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[This module is designed by author mentioned above with full copyright to demonstrante the concept of stability to third year mechanical engineering students while teaching subject Instrumentation and Control-(COEG-304) at Kathmandu University]

Stability

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Stable system

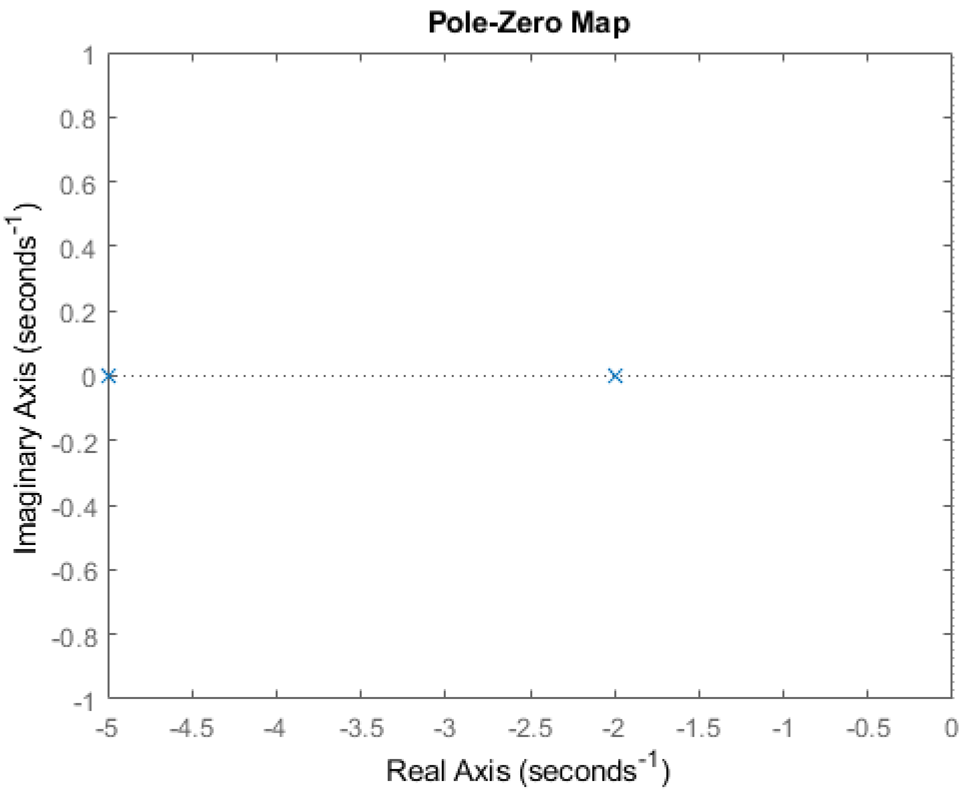
A linear, time-invariant system is ***stable*** if the natural response approaches zero as time approaches infinity.

Consider the following second order system $G_1(s)$

$$G_1(s) = \frac{15}{(s+2)(s+5)}$$

Pole zero plot of $G_1(s)$

```
N1=[0 15];
D1=poly([-2 -5]) ; %calculates the coefficient of the polynomial when roots are given
figure
pzmap(tf(N1,D1))
```



Plot the natural and step response of $G_1(s)$

```
syms s
R1=1/s; %step input
G1=15/((s+2)*(s+5));
C1=R1*G1 % step response
```

C1 =

$$\frac{15}{s\ (s+2)\ (s+5)}$$

```
c1=ilaplace(C1)
```

c1 =

$$e^{-5\,t} - \frac{5\,e^{-2\,t}}{2} + \frac{3}{2}$$

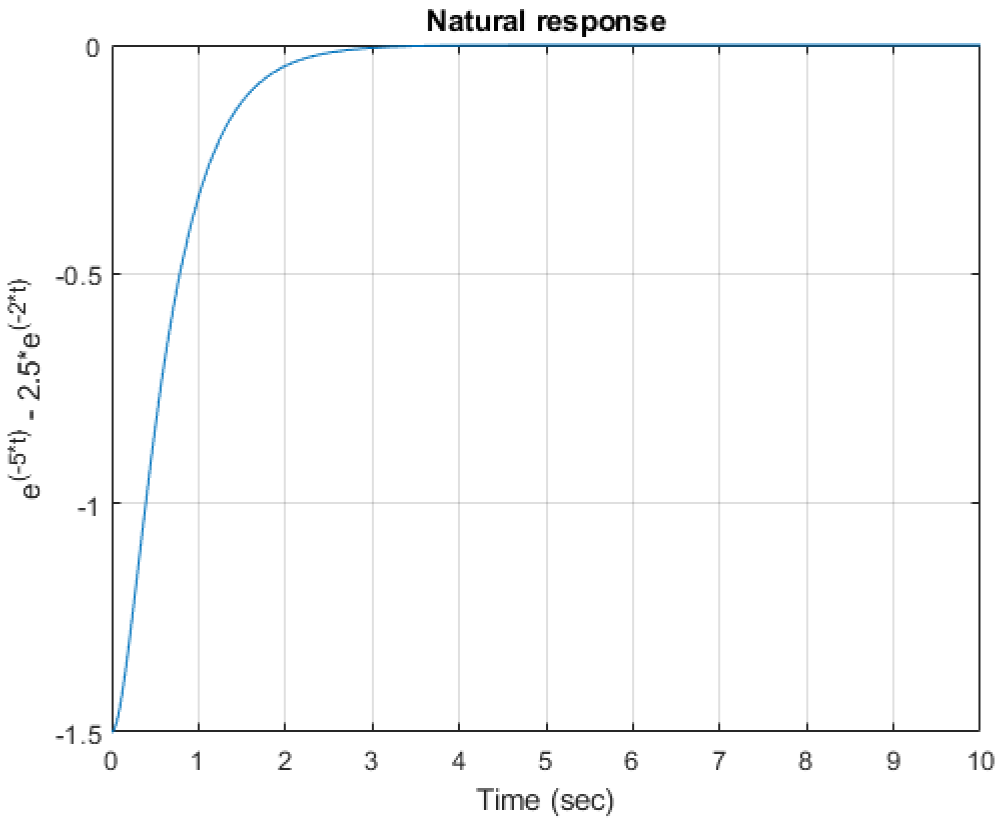
```
n1=c1-3/2 %natural response
```

n1 =

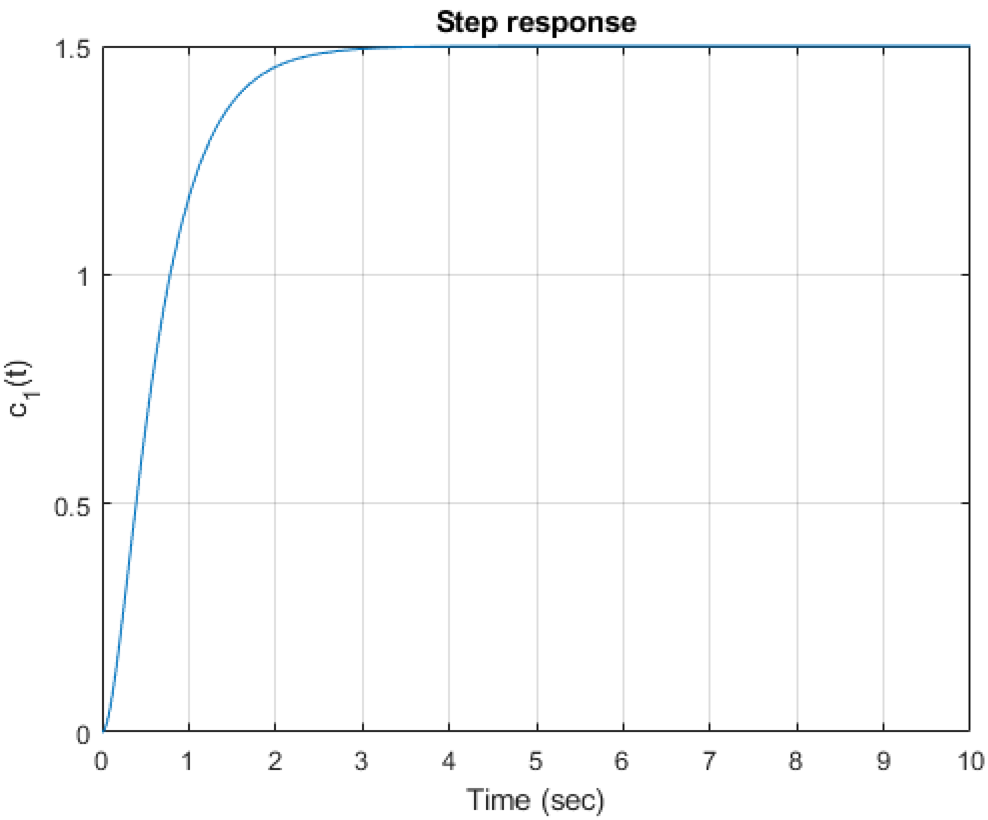
$$e^{-5\,t} - \frac{5\,e^{-2\,t}}{2}$$

```
figure
fplot(n1,[0 10])
title('Natural response')
```

```
xlabel('Time (sec)')
ylabel('e^{(-5*t)} - 2.5*e^{(-2*t)}')
grid on
```




```
figure
fplot(c1,[0 10])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{1}(t)')
grid on
```

