COEG 304: Instrumentation and Control

Chapter 5: Stability Analysis

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Course content

1. Introduction
2. Mathematical Modeling
3. System Reduction Technique
4. Transient and Steady-State-Response Analysis
5. Stability Analysis
6. Introduction to Process Control
7. Sensors
8. Analog Signal Conditioning
9. Digital Signal Conditioning
10. Final Control

Contents

5.1 Definition of stability and stable system

- Stability is the most important system specification.
- If a system is unstable, transient response and steady-state error are moot points.
- An unstable system cannot be designed for a specific transient response or steady-state error requirements.
- If so, What is stability?
- If we may recall, the total response of the system is the sum of the forced and natural response, i.e.

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

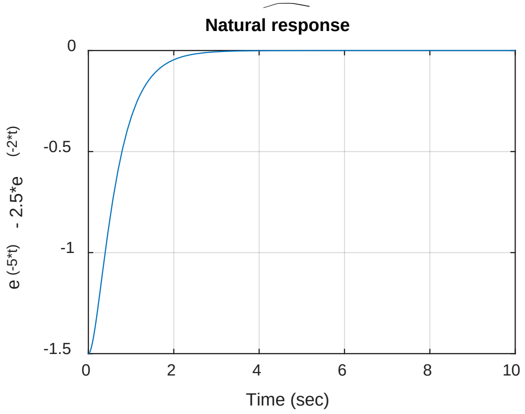
- A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.

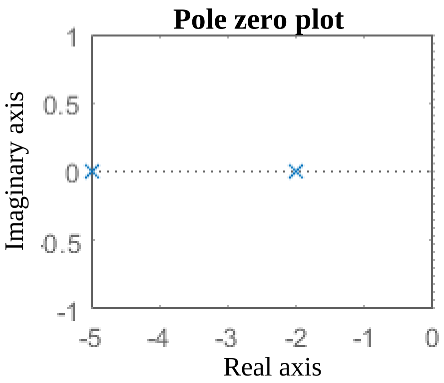
5.1 Definition of stability and stable system

- A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.
- Consider the following second-order systems:

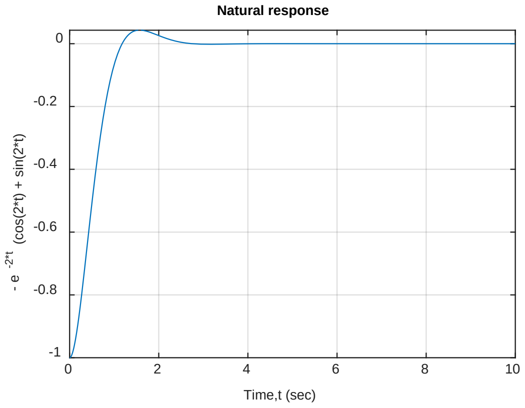
$G_1(s) = \frac{15}{(s+2)(s+5)}$	$G_2(s) = \frac{8}{(s^2+4s+8)}$

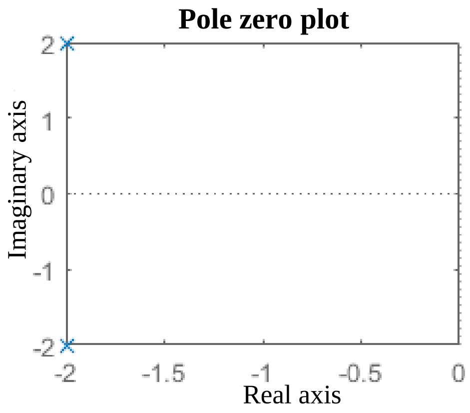
- $G_1(s)$ has two real poles, i.e. $s_1 = -2$ and $s_2 = -5$.
- The step response of $G_1(s)$ is:
- $c_1(t) = \underbrace{e^{-5t} - 2.5e^{-2t}}_{\text{Natural response}} + \underbrace{1.5}_{\text{Forced response}}$





- $G_2(s)$ has a complex-conjugate pair of poles, i.e. $s_{1,2} = -2 \pm j2$
- The step response of $G_2(s)$ is:
- $c_2(t) = \underbrace{-e^{-2t}[cos(2t) + \sin(2t)]}_{\text{Natural response}} + \underbrace{1}_{\text{Forced response}}$

