Chapter 1

Matrix Algebra

SYNOPSIS

1. MATRIX

A matrix is a rectangular array of numbers. The numbers may be real or complex. It may be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

or as $A = [a_{ij}]_{m \times n}$ where $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$

A matrix with m rows and n columns is called as $m \times n$ matrix.

The numbers $a_{11}, a_{12}, \dots, a_{1n}$ are called the elements of the matrix. In the matrix, the horizontal lines are called rows or row vectors and the vertical lines are called columns or column vectors. The number a_{ij} indicates the element present in the ith row and jth column.

2. TYPES OF MATRICES

A matrix $A = [a_{ij}]_{m \times n}$ is said to be a

(i) Rectangular matrix if $m \neq n$

(ii) Square matrix if $m = n$

(iii) Row matrix if $m = 1$

(iv) Column matrix if $n = 1$

(v) Null or zero matrix if $a_{ij} = 0, \ \forall \ i$ and j

(vi) Diagonal matrix if $m = n$ and $a_{ij} = 0, \ \forall \ i \neq j$

(vii) Scalar matrix if $m = n$ and $a_{ij} = 0, \ \forall \ i \neq j$ and $a_{ii} = \lambda$(scalar) $\forall \ i$

(viii) Unit or Identity matrix if $m = n$ and $a_{ij} = 0, \quad \forall \quad i \neq j$ and $a_{ii} = 1 \;\; \forall \;\; i$

(ix) Upper triangular matrix if $m = n$ and $a_{ij} = 0, \;\; \forall \;\; i > j$

(x) Lower triangular matrix if $m = n$ and $a_{ij} = 0, \;\; \forall \;\; i < j$

(xi) A matrix is said to be triangular if it is either lower or upper triangular matrix.

(xii) Sparse matrix if most of the elements of the matrix are zero.

(xiii) Complex matrix if atleast one element is imaginary.

3. Algebra of Matrices

(i) **Equality of Matrices**: Two matrices are said to be equal provided they are of the same order and corresponding elements are equal.

(ii) **Addition of Matrices**: Two matrices A and B can be added if and only if they are of the same order and the matrix $(A + B)$ is obtained by adding the corresponding elements of A and B. Addition is not defined for matrices of different sizes. The additive inverse of A, denoted by $-A$.

If A and B are two matrices of the same order, then the differences between A and B is defined by $A - B = A + (-B)$.

Properties of Addition: If A, B and C are three matrices of the same size, then

$A + B = B + A$ (commutative law)

$(A + B) + C = A + (B + C)$ (Associative law)

$A = A + O = O + A$ (Additive property of zero)

$A + (-A) = O$ (Additive inverse)

$A + B = A + C \Rightarrow B = C$ (Left cancellation law)

$B + A = C + A \Rightarrow B = C$ (Right cancellation law)

(iii) **Scalar Multiplication:** If A is a matrix and K is a scalar, then KA is defined as the matrix obtained by multiplying every element of A by K.

Properties of Scalar Multiplication: If A,B are two matrices of the same order and k, k_1, k_2 are scalars, then

$(k_1 + k_2)A = k_1A + k_2A$

$(k_1k_2)A = k_1(k_2A)$

$k(A \pm B) = kA \pm kB$

$(-kA) = -(kA) = k(-A)$

(iv) **Multiplication of Matrices:** The product of two matrices A and B is possible only if the number of columns of A is equal to the number of rows of B and these types of matrices are called conformable for multiplication.

Properties of Matrix Multiplication:

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{p\times q}$ then

(i) In general $AB \neq BA$ (commutative law)

(ii) $(AB)C = A(BC)$ (Associative law)

(iii) $A(B + C) = AB + AC$ and $(B + C)A = BA + BC$ (Distributive law)

(iv) $AB = AC \Rightarrow B = C$ (Cancellation law). It is possible only when A is non-singular matrix.

(v) $AI_n = I_mA = A$

(vi) $k(AB) = (kA)B = A(KB)$, where k is a scalar

(vii) If A is a square matrix, then
$A^m.A^n = A^{m+n} \; \forall \; m, n \in N$
$(A^m)^n = A^{mn} \; \forall \; m, n \in N$

4. Trace of a Matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n.

Then the sum of the elements lying along the principal diagonal is called the trace of A and denoted by $tr(A)$.

Thus $\quad tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$

Properties of Trace of Matrix:

Let A and B be any two square matrices of order n and k is a scalar. Then

(i) $tr(kA) = k \;\; tr(A)$

(ii) $tr(A + B) = tr(A) + tr(B)$

(iii) $tr(A - B) = tr(A) - tr(B)$

(iv) $tr(AB) = tr(BA)$

5. Involutory matrix

If a square matrix $'A'$ is such that $A^2 = I$, then A is called Involutory. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is Involutory.}$$

Note:

1. Identity matrix is always Involutory.

2. A is Involutory matrix iff $(A - I)(A + I) = O$

6. Nilpotent Matrix

For any square matrix $'A'$, if there exists a positive integer m such that $A^m = O$, then A is a nilpotent matrix. The index m of the nilpotent matrix A is the least positive integer such that $A^m = O$.

For example, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is a nilpotent matrix of index 2 since } A^2 = O.$$

7. Transpose of a matrix

The matrix obtained by interchanging the rows and columns of a matrix A is called transpose of A denoted by A^T or A'.

Properties of Transpose of a Matrix:

(i) $(A + B)^T = A^T + B^T$

(ii) $(kA)^T = KA^T$, where k is a scalar

(iii) $(AB)^T = B^T A^T$

(iv) $(A^T)^T = A$

(v) If A is an invertible matrix, then $(A^{-1})^T = (A^T)^{-1}$

8. Determinant Of a square Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then the determinant of A is denoted by det A or $|A|$ and defined as

$$detA = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}_{n\times n}$$

The determinant has always a real finite value. If we define a 3×3 determinant, then it has three rows and three columns and its value is given as follows.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{n\times n}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

This is called expanding the determinant by first row. A determinant can be expanded in terms of any row or column.

NOTE:

1. If A is a square matrix of order n, then $|A| = |A^T|$.

2. If A and B are two square matrices of the same order, then $|AB| = |A||B|$

3. If A is a square matrix of order n, then $|kA| = k^n|A|$, for any scalar k.

4. $|A^n| = (|A|)^n$

Minors and Cofactors

The minor of an element in a determinant is the determinant obtained by deleting the row and column containing that element.

The cofactor of any element in a determinant is its minor with the proper sign. The sign of an element in the ith row and jth column is $(-1)^{i+j}$. The cofactor of an element is usally denoted by the corresponding capital letter.

Thus a determinant is the sum of the products of the elements of any row (or column) by the corresponding cofactors. This is known as **Laplace's expansion**.

Properties of Derterminants

(i) A determinant remains unaltered if its rows and columns are interchanged.

(ii) If any two rows (or columns) of a determinant are interchanged, the determinant changes its sign.

(iii) A determinant vanishes if two of its rows (or columns) are identical or proportional.

(iv) If each element of a row (or column) is multiplied by a scalar, then the determinant is multiplied by that scalar.

(v) If to each element of a row (or column) be added equi-multiples of the corresponding elements of two or more rows (or columns), the determinant remains unaltered.

9. SINGULAR AND NONSINGULAR MATRICES

A square matrix is said to be singular matrix if determinant of the matrix is zero. Otherwise, it is called non-singular matrix.

10. ADJOINT OF A MATRIX

The transpose of the matrix of cofactors of A is known as adjoint of a matrix and denoted by adj.(A).

Thus **adj.A = (Cofactor matrix)^T**

For example,if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$$

Properties of Adjont

(i) $A(\text{adj}A) = (\text{adj}A)A = |A|I_n$

(ii) $\text{adj}(KA) = K^{n-1}(\text{adj}A)$ where K is a scalar and A is a square matrix of order n.

(iii) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

10. INVERSE OF A MATRIX

Let A be a square matrix.If there exists another matrix B exists such that $AB = BA = I$, where I is a Unit matrix, then the matrix B is called inverse of A and denoted byA^{-1}. It is defined as

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Properties of Inverse Matrix

(i) Inverse of a matrix if it exists is unique.