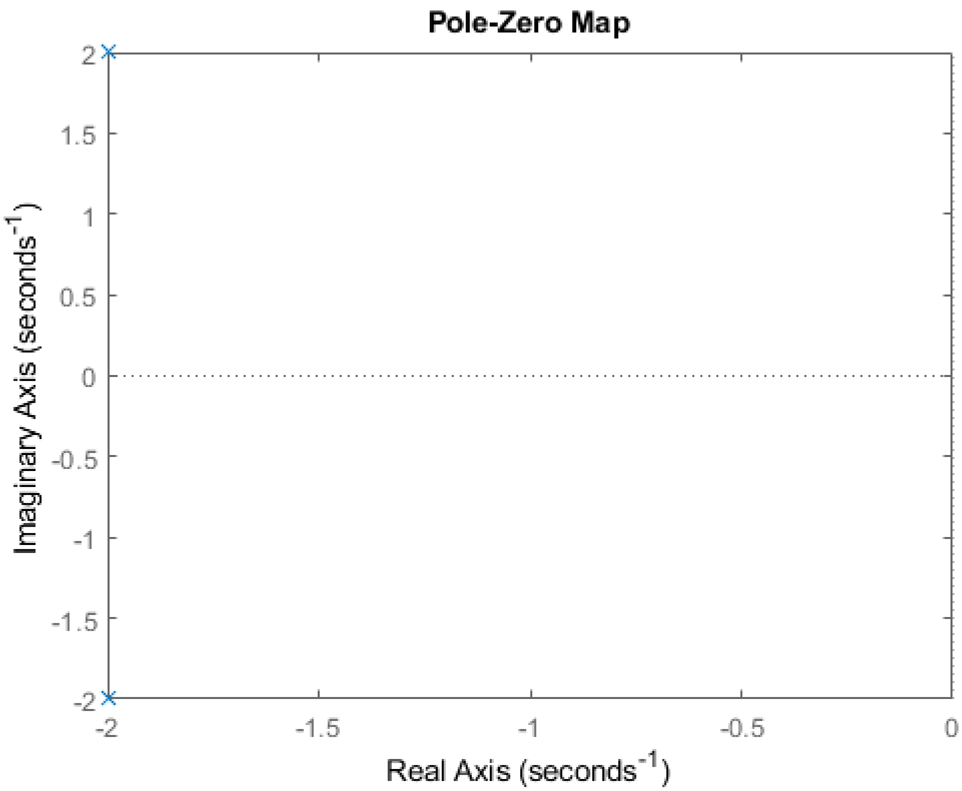


Consider the following second order system $G_2(s)$

$$G_2(s) = \frac{8}{(s^2 + 4s + 8)}$$

Pole zero plot of $G_2(s)$

```
N2=[0 8];
D2=[1 4 8];
figure
pzmap(tf(N2,D2))
```



Plot the natural and step response of $G_2(s)$

```
syms s
R2=1/s; % step input
G2=8/(s^2+4*s+8)
```

G2 =

$$\frac{8}{s^2+4\,s+8}$$

```
C2=R2*G2 % step response
```

C2 =

$$\frac{8}{s\,(s^2+4\,s+8)}$$

```
c2=ilaplace(C2)
```

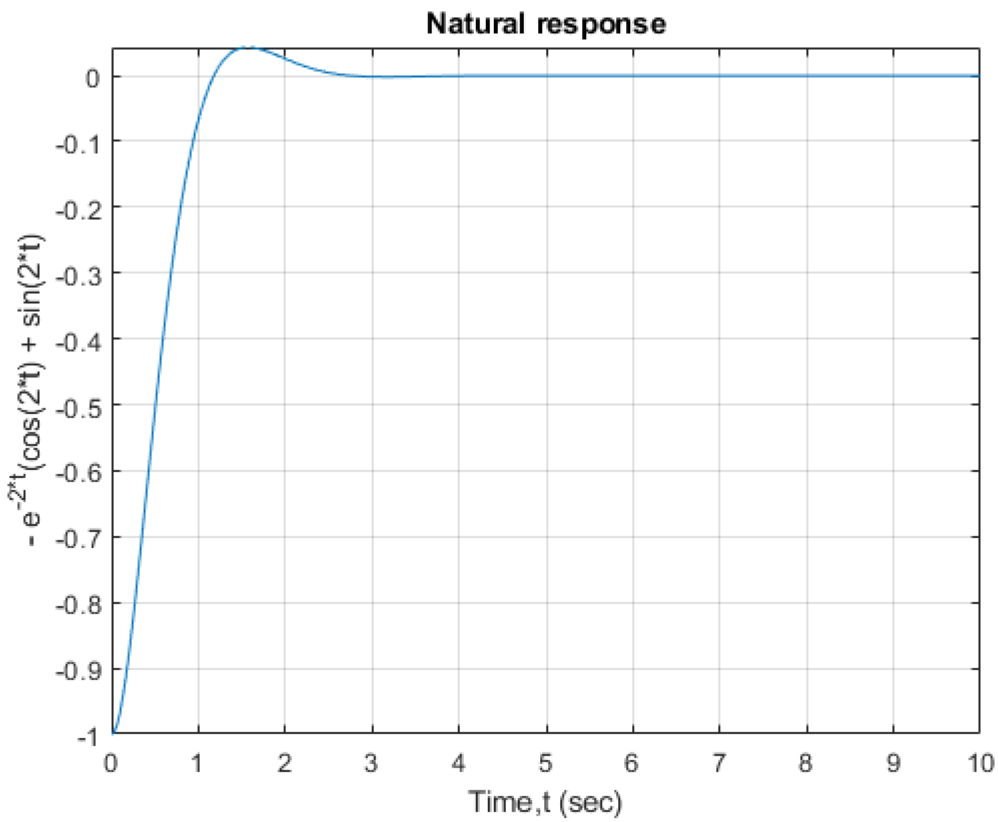
c2 = $1-\mathrm{e}^{-2\,t}\,(\cos(2\,t)+\sin(2\,t))$

```
n2=c2-1 % natural response
```

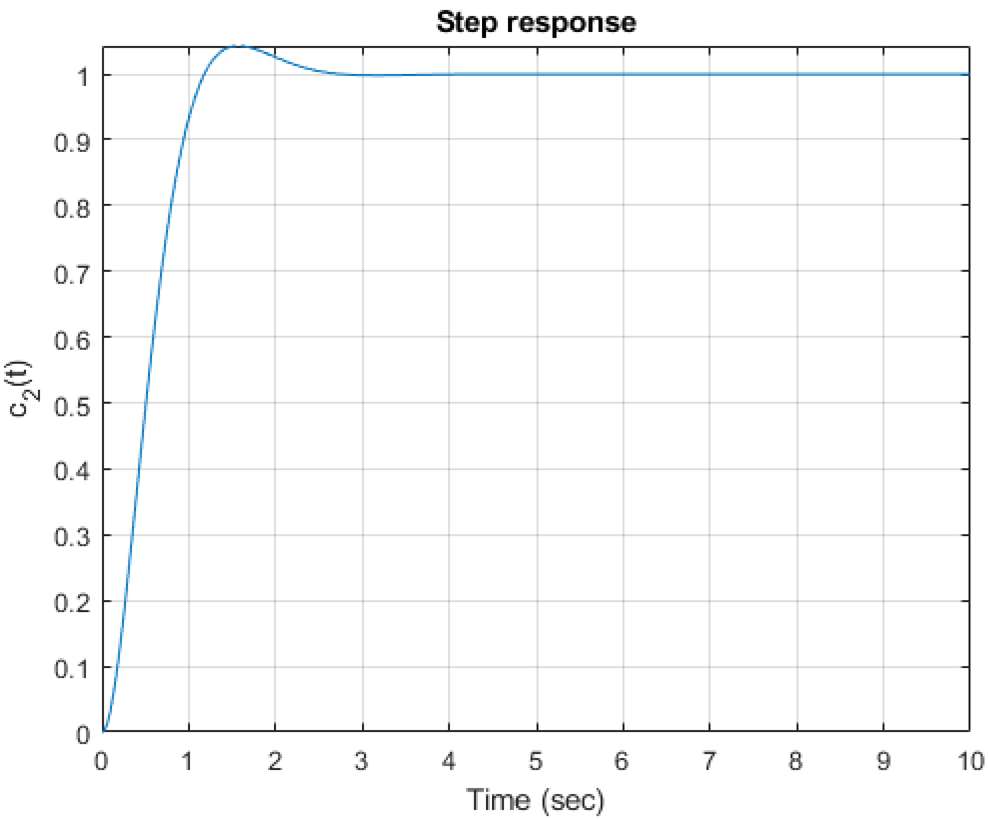
n2 = $-\mathrm{e}^{-2\,t}\,(\cos(2\,t)+\sin(2\,t))$

```
figure
```

```
fplot(n2, [0 10])
title('Natural response')
xlabel('Time,t (sec)')
ylabel('- e^{-2*t}(cos(2*t) + sin(2*t)')
grid on
```

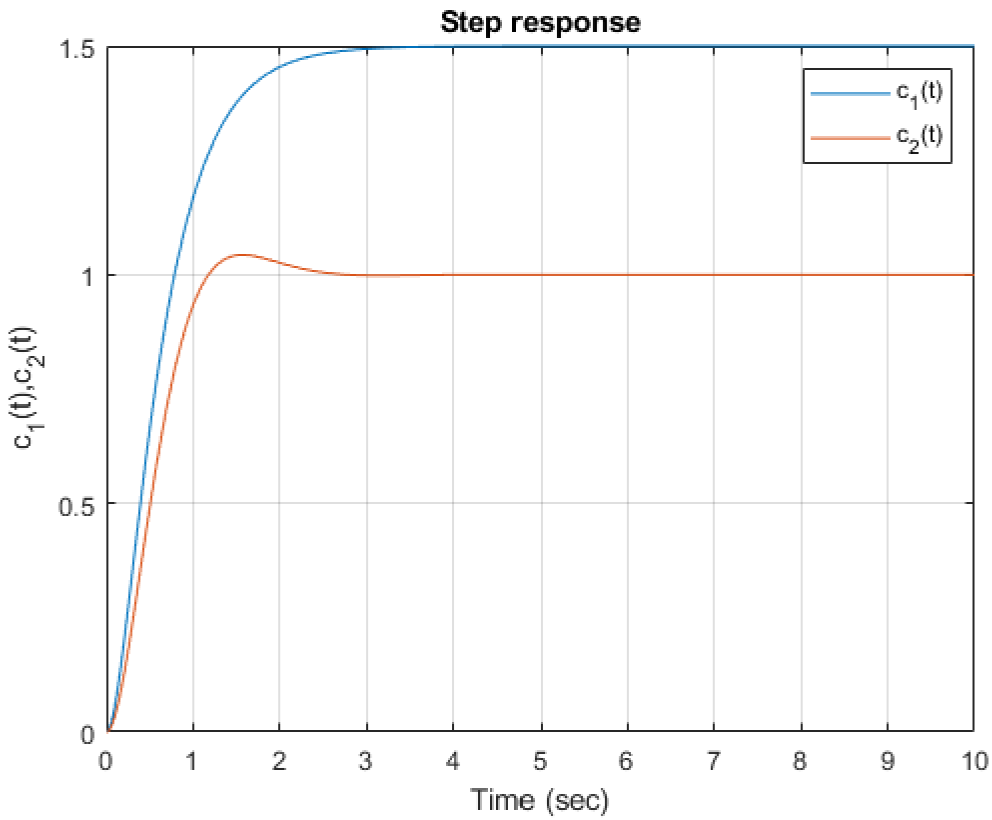



```
figure
fplot(c2,[0 10])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{2}(t)')
grid on
```



