The Role of Energy Efficiency in Productivity: Evidence from Canada*

by Anil Gogebakan†

August 26, 2025

Abstract

This paper quantifies how the misallocation of energy, capital, and labor across provinces and sectors reduces productivity in Canada. Using annual provincial input–output data (2014–2020) and a standard Hsieh–Klenow–style framework, I decompose the loss into interprovincial (within-sector) and intersectoral (within-province) components and measure each input's contribution. Unlike most studies focusing on firm-level variation within manufacturing, I examine the full economy at the province–sector level. The results show that misallocation drags aggregate productivity 5–8% below its efficient potential, depending on the assumed substitutability of goods between provinces. Within-sector misallocation drives most of the loss, reaching 4.3%, while between-sector misallocation contributes up to 4%, consistently less than the within-sector component, indicating interprovincial misallocation is the primary source. While energy accounts for only 8% of input costs, it makes up as much as 1.5% of the gap—comparable to capital's 1.8% peak and substantially exceeding that of labor's 0.8%—highlighting energy's key role in productivity.

Keywords: Misallocation; Total Factor Productivity; Distortions; Energy; Canada.

JEL classification: O11, O13, O41, O47, O51, D24, D61.

*This project has been supported by Environment and Climate Change Canada's Economics and Environmental Policy Research Network (EEPRN).

†University of Calgary, anil.gogebakan@ucalgary.ca

1 Introduction

How large are the productivity losses from energy misallocation in Canada? While the inefficient use of resources has long been recognized as a major source of substantial economic output loss and low productivity Hsieh and Klenow (2009), Restuccia and Rogerson (2017), Brandt et al. (2013), Bartelsman et al. (2013), Chen and Irarrazabal (2015), most existing studies focus on capital and labor misallocation across firms within the manufacturing sector Bartelsman et al. (2013), Chen and Irarrazabal (2015). In contrast, energy—a key input in nearly all economic activities—has received relatively little attention Asker et al. (2019), Choi (2020), Tombe and Winter (2015), despite its growing relevance in both productivity and environmental policy debates.

Energy differs from capital and labor in ways that make its misallocation particularly relevant in the context of aggregate productivity analysis. First, energy is far less mobile than capital or labor, as its availability and cost vary significantly across provinces due to differences in natural endowments and infrastructure—challenges that are further amplified by interprovincial trade barriers. Second, energy markets are heavily shaped by regulation, ownership, and policy—resulting in persistent price gaps across provinces that do not adjust through market mechanisms, unlike wages or returns to capital. Third, although energy represents only about 8% of input costs, its misallocation contributes up to 2% of aggregate output loss—making it more distortionary per dollar than either labor or capital. Furthermore, improving the efficiency of energy use is not only economically beneficial but also environmentally strategic. Achieving higher output with the same energy input can reduce the economic cost of environmental regulations, making it easier to meet climate targets without sacrificing growth. These features make energy a critical, yet mostly disregarded, factor in understanding allocative efficiency.

This paper quantifies the productivity loss in Canada from the misallocation of energy in addition to capital and labor at the sector level across provinces. Using detailed annual provincial input-output data from Statistics Canada for the period 2014–2020, I extend the standard Hsieh and Klenow (2009) framework to incorporate energy as a third input alongside capital and labor. I measure the marginal revenue products of each input at the sector-province level and compare them to an efficient benchmark, allowing me to compute both the magnitude and sources of allocative inefficiency.

Canada is an especially relevant case for this analysis. Its provinces operate with substantial autonomy over energy policy, resulting in significant variation in prices, regulatory regimes, and energy mix. These differences, combined with fragmented infrastructure and limited interprovincial trade, make the Canadian economy particularly vulnerable to spatial misallocation of energy. Quantifying these inefficiencies is essential for designing better policies that promote both economic productivity and energy efficiency.

A growing Canadian literature documents the productivity costs of interprovincial trade barriers and frictions, which are often estimated to be significant. Studies such as Albrecht and Tombe (2016) and Alvarez et al. (2019) show that barriers to internal trade can reduce aggregate productivity by 3 to 7%, with the greatest potential gains from integration occurring in resource-intensive and infrastructure-dependent sectors. While most of the existing literature focuses on final goods markets, my analysis reveals similar frictions for production inputs, particularly energy, which lead to persistent interprovincial distortions and inefficient allocation. This paper therefore complements and extends the internal trade literature by quantifying the misallocation of a key production input—energy—and linking it explicitly to both within- and between-sector distortions arising from provincial trade and regulatory fragmentation.

This analysis is further relevant to Canada's growing environmental concerns and ongoing discussions on environmental policy. Energy efficiency gains reduce both production costs and greenhouse gas emissions, creating a "double dividend" Goulder (1995). Given that energy misallocation accounts for about 2% of aggregate loss (within- and between-components combined), these gains have the potential to significantly lower the economic cost of meeting climate goals. In a context where provinces face different carbon pricing regulations, Tombe and Winter (2015) and MacNab (2017), as well as varying renewable energy potentials, reallocating energy to its most productive uses can help meet national climate targets at lower economic cost. In addition, improving interprovincial energy flows could reduce reliance on local sources with higher emissions, providing environmental benefits that GDP-based measures alone fail to capture. These concerns position energy misallocation not only as an economic challenge, but also as an environmental policy issue, with implications for both infrastructure planning and climate strategy.

The empirical results reveal substantial inefficiencies in the allocation of energy, capital, and labor across provinces and sectors. Using a constant elasticity of substitution parameter, I find that aggregate output could be up to 9.4% higher under a more elastic assumption ($\sigma = 7$) and about 8% higher under a conservative elasticity assumption ($\sigma = 3$). These elasticity values mirror the existing literature, which typically estimates substitutability across production units within sectors. Since the model here employs a more aggregated provincial–sectoral level data, these values should be interpreted as illustrative benchmarks rather than precise estimates: they span the plausible range of substitution possibilities, but actual substitutability across provinces may be lower or higher because of additional frictions and heterogeneity at this higher level of aggregation. Presenting results for both lower and higher elasticity scenarios allows us to quantify the bounds of the range of the potential efficiency gains and show how sensitive the estimates are to the assumed substitutability parameter.

I further decompose the potential gains into within-sector (interprovincial) and between-sector (intersectoral) components. In the benchmark case ($\sigma = 3$), within-sector misallocation—driven by regulatory fragmentation and limited energy trade—accounts for roughly 3.4 to 4.3 percentage points of the total loss, while between-sector misallocation ranges from about 1.3 to 4.0 percentage points. In the between-sector dimension, energy consistently emerges as the second-largest contributor—exceeding labor in all years and, in some cases, approaching capital's contribution—despite representing only about 8% of input costs. Furthermore, in the within-sector component, energy's role remains substantial, accounting for 1.0 to 1.6 percentage points of the loss. These results underscore the disproportionate role of energy and the importance of interprovincial factors—such as trade barriers and regulatory differences—in driving productivity loss.

To model interprovincial flows and their contribution to misallocation, I adopt an approach analogous to the Armington model of trade, which assumes imperfect substitutability between similar goods produced in different locations. This assumption reflects the observed persistence of cross-provincial price differences, suggesting that provincial outputs are not perfect substitutes and that interprovincial frictions limit reallocation. Framing the model in this way provides a theoretically consistent and empirically relevant basis for analyzing the role of interprovincial trade frictions in aggregate productivity losses. In short, findings from this paper highlight the need to integrate energy policy and interprovincial trade reforms more centrally into productivity-enhancing strategies.

This paper makes three main contributions. First, it provides the first comprehensive estimate of energy misallocation in Canada using sector-by-province data, offering insights that go beyond the manufacturing sector and firm-level analyses common in the literature. Second, it quantifies the welfare cost of energy distortions across geographic and sectoral dimensions, emphasizing the role of spatial frictions in depressing productivity. Third, it offers a tractable and generalizable framework for evaluating allocative efficiency in energy use, which can inform policy debates around energy pricing, interprovincial infrastructure, and climate policy.

The remainder of the paper is organized as follows: Section 2 describes the data and measurement approach. Section 3 presents the theoretical framework. Section 4 outlines the main findings. Section 5 concludes.

2 Data

This study examines the data from the Provincial Symmetric Input-Output Tables (Catalogue no. 15-211-X) published by Statistics Canada's Industry Accounts Division. These tables provide a comprehensive, annually consistent depiction of inter-industry transactions at the provincial level in Canada. Specifically, I utilize the detailed aggregation level for the years from 2014 to 2020 inclusive, which offers a high-resolution view of economic flows across provinces and sectors.

The symmetric input-output tables reformat the standard supply and use tables into an industry-by-industry framework, allowing for clearer identification of the production structure and intermediate demand relationships. The data captures all inter-sectoral purchases—including expenditures on imports, inventory withdrawals, and primary inputs—making them well-suited for structural and efficiency analyses. The final demand tables similarly record all purchases by final demand categories from provincial and imported sources.

The data used reflect Statistics Canada's most detailed industry classifications and are harmonized across years, enabling consistent cross-provincial and intertemporal comparisons. The version of the tables used in this study corresponds to the level of aggregation that was previously known as "Aggregation Level S," which was renamed "Detailed" in 2019.

For methodological transparency and further technical detail, the construction of these tables is documented by Statistics Canada and available through direct inquiry with the Industry Accounts Division.

3 Model

3.1 Aggregate Output and Sectoral Shares

I consider a standard model of monopolistic competition with heterogeneous provinces, indexed by i. I closely follow the framework of (Hsieh and Klenow, 2009) with a natural extension of energy as an input in the production function. In the economy, a single aggregate output Y is produced by aggregating all sector contributions at the national level:

$$Y = \prod_{s=1}^{S} Y_s^{\theta_s}, \quad \text{where } \sum_{s=1}^{S} \theta_s = 1. \tag{3.1}$$

θ_s is the share of each sector within the national economic output. Each sector's output Y_s is given by:

$$Y_s = \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{3.2}$$

This is the standard constant elasticity of substitution (CES) function over provinces with elasticity of substitution parameter σ.