(ii) $AA^{-1} = A^{-1}A = I$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^{-1})^{-1} = A$, where A is non-singular matrix.

(v) $(A^T)^{-1} = (A^{-1})^T$,where A is non-singular matrix.

NOTE: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then adj$(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

11. SPECIAL MATRICES

(i) Symmetric and Skew-Symmetric Matrices

A square matrix A is said to be symmetric if $A' = A$ and Skew-Symmetric if $A' = -A$.

RESULTS:

(i) The main diagonal elements of a skew-symmetric matrix are zero's i.e., $a_{ii} = 0 \;\; \forall \;\; i$

(ii) Determinant of a Skew-symmetric matrix of odd order is zero and determinant of a Skew-symmetric matrix of even order is a perfect square.

(iii) Every square matrix A can be written uniquely as a sum of symmetric matrix and a Skew-symmetric matrix. The symmetric part is $\frac{1}{2}(A + A')$ and Skew-symmetric part is $\frac{1}{2}(A - A')$.

(iv) If A is symmetric (or skew-symmetric) then kA is also symmetric(or skew-symmetric) for any scalar k.

(ii) Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA' = A'A = I$. In other words, A is orthogonal matrix if and only if $A' = A^{-1}$.

RESULTS:

(i) If A is an orthogonal matrix, then $|A| \neq 0$. Infact $|A| = \pm 1$

(ii) If A is an orthogonal matrix, then A' is also an orthogonal matrix.

(iii) If A and B are orthogonal matrices, then AB and BA are both orthogonal matrices.

(iii) Conjugate of A Matrix

The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex number is called the conjugate of A denoted by $\bar{A}$.

Thus, if $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$.

Properties of Conjugate of a Matrix :

(i) $(\overline{\bar{A}}) = A$

(ii) $(\overline{A+B}) = \bar{A} + \bar{B}$

(iii) $(\overline{kA}) = \bar{k}\bar{A}$, where k is a scalar

(iv) $\overline{AB} = \bar{A}\bar{B}$

(v) $(\overline{A^n}) = (\bar{A})^n$

(vi) $\bar{A} = A$ if and only if A is purely real matrix

(g) $\bar{A} = -A$ if and only if A is purely imaginary matrix.

(iv) Transposed of Conjugate of a Matrix

It is the transpose of a conjugate of a matrix A i.e., $(\bar{A})'$ or $(\bar{A}')$ and denoted by A^θ.

Thus $\quad A^\theta = (\bar{A})' = (\bar{A}')$

Properties:

(i) $(A^\theta)^\theta = \theta$

(ii) $(A+B)^\theta = A^\theta + B^\theta$

(iii) $(KA)^\theta = \bar{k}A^\theta$

(iv) $(AB)^\theta = B^\theta . A^\theta$

(v) $(A^n)^\theta = (A^\theta)^n$

(v) Unitary Matrix

A square matrix A is said to be unitary matrix, if $AA^\theta = A^\theta A = I$.

Properties:

(i) If A is an unitary matrix, then A' and A^{-1} are also unitary matrices.

(ii) If A and B are two unitary matrices of same order, then AB and BA are also unitary matrices of same order.

(vi) Hermitian and Skew-Hermitian Matrices

A square matrix A is said to be Hermitian if $A^\theta = A$ and Skew-Hermitian if $A^\theta = -A$.

RESULTS:

(i) Every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix. The Hermitian part is $\frac{1}{2}(A + A^\theta)$ and Skew-Hermitian part is $\frac{1}{2}(A - A^\theta)$.

(ii) If A is Hermitian matrix, then iA is skew-Hermitian and if A is Skew-Hermitian then iA is Hermitian

(iii) If A is a Hermitian (or Skew-Hermitian), then kA is also Hermitian (or Skew-Hermitian) for any scalar k.

12. SUB MATRIX

A matrix obtained fram a given matrix by deleting some rows or columns or both is called a submatrix.

If $A = [a_{ij}]_{m\times n}$ is a matrix and B is its submatrix of order r, then $|B|$, the determinant is called the minor of A of order r. Clearly there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

13. RANK OF A MATRIX

A matrix $A = [a_{ij}]_{m\times n}$is said to be of rank r, if it satisfies the following properties:

(i) There is atleast one square submatrix of order r whose determinant is not equal to zero.

(ii) The determinant of order higher than r, i.e.,$(r + 1)$ should be zero. In other words, the rank of a matrix is the largest order of any non-vanishing minor of the matrix. The rank of a matrix A is denoted by $\rho(A)$ or $r(A)$.

NOTE: If A is a non-singular matrix of order n, then rank of $A = n$. i.e., $\rho(A) = n$.

Properties of Rank:

(i) The rank of a matrix doesnot change when the following elementary row operations are applied to the matrix.

(a) The interchange of any two rows $(R_i \leftrightarrow R_j)$

(b) The multiplication of any row by a non-zero constant $(R_i \to kR_i)$

(c) A constant multiple of another row is added to the corresponding elements of any other row $(R_i \to R_i + KR_j$, where $\quad i \neq j)$

NOTE:

1. The arrow $\rightarrow$ means "replaced by"
2. When the above three operations are applied to columns, then they are called elementary column operations.

(ii) If $A = [a_{ij}]_{m\times n}$, then $\rho(A) \leq \min\{m, n\}$

Thus $\rho(A) \leq m$ and $\rho(A) \leq n$.

(iii) If A and B are matrices of same order, then $\rho(A + B) \leq \rho(A) + \rho(B)$

(iv) (a) $\rho(A') = \rho(A)$ and $\rho(AA') = \rho(A)$

(b) $\rho(A^\theta) = \rho(A)$ and $\rho(AA^\theta) = \rho(A)$

(v) The rank of a matrix A does not change by pre-multiplication or post- multiplication with any non-singular matrix.

(vi) If A and B are matrices of same order, then $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.

Thus $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$.

(vii) The rank of a skew-symmetric matrix cannot be equal to one.

(viii) The rank of a matrix is same as the number of linearly independent row vectors in the matrix as well as the number of linearly independent column vectors in the matrix.

14. EQUIVALENT MATRIX

A matrix obtained from a given matrix by applying any of the elementary row operations is said to be equivalent to it. If A and B are two equivalent matrices, we write $A \sim B$. Note that if $A \sim B$, then $\rho(A) = \rho(B)$.

15. ECHELON FORM (OR) TRIANGULAR FORM

A matrix is said to be in Echelon form if

(i) All the non-zero rows, if any precede the zero rows.

(ii) The number of zeros preceding the first non-zero element in a row is less than the number of such zeros in the next row.

(iii) The first non-zero element in every row is unity, i.e., the elements of principal diagonal must be unity if possible.

Thus by applying the elementary row operations, we shall try to transform the given matrix into the following form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & * \end{bmatrix}$$

where $*$ stands for zero or non-zero element. That is, we shall try to make a_{ii} as 1 and all the elements below a_{ii} as zero.

Definition: The number of non-zero rows in Echelon Form of a given matrix is defined as the rank of given matrix.

i.e., $\rho(A)$ = number of non-zero rows in Echelon form of matrix A.

16. Normal Form or Canonical Form

By applying elementary row and column operations, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A:

(i) I_r (ii) $[I_r \quad O]$ (iii) $\begin{bmatrix} I_r \\ O \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

The number r so obtained from above is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I & O \\ O & O \end{bmatrix}$ is called first canonical form of A.

17. Determination of Linearly Dependent and Linearly Independent sets of Vectors by Rank Method

Let $X_1, X_2, \ldots, X_n$ be the given vectors. Construct a matrix with the given vectors as its rows.

1. If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.
2. If the rank of the matrix of the given vectors is lessthan the number of vectors, then the vectors are linearly dependent.

18. Orthogonality of Vectors

(i) Two non-zero vectors X_1 and X_2 are orthogonal if and only if $X_1^T X_2 = 0$.