Plot the step response of $G_1(s)$ and $G_2(s)$ in same figure

```
figure
fplot(c1,[0 10])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{1}(t),c_{2}(t)')
grid on
hold on
fplot(c2,[0 10])
legend('c_{1}(t)','c_{2}(t)')
hold off
```



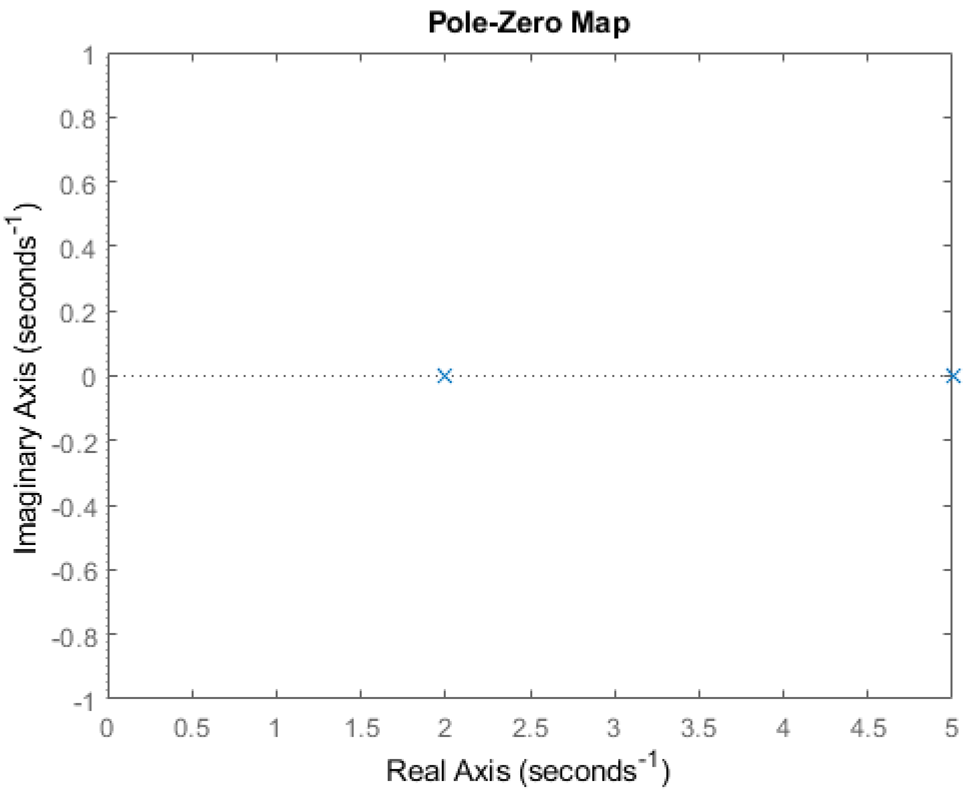
Unstable system

A linear, time-invariant system is ***unstable*** if the natural response grows without bound as time approaches infinity.

$$G_3(s) = \frac{15}{(s-2)(s-5)}$$

Pole zero plot of $G_3(s)$

```
N3=[0 15];
D3=poly([2 5]) ;
figure
pzmap(tf(N3,D3))
```



Plot the natural and step response of $G_3(s)$

```
syms s
R3=1/s; % step input
G3=15/((s-2)*(s-5))
```

G3 =

$$\frac{15}{(s-2)\ (s-5)}$$

```
C3=R3*G3 % step response
```

C3 =

$$\frac{15}{s\ (s-2)\ (s-5)}$$

```
c3=ilaplace(C3)
```

c3 =

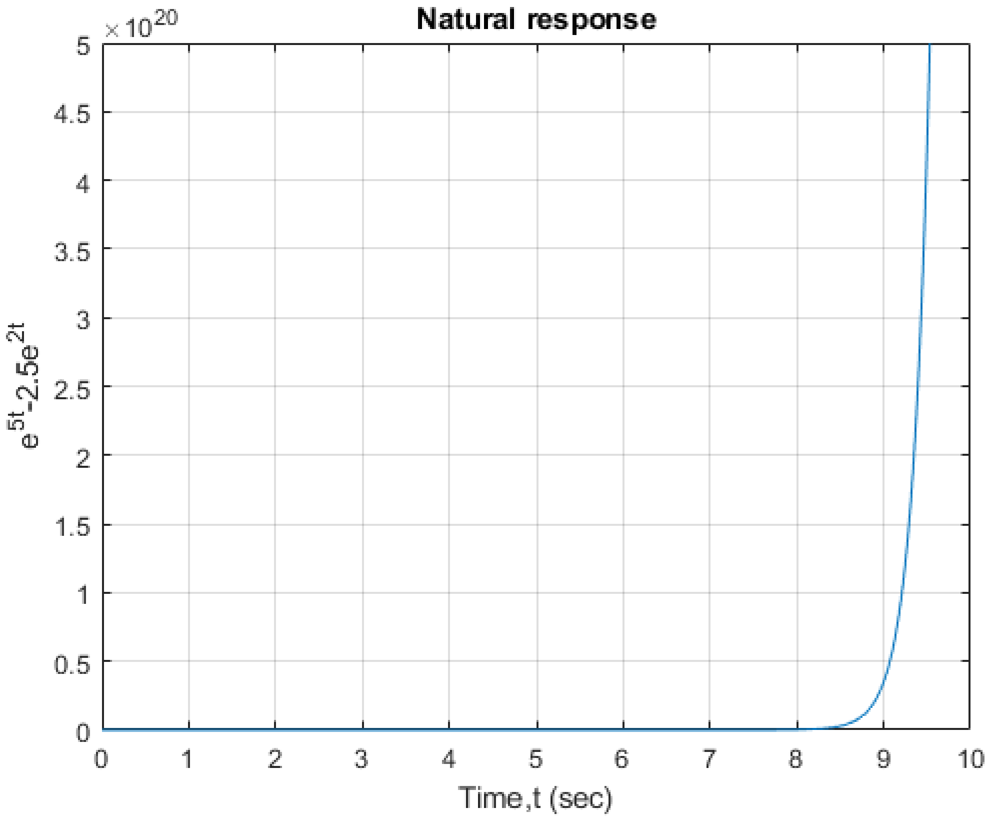
$$e^{5t} - \frac{5\,e^{2t}}{2} + \frac{3}{2}$$

```
n3=c3-1.5 % natural response
```

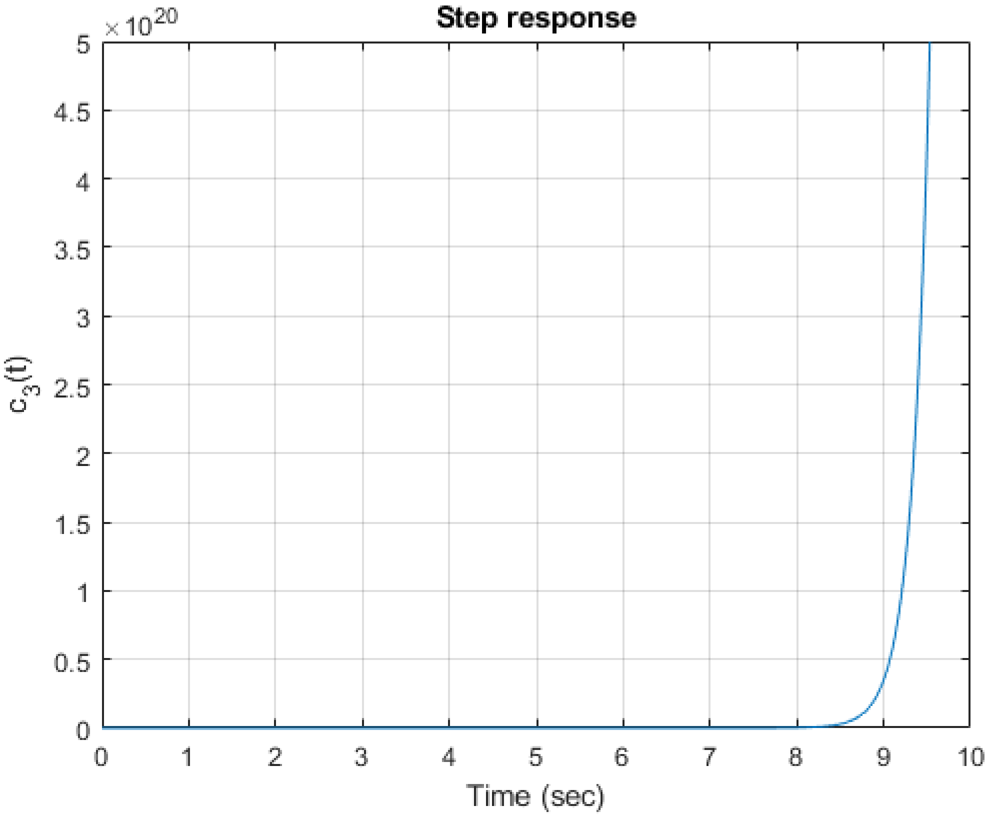
n3 =

$$e^{5t} - \frac{5e^{2t}}{2}$$

```
figure
fplot(n3, [0 10])
title('Natural response')
xlabel('Time,t (sec)')
ylabel('e^{5t}-2.5e^{2t}')
grid on
```