The sectoral profit maximization problem yields the aggregate price index P given by:

$$P = \prod_{s=1}^{S} \left(\frac{P_s}{\theta_s} \right)^{\theta_s} \tag{3.3}$$

Intuitively sectoral prices are scaled to their shares in the national economy and then aggregated based on the same shares.

Also, province- and sector-level profit maximization gives us the revenue equation for each province-by-sector level of revenue.

$$P_{si} Y_{si} = P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}} = P_{si}^{1-\sigma} P_s^{\sigma} Y_s. \tag{3.4}$$

Where the second part of the equality follows from simple algebra, where we take the power of σ on both sides of the first equality.

The sectoral expenditure minimization problem gives us the sectoral price index given by:

$$P_s = \left(\sum_i P_{si}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{3.5}$$

Now, I turn to the terms of the production function and productivity. I start with a usual profit maximization for sector s in province i. Define the production function as the Cobb-Douglas form with three inputs to production, namely capital (K), labor (L), and energy (E).

$$Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}, \quad \text{where } \alpha_s + \beta_s + \gamma_s = 1. \tag{3.6}$$

A_{si} represents total physical factor productivity (TFPQ). Each sector s in province i solves the following problem:

$$\max_{K_{si},L_{si},E_{si}} \quad P_{si}Y_{si} - (1+\tau_{K_{si}})rK_{si} - (1+\tau_{L_{si}})wL_{si} - (1+\tau_{E_{si}})p_E E_{si}. \tag{3.7}$$

Each input is subject to input distortions $\tau_{K_{si}}, \tau_{L_{si}}, \tau_{E_{si}}$, so that sectors in each province face distorted input prices.

By plugging Y_{si} and $P_{si}Y_{si}$ expressions above, we can solve this standard problem to get the marginal revenue product for each input.

$$MRPK_{si} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{K_{si}} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{K_{si}} = (1+\tau_{K_{si}})r, \tag{3.8}$$

$$MRPL_{si} = \beta_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{L_{si}} = \beta_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{L_{si}} = (1+\tau_{L_{si}})w, \tag{3.9}$$

$$MRPE_{si} = \gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{E_{si}} = \gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{E_{si}} = (1+\tau_{E_{si}})p_E. \tag{3.10}$$

Where $MRPK_{si}, MRPL_{si}, MRPE_{si}$ are the Marginal Revenue Product of capital, labor, and energy, respectively.

Define the following,

$$TFPQ_{si} = A_{si} = \frac{Y_{si}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{3.11}$$

$$TFPR_{si} = P_{si}A_{si} = \frac{P_{si}Y_{si}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{3.12}$$

where TFPQ is total physical factor productivity, which naturally can be different for each sector and wouldn't mean any distortion. On the other hand TFPR indicates total factor revenue productivity, and it should be equalized across provinces and sectors if it were not for distortions. Any dispersion in TFPR would translate into lower output and would mean misallocation of resources.

It is straightforward to see that the geometric average of marginal revenue products would be proportional to TFPR, and also it is proportional to the geometric average of distortion (τ) terms.

Hence,

$$TFPR_{si} \propto (MRPK_{si})^{\alpha_s}(MRPL_{si})^{\beta_s}(MRPE_{si})^{\gamma_s} \propto (1+\tau_{K_{si}})^{\alpha_s}(1+\tau_{L_{si}})^{\beta_s}(1+\tau_{E_{si}})^{\gamma_s} \tag{3.13}$$

Defining sectoral weighted average marginal revenue product for inputs as follows

$$\overline{MRPK_s} = \frac{\sum_i K_{si} MRPKsi}{\sum_i K_{si}} \tag{3.14}$$

gives us

$$\frac{\overline{MRPK_s}}{MRPK_{si}} = \frac{1}{(1+\tau_{K_{si}})\sum_i \frac{1}{(1+\tau_{K_{si}})}\frac{P_{si}Y_{si}}{P_sY_s}} \tag{3.15}$$

$$\frac{\overline{MRPL_s}}{MRPL_{si}} = \frac{1}{(1+\tau_{L_{si}})\sum_i \frac{1}{(1+\tau_{L_{si}})}\frac{P_{si}Y_{si}}{P_sY_s}} \tag{3.16}$$

$$\frac{\overline{MRPE_s}}{MRPE_{si}} = \frac{1}{(1+\tau_{E_{si}})\sum_i \frac{1}{(1+\tau_{E_{si}})}\frac{P_{si}Y_{si}}{P_sY_s}} \tag{3.17}$$

Intuitively, these are the deviations from the optimal allocation of resources across sectors and provinces.

With a bit more algebra, we arrive at the expression below.

$$A_s = \left[\sum_i \left(A_{si}\left(\frac{\overline{MRPK_s}}{MRPK_{si}}\right)^{\alpha}\left(\frac{\overline{MRPL_s}}{MRPL_{si}}\right)^{\beta}\left(\frac{\overline{MRPE_s}}{MRPE_{si}}\right)^{\gamma}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}. \tag{3.18}$$

Which is the total factor productivity at the sector level. To arrive at the output, we need to multiply each sector's productivity by based on their sector share θ_s to get the aggregate productivity level.

3.2 Aggregate output

Now we have all the ingredients to calculate aggregate output in the economy. If there were **no distortions** ($\tau_K = \tau_L = \tau_E = 0$), TFP_s would reach to its efficient level TFP_s^* - When distortions exist, provinces with higher distortions contribute less to output, reducing aggregate TFP.

$$A_s^* = TFP_s^* = \left(\sum_i A_{si}^{\sigma-1}\right)^{\frac{1}{\sigma-1}} \tag{3.19}$$

$$\frac{TFP_s}{TFP_s^*} = \left[\sum_i \left(\frac{A_{si}}{A_s^*} \left(\frac{\overline{MRPK_s}}{MRPK_{si}} \right)^{\alpha} \left(\frac{\overline{MRPL_s}}{MRPL_{si}} \right)^{\beta} \left(\frac{\overline{MRPE_s}}{MRPE_{si}} \right)^{\gamma} \right)^{\sigma-1} \right]^{\frac{1}{\sigma-1}}. \tag{3.20}$$

Finally, it is straightforward to compare the efficient level of aggregate output with the actual level of output.

$$\frac{Y}{Y^*} = \prod_s \left(\frac{TFP_s}{TFP_s^*} \right)^{\theta_s} \tag{3.21}$$

3.3 Productivity Decomposition

To understand which input distortion or which dimension (i.e., province or sector) contributes to welfare loss, we want to break down the equation. Let $\hat{x} = x/x^*$ be the comparison term between two levels of a variable. Here we are comparing the actual productivity level to the optimal level of productivity (i.e. no distortions). We start writing down the national level productivity TFP/TFP^* or A/A^*.

$$\frac{A}{A^*} = \underbrace{\prod_s \left(\frac{A_s}{A_s^*} \right)^{\theta_s}}_{\text{Within-sector misallocation}} \times \underbrace{\prod_s \left(\left(\frac{k_s}{k_s^*} \right)^{\alpha_s} \left(\frac{l_s}{l_s^*} \right)^{\beta_s} \left(\frac{e_s}{e_s^*} \right)^{\gamma_s} \right)^{\theta_s}}_{\text{Between-sector misallocation}} \tag{3.22}$$

Within component can be explicitly expressed as:

$$\left(\frac{A}{A^*} \right)_{within} = \prod_s \left(\frac{ \left[\sum_i \left(A_{si} \left(\frac{R_{si}/(1+\tau_{K_{si}})}{\sum_i R_{si}/(1+\tau_{K_{si}})} \right)^{\alpha_s} \left(\frac{R_{si}/(1+\tau_{L_{si}})}{\sum_i R_{si}/(1+\tau_{L_{si}})} \right)^{\beta_s} \left(\frac{R_{si}/(1+\tau_{E_{si}})}{\sum_i R_{si}/(1+\tau_{E_{si}})} \right)^{\gamma_s} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} }{ \left[\sum_i A_{si}^{\sigma-1} \right]^{\frac{1}{\sigma-1}} } \right)^{\theta_s} \tag{3.23}$$

Where $R_{si} = P_{si}Y_{si}/P_sY_s$. *Between* component can be expressed as

$$\left(\frac{A}{A^*}\right)_{\text{between}} = \prod_s \left(\underbrace{\left(\frac{\left(\frac{1}{\overline{1+\tau_{Ks}}}\right)\cdot\sum_s \alpha_s\theta_s}{\sum_s \frac{\alpha_s\theta_s}{\overline{1+\tau_{Ks}}}}\right)^{\alpha_s}}_{\text{Capital Misallocation}} \times \underbrace{\left(\frac{\left(\frac{1}{\overline{1+\tau_{Ls}}}\right)\cdot\sum_s \beta_s\theta_s}{\sum_s \frac{\beta_s\theta_s}{\overline{1+\tau_{Ls}}}}\right)^{\beta_s}}_{\text{Labor Misallocation}} \times \underbrace{\left(\frac{\left(\frac{1}{\overline{1+\tau_{Es}}}\right)\cdot\sum_s \gamma_s\theta_s}{\sum_s \frac{\gamma_s\theta_s}{\overline{1+\tau_{Es}}}}\right)^{\gamma_s}}_{\text{Energy Misallocation}} \right)^{\theta_s} \tag{3.24}$$

where $\overline{1+\tau_{Ks}} = \left(\sum_i \frac{R_{si}}{1+\tau_{K_{si}}}\right)^{-1}$ denotes the harmonic mean of $1+\tau_{K_{si}}$ weighted by R_{si}. I present the full derivation of these expressions in the appendix section. Now, we are ready to calculate the productivity loss from misallocation of various inputs, as well as the within-sector and between-provinces. This breakdown allows us to identify how much of the overall productivity gap is due to misallocation of each specific input, as well as inefficiencies within sectors across provinces.