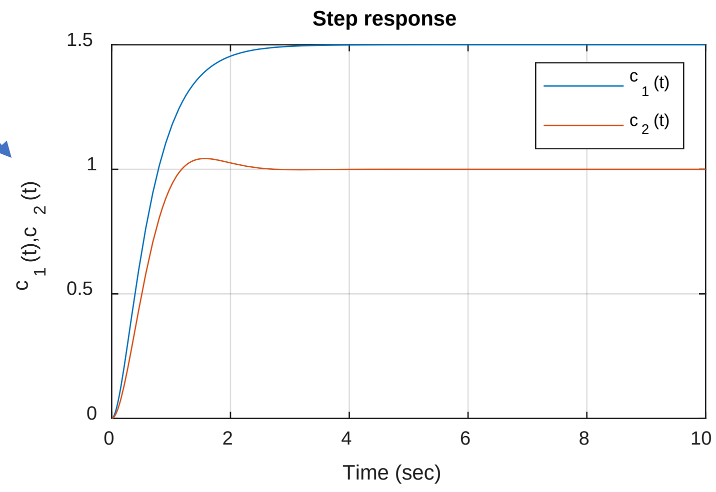




5.1 Definition of stability and stable system

$G_1(s) = \frac{15}{(s+2)(s+5)}$	$G_2(s) = \frac{8}{(s^2+4s+8)}$
• $c_1(t) = e^{-5t} - 2.5e^{-2t} + 1.5$	• $c_2(t) = -e^{-2t}[cos(2t) + \sin(2t)] + 1$

- Step response in both cases are superposition of natural (transient) and forced (steady-state) response.
- In both cases, the natural response decays to zero as $t \to \infty$.
- Step responses characteristic of stable system.

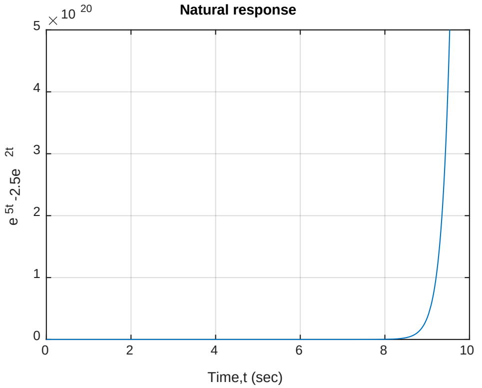


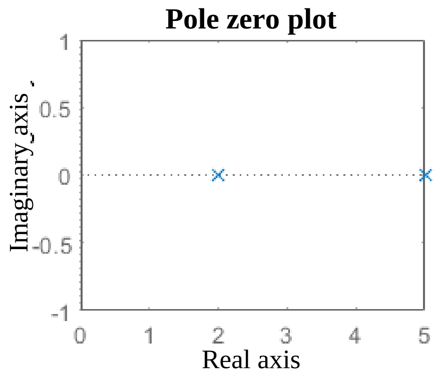
5.1 Definition of stability and stable system

- Now, consider the following similar-looking systems:

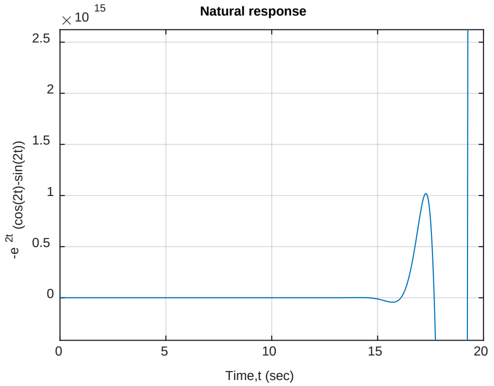
$G_3(s) = \frac{15}{(s-2)(s-5)}$	$G_4(s) = \frac{8}{(s^2-4s+8)}$

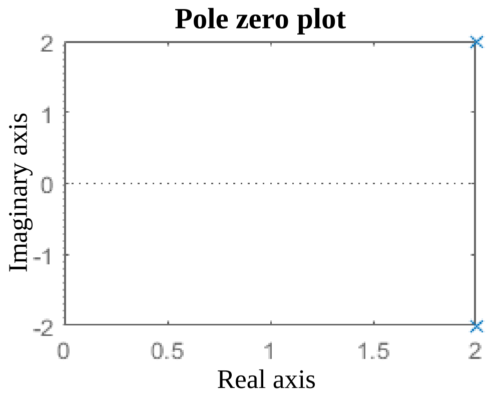
- $G_3(s)$ has two real poles, i.e. $s_1 = 2$ and $s_2 = 5$.
- The step response of $G_1(s)$ is:
- $c_3(t) = \underbrace{e^{5t} - 2.5e^{2t}}_{\text{Natural response}} + \underbrace{1.5}_{\text{Forced response}}$