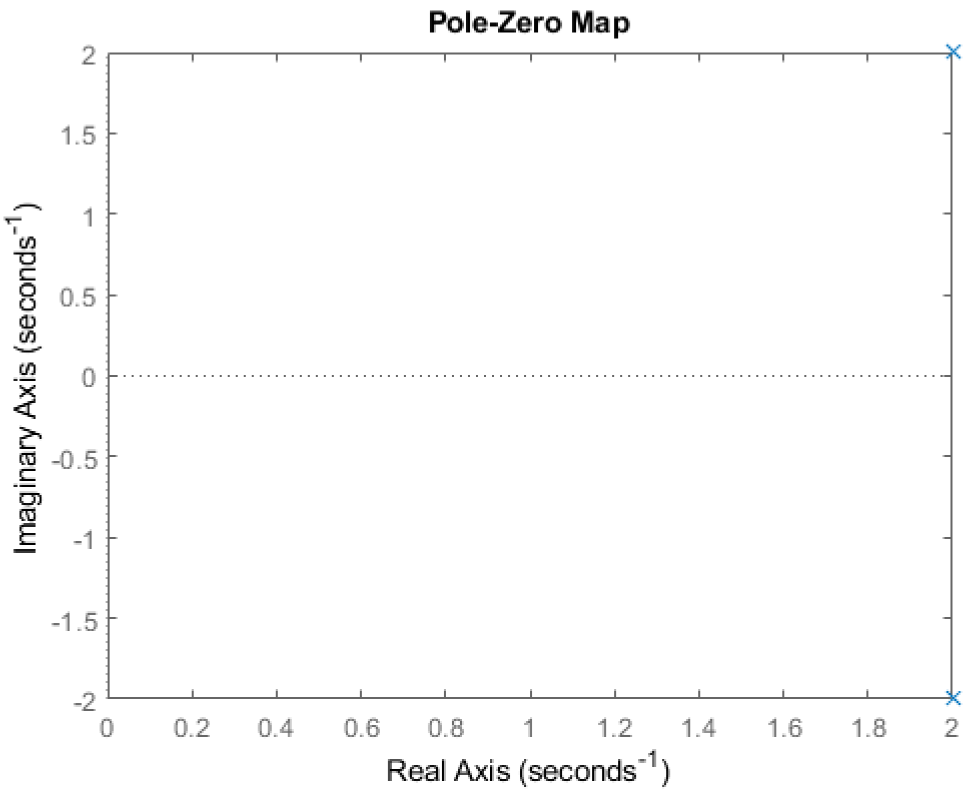

```
figure
fplot(c3,[0 10])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{3}(t)')
grid on
```



$G_4(s) = \frac{8}{(s^2 - 4s + 8)}$

Pole zero plot of $G_4(s)$

```
N4=[0 8];
D4=[1 -4 8];
figure
pzmap(tf(N4,D4))
```



Plot the natural and step response of $G_4(s)$

```
syms s
R4=1/s; %step input
G4=8/(s^2-4*s+8)
```

G4 =

$$\frac{8}{s^2 - 4\,s + 8}$$

```
C4=R4*G4
```

C4 =

$$\frac{8}{s\,(s^2 - 4\,s + 8)}$$

```
c4=ilaplace(C4)
```

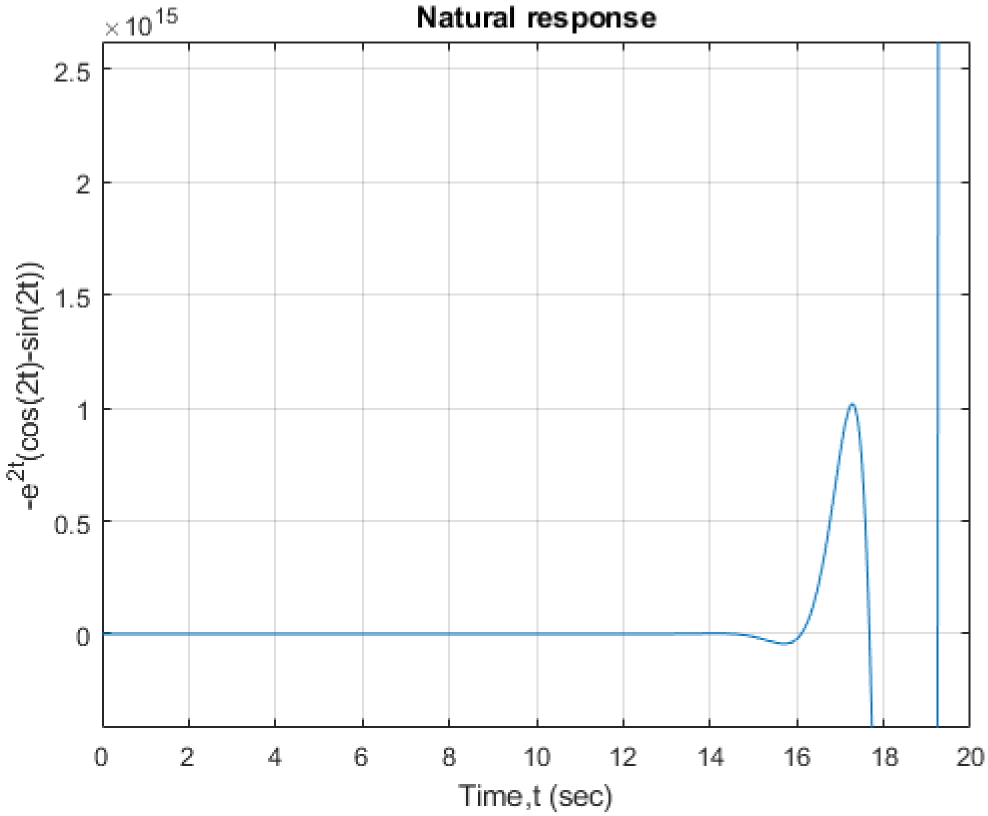
c4 = $1 - e^{2\,t}\,(\cos(2\,t) - \sin(2\,t))$

```
n4=c4-1% natural response
```

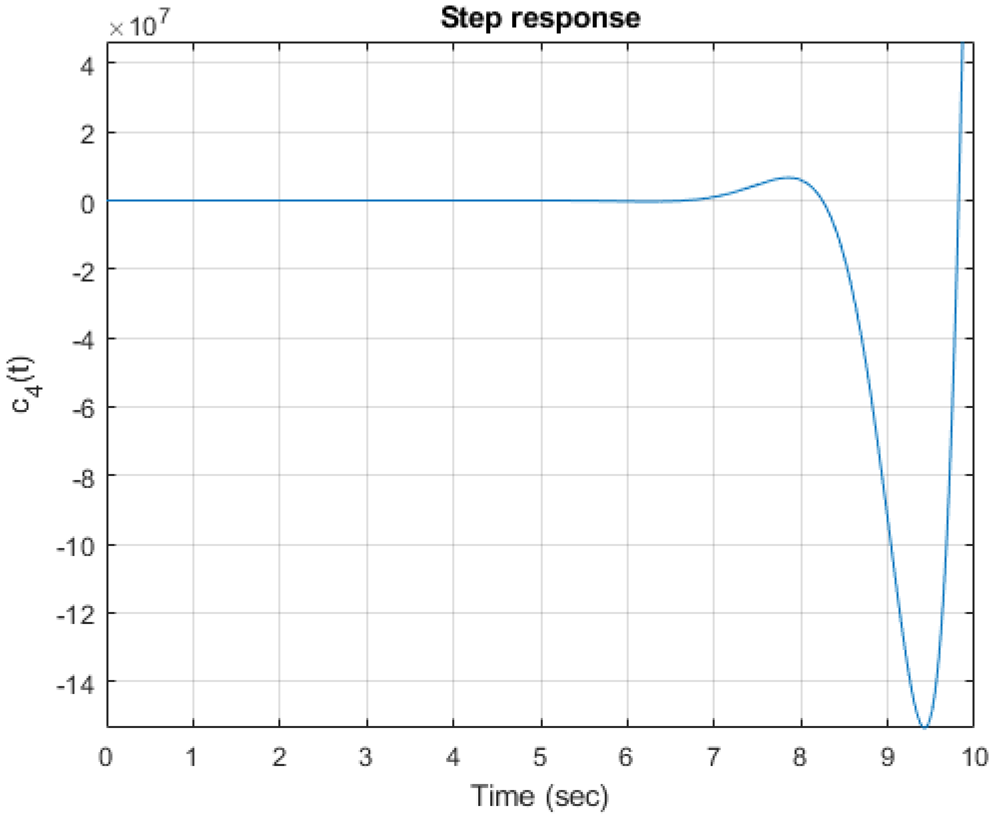
n4 = $-e^{2\,t}\,(\cos(2\,t) - \sin(2\,t))$

```
figure
fplot(n4, [0 20])
```

```
title('Natural response')
xlabel('Time,t (sec)')
ylabel('-e^{2t}(cos(2t)-sin(2t))')
grid on
```

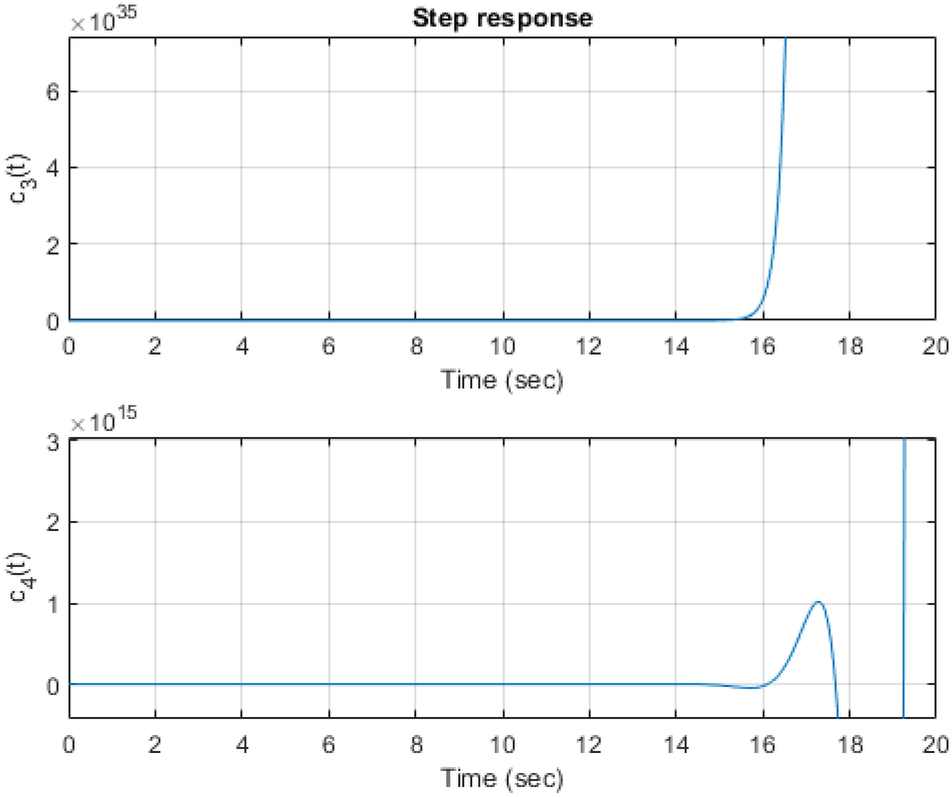



```
figure
fplot(c4,[0 10 ])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{4}(t)')
grid on
```