(ii) Three non-zero vectors x_1, X_2, X_3 are orthogonal if and only if they are pairwise orthogonal.

PREVIOUS GATE QUESTIONS

1. The rank of $(m \times n)$ matrix $(m < n)$ cannot be more than

[GATE 1994(EC)]

(A) m (B) n (C) mn (D) None

Ans. A or B or C

SOLUTION: We know that $\rho(A_{m\times n}) \leq \min\{m, n\}$

But it is given that $m < n$

$\therefore \rho(A_{m\times n}) \leq m$. Hence $\rho(A_{m\times n})$ cannot be more than m or n or mn.

2. A 5×7 matrix has all its entries equal to 1. Then the rank of a matrix is

[GATE 1994(EE)]

(A) 7 (B) 5 (C) 1 (D) zero

Ans. (C)

SOLUTION: Let the given matrix be

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1, R_5 \to R_5 - R_1)$

This is in Echelon form.

$\therefore$ rank of A = number of non-zero rows = 1

3. The rank of the matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is **[GATE 1994(CS)]**

(A) 0 (B) 1 (C) 2 (D) 3

Ans. (C)

SOLUTION: Let $A = \begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$. Then

$$|A| = \begin{vmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{vmatrix} \qquad \text{[Expand by } R_1\text{]}$$

$$= 0 + 0 - 3(9 - 9) = 0$$

$\therefore$ Rank of $A \neq 3$ i.e., $\rho(A) \leq 2$.

Since the submatrix $\begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$ is non-singular, therefore, the rank of $A = 2$

4. If A and B are real symmetric matrices of order n then which of the following is true

[GATE 1994(CS)]

(A) A A^T = I
(B) A = A^{-1}
(C) AB = BA
(D) $(AB)^T$ = BA

Ans. (D)

SOLUTION: Since A and B are symmetric, we have $A^T = A$ and $B^T = B$. By the properties of transpose of matrices, we have $(AB)^T = B^T A^T = BA$

5. The rank of the matrix $\begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$ is 3 **[GATE 1994(ME)]**

(A) True (B) False

Ans. (B)

SOLUTION: Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$. Then

$$|A| = \begin{vmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{vmatrix}. \text{ [Expand by } R_1\text{]}$$

$$= 0 - 2(-28 + 56) + 2(0 + 28)$$

$$= -56 + 56 = 0$$

$\therefore$ Rank of $A < 3$ i.e., $\rho(A) \leq 2$.

Since the submatrix $\begin{bmatrix} 0 & 2 \\ 7 & 4 \end{bmatrix}$ is non-singular, therefore, the rank of A is 2.

6. The matrix $\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$. **[GATE 1994(PI)]**

(A) True (B) False

Ans. (A)

SOLUTION: Let $A = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$. Then

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{-5+4}\begin{bmatrix} -1 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$$

Alternate Method:

$$\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5-4 & -4+4 \\ 5-5 & -4+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$

7. The value of the determinant $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$ is **[GATE 1994(PI)]**

(A) 8 (B) 12 (C) -12 (D) -8

Ans. (D)

SOLUTION: Let $\Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

$$= 1(225 - 256) - 4(100 - 144) + 9(64 - 81)$$
$$= -31 + 176 - 153 = -184 + 176 = -8$$

8. If for a matrix, rank equals both the number of rows and number of columns, then the matrix is called **[GATE 1994(PI)]**

(A) non-singular (B) Singular (C) Transpose (D) Minor

Ans. (B)

SOLUTION: It is given that

rank of A = No. of rows of A = No. of columns of A

i.e., $\rho(A)$ = order of the square matrix.

Hence the matrix is non-singular.

9.The inverse of the matrix $S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is **[GATE 1995(EE)]**

(A) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 2 & -2 \\ -2 & 2 & -2 \\ 0 & 2 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$

Ans: (D)

SOLUTION: We can compute S^{-1} by using the formula $S^{-1} = \dfrac{\text{adj } S}{|S|}$

Alternate Method

An easier method for finding S^{-1} is by multiplying S with each of the choices (A), (B), (C) and (D) and finding out which one gives the product as Identity matrix. For examples, we multiply S with the option (D).

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2+1/2+0 & 1/2-1/2+0 & -1/2+1/2+0 \\ 1/2-1/2+0 & 1/2+1/2+0 & -1/2-1/2+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\text{Hence} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

10. The rank of the following $(n+1) \times (n+1)$ matrix, where $'a'$ is a real number is

[GATE 1995(CS)]

$$\begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ 1 & a & a^2 & \dots & a^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a & a^2 & \dots & a^n \end{bmatrix}$$

(A) 1 (B) 2 (C) n (D) depends on the value of a

Ans. (A)

SOLUTION: Applying $R_2-R_1, R_3-R_1, R_4-R_1, \dots, R_{n+1}-R_1$, the given matrix reduces to the Echelon form

$$\begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Here the number of non-zero rows = 1

Hence the rank of the given matrix is 1.

Alternate Method: All the rows of the given matrix is same. So the matrix has only one independent row. Rank of the matrix = No. of independent rows of the matrix = 1.

11. Given matrix $L = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$ and $M = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ then $L \times M$ is **[GATE 1995(PI)]**

(A) $\begin{bmatrix} 8 & 1 \\ 13 & 2 \\ 22 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 8 \\ 2 & 13 \\ 5 & 22 \end{bmatrix}$ (D) $\begin{bmatrix} 6 & 2 \\ 9 & 4 \\ 0 & 5 \end{bmatrix}$

Ans. (B)

SOLUTION:

$$LM = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}_{3\times2} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}_{2\times2} = \begin{bmatrix} 6+0 & 4+1 \\ 9+0 & 6+2 \\ 12+0 & 8+5 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{bmatrix}$$

12. The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication

[GATE 1996 (CS)]

(A) If $a = b$ (or) $\theta = n\pi$, n is an integer
(B) always
(C) never
(D) If a $\cos\theta \neq b\sin\theta$

Ans. (A)

SOLUTION: Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix} \text{ and}$$

$$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ b\sin\theta & b\cos\theta \end{bmatrix}$$

$$AB = BA \Rightarrow \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix} = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ b\sin\theta & b\cos\theta \end{bmatrix}$$

By equality of matrices, $-b\sin\theta = -a\sin\theta \Rightarrow a\sin\theta - b\sin\theta \Rightarrow sin\theta(a-b) = 0$

$\therefore \quad a - b = 0 \quad \text{or} \quad \sin\theta = 0 \Rightarrow a = b \quad \text{or} \quad \theta = n\pi$

Hence A and B commute when $a = b$ or $\theta = n\pi$, n is an integer.

13. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be two matrices such that $AB = I$. Let $C = A\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $CD = I$. Express the elements of D in terms of the elements of B.

[GATE 1996(CS)]

SOLUTION: We have $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Given $CD = I \Rightarrow D = C^{-1}$ ...(1)

$C = A\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ...(2)

and $AB = I \Rightarrow B = A^{-1}$ ...(3)

From (1) and (2), we have

$$D = \left(A\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{B, by (3)}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{22} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ -b_{11} + b_{21} & -b_{12} + b_{22} \end{bmatrix}$$

14. The determinant of the matrix $\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is **[GATE 1997(CS)]**

(A) 11 (B) -48 (C) 0 (D) -24

Ans. (D)

SOLUTION: The given matrix A is an upper triangular matrix.

$\therefore$ $|A|$ = Product of the diagonal elements $= 6(2)(4)(-1) = -24$

15. Let $A_{n\times n}$ be matrix of order n and I_{12} be the matrix obtained by interchanging the first and second rows of I_n. Then AI_{12} is such that its first **[GATE 1997 (CS)]**

(A) row is the same as its second row
(B) row is the same as the second row of A
(C) column is the same as the second column of A
(D) row is a zero row.