- $G_4(s)$ has a complex-conjugate pair of poles, i.e. $s_{1,2} = 2 \pm j2$
- The step response of $G_2(s)$ is:
- $c_4(t) = \underbrace{-e^{2t}[\cos(2t) - \sin(2t)]}_{\text{Natural response}} + \underbrace{1}_{\text{Forced response}}$

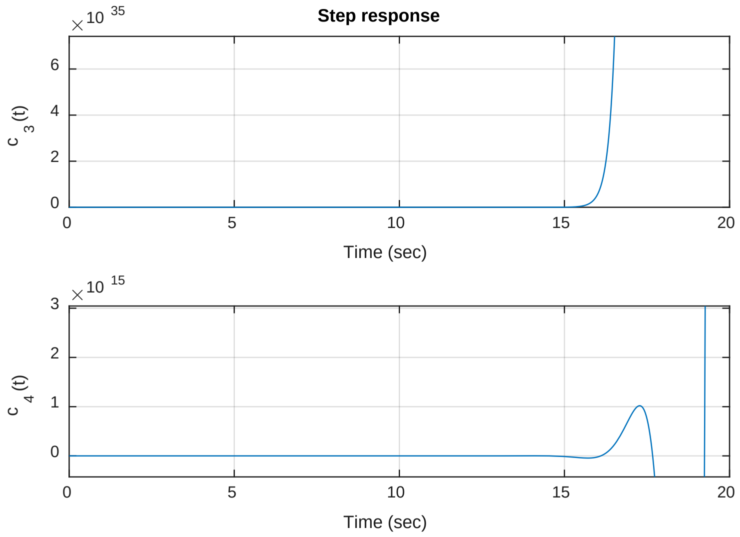




5.1 Definition of stability and stable system

$G_3(s) = \frac{15}{(s-2)(s-5)}$	$G_4(s) = \frac{8}{(s^2-4s+8)}$
• $c_3(t) = e^{5t} - 2.5e^{2t} + 1.5$	• $c_4(t) = -e^{2t}[\cos(2t) - \sin(2t)] + 1$

- Again, step response consists of a natural response and a forced response component.
- However, as $t \to \infty$, the natural response do not decay to zero
  - Such system blow up-Why?
  - Since, exponential terms are positive.
- Step response characteristic of unstable systems



- A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity

5.1 Definition of stability and stable system

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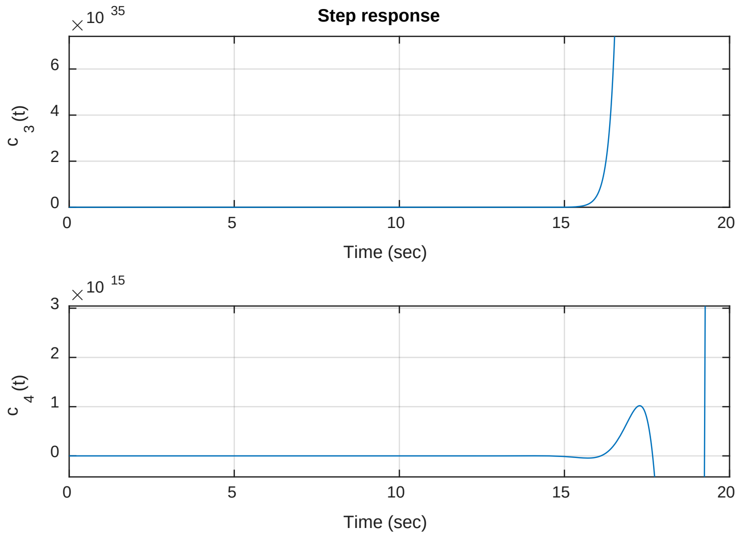