4 Results

4.1 Measuring Input Specific Distortions

To measure the marginal revenue products, I start with the marginal revenue products under perfect competition and Cobb-Douglas production function assumptions. Recall that

$$MRPK_{si} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{K_{si}} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{K_{si}} = (1+\tau_{K_{si}})r \tag{4.1}$$

By taking the logarithm of each side and subtracting $ln(r)$, I can write the following equation.

$$ln(MRPK_{si}) - ln(r) = ln(\alpha_s \frac{(\sigma-1)}{\sigma}) + ln(\frac{P_{si}Y_{si}}{K_{si}}) - ln(r) = ln(1+\tau_{K_{si}}) \tag{4.2}$$

or alternatively,

$$ln(MRPK_{si}) - ln(r) = ln(\alpha_s \frac{(\sigma-1)}{\sigma}) + ln(\frac{P_{si}Y_{si}}{rK_{si}}) = ln(1+\tau_{K_{si}}) \tag{4.3}$$

This implies the equation below

$$\underbrace{\ln(MRPK_{si})}_{\epsilon_{si}} - \underbrace{ln(r) - ln(\frac{\sigma - 1}{\sigma})}_{\beta_0} - \underbrace{ln(\alpha_s)}_{\sum_s \beta_s \gamma_s} = ln(\frac{P_{si} Y_{si}}{r K_{si}}) \tag{4.4}$$

This expression motivates the following regression to recover the dispersion of marginal revenue products.

$$ln(\frac{P_{si} Y_{si}}{r K_{si}}) = \beta_0 + \sum_s \beta_s \gamma_s + \epsilon_{si} \tag{4.5}$$

The interpretation of the regression is intuitive. The dependent variable measures the ratio of revenue to capital expenditure. The intercept term captures common parameters, including the rent of capital and the elasticity of substitution. Sector-fixed effects absorb sector-level averages, while the error term reflects deviations from these averages. These residuals represent the unexplained variation and therefore provide information on the distribution of marginal revenue products. Formally, this implies $Var(ln(MRPK_{si}) = Var(ln(\hat{\epsilon}_{si}))$

This expression provides an estimate of the extent of misallocation. In the absence of distortions, MRPs would be equalized across units within a sector, and the variance would approach zero. Therefore, a larger residual variance indicates greater misallocation and potential productivity gains from reallocation.

The same procedure can be applied to labor and energy to obtain measures of misallocation across all major inputs. In the empirical analysis below, these residual variances are used to document the degree of factor misallocation across Canadian sectors and provinces.

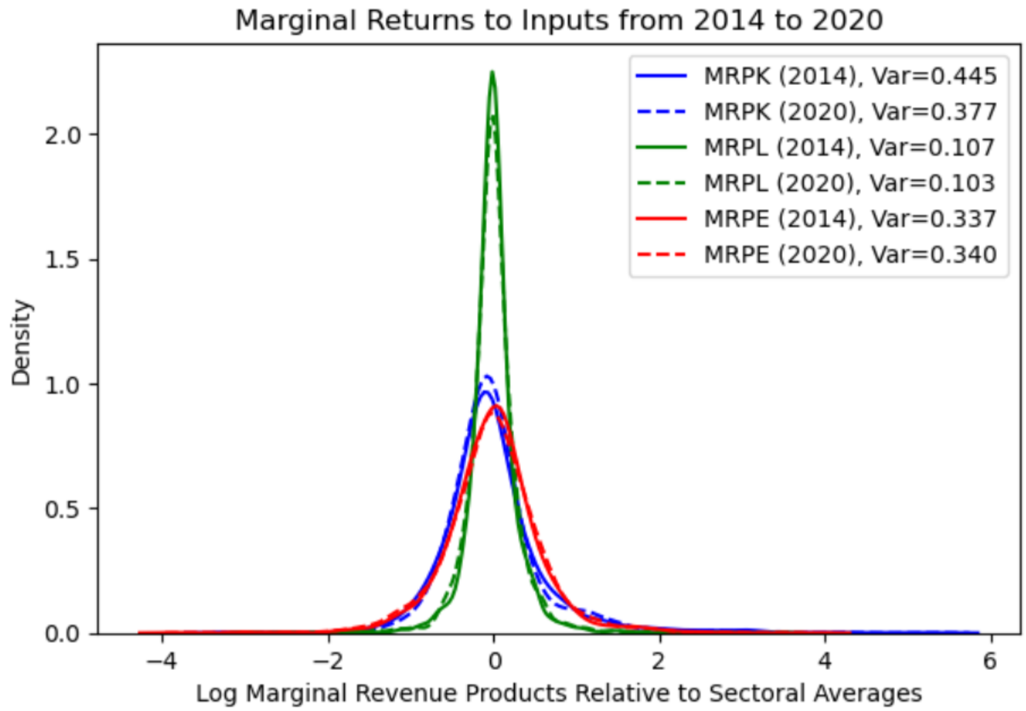

Figure 1: Marginal returns to inputs, 2014 vs. 2020.

In Figure 1, I present the distribution of marginal returns to labor, capital and energy for 2014 and 2020. The comparison highlights how marginal revenues of these inputs are allocated differently between provinces and sectors. Labor exhibits the lowest dispersion in both years, suggesting that misallocation of labor is relatively limited. This is consistent with much of the literature, which often finds that labor markets adjust more flexibly across regions and sectors compared to capital and energy. In contrast, capital and energy show much greater variation. Interestingly, while capital allocation appears to improve over time—with its distribution becoming more compressed—energy shows little sign of convergence. The persistence of wide dispersion in energy's marginal returns points to a structural inefficiency that has not diminished over the six-year period. This makes energy stand out as a central bottleneck: even as capital markets become more efficient, the lack of progress in energy allocation means that aggregate productivity gains remain constrained. Put differently, efficiency gains from better capital allocation risk being offset by persistent inefficiencies in energy usage across provinces and sectors.

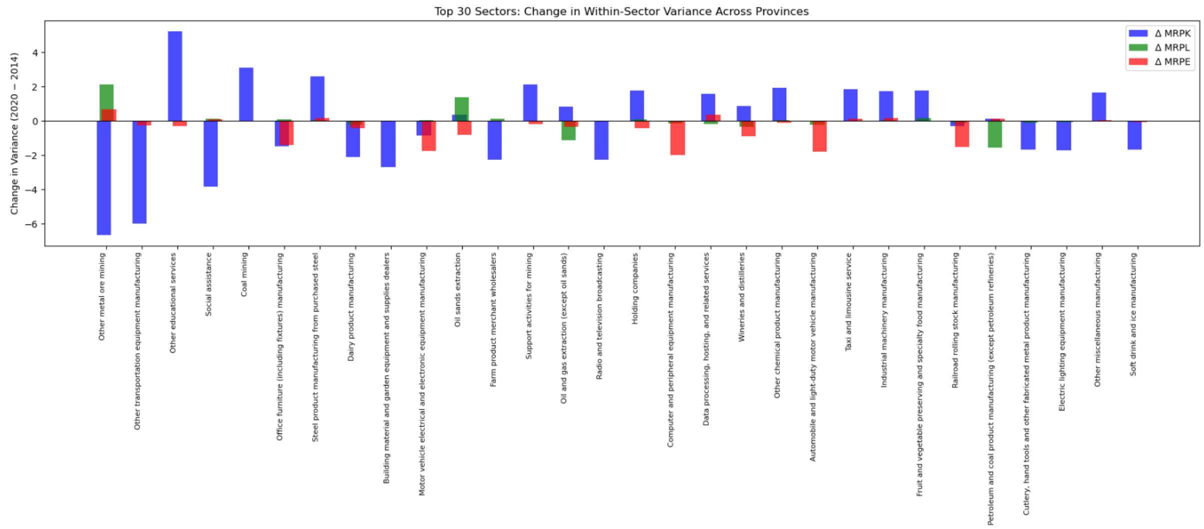

FIGURE 2: Changes in the variation of marginal returns to inputs by sector, 2014 vs. 2020.

Figure 2 investigates the distribution changes for each input further by showing how the variation in marginal revenue products of inputs change between 2014 and 2020 by sectors. For visual clarity, I rank sectors by the magnitude of these changes and plot the top 30 sectors where variation changed the most. The interpretation is straightforward: a decline in variance suggests an improvement in allocative efficiency, while an increase indicates a worsening misallocation problem. What emerges is a mixed picture—some sectors exhibit clear improvement, while others face worsening inefficiencies, particularly in energy inefficiency. Also, it is common to observe that a sector improves its efficiency in one input and worsens its efficiency in another input. Labor seems to exhibit the lowest amount of change in dispersion, while capital and energy leads most of the changes from 2014 to 2020. This sectoral heterogeneity underscores the idea that misallocation is not a uniform economy-wide phenomenon but one that is deeply shaped by sector-specific characteristics and dynamics. In the broader misallocation literature, such persistent dispersion is often tied to frictions—whether regulatory, infrastructural, or institutional—that prevent resources from flowing to their most productive uses. The evidence here suggests that energy is precisely where these frictions are most relevant in this context.

