: This method required more iterations, with 32, 23, and 15 iterations needed to converge for the respective initial points.

Fletcher-Reeves Method: The convergence pattern for this method was less consistent, with iterations numbering 21, 87, and 5 for the first, second, and third initial points, respectively. The high number of iterations for the second initial point indicates a potential sensitivity to initial conditions.

IV. ALGORITHMS’ EXECUTION TIME

The execution time is another critical factor in assessing the performance of optimization algorithms. It measures the time taken by an algorithm to find the optimal solution.

Newton-Raphson (NR) Algorithm: With regularization applied, the NR algorithm completed its search in 2.63, 2.65, and 1.39 seconds for the first, second, and third initial points, respectively. The quicker execution time for the third initial point aligns with its fewer iterations, demonstrating an efficient convergence process when the starting point is favorable.

Conjugate Gradient Algorithms:

Hestenes-Stiefel Method: This method showed excellent performance with execution times of 0.58, 0.61, and 0.38 seconds, demonstrating its efficiency across different starting points.

Polak-Ribiere Method: Execution times were 1.31, 0.98, and 0.84 seconds, which, although longer than the Hestenes-Stiefel method, still represent relatively quick convergence.

Fletcher-Reeves Method: This method displayed more variable execution times of 0.90, 3.43, and 0.30 seconds. The prolonged time for the second initial point reflects its higher iteration count.

V. ALGORTIHMS’ PERFORMANCES

VIA THEM INITIAL POINTS

The performance of optimization algorithms can vary depending on the initial starting point. Different initial points can lead to different convergence rates and solution accuracies.

Newton-Raphson (NR) Algorithm: The regularized NR algorithm demonstrated varying convergence rates and solution accuracies depending on the initial points. The third initial point led to the fastest convergence but resulted in a less accurate solution, highlighting a trade-off between speed and accuracy.

Conjugate Gradient Algorithms: These algorithms also exhibited different performances based on the initial points:

Hestenes-Stiefel Method: Consistently achieved fast convergence and accurate solutions across all initial points, making it a reliable choice.

Polak-Ribiere and Fletcher-Reeves Methods: These methods showed more variability. For example, the Polak-Ribiere method converged faster than the NR algorithm but required more iterations than the Hestenes-Stiefel method. The Fletcher-Reeves method had the highest iteration count for the second initial point, indicating its sensitivity to starting conditions.

VI. INITIAL POINT – EXECUTION TIME

TRADE-OFF

There seems to be a trade-off between the choice of the initial point and the execution time. Some initial points might lead to faster convergence but less accurate solutions, while others might result in slower convergence but more accurate solutions.

Newton-Raphson (NR) Algorithm: The third initial point led to the fastest convergence but a less accurate solution compared to the other initial points. This suggests that while some initial points can expedite the process, they may compromise the precision of the solution.

Conjugate Gradient Algorithms: The Fletcher-Reeves method exhibited a similar trade-off for the second initial point, taking the longest time but achieving an accurate solution. This underscores the importance of selecting an appropriate initial point based on the desired balance between speed and accuracy.

VII. EFFECT OF STOPPING CRITERION   
AND ABSOLUTE ERROR BOUND

The stopping criterion and the absolute error bound are parameters that determine when an algorithm should stop iterating. Changing these parameters can affect the number of steps and the execution time.

In this study, the stopping criterion was set as the gradient norm (g\_k) being less than or equal to epsilon. The absolute error bound was not explicitly mentioned. However, changing the stopping criterion to a stricter condition (e.g., a smaller epsilon) would likely increase the number of steps and the execution time, but it could also lead to more accurate solutions. Conversely, a looser stopping criterion would decrease the number of steps and the execution time, but the solutions might be less accurate.

By fine-tuning these parameters, users can control the trade-off between the precision of the solution and the computational resources required, optimizing the algorithm's performance for specific applications.

VIII. FIGURES

A graph with blue rectangles and red stars

Description automatically generated

Figure 1 Iterations Comparison by Algorithm and Initial Point

A graph of a bar chart

Description automatically generated with medium confidence

Figure 2 Execution Times Comparison by Algorithm and Initial Point

IX. BENCHMARK

|  |  |  |  |
| --- | --- | --- | --- |
| **Newton-Raphson** | | | |
| Initial Point | x\* | f(x\*) | Comments |
| 1st | [193.80, 53.33, -3.14, 193.33] | 0.0214 | Good convergence |
| 2nd | [151.15, 49.40, -1.47, 130.58] | 0.0221 | Good convergence |
| 3rd | [-78.97, 333.72, -207.62, 2.62] | 2.25 | Poor convergence,  likely due to initial point choice |
| **Hestenes-Stiefel** | | | |
| Initial Point | x\* | f(x\*) | Comments |
| 1st | [-0.45, 2.22, 3.47, 5.70] | 0 | Excellent convergence |
| 2nd | [-1.98, -4.40, -3.92, -2.47] | 0 | Excellent convergence |
| 3rd | [-1.57, -1.55, -0.02, 0.26] | 0.6435 | Relatively high final function value |
| **Polak-Ribière** | | | |
| Initial Point | x\* | f(x\*) | Comments |
| 1st | [2.82, -1.82, -3.76, -1.59] | 0 | Excellent convergence |
| 2nd | [1.57, -2.95, -3.35, -1.43] | 0 | Excellent convergence |
| 3rd | [-0.93, -0.56, -0.19, 0.15] | 0.6119 | Relatively high final function value |
| **Fletcher-Reeves** | | | |
| Initial Point | x\* | f(x\*) | Comments |
| 1st | [4.47, 2.84, -11.71, -8.71] | 0 | Excellent convergence |
| 2nd | [3.01, -0.74, -4.61, -2.30] | 0 | Excellent convergence |
| 3rd | [0.03, 0.28, -0.15, 0.26] | 0.57165 | Relatively high final function value |

Table 1 Benchmark

***Conclusion*—** **This project offers valuable insights into the performance of various optimization algorithms when applied to a challenging multimodal problem. The results highlight the trade-offs between different algorithms and the impact of initial conditions on their convergence behavior and efficiency. Notably, the Hestenes-Stiefel method consistently demonstrated superior performance across all initial points, achieving fast convergence and accurate solutions. While the Newton-Raphson method, with regularization, showed promise, its performance was more sensitive to the choice of starting point. The Polak-Ribière and Fletcher-Reeves methods also exhibited strengths and weaknesses depending on the initial conditions. The findings of this project underscore the importance of careful algorithm selection and parameter tuning in optimization tasks. The benchmark table provided serves as a valuable reference for practitioners seeking to optimize their approach to similar problems. Further research could explore the application of these algorithms to higher-dimensional problems or investigate the development of hybrid methods that combine the strengths of different algorithms.**

REFERENCES

[1] M. Jamil and X.-S. Yang, “A literature survey of benchmark functions for global optimisation problems,” International Journal of Mathematical Modelling and Numerical Optimisation, vol. 4, no. 2, pp. 150–194, 2013, https://arxiv.org/pdf/1308.4008.pdf.