- A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity

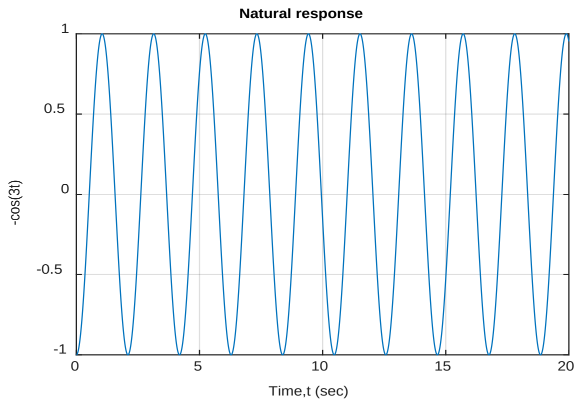
5.1 Definition of stability and stable system

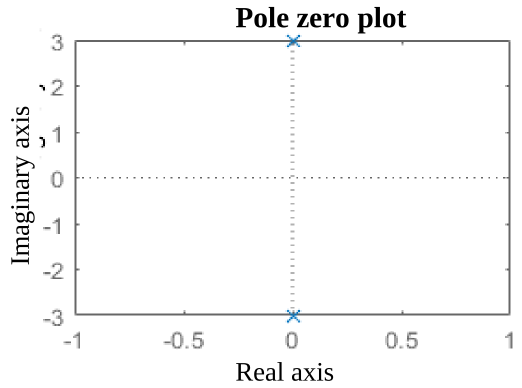
- So, poles in LHP corresponds to stable system while poles in RHP correspond to unstable system.
- It seems that the imaginary axis is the boundary for stability.
- What if poles are on the imaginary axis?
- Consider the following system:

$$G_5\,(s) = \frac{9}{(s^2+9)}$$

- Two purely imaginary poles associated with above system are: $s_{1,2} = \pm j3$
- Step response of $G_5(s)$ is $c_5(t) = -\cos(3t) + 1$

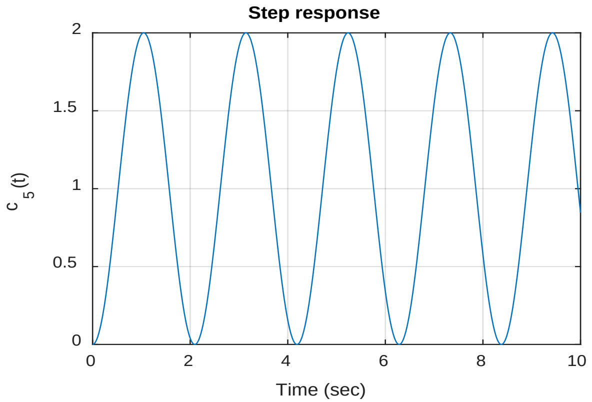
Natural response Forced response





5.1 Definition of stability and stable system

- We can observe that natural response neither decays to zero, nor grows without bound.
  - It oscillates indefinitely
  - Such system is marginally stable
- Step response characteristics of marginally stable system.



- A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillate as time approaches infinity.

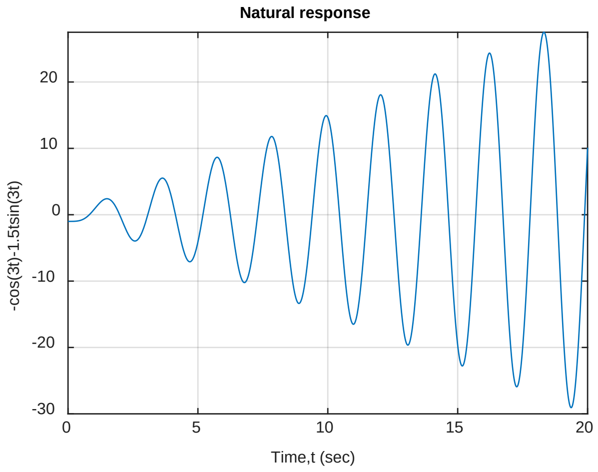
5.1 Definition of stability and stable system