Plot the step response of $G_3(s)$ and $G_4(s)$ in same figure

```
figure
subplot(2,1,1)
fplot(c3,[0 20])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{3}(t)')
grid on
subplot(2,1,2)
fplot(c4,[0 20])
xlabel('Time (sec)')
ylabel('c_{4}(t)')
grid on
```



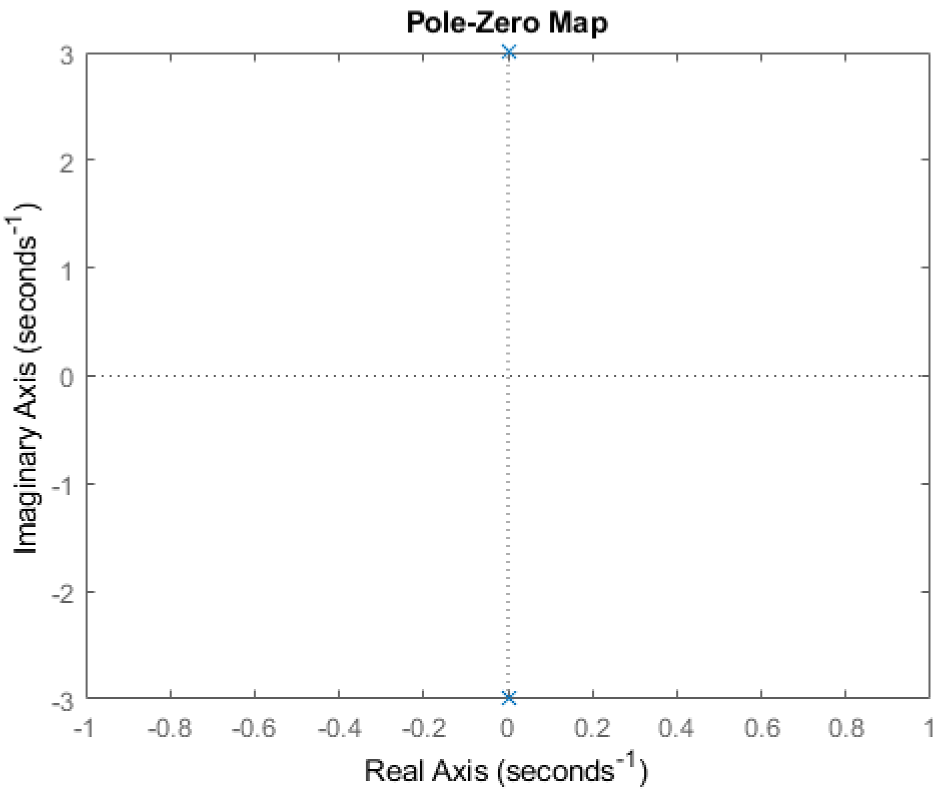
Marginally stable system

A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillate as time approaches infinity.

$$G_5(s) = \frac{9}{(s^2 + 9)}$$

Pole zero plot of $G_5(s)$

```
N5=[0 9];
D5=[1 0 9] ;
figure
pzmap(tf(N5,D5))
```



Plot the natural and step response of $G_5(s)$

```
syms s
R5=1/s; %step input
G5=9/(s^2+9)
```

G5 =

$$\frac{9}{s^2+9}$$

```
C5=R5*G5
```

C5 =

$$\frac{9}{s\ (s^2+9)}$$

```
c5=ilaplace(C5)
```

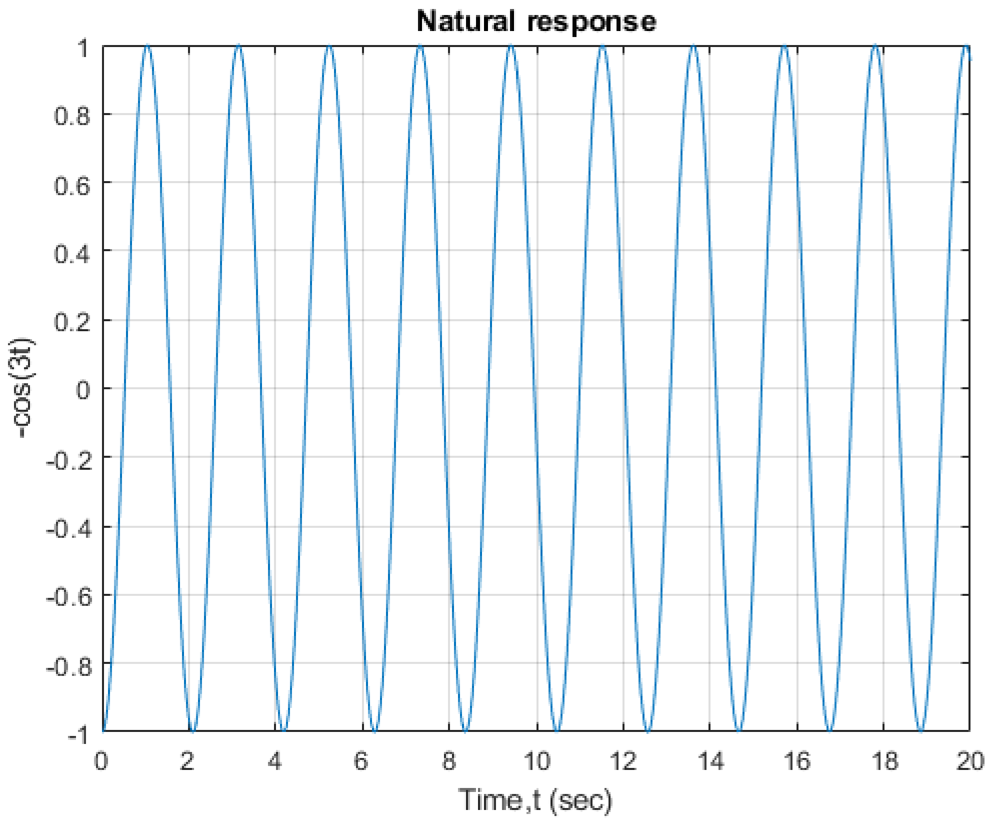
c5 = $1-\cos(3\,t)$

```
n5=c5-1% natural response
```

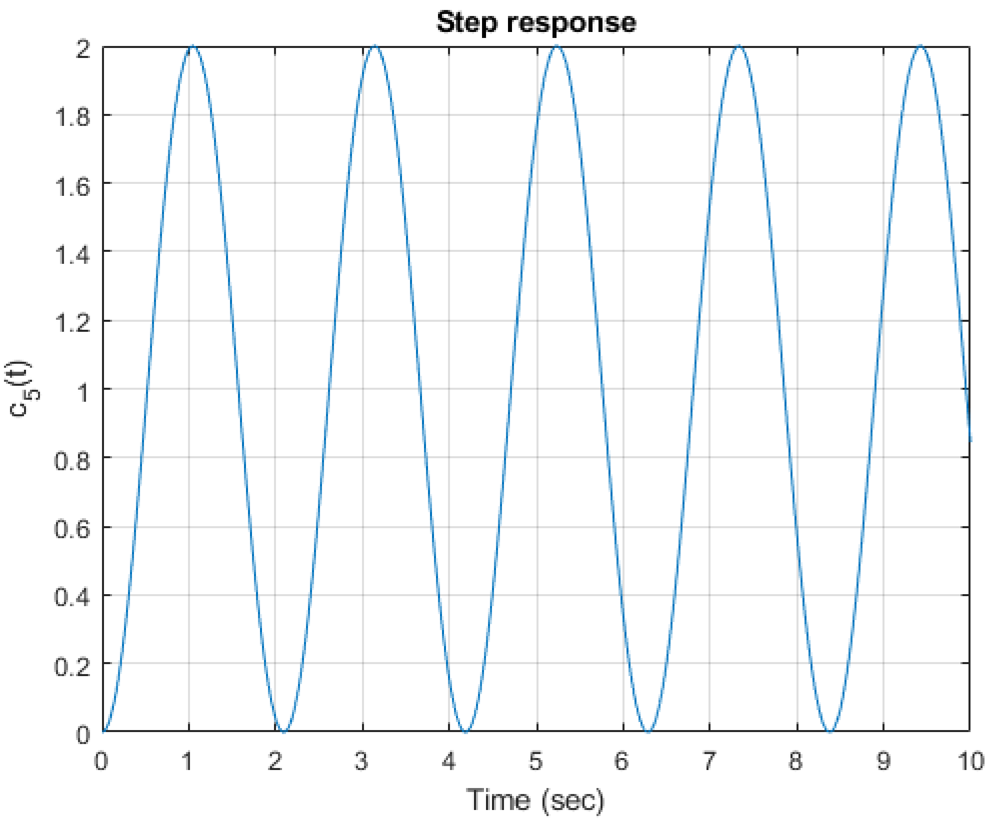
n5 = $-\cos(3\,t)$

```
figure
fplot(n5, [0 20])
title('Natural response')
```

```
xlabel('Time,t (sec)')
ylabel('-cos(3t)')
grid on
```




```
figure
fplot(c5,[0 10 ])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{5}(t)')
grid on
```