4.2 Relative TFPR Dispersion by Province

Next, I examine the dispersion of the total factor productivity revenue (TFPR) relative to the sectoral averages. A higher level of dispersion indicates greater productivity losses associated with misallocation in the economy. By definition, TFPR corresponds to the geometric average of the marginal revenue products under a Cobb–Douglas production function. This allows us to derive a simple expression for TFPR relative to its sectoral mean. To capture the extent of variation, I calculate the variance of the logarithm of this ratio, which measures the percentage deviation from the sectoral average. I present the results for the initial and final years of the sample period—2014 and 2020—at the province level.

$$\frac{TFPR_{si}}{\overline{TFPR_s}} = \left[\left(\frac{\overline{MRPK_s}}{MRPK_{si}} \right)^{\alpha_s} \left(\frac{\overline{MRPL_s}}{MRPL_{si}} \right)^{\beta_s} \left(\frac{\overline{MRPE_s}}{MRPE_{si}} \right)^{\gamma_s} \right]^{-1} \tag{4.6}$$

This equation implies the following explicit formula that we can use to illustrate the inefficiencies at the province level

$$\frac{TFPR_{si}}{\overline{TFPR_s}} = \frac{(1+\tau_{K_{si}})^{\alpha_s}(1+\tau_{L_{si}})^{\beta_s}(1+\tau_{E_{si}})^{\gamma_s}}{\left(\frac{1}{\sum_i (1+\tau_{K_{si}}) \frac{P_{si}Y_{si}}{P_sY_s}} \right)^{\alpha_s} \left(\frac{1}{\sum_i (1+\tau_{L_{si}}) \frac{P_{si}Y_{si}}{P_sY_s}} \right)^{\beta_s} \left(\frac{1}{\sum_i (1+\tau_{E_{si}}) \frac{P_{si}Y_{si}}{P_sY_s}} \right)^{\gamma_s}} \tag{4.7}$$

Figures 3 and 4 present the dispersion of TFPR relative to the sectoral benchmark for 2014 and 2020 across provinces. The provinces in the legend are ordered by the degree of dispersion around the benchmark. Ontario and Quebec exhibit the lowest levels of misallocation-related productivity loss in both years, although Quebec's position deteriorates noticeably between 2014 and 2020. In contrast, Alberta and British Columbia show some improvement over the same period, whereas New Brunswick experiences a clear decline. Manitoba and Saskatchewan, despite some improvement, remain among the provinces most affected by misallocation. To ensure a clearer presentation of the distributions, I trim the top and bottom 1% of observations to remove outliers. Overall, these figures highlight the relative positions of provinces in terms of efficiency and misallocation over time.

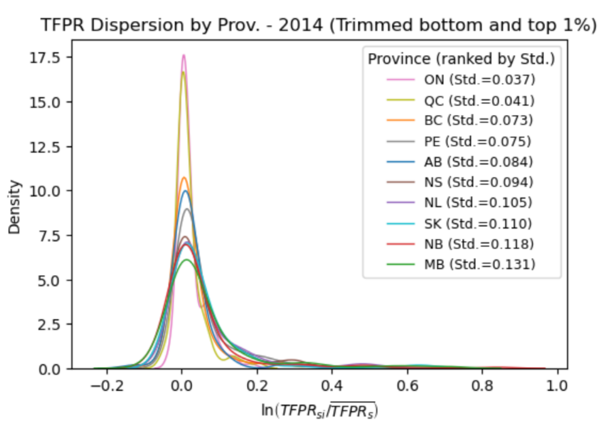

Figure 3: Relative TFPR dispersion by provinces in 2014

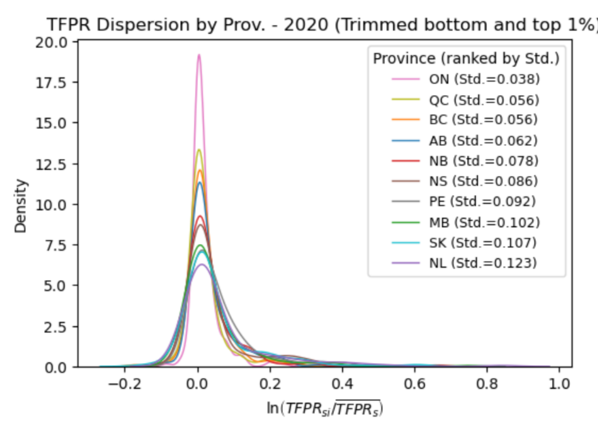

Figure 4: Relative TFPR dispersion by provinces in 2020

Overall, the dispersion of TFPR relative to the benchmark can differ by as much as 20% from the sectoral average, suggesting that misallocation is a significant contributor to productivity losses in Canada.

4.3 Total Misallocation and Decomposition

I decompose aggregate total factor productivity (TFP) losses at the province-sector level into two components: within-sector across provinces and between-sector within provinces, along with input-specific productivity loss measures from misallocation of capital, labor, and energy separately. Tables 1 and 2 report these results for the years 2014-2020, under two reference elasticities of substitution between sectors, $\sigma = 3$ and $\sigma = 7$ respectively.

Table 1: Potential TFP Gains from Input Reallocation (in %), 2014–2020, $\sigma = 3$

Component	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Total Misallocation	8.05	6.46	4.90	4.84	5.28	5.74	5.08
Between-sector Misallocation	3.96	2.25	1.27	1.53	1.53	1.96	1.63
Capital	1.80	1.22	0.55	0.66	0.71	0.88	0.83
Labor	0.78	0.50	0.36	0.37	0.39	0.37	0.46
Energy	1.43	0.55	0.36	0.50	0.45	0.73	0.34
Within-sector Misallocation	4.26	4.31	3.67	3.37	3.81	3.86	3.50
Capital	1.33	1.27	1.75	1.36	1.71	1.85	1.33
Labor	2.55	2.76	1.26	1.54	1.69	1.54	1.73
Energy	1.53	1.67	1.14	0.93	1.09	0.98	0.81

Table 1 reports the estimated potential total factor productivity (TFP) gains by eliminating input misallocation across provinces and sectors in Canada over the period 2014–2020 when $\sigma = 3$. The numbers are expressed as percentage deviations from the efficient benchmark, that is, $(1 - \frac{A}{A*}) \times 100$, where A denotes observed TFP and A* denotes counterfactual TFP under efficient input allocation. These values measure the extent to which distortions in capital, labor, and energy allocation lower aggregate productivity.

Under a conservative elasticity of substitution between provinces, $sigma = 3$, the total potential productivity gains from eliminating input misallocation range from 8% in 2014 to 5% in 2020. This gradual decline suggests a modest but consistent improvement in allocative efficiency over the period. The decomposition of these potential gains into within- and between-sector reveals a clear pattern: most of the productivity loss is sourced from within-sector misallocation—that is, inefficient allocation of inputs across provinces within the same sector. In 2014, for example, within-sector misallocation accounted for 4.26% of the total 8.05% potential gain, while between-sector misallocation contributed 3.96%.

This relationship persists throughout the sample period, highlighting the importance of interprovincial distortions within the same sectors.

Breaking down the sources of between-sector misallocation by input type reveals that capital is the largest contributor. In 2014, for example, capital misallocation alone accounted for 1.80 percentage points of the between-sector productivity loss. Energy followed closely with 1.43 percentage points, while labor accounted for 0.78 percentage points. These patterns are broadly consistent over time: capital misallocation remains the primary factor, contributing between 0.55 and 1.80 percentage points per year. However, despite making up only around 8% of total input use, energy is consistently the second-largest source of misallocation, with potential contributions ranging from 0.34 to 1.43 percentage points annually. This disproportionate impact highlights the critical role of energy in shaping allocative efficiency. In contrast, labor misallocation is relatively modest—typically below 0.5 percentage points—indicating a more efficient allocation of labor across sectors. The outsized role of energy, despite its relatively small input share, highlights why energy misallocation deserves focused attention alongside capital in discussions of productivity and resource use.

Within-sector misallocation presents a more complex picture, particularly in terms of input-specific contributions. Notably, the sum of the input-specific within-sector misallocation terms—capital, labor, and energy—does not equal the total within-sector misallocation. This is by construction: each input-specific figure results from a separate counterfactual scenario in which distortions in that particular input are removed, while other distortions are held constant. Due to the non-linear, complementary nature of input use in production, the total gains from removing all distortions simultaneously are not equal to the sum of individual gains. That said, we observe several patterns. Labor misallocation plays a prominent role within sectors, especially in earlier years, indicating that labor mobility is limited due to provincial differences. For instance, inefficiencies in labor allocation contributed 2.55 percentage points to within-sector (across provinces) misallocation in 2014 and 2.76 points in 2015. Capital also remains a persistent and important source of within-sector inefficiency, particularly in later years in the sample—for example, accounting for 1.85 percentage points in 2019. Energy misallocation within sectors is also sizeable, despite its low input share, with contributions of 1.53 percentage points in 2014 and 0.83 % in 2015. While the results suggest that energy efficiency within sectors across provinces is improving, it remains a significant source of productivity loss, emphasizing the need for further research focus.