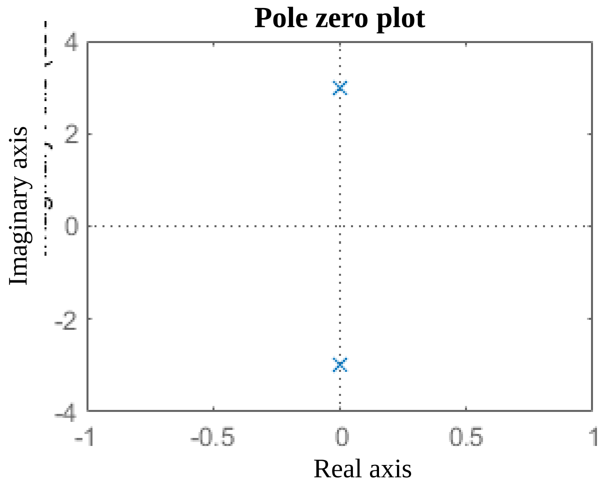
- We'll look at one more interesting case.
- Consider the following system [with repeated imaginary poles]

$$G_6(s) = \frac{81}{(s^2+9)^2}$$

- Repeated poles on the imaginary axis are: $s_{1,2} = \pm j3$ and $s_{3,4} = \pm j3$
- The step response of the system is : $c_6(t) = -\cos(3t) - \frac{3t\sin(3t)}{2} + 1$

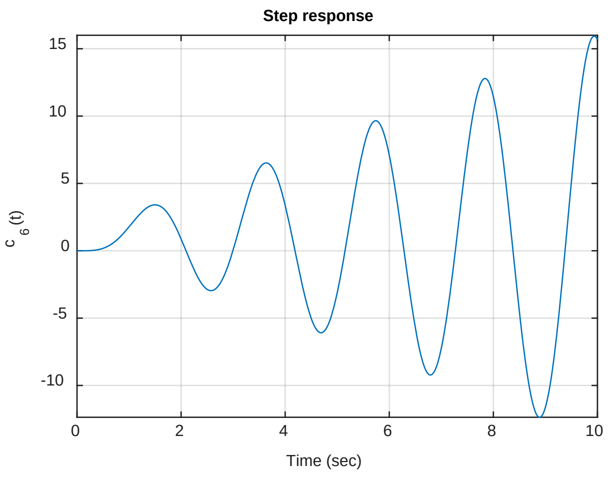
Natural response　　Forced response





5.1 Definition of stability and stable system

- $c_6(t) = -\cos(3t) - \dfrac{3t\sin(3t)}{2} + 1$
- As observed, multiplying time factor causes the natural response to grow without bound.
  - An unstable system
  - Multiple identical poles on the imaginary axis implies an unstable system
- Step response characteristics of unstable system due to repeated poles in imaginary axis:

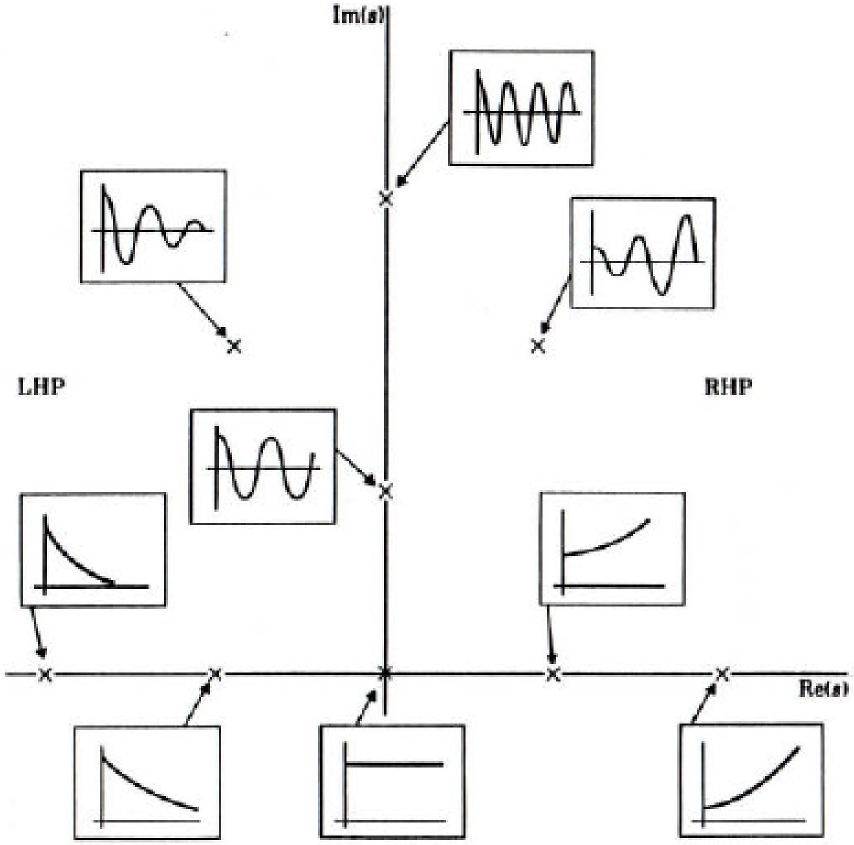


5.1 Definition of stability and stable system

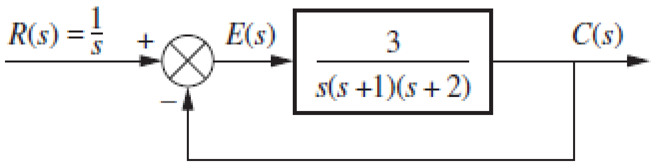
- Summary

- Let us summarize our definitions of stability for linear, time-invariant systems Using the natural response :
  - A system is stable if the natural response approaches zero as time approaches infinity.
  - A system is unstable if the natural response approaches infinity as time approaches infinity.
  - A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

- Let us summarize our definitions of stability for linear, time-invariant systems Using the total response :
  - A system is stable if *every* bounded input yields a bounded output.
  - A system is unstable is *any* bounded input yields an unbounded output.



5.1 Definition of stability and stable system [based on closed loop transfer function]



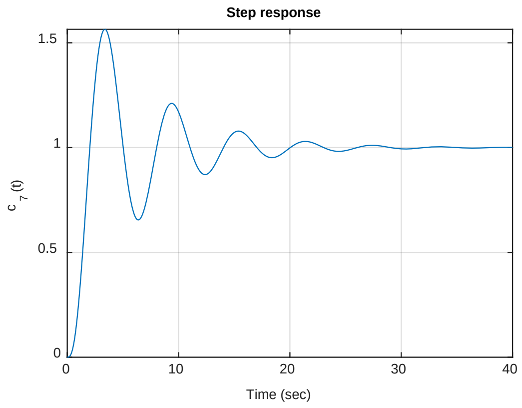
Stable system

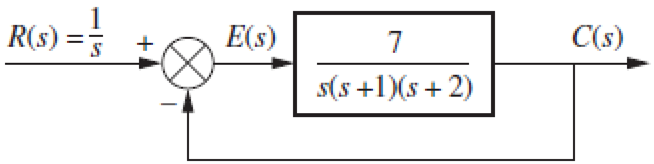
- Closed loop system transfer function will be:

$$G_7(s) = \frac{C(s)}{R(s)} = \frac{3}{(s(s+1)(s+2)+3)}$$

- The poles of the above system are:

$$s_1 = -2.6717, s_{2,3} = -0.1642 \pm 1.0469i$$





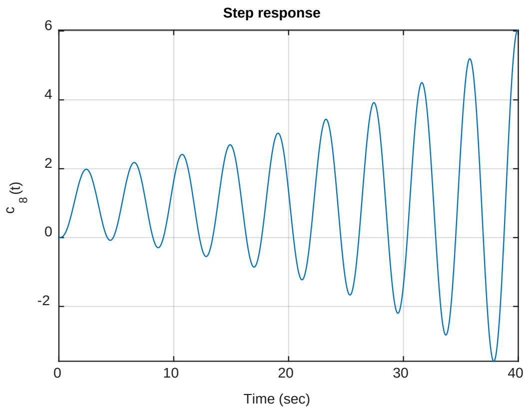
Unstable system

- Closed loop system transfer function will be:

$$G_8(s) = \frac{C(s)}{R(s)} = \frac{7}{(s(s+1)(s+2)+7)}$$

- The poles of the above system are:

$$s_1 = -3.0867, s_{2,3} = 0.0434 \pm 1.5053i$$



5.1 Definition of stability and stable system [based on closed loop transfer function]

- Stable systems has closed-loop transfer functions with poles only in the left half-plane.
- Unstable system have closed-loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.
- Marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.

References

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[3] Reymond T. Stefani, Design of Feedback Control Systems, 4th ed, Oxford, 2004

काठमाडौं विश्वविद्यालय

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Thank You !

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