Ans. (C)

SOLUTION: For instance, take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Now $I_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (by $R_1 \to R_2$)

$\therefore AI_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+2 & 1+0 \\ 0+4 & 3+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

16. If the determinant of the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -6 \\ 2 & 7 & 8 \end{bmatrix}$ is 26 then the determinant of the matrix $\begin{bmatrix} 2 & 7 & 8 \\ 0 & 5 & -6 \\ 1 & 3 & 2 \end{bmatrix}$ is **[GATE 1997 (CS)]**

(A) -26 (B) 26 (C) 0 (D) 52

Ans. (B)

SOLUTION: Let $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 5 & -6 \\ 2 & 7 & 8 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 3 & 2 \\ 2 & 7 & 8 \\ 0 & 5 & -6 \end{vmatrix}$ (Applying $R_2 \leftrightarrow R_3$)

Given $\Delta = 26 \Rightarrow 26 = (-1)^2 \begin{vmatrix} 2 & 7 & 8 \\ 1 & 3 & 2 \\ 0 & 5 & -6 \end{vmatrix}$ (Applying $R_1 \leftrightarrow R_2$) $= \begin{vmatrix} 2 & 7 & 8 \\ 1 & 3 & 2 \\ 0 & 5 & -6 \end{vmatrix}$

17. If A and B are two matrices and if AB exist then BA exists **[GATE 1997 (CE)]**

(A) only if A has as many rows as B has columns
(B) only if both A and B are square matrices
(C) only if A and B are skew matrices
(D) only if both A and B are symmetric.

Ans. (A)

SOLUTION: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$

Both AB and BA exist only if $m = q$ and $n = p$.

18. Inverse of matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is **[GATE 1997 (CE)]**

(A) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Ans. (A)

SOLUTION: Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Then $|A| = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ [Expand by C_1]

$$= 0 - 0 + 1(1 - 0) = 1 \neq 0$$

$\therefore$ A^{-1} exists. Now adj $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Hence $A^{-1} = \dfrac{\text{adj}A}{|A|} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

19. Let $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ Then A^{-1} = **[GATE 1998 (EE)]**

(A) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & 0 & 2 \\ 0 & -1/3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1/5 & 0 & 1/2 \\ 0 & 1/3 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1/5 & 0 & -1/2 \\ 0 & 1/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Ans. (A)

SOLUTION:

Method 1: Proceed as in the above example.

Method 2: An easier method for finding A^{-1} is by multiplying A with each of the choices $(A), (B), (C) and (D)$ and finding out which one gives the product as identity matrix.

For example, multiply the matrix A with the choice (A)

$$\begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/3 & 0 \\ -2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 5+0-4 & 0+0+0 & -10+0+10 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0-2 & 0+0+0 & -4+0+5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence $\begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$

20. If $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ then which of the following is a factor of Δ **[GATE 1998(CS)]**

(A) $a+b$ (B) $a-b$ (C) abc (D) $a+b+c$

Ans. (B)

SOLUTION We know that if a determinant Δ becomes zero when we put $x = \alpha$, then $(x - \alpha)$is a factor of Δ.

$$\text{Given } \Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b-a & c(b-a) \\ 1 & c-a & b(c-a) \end{vmatrix} \text{(Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & c \\ 0 & 1 & b \end{vmatrix}$$

If $a = b$ or $c = a$ then Δ becomes zero $\therefore (a - b)$ is a factor of Δ.

21. The rank of the matrix $\begin{vmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{vmatrix}$ is **[GATE 1998(CS)]**

(A) 3 (B) 1 (C) 2 (D) 4

Ans. (D)

SOLUTION

$$\text{Let } A = \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 0 & -19 \end{bmatrix} \text{(Applying } R_3 \to R_3 - 4R_1, R_4 \to R_4 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -19 \end{bmatrix} \text{(Applying } R_2 \leftrightarrow R_3)$$

Now $|A| = (-19)\begin{vmatrix} 1 & 4 & 8 \\ 0 & -14 & -29 \\ 0 & 0 & 3 \end{vmatrix}$ [Expanded by R_4]

$= (-19)[0 + 0 + 3(-14 - 0)]$ [Expanded by R_3]

$= (-19)(-42) \neq 0$

Hence rank of $A = 4$ i.e., $\rho(A) = 4$

22. If A is a real square matrix then AA^T is **[GATE 1998(CE)]**

(A) unsymmetric (B) always symmetric

(C) skew-symmetric (D) sometimes symmetric

Ans. (B)

SOLUTION: We are given A is a real square matrix. We know that the matrix A is symmetric if $AA^T = A$.

Now $(AA^T)^T = (A^T)^T A^T [\because (AB)^T = B^T A^T$ (Reversal Law)]

$= AA^T \quad [\because (A^T)^T = A]$

Hence AA^T is symmetric.

23. In matrix algebra $AS = AT$ (A, S, T are matrices of appropriate order) implies $S = T$ only if **[GATE 1998(CE)]**

(A) A is symmetric (B) A is singular

(C) A is non-singular (D) A is skew-symmetric

Ans. (C)

SOLUTION: If A is non-singular, then A^{-1} exists. Thus

$AS = AT \Rightarrow A^{-1}(AS) = A^{-1}(AT) \Rightarrow (A^{-1}A)S = (A^1 A)T \Rightarrow IS = IT \Rightarrow S = T$

Hence $AS = AT$ implies $S = T$ only if A is non-singular.

24. If $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ and $\text{adj}A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix}$ then $k =$ **[GATE 1999(CS)]**

(A) -5 (B) 3 (C) -3 (D) 5

Ans. (A)

SOLUTION: We know that if $A = [a_{ij}]_{n \times n}$ then adj$(A) = [b_{ij}]_{n \times n}$ where $b_{ij} = A_{ij}$ where A_{ij} is the cofactor of $(j, i)th$ element of A.

$$\therefore \quad K = b_{32} = A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = (-1)(5-0) = -5$$

25. If A is any $n \times n$ matrix and K is a scalar then $|KA| = \alpha |A|$ where α is

[GATE 1999(CE)]

(A) kn (B) n^k (C) k^n (D) $\frac{k}{n}$

Ans. (C)

SOLUTION: Using Scalar Multiple Property of deteninant of matrices, we have

$|KA| = K^n |A|$ $\quad [\because A$ is $n \times n$ matrix$] \quad \therefore \quad \alpha = K^n$ where k is a scalar

26. The number of terms in the expansion of general determinant of order n is

[GATE 1999 (CE)]

(A) n^2 (B) $n!$ (C) n (D) $(n+1)^2$

Ans. (B)

SOLUTION: We know that the number of terms in the expansion of a determinant of order 2 is $2(= 2!)$ and of order 3 is $6(= 3!)$.

Similarly the number of terms in the expansion of a determinant of order n is $n!$.

27. The equation $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ y & x^2 & x \end{bmatrix} = 0$ represents a parabola passing through the points

(A) (0,1), (0,2), (0,−1) (B) (0,0), (−1,1), (1,2)
(C) (1,1), (0,0), (2,2) (D) (1,2), (2,1), (0,0)

Ans. (B)

SOLUTION: We have $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ y & x^2 & x \end{vmatrix} = 0$ [Expand by R_1]

$\Rightarrow 2(x+x^2)-1(x+y)+1(x^2-y) = 0 \Rightarrow 2x+2x^2-x-y+x^2-y = 0 \Rightarrow 3x^2+x-2y = 0$ which is a parabola passing through the origin.

The easier method for finding the points through which the parabola passing, substitute each of the choices (A), (B), (C) and (D) one by one and find out which one satisfies the equation of the parabola.

For example, consider the choice (B). Since all the three points $(0,0), (-1,1), (1,2)$ satisfies the equation $3x^2+x-2y = 0$, the correct answer is (B).

28. An $n \times n$ array V is defined as follows:

$V[i,j] = i - j$ for all $i, j, 1 \leq i, j \leq n$

Then the sum of the elements of the array V is **[GATE 2000 (CS)]**

(A) 0 (B) $n-1$ (C) n^2-3n+2 (D) $n(n+1)$

Ans. (A)

SOLUTION: We have $V[i,j] = i - j, 1 \leq i, j \leq n$

i.e., For $i = 1, j = 1, 2, \ldots, n$

For $i = 2, j = 1, 2, \ldots, n$

For $i = 3, j = 1, 2, \ldots, n$.