To examine the range of potential productivity gains under a less conservative elasticity of substitution, I repeat the analysis assuming $\sigma = 7$. Table 2 reports the results from this exercise, indicating that potential aggregate productivity gains are higher, ranging from 9.40% in 2014 to 5.81% in 2020. This higher estimate arises because a larger σ allows

Table 2: Potential TFP Gains from Input Reallocation (in %), 2014–2020, $\sigma = 7$

Component	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Total Misallocation	9.40	7.51	5.81	5.72	6.85	6.52	5.81
Between-sector Misallocation	3.96	2.25	1.27	1.53	1.53	1.96	1.63
Capital	1.80	1.22	0.55	0.66	0.71	0.88	0.83
Labor	0.78	0.50	0.36	0.37	0.39	0.37	0.46
Energy	1.43	0.55	0.36	0.50	0.45	0.73	0.34
Within-sector Misallocation	5.66	5.39	4.60	4.26	5.40	4.65	4.25
Capital	5.12	3.30	4.89	4.29	7.28	4.41	3.51
Labor	5.85	5.51	3.35	4.21	5.49	3.75	3.82
Energy	3.27	3.36	2.36	2.23	3.37	2.07	1.85

for greater substitutability of inputs across provinces, which amplifies the productivity gap between the observed allocation and the efficient frontier. Under this higher elasticity, the within-sector misallocation component rises to the range of 4.25 to 5.66 percentage points per year. In contrast, the between-sector component remains unchanged from the $\sigma = 3$ case. This is consistent with the model's structure, in which the between-sector term is independent of σ. The increase in potential aggregate productivity gains is therefore sourced entirely from larger distortions within sectors across provinces. I further break it down by input type to explore the role of each input; the potential productivity gains from reallocating capital within sectors reach 7.28 percentage points in 2018. Labor-related distortions also remain sizable, accounting for 5.85 and 5.51 percentage points of within-sector misallocation in 2014 and 2015, respectively. While energy's share in production is relatively small, its misallocation is disproportionately large, with potential productivity gains peaking at 3.37 percentage points in 2018. These patterns indicate that capital and energy misallocation persistently account for a substantial share of the measured productivity gap when σ is higher.

The results indicate two key sources of aggregate productivity loss: (1) interprovincial input distortions within sectors, and (2) misallocation of energy use despite its low share of production as an input, highlighting the need to focus on energy as an important source of aggregate productivity loss. Labor appears to be more efficiently allocated across sectors within a province, whereas there is still significant room for improvement across provinces, highlighting the significance of interprovincial distortions in the labor market as well. The disproportionate role of energy in driving both within- and between-sector inefficiencies suggests that policies aimed at improving pricing, coordination, and investment in energy markets could generate sizable productivity gains.

5 Conclusion

This paper quantifies the productivity loss in Canada resulting from the misallocation of energy, labor, and capital across sectors and provinces. By extending the standard misallocation framework to include energy as a distinct input and using detailed provincial input–output data, I estimate the potential gains from reallocating inputs to their most productive uses at the province-by-sector level and decompose these gains into within-sector (interprovincial) and between-sector (intersectoral) components, with both terms further broken down by input contributions relative to an efficient benchmark.

The results reveal significant inefficiencies. In the benchmark case, under a conservative assumption of ($\sigma = 3$), efficient reallocation of inputs between provinces and sectors could increase aggregate output by approximately 5- 8%. Within-sector (interprovincial) misallocation—driven by regulatory fragmentation and limited energy trade—accounts for roughly 3.4 to 4.3 percentage points of this loss, while between-sector (intersectoral) misallocation contributes about 1.3 to 4.0 percentage points. In the between-sector dimension, energy consistently emerges as the second-largest contributor—exceeding labor in all years and, in some cases, approaching capital's contribution—despite representing only about 8% of input costs. In the within-sector dimension, the role of energy remains substantial, accounting for 1.0 to 1.7 percentage points of total loss.

These results highlight two key points. First, energy plays a disproportionately large role in shaping aggregate productivity, despite its relatively small share in production—a characteristic that has often led to its omission in misallocation analyses. This highlights the need to recognize energy as a critical driver of aggregate productivity loss and to incorporate it more systematically into future misallocation-related studies, while also integrating energy policy more centrally into broader productivity and growth strategies. Second, interprovincial fragmentation, including trade barriers and regulatory differences, remains a major obstacle to efficient resource allocation, suggesting that greater coordination between provincial regulations and investment in interprovincial trade infrastructure could generate significant economic gains.

In conclusion, the results underscore the importance of considering both spatial and input-specific dimensions of misallocation when evaluating aggregate productivity. By explicitly incorporating energy as a distinct factor of production, this paper adds a critical layer to the misallocation literature and demonstrates that even relatively small cost inputs can generate sizable distortions when poorly allocated. The Canadian context—with its provincial regulatory heterogeneity and limited internal trade—further illustrates how institutional frictions can amplify inefficiencies. While focused on Canada, the analysis offers broader lessons for federal systems with fragmented energy regulation or limited interregional energy trade, such as the United States, Australia, or European countries. These insights provide a foundation for future research on province-sector-level misallocation and the role of coordination in national productivity strategies. Addressing these inefficiencies is not only essential for uncovering Canada's growth potential, but also for aligning economic and climate objectives in a coherent policy framework.

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Appendix

5.1 Derivation of Sectoral Shares

To determine θ_s, we solve the maximization problem:

$$\max_{Y_s} \quad PY - \sum_s P_s Y_s. \tag{5.1}$$

Plugging in the production function:

$$Y = \prod_{s=1}^{S} Y_s^{\theta_s}, \tag{5.2}$$

the first-order condition gives:

$$P\theta_s Y_s^{-1} \prod_{s=1}^{S} Y_s^{\theta_s} = P_s. \tag{5.3}$$

Multiplying both sides by Y_s yields:

$$P\theta_s \prod_{s=1}^{S} Y_s^{\theta_s} = P_s Y_s. \tag{5.4}$$

Solving for θ_s:

$$\theta_s = \frac{P_s Y_s}{PY}. \tag{5.5}$$

Now, plugging Y_s into the expression for Y would give us

$$Y = \prod_{s=1}^{S} \left(\frac{\theta_s PY}{P_s}\right)^{\theta_s} = PY^{\sum \theta_s} \prod_{s=1}^{S} \left(\frac{\theta_s}{P_s}\right)^{\theta_s} \tag{5.6}$$

As $\sum_{s=1}^{S} \theta_s = 1$, we get

$$P = \prod_{s=1}^{S} \left(\frac{P_s}{\theta_s}\right)^{\theta_s} \tag{5.7}$$

5.2 Province-Level Pricing and Revenue

Provinces solve:

$$\max_{Y_{si}} \quad P_s \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - \sum_i P_{si} Y_{si}. \tag{5.8}$$

FOC yields:

$$P_{si} = P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{-1}{\sigma}}, \tag{5.9}$$

and thus,

$$P_{si} Y_{si} = P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}}. \tag{5.10}$$

5.3 Derivation of the Sectoral Price Index

We derive the sectoral price index P_s using the cost minimization problem. The total output in sector s is given by a CES aggregator of individual variety outputs Y_{si}:

$$Y_s = \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{5.11}$$

where $\sigma > 1$ is the elasticity of substitution between varieties.

To derive P_s, consider a cost-minimizing province solving the problem:

$$\min_{Y_{si}} \sum_i P_{si} Y_{si} \quad \text{subject to} \quad Y_s = \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{5.12}$$

We form the Lagrangian:

$$\mathcal{L} = \sum_i P_{si} Y_{si} + \lambda_s \left(Y_s^{\frac{\sigma-1}{\sigma}} - \sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} \right). \tag{5.13}$$

The first-order condition with respect to Y_{si} is:

$$P_{si} - \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} = 0. \tag{5.14}$$

Total costs are given by

$$\sum_i P_{si} Y_{si} = \sum_i \lambda_s \frac{\sigma-1}{\sigma} Y_{si}^{-\frac{1}{\sigma}} Y_{si} = \lambda_s \frac{\sigma-1}{\sigma} \sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} = \lambda_s \frac{\sigma-1}{\sigma} Y_s^{\frac{\sigma-1}{\sigma}} \tag{5.15}$$

Rearranging yields the demand function,

$$Y_{si}^{\frac{\sigma-1}{\sigma}} = \left(\lambda_s \frac{1}{P_{si}} \frac{\sigma-1}{\sigma} \right)^{\sigma-1}. \tag{5.16}$$

Now we solve for λ_s,

$$Y_s = \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(\lambda_s \frac{\sigma-1}{\sigma}\right)^{\sigma} \left(\sum_i \left(\frac{1}{P_{si}}\right)^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}} \tag{5.17}$$

$$\lambda_s = \frac{\sigma}{\sigma-1} Y_s^{\frac{1}{\sigma}} \left(\sum_i \left(\frac{1}{P_{si}}\right)^{\sigma-1}\right)^{\frac{-1}{\sigma-1}} \tag{5.18}$$

Plugging this into our total cost expression yields

$$\sum_i P_{si} Y_{si} = \lambda_s \frac{\sigma-1}{\sigma} Y_s^{\frac{\sigma-1}{\sigma}} = Y_s^{\frac{1}{\sigma}} \left(\sum_i \left(\frac{1}{P_{si}}\right)^{\sigma-1}\right)^{\frac{-1}{\sigma-1}} Y_s^{\frac{\sigma-1}{\sigma}} = Y_s \left(\sum_i P_{si}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{5.19}$$

Therefore, based on the following equation.

$$\sum_i P_{si} Y_{si} = P_s Y_s \tag{5.20}$$

we conclude that

$$P_s = \left(\sum_i P_{si}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{5.21}$$

5.4 Production Function

The Cobb-Douglas production function at the province level is:

$$Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}, \quad \text{where } \alpha_s + \beta_s + \gamma_s = 1. \tag{5.22}$$

5.5 Distortions in Input Markets

Each input is subject to a distortion $\tau_{K_{si}}, \tau_{L_{si}}, \tau_{E_{si}}$, so that firms face distorted input prices. The firm's problem is:

$$\max_{K_{si}, L_{si}, E_{si}} \quad P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}} - (1+\tau_{K_{si}}) r K_{si} - (1+\tau_{L_{si}}) w L_{si} - (1+\tau_{E_{si}}) p_E E_{si}. \tag{5.23}$$