Special case unstable system

Repeated imaginary poles

$$G_6(s) = \frac{81}{(s^2+9)^2}$$

Pole zero plot of $G_6(s)$

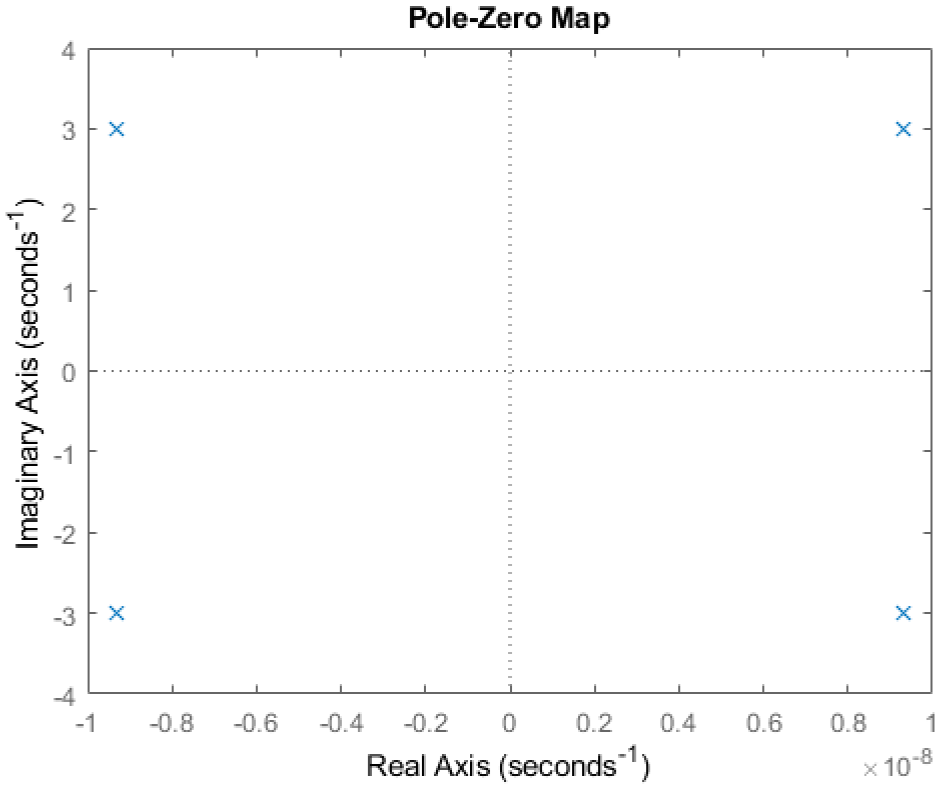
```
N6=[0 81];
D6=[1 0 18 0 81]
```

```
D6 = 1×5
     1     0    18     0    81
```

```
roots(D6)
```

```
ans = 4×1 complex
  -0.0000 + 3.0000i
  -0.0000 - 3.0000i
   0.0000 + 3.0000i
   0.0000 - 3.0000i
```

```
figure
pzmap(tf(N6,D6))
```



Plot the natural and step response of $G_6(s)$

```
syms s
R6=1/s; %step input
G6=81/((s^2+9)^2)
```

G6 =

$$\frac{81}{(s^2+9)^2}$$

```
C6=R6*G6
```

C6 =

$$\frac{81}{s\ (s^2+9)^2}$$

```
c6=ilaplace(C6)
```

c6 =

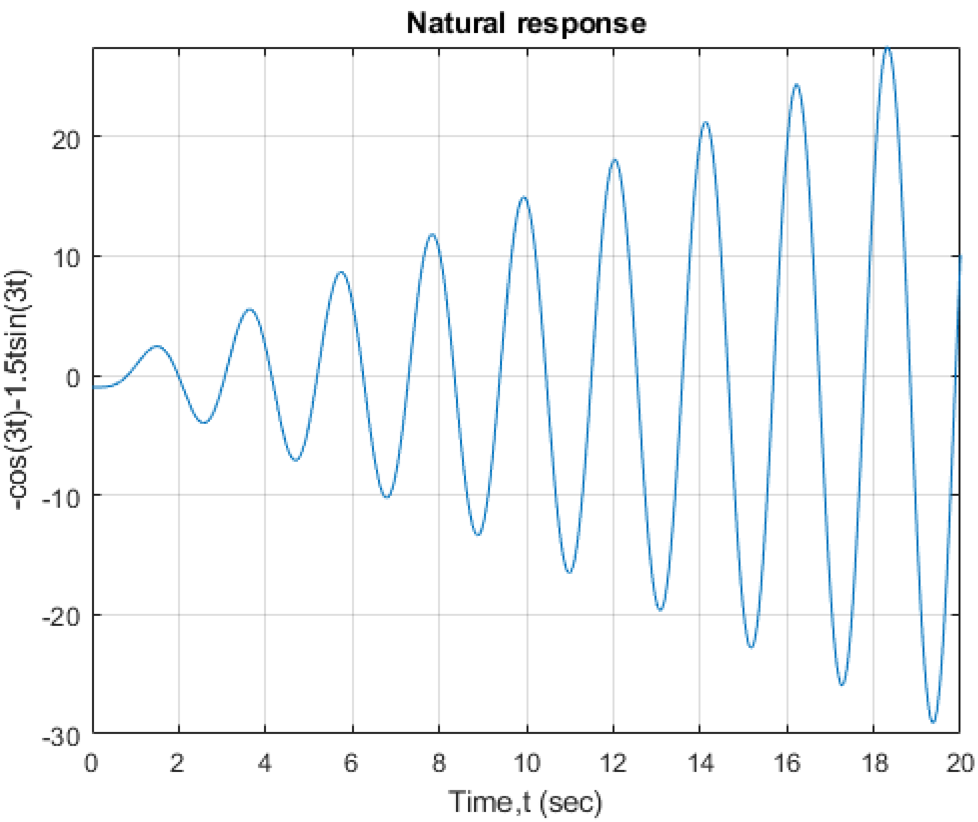
$$1-\frac{3\,t\sin(3\,t)}{2}-\cos(3\,t)$$

```
n6=c6-1% natural response
```

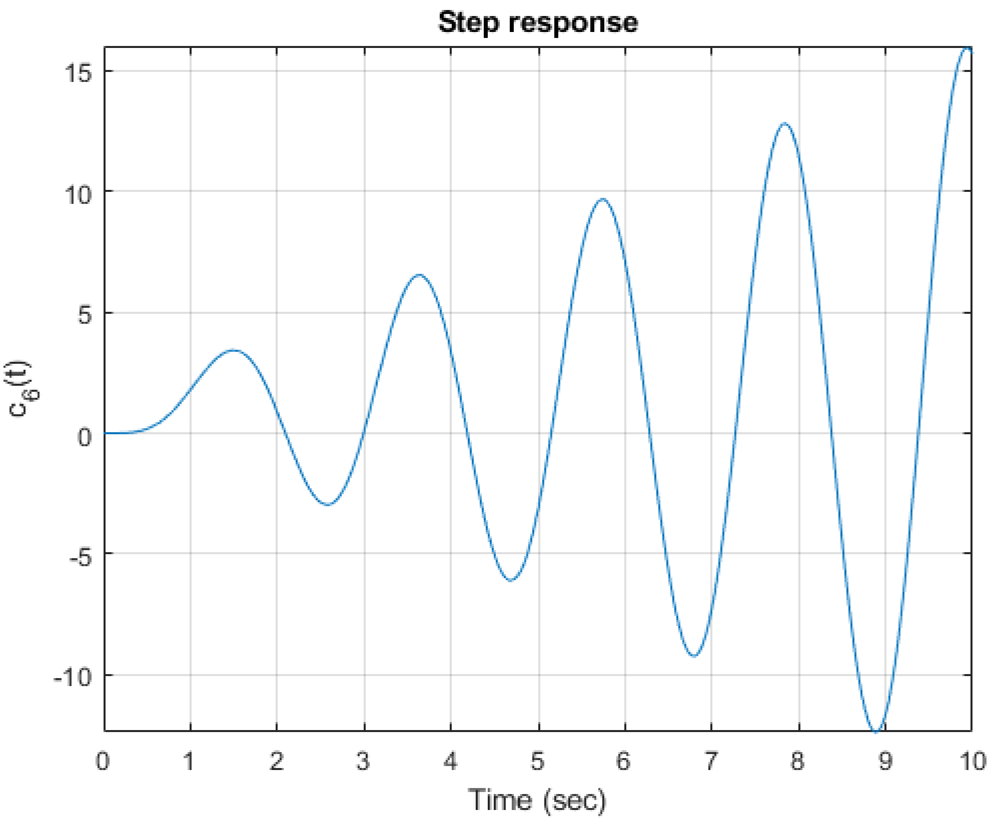
n6 =

$-\cos(3\,t) - \frac{3\,t\,\sin(3\,t)}{2}$

```
figure
fplot(n6, [0 20])
title('Natural response')
xlabel('Time,t (sec)')
ylabel('-cos(3t)-1.5tsin(3t)')
grid on
```

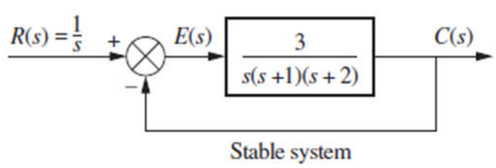



```
figure
fplot(c6,[0 10 ])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{6}(t)')
grid on
```