$\ldots \quad \ldots \quad \ldots$

For $i = n, j = 1, 2, \ldots, n$

$$\therefore \qquad V = \begin{bmatrix} 0 & -1 & -2 & -3 & \ldots & 1-n \\ 1 & 0 & -1 & -2 & \ldots & 2-n \\ 2 & 1 & 0 & -1 & \ldots & 3-n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n-2 & n-3 & n-4 & \ldots & 0 \end{bmatrix}_{n \times n}$$

Here V is a skew-symmetric matrix since all main diagonal elements are zeros.

$\therefore (i,j)$ th element of $V = -(j,i)$ th element of V.

Hence sum of all the elements of $V = 0$.

29. The determinant of the matrix $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$ is **[GATE 2000(CS)]**

(A) 4 (B) 0 (C) 15 (D) 20

Ans. (A)

SOLUTION: Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$ [Expand by R_1]

$$= 2\begin{vmatrix} 1 & 7 & 2 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{vmatrix} = 2[1(2-0) - 0 + 0] \text{ (By } C_1) = 4$$

30. If A,B,C are square matrices of the same order then (ABC)$^{-1}$ is equal to

[GATE 2000(CE)]

(A) $C^{-1}B^{-1}A^{-1}$ (B) $B^{-1}C^{-1}A^{-1}$
(C) $A^{-1}B^{-1}C^{-1}$ (D) $A^{-1}C^{-1}B^{-1}$

Ans. (A)

SOLUTION: By the property of the reversal law of inverse of product of three matrices A,B,C, we have $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ $[\because (AB)^{-1} = B^{-1}A^{-1}]$

31. Consider the following two statements:

(I) The maximum number of linearly independent column vectors of a matrix A is called the rank of A.

(II) If A is $n \times n$ square matrix then it will be non-singular if rank of $A = n$.

[GATE 2000(CE)

(A) Both the statements are false (B) Both the statements are true
(C) (I) is true but (II) is false (D) (I) is false but (II) is true

Ans. (B)

SOLUTION: We know that rank of a matrix is same as the number of linearly independent row vectors in the matrix as well as the number of linearly independent column vectors in the matrix.

Hence (I) is true

Also Rank of $A = n$ = order of the square matrix $\Rightarrow |A| \neq 0 \quad \therefore A$ is a non-singular.

Hence (II) is also true.

32. The rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$ is **[GATE 2000 (IN)]**

(A) 0 (B) 1 (C) 2 (D) 3

Ans. (C)

SOLUTION: $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{vmatrix} \text{(Applying } R_2 \to R_2 - 3R_1, R_3 \to R_3 - 4R_1)$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{vmatrix} \text{(Applying } R_3 \to R_3 - R_2)$$

$= 0$

$\therefore$ Rank of $A \neq 3$. So $\rho(A) \leq 2$

Since the submatrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is non-singular, the rank of $A = 2$.

33. Consider the following statements:

S1: The sum of two singular matrices may be singular

S2: The sum of two non-singular may be non-singular

which of the following statements is true **[GATE 2001(CS)]**

(A) S_1 and S_2 are both true
(B) S_1 and S_2 are both false
(C) S_1 is true and S_2 is false
(D) S_1 is false and S_2 is true

Ans. (A)

SOLUTION:

S1: (1) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix}$ where $|A| = 0$ & $|B| = 0$.

Then $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow |A + B| = 0$

(2) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ where $|A| = 0, |B| = 0$.

S2: (1) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ where $|A| \neq 0,$ & $|B| \neq 0$.

Then $A + B = \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix} \Rightarrow |A + B| \neq 0$

(2) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$ where $|A| \neq 0, |B| \neq 0$.

Then $A + B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow |A + B| = 0$

34. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$ is **[GATE 2002(EE)]**

(A) 100 (B) 200 (C) 1 (D) 300

Ans. (C)

SOLUTION: Given matrix is a lower triangular matrix

$\therefore$ Determinant of the matrix = product of the main diagonal elements =(1)(1)(1)(1) = 1.

35. The rank of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is **GATE 2002(CS)]**

(A) 4 (B) 2 (C) 1 (D) 0

Ans. (C)

SOLUTION: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

This is in Echelon form. Number of non-zero rows = 1

$\therefore \quad \rho(A) = 1$

Alternate Method: Since $|A| = 0$, the rank of $A \neq 2$. But A is a non-zero matrix. Hence $\rho(A) = 1$.

36. Given matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ the rank of the matrix is **[GATE 2003(CE)]**

(A) 4 (B) 3 (C) 2 (D) 1

Ans. (C)

SOLUTION: Since A is 3×4 matrix, therefore, the rank cannot exceed 3. Also each of the minor of order 3 is zero. Hence the rank is lessthan 3. Consider the minors of order 3.

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 2(4-0) - 1(3-7) + 3(0-4) = 8 + 4 - 12 = 0$$

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 4(0-4) - 2(0-8) + 1(6-6) = -16 + 16 = 0$$

Similarly $\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 3 \\ 2 & 0 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 4 \\ 2 & 1 & 1 \end{vmatrix} = 0$

Since $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \neq 0$, the rank of A is 2.

Alternate Method:

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{(Applying } R_2 \to 4R_2 - 6R_1, R_3 \to 2R_3 - R_1)$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 \to 10R_3 + R_2)$$

This is in Echelon form. Number of non-zero rows = 2. Hence $\rho(A) = 2$.

37. If the matrix $X = \begin{bmatrix} a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix}$ and $X^2 - X + I = O$ then the inverse of X is **[GATE 2004(EC)]**

(A) $\begin{bmatrix} 1-a & -1 \\ a^2 & a \end{bmatrix}$ (B) $\begin{bmatrix} 1-a & -1 \\ a^2 - a + 1 & a \end{bmatrix}$

(C) $\begin{bmatrix} -a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix}$ (D) $\begin{bmatrix} a^2 - a + 1 & a \\ 1 & 1 - a \end{bmatrix}$

Ans. (B)

SOLUTION: We have $X^2 - X + I = O$ ...(1)

Multiplying on both sides of (1) by X^{-1}, we get

$$X^{-1}(X^2 - X + I) = X^{-1}(O) \Rightarrow (X^{-1}X)X - X^{-1}X + X^{-1} = O$$

$$\Rightarrow IX - I + X^{-1} = O \Rightarrow X^{-1} = I - X$$

$$\therefore \quad X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix} \begin{bmatrix} 1 - a & -1 \\ a^2 - a + 1 & a \end{bmatrix}$$

38. The number of different $n \times n$ symmetric matrices with each element being either 0 or 1 is **[GATE 2004(CS)]**

(A) 2^n (B) 2^{n^2} (C) $2^{\frac{n^2+n}{2}}$ (D) $2^{\frac{n^2-n}{2}}$

Ans. (C)

SOLUTION: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$ be the symmetric matrix.

Then the total number of different elements in A is $\dfrac{n^2+n}{2}$ and each element can be filled in 2 ways with 0 or 1.

Hence the total number of different $n \times n$ symmetric matrices is $2^{(n^2+n)/2}$.

39. Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$ then B^{-1} is **[GATE 2004(CS)]**

(A) $D^{-1}C^{-1}A^{-1}$ (B) CDA (C) ADC (D) does not necessarily exist

Ans. (B)

SOLUTION: We have $ABCD = I$

$$\Rightarrow (ABCD)D^{-1}C^{-1} = ID^{-1}C^{-1} \Rightarrow ABCIC^{-1} = D^{-1}C^{-1}$$

$$\Rightarrow AB = D^{-1}C^{-1} \Rightarrow A^{-1}(AB) = A^{-1}(D^{-1}C^{-1})$$

$$\Rightarrow A^{-1}AB = A^{-1}D^{-1}C^{-1} \Rightarrow IB = A^{-1}D^{-1}C^{-1}$$

$$\Rightarrow B = A^{-1}D^{-1}C^{-1} \Rightarrow B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$$

$$= (C^{-1})^{-1}.(D^{-1})^{-1}.(A^{-1})^{-1} = CDA$$

40. In an $m \times n$ matrix such that all non-zero entries are covered in $'a'$ rows and $'b'$ columns. Then the maximum number of non-zero entries, such that no two are on the same row or column is **[GATE 2004(CS)]**

(A) $\leq a + b$ (B) $\leq \max(a, b)$ (C) $\leq \min[m - a, n - b]$ (D) $\leq \min\{a, b\}$

Ans.(D)

SOLUTION: Every entry will remove one row and one column from further consideration of availability, since no two entries should be in same row or column. Proceeding in this way we can add a maximum of either $'a'$ entries or $'b'$ entries depending on which is lesser.