We can rewrite this problem by using $Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}$:

$$\max_{K_{si}, L_{si}, E_{si}} \quad P_s Y_s^{\frac{1}{\sigma}} \left(A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}} - (1+\tau_{K_{si}}) r K_{si} - (1+\tau_{L_{si}}) w L_{si} - (1+\tau_{E_{si}}) p_E E_{si}. \tag{5.24}$$

The first-order conditions (FOCs) for optimal input choices are:

$$MRPK_{si} = P_s Y_s^{\frac{1}{\sigma}} \frac{(\sigma-1)}{\sigma} Y_{si}^{-\frac{1}{\sigma}} \alpha_s \frac{Y_{si}}{K_{si}} = (1+\tau_{K_{si}})r, \tag{5.25}$$

$$MRPL_{si} = P_s Y_s^{\frac{1}{\sigma}} \frac{(\sigma-1)}{\sigma} Y_{si}^{-\frac{1}{\sigma}} \beta_s \frac{Y_{si}}{L_{si}} = (1+\tau_{L_{si}})w, \tag{5.26}$$

$$MRPE_{si} = P_s Y_s^{\frac{1}{\sigma}} \frac{(\sigma-1)}{\sigma} Y_{si}^{-\frac{1}{\sigma}} \gamma_s \frac{Y_{si}}{E_{si}} = (1+\tau_{E_{si}})p_E. \tag{5.27}$$

Marginal revenue products:

$$MRPK_{si} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{K_{si}} = \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{K_{si}} = (1+\tau_{K_{si}})r, \tag{5.28}$$

$$MRPL_{si} = \beta_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{L_{si}} = \beta_s \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{L_{si}} = (1+\tau_{L_{si}})w, \tag{5.29}$$

$$MRPE_{si} = \gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}}{E_{si}} = \gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{E_{si}} = (1+\tau_{E_{si}})p_E. \tag{5.30}$$

Now,

$$TFPQ_{si} = A_{si} = \frac{Y_{si}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{5.31}$$

$$TFPR_{si} = P_{si} A_{si} = \frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{5.32}$$

Now, lets take the geometric average of Marginal revenue product of each input with their sector shares

$$\begin{aligned} &(MRPK_{si})^{\alpha_s}(MRPL_{si})^{\beta_s}(MRPE_{si})^{\gamma_s} \\ &= (\alpha \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{K_{si}})^{\alpha_s} (\beta \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{L_{si}})^{\beta_s} (\gamma \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{E_{si}})^{\gamma_s} \\ &= ((1+\tau_{K_{si}})r)^{\alpha_s}((1+\tau_{L_{si}})w)^{\beta_s}((1+\tau_{E_{si}})p_E)^{\gamma_s} \\ &= {\alpha_s}^{\alpha_s} {\beta_s}^{\beta_s} {\gamma_s}^{\gamma_s} \frac{(\sigma-1)}{\sigma} \frac{P_{si} Y_{si}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \\ &= {\alpha_s}^{\alpha_s} {\beta_s}^{\beta_s} {\gamma_s}^{\gamma_s} \frac{(\sigma-1)}{\sigma} TFPR_{si} \end{aligned} \tag{5.33}$$

Hence,

$$TFPR_{si} \propto (MRPK_{si})^{\alpha_s}(MRPL_{si})^{\beta_s}(MRPE_{si})^{\gamma_s} \propto ((1+\tau_K))^{\alpha_s}((1+\tau_L))^{\beta_s}((1+\tau_E))^{\gamma_s} \tag{5.34}$$

This formulation explicitly shows how TFPR is the geometric mean of marginal revenue products.

Now, to recover distortions we will go back to equations of marginal revenue product of inputs and normalize them in a specific way to get distortions explicitly so we can calculate it with data.

Recall that

$$\alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{rK_{si}} = (1+\tau_{K_{si}}) \tag{5.35}$$

$$\beta_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{wL_{si}} = (1+\tau_{L_{si}}) \tag{5.36}$$

$$\gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{p_E E_{si}} = (1+\tau_{E_{si}}) \tag{5.37}$$

Take the average of both sides and set the average distortions to 0 to get.

$$\sum_i \tau_{K_{si}} = \sum_i \tau_{L_{si}} = \sum_i \tau_{E_{si}} = 0 \tag{5.38}$$

To identify province-level distortions relative to a sectoral benchmark, we normalize the average distortion to zero:

$$\frac{1}{I}\sum_i^I \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{rK_{si}} = \frac{1}{I}\sum_i^I (1+\tau_{K_{si}}) = 1 \tag{5.39}$$

$$\frac{1}{I}\sum_i^I \beta_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{wL_{si}} = \frac{1}{I}\sum_i^I (1+\tau_{L_{si}}) = 1 \tag{5.40}$$

$$\frac{1}{I}\sum_i^I \gamma_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{p_E E_{si}} = \frac{1}{I}\sum_i^I (1+\tau_{E_{si}}) = 1 \tag{5.41}$$

This allows us to interpret each $\tau_{K_{si}}, \tau_{L_{si}}, \tau_{E_{si}}$ as a deviation from the sectoral mean, effectively treating the average province as undistorted.

Now,

$$\frac{\alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{rK_{si}}}{\frac{1}{I}\sum_i^I \alpha_s \frac{(\sigma-1)}{\sigma} \frac{P_{si}Y_{si}}{rK_{si}}} = (1+\tau_{K_{si}}) \tag{5.42}$$

$$\frac{\frac{P_{si}Y_{si}}{rK_{si}}}{\frac{1}{I}\sum_i^I \frac{P_{si}Y_{si}}{rK_{si}}} = (1+\tau_{K_{si}}) \tag{5.43}$$

$$\frac{\frac{P_{si}Y_{si}}{wL_{si}}}{\frac{1}{I}\sum_i^I \frac{P_{si}Y_{si}}{wL_{si}}} = (1+\tau_{L_{si}}) \tag{5.44}$$

$$\frac{\frac{P_{si}Y_{si}}{p_E E_{si}}}{\frac{1}{I}\sum_i^I \frac{P_{si}Y_{si}}{p_E E_{si}}} = (1+\tau_{E_{si}}) \tag{5.45}$$

With this simple trick we get rid of σ and sector share constants $(\alpha_s, \beta_s, \gamma_s)$

As we are already having distortions the next step is to calculate sector-level weighted averages of marginal revenue products.

we can start with

$$\overline{MRPK_s} = \frac{\sum_i K_{si} MRPKsi}{\sum_i K_{si}} = \frac{\sum_i \alpha_s \frac{\sigma-1}{\sigma} P_{si}Y_{si}}{\sum_i \alpha_s \frac{\sigma-1}{\sigma} \frac{P_{si}Y_{si}}{r(1+\tau_{K_{si}})}} = \frac{\sum_i P_{si}Y_{si}}{\sum_i \frac{P_{si}Y_{si}}{r(1+\tau_{K_{si}})}} \tag{5.46}$$

Given that sectoral revenue $P_sY_s = \sum_i P_{si}Y_{si}$ we can write that

$$\overline{MRPK_s} = \frac{r}{\sum_i \frac{1}{(1+\tau_{K_{si}})} \frac{P_{si}Y_{si}}{P_sY_s}} \tag{5.47}$$

Finally,

$$\frac{\overline{MRPK_s}}{MRPK_{si}} = \frac{\frac{r}{\sum_i \frac{1}{(1+\tau_{K_{si}})} \frac{P_{si}Y_{si}}{P_sY_s}}}{r(1+\tau_{K_{si}})} = \frac{1}{(1+\tau_{K_{si}}) \sum_i \frac{1}{(1+\tau_{K_{si}})} \frac{P_{si}Y_{si}}{P_sY_s}} \tag{5.48}$$

Similar algebra yields,

$$\frac{\overline{MRPL_s}}{MRPL_{si}} = \frac{1}{(1+\tau_{L_{si}}) \sum_i \frac{1}{(1+\tau_{L_{si}})} \frac{P_{si}Y_{si}}{P_sY_s}} \tag{5.49}$$

$$\frac{\overline{MRPE_s}}{MRPE_{si}} = \frac{1}{(1+\tau_{E_{si}}) \sum_i \frac{1}{(1+\tau_{E_{si}})} \frac{P_{si}Y_{si}}{P_sY_s}} \tag{5.50}$$

It is straightworward to see these equaitons are equal to 1 if there were no distortions.

Given that we have explicit formulas for distortions and marginal revenue products compared to sector averages we can move forward to calculate the output implications of these. Now, recall that we have derived the expression

$$P_{si}Y_{si} = P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}}. \tag{5.51}$$

if we divide each side by $P_s Y_s$ we would get

$$\frac{P_{si}Y_{si}}{P_s Y_s} = \left(\frac{Y_{si}}{Y_s}\right)^{\frac{\sigma-1}{\sigma}}, \tag{5.52}$$

Taking the geometric average across all factors K, L, and E:

$$\begin{aligned}\left(\frac{\overline{MRPK_s}}{MRPK_{si}}\right)^{\alpha} \left(\frac{\overline{MRPL_s}}{MRPL_{si}}\right)^{\beta} \left(\frac{\overline{MRPE_s}}{MRPE_{si}}\right)^{\gamma} \\ = \left(\frac{\sum_i K_{si} MRPK_{si}}{MRPK_{si} \sum_i K_{si}}\right)^{\alpha} \left(\frac{\sum_i L_{si} MRPL_{si}}{MRPL_{si} \sum_i L_{si}}\right)^{\beta} \left(\frac{\sum_i E_{si} MRPE_{si}}{MRPE_{si} \sum_i E_{si}}\right)^{\gamma}\end{aligned} \tag{5.53}$$