Closed loop stability of the system

System 1



The closed loop transfer function of the above system will be:

$$G_7(s) = \frac{C(s)}{R(s)} = \frac{3}{(s(s+1)(s+2)+3)}$$

Pole zero plot of $G_7(s)$

```
N7=[0 3];
D7=[1 3 2 3]
```

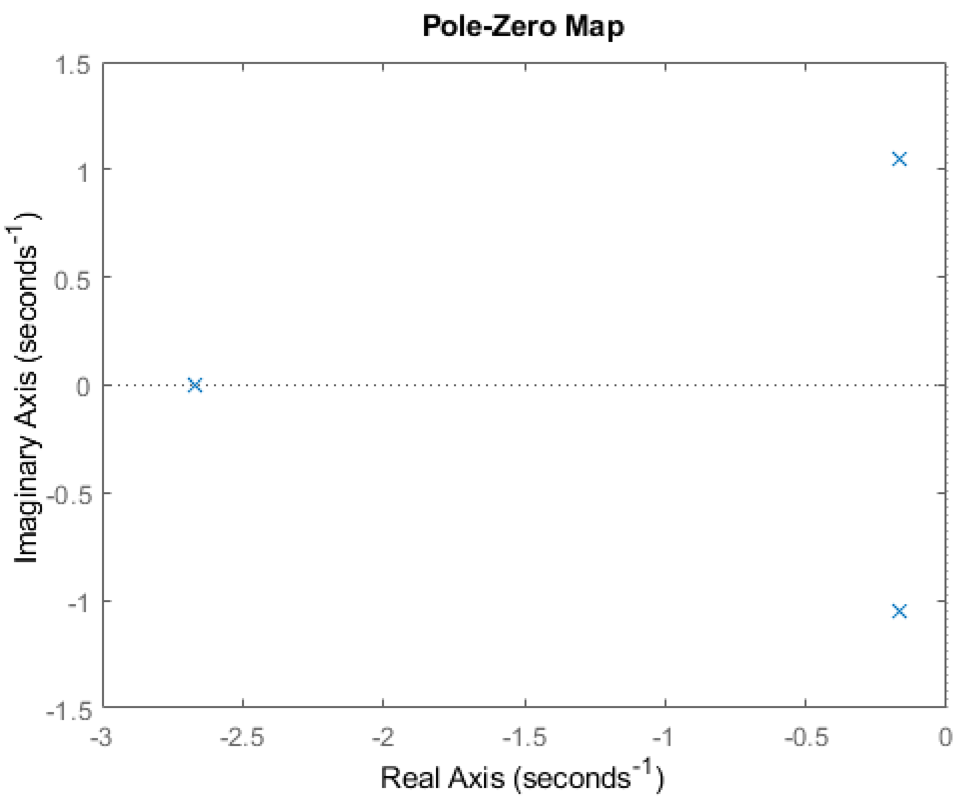
```
D7 = 1×4
     1     3     2     3
```

```
roots(D7)
```

```
ans = 3×1 complex
  -2.6717 + 0.0000i
  -0.1642 + 1.0469i
```

```
  -0.1642 - 1.0469i
```

```
figure
pzmap(tf(N7,D7))
```



Plot the natural and step response of $G_7(s)$

```
syms s
R7=1/s; %step input
G7=3/((s*(s+1)*(s+2))+3)
```

```
G7 =
```

$$\frac{3}{s\ (s+1)\ (s+2)+3}$$

```
C7=R7*G7
```

```
C7 =
```

$$\frac{3}{s\ (s\ (s+1)\ (s+2)+3)}$$

```
c7=ilaplace(C7)
```

```
c7 =
```

$$1-2\left(\sum_{k=1}^{3}\frac{e^{t\,\sigma_1}}{3\,\sigma_1^2+6\,\sigma_1+2}\right)-3\left(\sum_{k=1}^{3}\frac{e^{\sigma_1\,t}\,\sigma_1}{6\,\sigma_1+3\,\sigma_1^2+2}\right)-\left(\sum_{k=1}^{3}\frac{e^{t\,\sigma_1}\,\sigma_1^2}{3\,\sigma_1^2+6\,\sigma_1+2}\right)$$

where

$$\sigma_1 = \mathrm{root}\left(s_3^3+3\,s_3^2+2\,s_3+3, s_3, k\right)$$

```
n7=c7-1% natural response
```

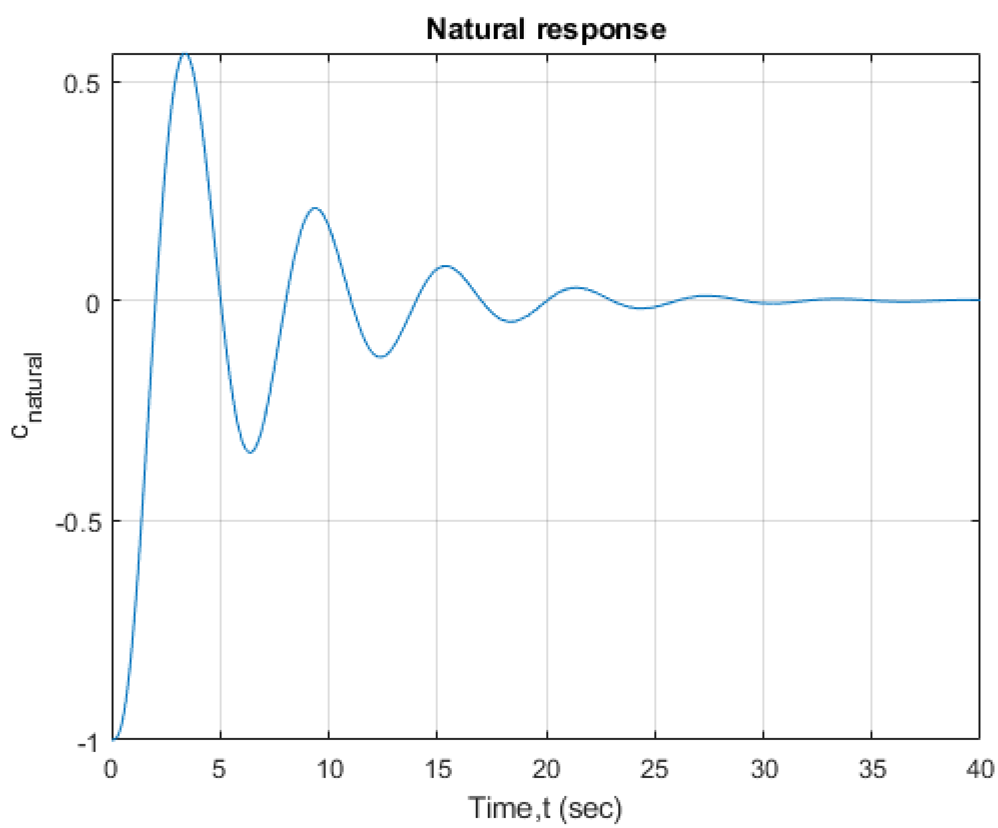
```
n7 =
```

$$-\left(\sum_{k=1}^{3}\frac{e^{t\,\sigma_1}\,\sigma_1^2}{3\,\sigma_1^2+6\,\sigma_1+2}\right)-2\left(\sum_{k=1}^{3}\frac{e^{t\,\sigma_1}}{3\,\sigma_1^2+6\,\sigma_1+2}\right)-3\left(\sum_{k=1}^{3}\frac{e^{\sigma_1\,t}\,\sigma_1}{6\,\sigma_1+3\,\sigma_1^2+2}\right)$$

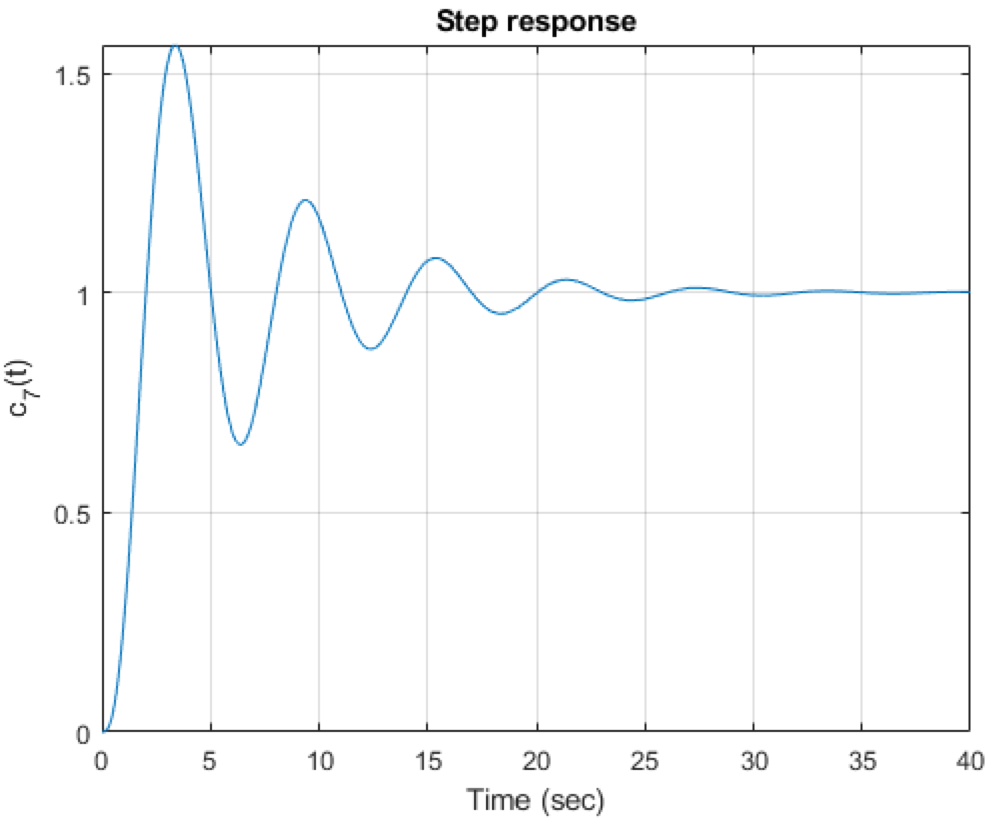
where

$$\sigma_1 = \mathrm{root}\left(s_3^3+3\,s_3^2+2\,s_3+3, s_3, k\right)$$

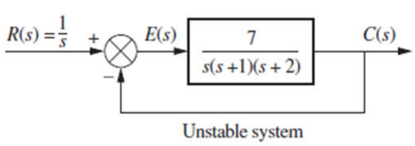
```
figure
fplot(n7, [0 40])
title('Natural response')
xlabel('Time,t (sec)')
ylabel('c_{natural}')
grid on
```



```
figure
fplot(c7,[0 40])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{7}(t)')
grid on
```