If $a < b$ we will run out of rows first and if $b < a$ we will run out of columns first and if $a = b$ then we run out of both rows and columns. Therefore maximum entries that can be added $\leq$ $\min\{a, b\}$.

41. For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

[GATE 2004(ME)]

(A) 4 (B) 6 (C) 8 (D) 12

Ans. (A)

SOLUTION: For singular matrix A, we have $|A| = 0 \Rightarrow \begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$$\Rightarrow 8(0 - 12) - x(0 - 24) + 0 = 0 \Rightarrow -96 + 24x = 0$$

$\therefore \quad x = 4.$

42. Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$,$[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$, and $[F]_{5\times1}$ are given. Matrices $[B]$ and $[E]$ are symmetric. Following statements are made with respect to these matrices.

(I) Matrix product $[F]^T[C]^T[B][C][F]$ is scalar.

(II) Matrix product $[D]^T[F][D]$ is always symmetric with reference to above statements, which of the following applies? **[GATE 2004(CE)]**

(A) Statement (I) is true but (II) is false
(B) Statement (I) is false but (II) is true

(C) Both statements are true
(D) Both statements are false.

Ans.(D)

SOLUTION: Both the statements are false. Statement(I) is false because the product of two or more matrices is always a matrix and not a scalar. Statement(II) is also false since the matrix product D^TFD doesnot exist because the matrices D^T, F and D are not compatible for matrix multiplication.

43. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$. Then $a + b$ = **[GATE 2005(EC)]**

(A) $\dfrac{7}{10}$ (B) $\dfrac{3}{20}$ (C) $\dfrac{19}{60}$ (D) $\dfrac{11}{120}$

Ans. (A)

SOLUTION: Given $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{6}\begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$

But given $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix} \Rightarrow \begin{bmatrix} \dfrac{3}{6} & \dfrac{0.1}{6} \\ 0 & \dfrac{2}{6} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & a \\ 0 & b \end{bmatrix}$$

By equality of matrices, we have

$$a = \frac{0.1}{6} \text{ and } b = \frac{2}{6}$$

$$\therefore \quad a + b = \frac{0.1}{6} + \frac{2}{6} = \frac{2.1}{6} = \frac{21}{60} = \frac{7}{20}$$

44. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $[AA^T]^{-1}$ is **[GATE 2005(EC)]**

(A) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (B) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

Ans. (C)

SOLUTION: Given A is orthogonal matrix. By definition, $AA^T = I \therefore (AA^T)^{-1} = I^{-1} = I$

45. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, the top row of R^{-1} is **[GATE 2005(EE)]**

(A) $[5 \quad 6 \quad 4]$ (B) $[5 \quad -3 \quad 1]$ (C) $[2 \quad 0 \quad -1]$ (D) $[2 \quad -1 \quad 1/2]$

Ans. (B)

SOLUTION: $|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix} = 1(2+3) - 0 - 1(6-2) = 5 - 4 = 1$

We know that $R^{-1} = \frac{\text{adj}(R)}{|A|} = \text{adj}(R) \qquad [\because |R| = 1]$

$= [\text{cofactor(R)}]^T.$

Since we need only the top row of R^{-1}, we need to find only cofactors of first column of (R) which after transpose will become the first row of adj(R).

Now cofactor of $1 = (-1)^{1+1} \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$

cofactor of $2 = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = (-1)^5(0+3) = -3$

cofactor of $2 = (-1)^{1+3} \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 0 + 1 = 1$

$\therefore$ Top row of R^{-1} is $[5 \quad -3 \quad 1]$

46. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is **[GATE 2006(EC)]**

(A) 0 (B) 1 (C) 2 (D) 3

Ans.(C)

SOLUTION: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

This is in Echelon form. Number of non-zero rows = 2.

Hence rank of A = 2.

Alternate Method: $|A| = 0$ and $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \neq 0 \quad \therefore \rho(A) = 2$

47. Multiplication of matrices E and F is G. Matrices E and G are

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F? **[GATE 2006(ME)]**

(A) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. (C)

SOLUTION: Given that $EF = G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i.e., $EF = I \quad \therefore \quad F = E^{-1}\ (\because AA^{-1} = I)$

$$\text{i.e.,} F = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \left[\because A^{-1} = \frac{\text{adj}A}{|A|}\right]$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Alternate Method:

An easier method for finding F is by multiplying E with each of the options (A),(B),(C) and (D) and finding out which one gives the product as identity matrix G.

Statement for Linked Questions 48 and 49.

$$P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T, \quad \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T \text{ and } R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T \text{ are three vectors.}$$

48. An orthogonal set of vectors having a span that contains P, Q, R is **[GATE 2006 (EE)]**

(A) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ (B) $\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(C) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ (D) $\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

Ans. (A)

SOLUTION: An easier method to find the orthogonal vectors having a span than contains P, Q, R is first determine whether the siven vectors are orthogonal or not with each of the choices (A), (B), (C) and (D).

First take the choice (A): $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

These vectors are othogonal since

$$X_1^T X_2 = -24 + 6 + 18 = 0$$

Notice that the choices (B), (C), (D) are not orthogonal.

49. The following vector is linearly dependent upon the solution to the previous problems

[GATE 2006 (EE)]

(A) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (B) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$ (C) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (D) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

Ans. (B)

SOLUTION: We know taht the vectors are linearly dependent if the rank of the matrix of the given vectors is less than the number of vectors.

The choice (B): $[-2 \quad -17 \quad 30]^T$ is linearly dependent upon the solution obtained in previous question namely $[-6 \quad -3 \quad 6]^T$ and $[4 \quad -2 \quad 3]^T$ since

$$\begin{vmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{vmatrix} = -6(-60+51)+3(120+6)+6(-68+4)=0$$

$\therefore$ Rank is less than 3. Hence the vectors are linearly dependent.

50. $q_1, q_2, q_3, \ldots, q_n$ are n-dimensional vectors with $m < n$. This set of vectors is linearly dependent. Q is the matrix with $q_1, q_2, \ldots, q_m$ as the columns. The rank of Q is

[GATE 2007]

(A) lessthan m (B) m (C) between m and n (D) n

Ans. (A)

SOLUTION We know that if the rank of the matrix of the given vectors is less than the number of vectors then the vectors are linearly dependent. We are given $Q = [q_1, q_2, q_3, \ldots, q_m]$ where $q_1, q_2, \ldots, q_m$ are dependent vectors.

Hence rank of $Q < m$(= no. of vectors).

51. It is given that $X_1, X_2, \ldots, X_M$ are M non-zero orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M$ is

[GATE 2007(EC)]

(A) 2M
(B) M + 1
(C) M
(D) dependent on the choice of $X_1, X_2, \ldots, X_M$

Ans. (C)

SOLUTION: Since $(X_1, X_2, \ldots, X_M)$ are orthogonal, they span a vector space of dimension M.

Since $(-X_1, -X_2, \ldots, -X_M)$ are linearly dependent on $(X_1, X_2, \ldots, X_M)$, the set $(X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M)$ will also span a vector space of dimension M only.

52. Consider the set of (column) vectors defined by $X = \left\{X \in R^3 | x_1 + x_2 + x_3 = 0, \text{ where } X^T = [x_1, x_2, x_3]^T\right\}$. which of the following is TRUE? **[GATE 2007(CS)]**

(A) $\left\{[1, -1, 0]^T, [1, 0, -1]^T\right\}$ is a basis for the subspace X.