Now recalling the formulas for Marginal revenue products we can see that

$$\sum_i K_{si} MRPK_{si} = \sum_i L_{si} MRPL_{si} = \sum_i E_{si} MRPE_{si} \propto \sum_i P_{si} Y_{si} = P_s Y_s \tag{5.54}$$

as $\alpha_s, \beta_s, and \gamma_s$ sums up to 1, we have $P_s Y_s$ in numerator. Also, note that

$$\sum_i K_{si} = K_s, \sum_i L_{si} = L_s, \sum_i E_{si} = E_s, \tag{5.55}$$

Finally, the geometric average of marginal revenue products is proportional to $TFPR_{si} = P_{si} A_{si}$

Then we have,

$$\begin{aligned}\left(\frac{\overline{MRPK_s}}{MRPK_{si}}\right)^{\alpha} \left(\frac{\overline{MRPL_s}}{MRPL_{si}}\right)^{\beta} \left(\frac{\overline{MRPE_s}}{MRPE_{si}}\right)^{\gamma} \\ = \frac{P_s Y_s}{P_{si} A_{si} K_s^{\alpha_s} L_s^{\beta_s} E_s^{\gamma_s}} \\ = \frac{P_s A_s}{P_{si} A_{si}}\end{aligned} \tag{5.56}$$

We have

$$P_s = \left(\sum_i P_{si}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{5.57}$$

To isolate A_s we can multiply the expression by A_{si} and take the power of $(\sigma - 1)$ and some over provinces.

$$\sum_i \left(\frac{\cancel{A_{si}} P_s A_s}{P_{si} \cancel{A_{si}}} \right)^{\sigma-1} = (P_s A_s)^{(\sigma-1)} \sum_i P_{si}^{(1-\sigma)} = A_s^{\sigma-1} \tag{5.58}$$

If we take the power of $1/(\sigma - 1)$ we arrive at $TFP_s = A_s$ by applying the same operations to the left-hand side we get an expression for TFP_s

$$A_s = \left[\sum_i \left(A_{si} \left(\frac{\overline{MRPK_s}}{MRPK_{si}} \right)^{\alpha} \left(\frac{\overline{MRPL_s}}{MRPL_{si}} \right)^{\beta} \left(\frac{\overline{MRPE_s}}{MRPE_{si}} \right)^{\gamma} \right)^{\sigma-1} \right]^{\frac{1}{\sigma-1}}. \tag{5.59}$$

Finally, we get an expression for A_{si} to bring this model into data. Recall that

$$P_{si} Y_{si} = P_s (Y_s)^{\frac{1}{\sigma}} Y_{si}^{\frac{\sigma-1}{\sigma}} \tag{5.60}$$

This implies

$$Y_{si} = (P_s (Y_s)^{\frac{1}{\sigma}})^{\frac{-\sigma}{1-\sigma}} (P_{si} Y_{si})^{\frac{\sigma}{\sigma-1}} \tag{5.61}$$

therefore,

$$A_{si} = \frac{(P_s Y_s)^{\frac{-1}{\sigma-1}}}{P_s} \frac{(P_{si} Y_{si})^{\frac{\sigma}{\sigma-1}}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{5.62}$$

So,

$$A_{si} \propto \frac{(P_{si} Y_{si})^{\frac{\sigma}{\sigma-1}}}{K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}} \tag{5.63}$$

5.6 Productivity Decomposition

We begin by defining sector-level total factor productivity A_s using a Cobb-Douglas production function, where output Y_s is produced using capital K_s, labor L_s, and energy E_s:

$$A_s = \frac{Y_s}{K_s^{\alpha_s} L_s^{\beta_s} E_s^{\gamma_s}} \tag{5.64}$$

Sectoral output Y_s aggregates province-level outputs Y_{si} through a constant elasticity of substitution (CES) aggregator with elasticity σ:

$$Y_s = \left(\sum_i Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{5.65}$$

Substituting province-level production functions $Y_{si} = A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}$ into the CES aggregator, we express sectoral TFP as:

$$\Rightarrow A_s = \frac{\left(\sum_i \left(A_{si} K_{si}^{\alpha_s} L_{si}^{\beta_s} E_{si}^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{K_s^{\alpha_s} L_s^{\beta_s} E_s^{\gamma_s}} \tag{5.66}$$

$$= \left[\sum_i \left(A_{si} \left(\frac{K_{si}}{K_s}\right)^{\alpha_s} \left(\frac{L_{si}}{L_s}\right)^{\beta_s} \left(\frac{E_{si}}{E_s}\right)^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{5.67}$$

Note that this expression is just normalizing province-level inputs by the sector total, for which we know explicitly what they are.

$$k_{si} = \frac{K_{si}}{K_s} = \frac{K_{si}}{\sum_i K_{si}} = \frac{\frac{\alpha_s}{r} \frac{\sigma-1}{\sigma} \frac{P_{si} Y_{si}}{(1+\tau_{K_{si}})}}{\sum_i \frac{\alpha_s}{r} \frac{\sigma-1}{\sigma} \frac{P_{si} Y_{si}}{(1+\tau_{K_{si}})}} = \frac{R_{si}/(1+\tau_{K_{si}})}{\sum_i R_{si}/(1+\tau_{K_{si}})} \tag{5.68}$$

$$\text{Let revenue shares } R_{si} = \frac{P_{si} Y_{si}}{P_s Y_s}, \quad k_{si} = \frac{K_{si}}{K_s}, \quad l_{si} = \frac{L_{si}}{L_s}, \quad e_{si} = \frac{E_{si}}{E_s} \tag{5.69}$$

$$\Rightarrow A_s = \left[\sum_i \left(A_{si} k_{si}^{\alpha_s} l_{si}^{\beta_s} e_{si}^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{5.70}$$

It is straightforward to see that the fully efficient allocation yields a sector-level TFP A_s^* expressed as:

$$A_s^* = \left[\sum_i A_{si}^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \tag{5.71}$$

Dividing the actual level of TFP A_s to efficient level of TFP A_s^* we get:

$$\frac{A_s}{A_s^*} = \frac{\left[\sum_i \left(A_{si} \left(\frac{R_{si}/(1+\tau_{K_{si}})}{\sum_i R_{si}/(1+\tau_{K_{si}})}\right)^{\alpha_s} \left(\frac{R_{si}/(1+\tau_{L_{si}})}{\sum_i R_{si}/(1+\tau_{L_{si}})}\right)^{\beta_s} \left(\frac{R_{si}/(1+\tau_{E_{si}})}{\sum_i R_{si}/(1+\tau_{E_{si}})}\right)^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_i A_{si}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}} \tag{5.72}$$

Moving to the national level, we express national TFP (i.e., A) as:

$$A = \frac{Y}{K^{\bar{\alpha}} L^{\bar{\beta}} E^{\bar{\delta}}} = \frac{\prod_s Y_s^{\theta_s}}{K^{\bar{\alpha}} L^{\bar{\beta}} E^{\bar{\delta}}}, \quad \bar{\alpha} = \sum_s \alpha_s \theta_s \tag{5.73}$$

$$= \prod_s \left(\frac{A_s K_s^{\alpha_s} L_s^{\beta_s} E_s^{\gamma_s}}{K^{\alpha_s} L^{\beta_s} E^{\gamma_s}} \right)^{\theta_s} \tag{5.74}$$

$$= \prod_s A_s^{\theta_s} \left(\frac{K_s}{K}\right)^{\alpha_s \theta_s} \left(\frac{L_s}{L}\right)^{\beta_s \theta_s} \left(\frac{E_s}{E}\right)^{\gamma_s \theta_s} \tag{5.75}$$

$$= \prod_s \left(A_s \left(\frac{K_s}{K}\right)^{\alpha_s} \left(\frac{L_s}{L}\right)^{\beta_s} \left(\frac{E_s}{E}\right)^{\gamma_s} \right)^{\theta_s} \tag{5.76}$$

$$K = \sum_s K_s = \sum_s \sum_i K_{si} \tag{5.77}$$

$$k_s = \frac{K_s}{K} = \frac{\sum_i K_{si}}{\sum_s \sum_i K_{si}}, \quad = \frac{\sum_i \frac{\alpha_s}{r} \left(\frac{\sigma-1}{\sigma}\right) \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}}{\sum_s \sum_i \frac{\alpha_s}{r} \left(\frac{\sigma-1}{\sigma}\right) \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}} = \frac{\sum_i \alpha_s \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}}{\sum_s \sum_i \alpha_s \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}} \tag{5.78}$$

$$= \frac{\alpha_s \sum_i \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}}{\sum_s \alpha_s \sum_i \frac{P_{si} Y_{si}}{1+\tau_{K_{si}}}} = \frac{\alpha_s P_s Y_s \sum_i \frac{R_{si}}{1+\tau_{K_{si}}}}{\sum_s \alpha_s P_s Y_s \sum_i \frac{R_{si}}{1+\tau_{K_{si}}}} \tag{5.79}$$

as $R_{si} = P_{si} Y_{si} / P_s Y_s$

$$\frac{K_s}{K} = \frac{\alpha_s P_s Y_s \sum_i R_{si}/(1+\tau_{K_{si}})}{\sum_s \alpha_s P_s Y_s \sum_i R_{si}/(1+\tau_{K_{si}})} = \frac{\alpha_s P_s Y_s \sum_i R_{si}/(1+\tau_{K_{si}})}{\sum_s \alpha_s P_s Y_s \sum_i R_{si}/(1+\tau_{K_{si}})} \quad \text{(Actual } k_s\text{)} \tag{5.80}$$

If we define (harmonic mean of sector-level distortions),

$$\overline{1+\tau_{Ks}} = \frac{1}{\sum_i \frac{R_{si}}{1+\tau_{K_{si}}}} \quad \text{(harmonic mean)} \tag{5.81}$$

and recalling that $\theta_s = P_s Y_s / PY$ then we can write

$$k_s = \frac{K_s}{K} = \frac{\alpha_s \theta_s / (\overline{1+\tau_{Ks}})}{\sum_s \alpha_s \theta_s / (\overline{1+\tau_{Ks}})} \tag{5.82}$$