System 2



The closed loop transfer function of the above system will be:

$$G_8(s) = \frac{C(s)}{R(s)} = \frac{7}{(s(s+1)(s+2)+7)}$$

Pole zero plot of $G_8(s)$

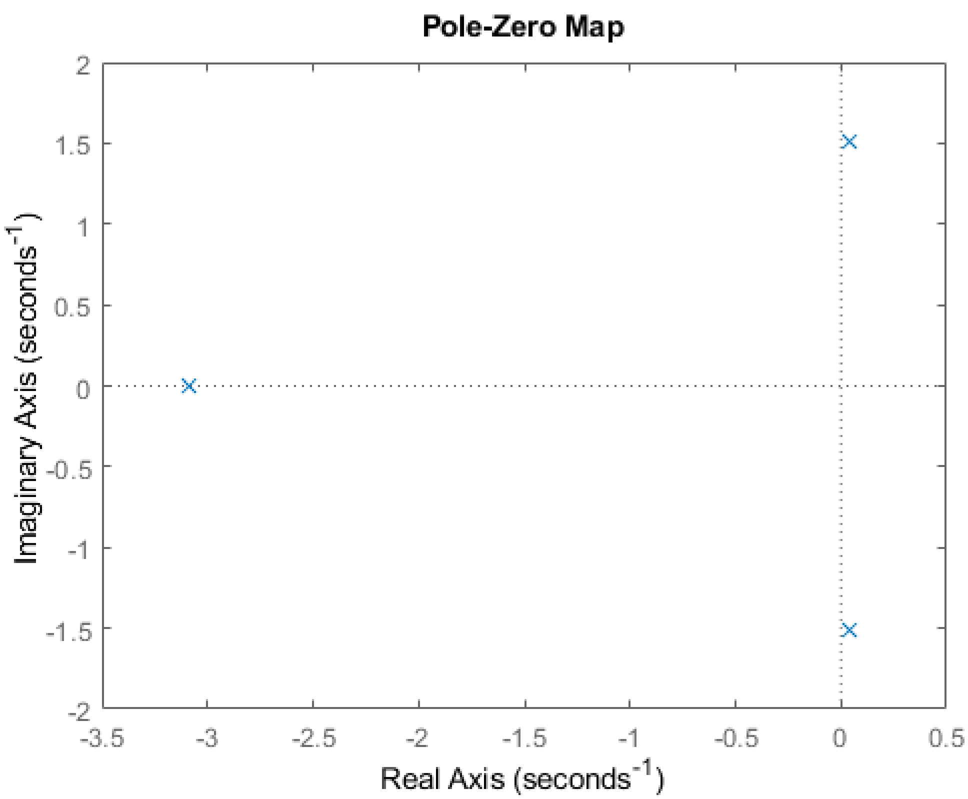
```
N8=[0 7];
D8=[1 3 2 7]
```

```
D8 = 1×4
     1     3     2     7
```

```
roots(D8)
```

```
ans = 3×1 complex
  -3.0867 + 0.0000i
   0.0434 + 1.5053i
   0.0434 - 1.5053i
```

```
figure
pzmap(tf(N8,D8))
```



Plot the natural and step response of $G_8(s)$

```
syms s
R8=1/s; %step input
G8=7/((s*(s+1)*(s+2))+7)
```

G8 =

$$\frac{7}{s\ (s+1)\ (s+2)+7}$$

```
C8=R8*G8
```

C8 =

$$\frac{7}{s\ (s\ (s+1)\ (s+2)+7)}$$

```
c8=ilaplace(C8)
```

c8 =

$$1-2\left(\sum_{k=1}^{3} \frac{e^{t\,\sigma_1}}{3\,\sigma_1{}^2+6\,\sigma_1+2}\right)-3\left(\sum_{k=1}^{3} \frac{e^{\sigma_1\,t}\,\sigma_1}{6\,\sigma_1+3\,\sigma_1{}^2+2}\right)-\left(\sum_{k=1}^{3} \frac{e^{t\,\sigma_1}\,\sigma_1{}^2}{3\,\sigma_1{}^2+6\,\sigma_1+2}\right)$$

where

$$\sigma_1 = \mathrm{root}\left(s_4{}^3+3\,s_4{}^2+2\,s_4+7, s_4, k\right)$$

```
n8=c8-1% natural response
```

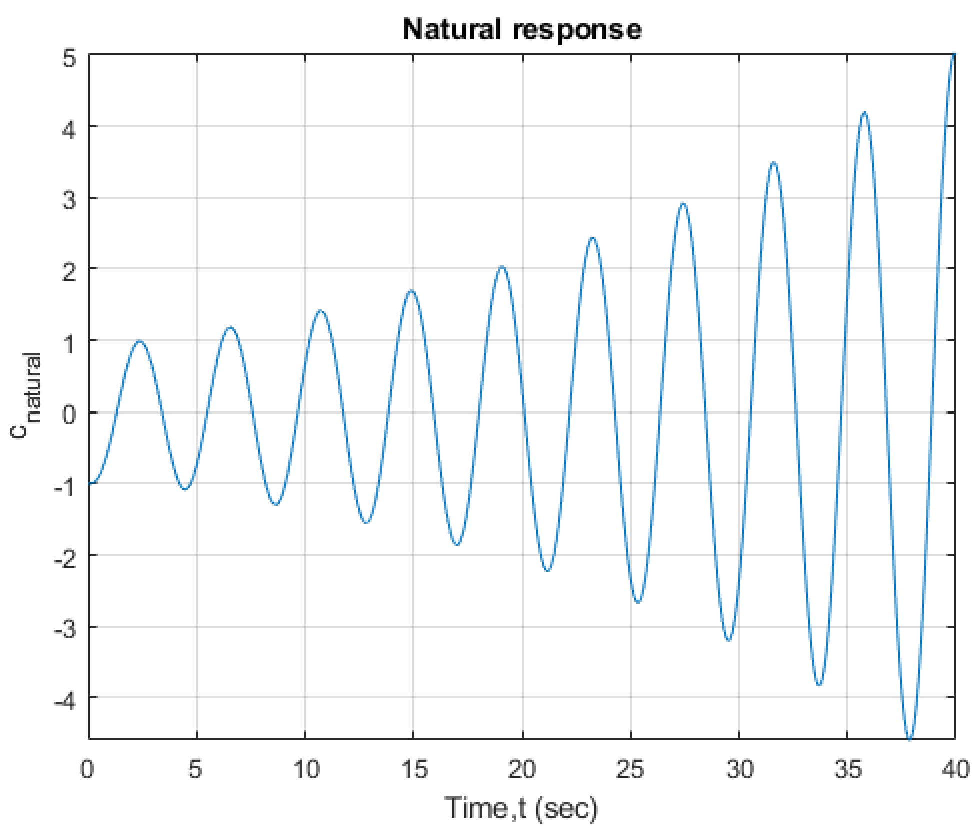
```
n8 =
```

$$-\left(\sum_{k=1}^{3} \frac{e^{t\,\sigma_1}\,\sigma_1{}^2}{3\,\sigma_1{}^2+6\,\sigma_1+2}\right)-2\left(\sum_{k=1}^{3} \frac{e^{t\,\sigma_1}}{3\,\sigma_1{}^2+6\,\sigma_1+2}\right)-3\left(\sum_{k=1}^{3} \frac{e^{\sigma_1\,t}\,\sigma_1}{6\,\sigma_1+3\,\sigma_1{}^2+2}\right)$$

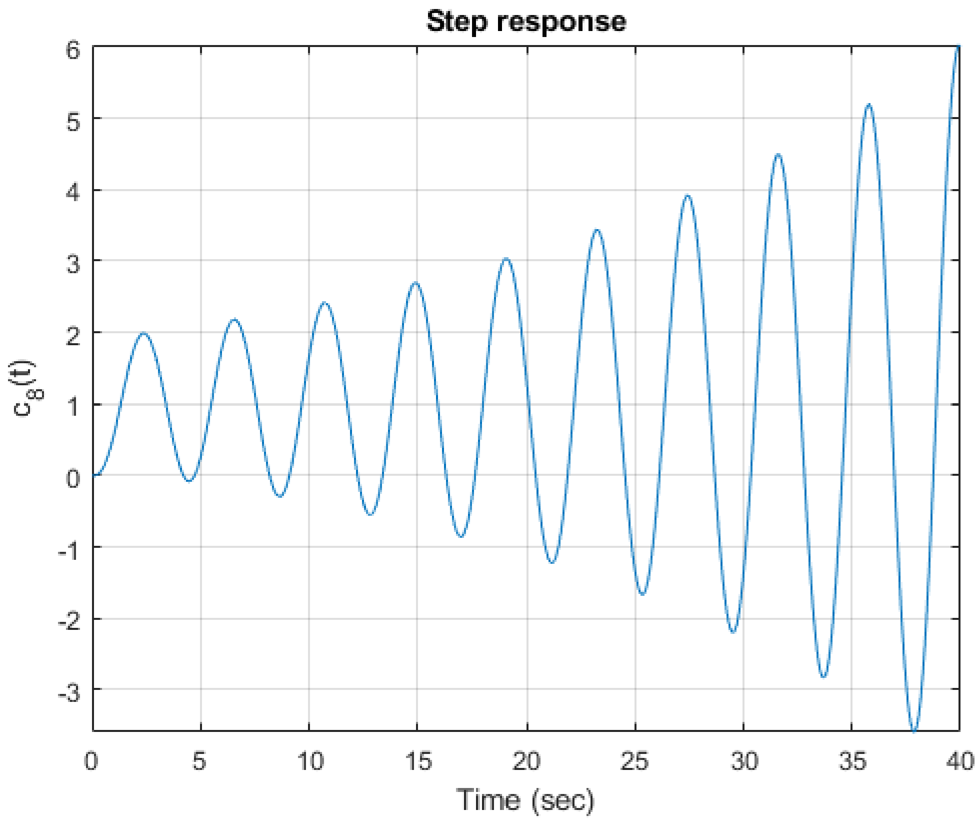
where

$$\sigma_1 = \mathrm{root}\left(s_4{}^3+3\,s_4{}^2+2\,s_4+7, s_4, k\right)$$

```
figure
fplot(n8, [0 40])
title('Natural response')
xlabel('Time,t (sec)')
ylabel('c_{natural}')
grid on
```



```
figure
fplot(c8,[0 40])
title('Step response')
xlabel('Time (sec)')
ylabel('c_{8}(t)')
grid on
```



References

[1] Nise, Norman S., *Control Systems Engineering,* 7th ed, Hoboken, NJ:Wiley, 2004.

[2] K. Webb, Class Lecture, Topic: "Section 6: Stability" ESE499, College of Engineering, OSU-Cascades, Oregon, *accesed on*:June 2s