(B) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X.

(C) X is not a subspace of R^3.

(D) None of the above.

Ans. (A)

SOLUTION By definition, a set of vectors is said to be a basis of subspace, if the set is linearly independent and the subspace is spanned by the set. Given set is $X = \{x \in R^3 | x_1 + x_2 + x_3 = 0\}$ and $X^T = [x_1, x_2, x_3]^T$

Now $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set because one cannot be obtained from another by scalar multiplication. An easier method to find the fact that it is independent is rank of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is 2.

Now we need to check if the set spans X, where $X = \{x \in R^3 | x_1 + x_2 + x_3 = 0\}$.

The general infinite solution of $X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$.

Choosing k_1, k_2 as $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$, we get two linearly independent solutions for X,

$$X = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} \quad \text{or} \quad X = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$$

Now the set spans X, since both of these can be generated by linear combinations of $[1, -1, 0]^T$ and $[1, 0, -1]^T$, Hence the set is a basis for the subspace X.

53. $X = [x_1, x_2, \ldots, x_n]^T$ is an n-tuple non-zero vector. The $n \times n$ matrix $V = XX^T$

[GATE 2007(EE)]

(A) has rank zero (B) has rank 1 (C) is orthogonal (D) has rank n

Ans. (B)

SOLUTION: We have $V = XX^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} [x_1 \ x_2 \ \ldots \ x_n]_{1 \times n}$

$$= \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ x_2x_1 & x_2^2 & x_2x_3 & \dots & x_2x_n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_nx_1 & x_nx_2 & x_nx_3 & \dots & x_n^2 \end{bmatrix}_{n\times n}$$

$$\sim \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

(Applying $R_2 \to \frac{R_2}{x_2} - \frac{R_1}{x_1}, R_3 \to \frac{R_3}{x_3} - \frac{R_1}{x_1}, \dots, R_n \to \frac{R_n}{x_n} - \frac{R_1}{x_1}$)

Hence $\rho(V) = 1$, since $x_1^2 \neq 0$

Alternate Method: We have $\rho(X_{n\times 1}) = 1$ and $\rho(X^T_{1\times n}) = 1$

$\therefore \rho(V) = 1$ [$\because \rho(AB) \leq \min\{\rho(A), \rho(B)\}$]

54. The inverse of the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is **[GATE 2007(CE)]**

(A) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (B) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$ (C) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (D) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

Ans.(A)

SOLUTION: We know that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}^{-1} = \frac{1}{7-10}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$$

55. Let $A = [a_{ij}], 1 \leq i, j \leq n$ with $n \geq 3$ and $a_{ij} = i.j$. Then the rank of A is

[GATE 2007(IN)]

(A) 0 (B) 1 (C) $n-1$ (D) n

Ans. (B)

SOLUTION: Given $A = [a_{ij}], 1 \leq i, j \leq n$ and $a_{ij} = i.j$.

$$\therefore \quad A = \begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & 4 & 6 & \ldots & 2n \\ 3 & 6 & 9 & \ldots & 3n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ n & 2n & 3n & \ldots & n^2 \end{bmatrix}$$

Hence $\rho(A) = 1$ since all rows are proportional.

56. The determinant $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$ equals to **[GATE 2007(PI)]**

(A) 0 (B) $2b(b-1)$ (C) $2(1-b)(1+2b)$ (D) $3b(1+b)$

Ans. (A)

SOLUTION: Let $\Delta = \begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$

$$= \begin{vmatrix} 2+2b & b & 1 \\ 2+2b & 1+b & 1 \\ 2+2b & 2b & 1 \end{vmatrix} \text{(Applying} C_1 \rightarrow C_1 + C_2 + C_3)$$

$$= (2+2b) \begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

$$= (2+2b)(0) \quad [\because \quad C_1 \text{ and } C_3 \text{ are identical}]$$

$= 0$

57. A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. let matrix $A^+ = (A^T A)^{-1} A^T$. Then which one of the following statement is FALSE?

[GATE 2008 (EE)]

(A) $AA^+A = A$ (B) $(AA^T)^2 = AA^+$ (C) $A^+A = I$ (D) $AA^+A = A^+$

Ans. (D)

SOLUTION: Given $A^+ = (A^T A)^{-1} A^T$. Consider the choice (A).

$$AA^+A = A[(A^TA)^{-1}A^T]A = A(A^TA)^{-1}(A^TA) = AP^{-1}P, \text{ where } P = A^TA$$
$$= AI = A$$

which is correct. So the choice (D) is not correct

58. If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?

[GATE 2008(EE)]

(A) Q will have four linearly independent rows and four linearly independent columns.
(B) Q will have four linearly independent rows and five linearly independent columns.
(C) QQ^T will be invertible.
(D) Q^TQ will be invertible.

Ans. (A)

SOLUTION: We know that the rank of a matrix is equal to the maximum number of its linearly independent rows and also to the maximum number of its linearly independent columns.

Since $\rho(Q) = 4$, therefore, Q will have four linearly independent rows and four linearly independent columns.

59. Let P be 2×2 real orthogonal matrix and $\bar{x}$ is a real vector $[x_1, x_2]^T$ with length $||x|| = (x_1^2 + x_2^2)^{1/2}$. Then which one of the following statement is correct? **[GATE 2008]**

(A) $||P\bar{x}|| \leq ||\bar{x}||$ where atleast one vector satisfies $||P\bar{x}|| < ||\bar{x}||$
(B) $||P\bar{x}|| \leq ||\bar{x}||$ for all vectors $\bar{x}$
(C) $||P\bar{x}|| = ||\bar{x}||$ when atleast one vector satisfies $||P\bar{x}|| < ||\bar{x}||$.
(D) No relationship can be established between $||\bar{x}||$ and $||P\bar{x}||$.

Ans.(B)

SOLUTION: Let us consider the orthogonal matrix, $P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Then

$$P\bar{x} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

Now $||P\bar{x}|| = \sqrt{(x_1\cos\theta + x_2\sin\theta)^2 + (-x_1\sin\theta + x_2\cos\theta)^2}$

$$= [x_1^2\cos^2\theta + 2x_1x_2\sin\theta\cos\theta + x_2^2\sin^2\theta + x_1^2\sin^2\theta$$

$$-2x_1x_2 \sin\theta\cos\theta + x_2^2\cos^2\theta]^{1/2}$$

$$= [x_1^2(\cos^2\theta + \sin^2\theta) + x_2^2(\sin^2\theta + \cos^2\theta]^{1/2}$$

$$= \sqrt{x_1^2 + x_2^2} = ||\bar{x}||.$$

Hence $||P\bar{x}|| = ||\bar{x}||$ for every vector.

60. The product of matrices $(PQ)^{-1}P$ is **[GATE 2008 (CE)]**

(A) P^{-1} (B) Q^{-1} (C) $P^{-1}Q^{-1}P$ (D) PQP^{-1}

Ans.(B)

SOLUTION: $(PQ)^{-1}P = (Q^{-1}P^{-1})P = Q^{-1}(P^{-1}P) = Q^{-1}I = Q^{-1}$

61. The inverse of matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is **[GATE 2008(PI)]**

(A) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

Ans. (A)

SOLUTION: We can find the inverse of the given matrix A by using the formula $A^{-1} = \frac{adjA}{|A|}$

Alternate Method: The given matrix A is an elementary matrix since it can be obtained from the unit matrix I_3 by interchanging R_1 and R_2. Hence the inverse matrix corresponding to the elementary matrix A is itself.