Also, as there is no distortions in optimal allocation we can simply write

$$k_s^* = \frac{K_s^*}{K^*} = \frac{\alpha_s \theta_s}{\sum_s \alpha_s \theta_s} \tag{5.83}$$

$$\Rightarrow \frac{k_s}{k_s^*} = \frac{\left(\frac{1}{1+\tau_{Ks}}\right) \cdot \sum_s \alpha_s \theta_s}{\sum_s \frac{\alpha_s \theta_s}{1+\tau_{Ks}}} \tag{5.84}$$

Similar algebra yields,

$$\Rightarrow \frac{l_s}{l_s^*} = \frac{\left(\frac{1}{1+\tau_{Ls}}\right) \cdot \sum_s \beta_s \theta_s}{\sum_s \frac{\beta_s \theta_s}{1+\tau_{Ls}}} \tag{5.85}$$

$$\Rightarrow \frac{e_s}{e_s^*} = \frac{\left(\frac{1}{1+\tau_{Es}}\right) \cdot \sum_s \gamma_s \theta_s}{\sum_s \frac{\gamma_s \theta_s}{1+\tau_{Es}}} \tag{5.86}$$

Now, if we divide the actual national TFP A to efficient level of national TFP A^* we get:

$$\frac{A}{A^*} = \prod_s \left(\left(\frac{A_s}{A_s^*}\right) \left(\frac{k_s}{k_s^*}\right)^{\alpha_s} \left(\frac{l_s}{l_s^*}\right)^{\beta_s} \left(\frac{e_s}{e_s^*}\right)^{\gamma_s} \right)^{\theta_s} \tag{5.87}$$

$$\frac{A}{A^*} = \underbrace{\prod_s \left(\frac{A_s}{A_s^*}\right)^{\theta_s}}_{\text{Within-sector misallocation}} \times \underbrace{\prod_s \left(\left(\frac{k_s}{k_s^*}\right)^{\alpha_s} \left(\frac{l_s}{l_s^*}\right)^{\beta_s} \left(\frac{e_s}{e_s^*}\right)^{\gamma_s} \right)^{\theta_s}}_{\text{Between-sector misallocation}} \tag{5.88}$$

$$\frac{A}{A^*} = \prod_s \left(\frac{A_s}{A_s^*}\right)^{\theta_s} \times \prod_s \left[\underbrace{\left(\frac{k_s}{k_s^*}\right)^{\alpha_s}}_{\substack{\text{Capital} \\ \text{misallocation}}} \cdot \underbrace{\left(\frac{l_s}{l_s^*}\right)^{\beta_s}}_{\substack{\text{Labor} \\ \text{misallocation}}} \cdot \underbrace{\left(\frac{e_s}{e_s^*}\right)^{\gamma_s}}_{\substack{\text{Energy} \\ \text{misallocation}}} \right]^{\theta_s} \tag{5.89}$$

We can write the *within* portion of the expression explicitly as:

$$\left(\frac{A}{A^*}\right)_{within} = \prod_s \left(\frac{\left[\sum_i \left(A_{si} \left(\frac{R_{si}/(1+\tau_{K_{si}})}{\sum_i R_{si}/(1+\tau_{K_{si}})} \right)^{\alpha_s} \left(\frac{R_{si}/(1+\tau_{L_{si}})}{\sum_i R_{si}/(1+\tau_{L_{si}})} \right)^{\beta_s} \left(\frac{R_{si}/(1+\tau_{E_{si}})}{\sum_i R_{si}/(1+\tau_{E_{si}})} \right)^{\gamma_s} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_i A_{si}^{\sigma-1} \right]^{\frac{1}{\sigma-1}}} \right)^{\theta_s} \tag{5.90}$$

To understand the each input's role in within sector inefficiency we can use the efficient benchmark for each input and can tell each input's contribution to within term. To do so, we use the standard result from perfect competition case.

$$P_{si} = \frac{A_s}{A_{si}} \tag{5.91}$$

by rearranging equation 5.51 we can get

$$Y_{si} = Y_s \left(\frac{P_{si}}{P_s}\right)^{-\sigma} \tag{5.92}$$

Plugging P_{si} term in gives us

$$Y_{si} = Y_s \left(\frac{A_s}{P_s}\right)^{-\sigma} A_{si}^{\sigma} \tag{5.93}$$

Now, we can express the revenue shares $R_{si} = \frac{P_{si}Y_{si}}{P_sY_s}$ in terms of A_{si} by plugging P_{si} and Y_{si} terms into the numerator.

$$R_{si} = \frac{P_{si}Y_{si}}{P_sY_s} = \frac{[\frac{A_s}{A_{si}}][Y_s \left(\frac{A_s}{P_s}\right)^{-\sigma} A_{si}^{\sigma}]}{P_sY_s} \tag{5.94}$$

It implies that $R_{si} \propto A_{si}^{\sigma-1}$ So the efficient benchmark revenue shares R_{si}^{*} can be expressed as

$$R_{si}^{*} = \frac{R_{si}}{\sum_j R_{sj}} = \frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}} \tag{5.95}$$

Now, we can write each input's contribution in the within term by assuming the other inputs allcoated efficiently. Therefore,

$$\left(\frac{A}{A^{*}}\right)_{withinK} = \prod_s \left(\frac{\left[\sum_i \left(A_{si} \left(\frac{R_{si}/(1+\tau_{K_{si}})}{\sum_i R_{si}/(1+\tau_{K_{si}})}\right)^{\alpha_s} \left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\beta_s} \left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_i A_{si}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}} \right)^{\theta_s} \tag{5.96}$$

$$\left(\frac{A}{A^{*}}\right)_{withinL} = \prod_s \left(\frac{\left[\sum_i \left(A_{si} \left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\alpha_s} \left(\frac{R_{si}/(1+\tau_{L_{si}})}{\sum_i R_{si}/(1+\tau_{L_{si}})}\right)^{\beta_s} \left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_i A_{si}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}} \right)^{\theta_s} \tag{5.97}$$

$$\left(\frac{A}{A^*}\right)_{withinE} = \prod_s \left(\frac{\left[\sum_i \left(A_{si}\left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\alpha_s}\left(\frac{A_{si}^{\sigma-1}}{\sum_j A_{sj}^{\sigma-1}}\right)^{\beta_s}\left(\frac{R_{si}/(1+\tau_{E_{si}})}{\sum_i R_{si}/(1+\tau_{E_{si}})}\right)^{\gamma_s}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_i A_{si}^{\sigma-1}\right]^{\frac{1}{\sigma-1}}} \right)^{\theta_s} \tag{5.98}$$

We can also write the *between* portion explicitly as:

$$\left(\frac{A}{A^*}\right)_{between} = \prod_s \left(\left(\frac{\left(\frac{1}{1+\tau_{Ks}}\right)\cdot \sum_s \alpha_s\theta_s}{\sum_s \frac{\alpha_s\theta_s}{1+\tau_{Ks}}} \right)^{\alpha_s} \left(\frac{\left(\frac{1}{1+\tau_{Ls}}\right)\cdot \sum_s \beta_s\theta_s}{\sum_s \frac{\beta_s\theta_s}{1+\tau_{Ls}}} \right)^{\beta_s} \left(\frac{\left(\frac{1}{1+\tau_{Es}}\right)\cdot \sum_s \gamma_s\theta_s}{\sum_s \frac{\gamma_s\theta_s}{1+\tau_{Es}}} \right)^{\gamma_s} \right)^{\theta_s} \tag{5.99}$$

To further decompose the between term to find each input misallocation contribution we can write the following expressions.

$$\left(\frac{A}{A^*}\right)_{betweenK} = \prod_s \left(\left(\frac{\left(\frac{1}{1+\tau_{Ks}}\right)\cdot \sum_s \alpha_s\theta_s}{\sum_s \frac{\alpha_s\theta_s}{1+\tau_{Ks}}} \right)^{\alpha_s} \right)^{\theta_s} \tag{5.100}$$

$$\left(\frac{A}{A^*}\right)_{betweenL} = \prod_s \left(\left(\frac{\left(\frac{1}{1+\tau_{Ls}}\right)\cdot \sum_s \beta_s\theta_s}{\sum_s \frac{\beta_s\theta_s}{1+\tau_{Ls}}} \right)^{\beta_s} \right)^{\theta_s} \tag{5.101}$$

$$\left(\frac{A}{A^*}\right)_{betweenE} = \prod_s \left(\left(\frac{\left(\frac{1}{1+\tau_{Es}}\right)\cdot \sum_s \gamma_s\theta_s}{\sum_s \frac{\gamma_s\theta_s}{1+\tau_{Es}}} \right)^{\gamma_s} \right)^{\theta_s} \tag{5.102}$$

Therefore,

$$\begin{aligned} \frac{A}{A^*} &= \left(\frac{A}{A^*}\right)_{\text{within}} \times \left(\frac{A}{A^*}\right)_{\text{between}} \\ &= \left(\frac{A}{A^*}\right)_{\text{within}} \times \left(\frac{A}{A^*}\right)_{\text{betweenK}} \times \left(\frac{A}{A^*}\right)_{\text{betweenL}} \times \left(\frac{A}{A^*}\right)_{\text{betweenE}} \end{aligned} \tag{5.103}$$

or more compactly,

$$\hat{A} = \hat{A}_{within} \times \hat{A}_{between} \tag{5.104}$$

Or,

$$\hat{A} = \hat{A}_{within} \times \hat{A}_{betweenK} \times \hat{A}_{betweenL} \times \hat{A}_{betweenE} \tag{5.105}$$