62. A square matrix B is skew-symmetric if **[GATE 2009 (CE)]**

(A) $B^T = -B$ (B) $B^T = B$ (C) $B^{-1} = B$ (D) $B^{-1} = B^T$

Ans.(A)

SOLUTION: A square matrix B is said to be skew-symmetric if $B^T = -B$

63. For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix $[M]^T = [M]^{-1}$. The value of x is given by **[GATE 2009(ME)]**

(A) $\frac{-4}{5}$ (B) $\frac{-3}{5}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$

Ans.(A)

SOLUTION: Given that

$$[M]=\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix} \text{ and } [M]^T=[M]^{-1} \Rightarrow MM^T=I \Rightarrow \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equality of matrices, we get

$$\frac{3}{5}x + \frac{12}{25} = 0 \Rightarrow \frac{3}{5}x = \frac{-12}{25} \therefore x = \frac{-12}{25}\left(\frac{5}{3}\right) = \frac{-4}{5}$$

64. The value of the determinant $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$ is **[GATE 2009(PI)]**

(A) -28 (B) -24 (C) 32 (D) 36

Ans. (B)

SOLUTION: Let $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$. Then

$$\Delta = 1(3-1) - 3(12-2) + 2(4-2) = 2 - 30 + 4 = -24$$

65. The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is **[GATE 2010(CE)]**

(A) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (B) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(C) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (D) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

Ans.(B)

SOLUTION: We know that the short-cut formula for a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence

$$\begin{bmatrix} 3+2i & i \\ i & 3-2i \end{bmatrix}^{-1} = \frac{1}{(3+2i)(3-2i)+i^2}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

$$= \frac{1}{9+4-1}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

66. X and Y are non-zero square matrices of size $n \times n$. If $XY = O_{n\times n}$. Then

[GATE 2010 (IN)]

(A) $|X| = 0$ and $|Y| \neq 0$ (B) $|X| \neq 0$ and $|Y| = 0$

(C) $|X| = 0$ and $|Y| = 0$ (D) $|X| \neq 0$ and $|Y| \neq 0$

Ans. (C)

SOLUTION: If product of two non-zero square matrices is zero matrix, then both matrices are singular.

67. The two vectors $[1,1,1]$ and $[1,a,a^2]$ where $a = \left(\frac{-1}{2} + j\frac{\sqrt{3}}{2}\right)$ are **[GATE 2011(EE)]**

(A) orthonormal (B) orthogonal (C) parallel (D) collinear

Ans.(B)

SOLUTION: Let $X_1 = [1 \quad 1 \quad 1]$ and $X_2 = [1 \quad a \quad a^2]$.Then

$$X_1^T X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & a & a^2 \end{bmatrix} = 1 + a + a^2 = 0 \quad \left[\because\ a = \frac{-1}{2} + j\frac{\sqrt{3}}{2}, a^2 = \frac{-1}{2} - j\frac{\sqrt{3}}{2}\right]$$

Hence X_1 and X_2 are orthogonal vectors.

68. [A] is a square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and differences of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$ respectively. which of the following statements is true? **[GATE 2011(CE)]**

(A) Both $[S]$ and $[D]$ are symmetric.

(B) Both $[S]$ and $[D]$ are skew-symmetric.

(C) $[S]$ is skew-symmetric and $[D]$ is symmetic.

(D) $[S]$ is symmetric and $[D]$ is skew-symmetic.

Ans. (D)

SOLUTION: We know that every square matric can be expressed as the sum of symmetric matrix and skew-symmetric matrix.

$$\therefore \quad A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = [S] + [D].$$

= symmetric matrix + skew-symmetric matrix.

69. What is the rank of the following matrix?

$$\begin{pmatrix} 5 & 3 & -1 \\ 6 & 2 & -4 \\ 14 & 10 & 0 \end{pmatrix}$$ **[GATE2012(BT)]**

(A) 0 (B) 1 (C) 2 (D) 3

Ans. (C)

SOLUTION: Let $A = \begin{pmatrix} 5 & 3 & -1 \\ 6 & 2 & -4 \\ 14 & 10 & 0 \end{pmatrix}$. Then

$$|A| = \begin{vmatrix} 5 & 3 & -1 \\ 6 & 2 & -4 \\ 14 & 10 & 0 \end{vmatrix} \text{[Expand by } R_1\text{]}$$

$$= 5(0 + 40) - 3(0 + 56) - 1(60 - 28) = 200 - 168 - 32 = 0$$

So rank$(A) \neq 3$. Consider 2×2 submatrix of A.

Since $\begin{vmatrix} 5 & 3 \\ 6 & 2 \end{vmatrix} = 10 - 18 = -8 \neq 0$

$\therefore$ rank $(A) = 2$

70. A square matrix is singular whenever: **[GATE 1987]**

(A) The rows are linearly independent
(B) The columns are linearly independent
(C) The rows are linearly dependent
(D) None of the above

Ans. (C)

SOLUTION: If the rows of a square matrix are linearly dependent, then the determinant of matrix becomes zero. Therefore, the matrix is singular if the rows are linearly dependent.

71. Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that

determinant $(I_m + AB)$ = determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

[GATE 2013 (EC)]

(A) 2 (B) 5 (C) 8 (D) 16

Ans.(B)

SOLUTION: Let us consider the matrices $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}_{1\times 4}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{4\times 1}$

Here $m = 1$ and $n = 4$

Now $AB = [1+1+1+1] = [4]$ and $BA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Given det $(I_m + AB) = \det(I_n + BA) \Rightarrow \det(I_1 + AB) = \det(I_4 + BA)$

$$\Rightarrow \det([1]+[4]) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right)$$

$$\Rightarrow \det([5]) = \det\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Hence $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 5$

72. Which one of the following does NOT equal $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$? **[GATE 2013(CS)]**

(A) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$ (B) $\begin{vmatrix} 1 & (x+1) & x^2+1 \\ 1 & (y+1) & y^2+1 \\ 1 & (z+1) & z^2+1 \end{vmatrix}$

(C) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$ (D) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

Ans.(A)

SOLUTION: By the property of the determinants, if the elements of a row of a determinant are added m times the corresponding elements of another row, the value of determinant thus obtained is equal to the value of original determinant.

With this property given determinant is equal to the determinants given in options $(B), (C) and (D)$.

73. The dimension of the null space of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ **[GATE 2013(IN)]**

(A) 0 (B) 1 (C) 2 (D) 3

Ans.(B)

SOLUTION: $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = 0 - 1(-1+0) + 1(0-1) = 1 - 1 = 0$

$\therefore \quad \rho(A) \leq 3.$

Since $\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \neq 0, \rho(A) = 2$

Hence dimension of the null space of $A = 3 - 2 = 1$.

74. There are three matrices $P(4 \times 2), Q(2 \times 4)$ and $R(4 \times 1)$. The minimum of multiplication required to compute the matrix PQR is **[GATE 2013(CE)]**

Ans. (16)

SOLUTION: The multiplications required to compute the matrix $Q_{2\times4} \times R_{4\times1}$ is 8.

$\therefore$ The minimum number of multiplication required to compute the matrix $P_{4\times2} \times QR_{2\times1}$

$$= 8 + 8 = 16.$$

75. If the A- matrix of the state space model of a SISO linear time invariant system is rank deficient, the transfer function of the system must have **[GATE 2013(IN)]**

(A) a pole with a positive real part (B) a pole with a negative real part
(C) a pole with a positive imaginary part (D) a pole at the origin

Ans. (D)

76. For matrices of same dimension M, N and scalar C, which one of these properties DOES NOT ALWAYS hold? **[GATE 2014(EC-Set 1)]**

(A) $(M^T)^T = M$ (B) $(CM)^T = C(M)^T$

(C) $(M+N)^T = M^T + N^T$ (D) $MN = NM$

Ans. (D)

SOLUTION: In general product of two matrices is not commutative i.e., $MN \neq NM$. But if M and N are two diagonal matrices of the same order, then $MN = NM$.

77. The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is **[GATE 2014(EC-Set 2)]**

Ans. 200

SOLUTION: Given $|A| = 5$ and $|B| = 40$

$\therefore |AB| = |A||B| = (5)(40) = 200$

78. Given that the deteninant of tha matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is -12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is **[GATE 2014(ME-Set 1)]**

(A) -96 (B) -24 (C) 24 (D) 96

SOLUTION: Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$. Then

we have $|A| = -12$ and $B = 2A$

$\therefore |B| = |2A| = 2^3|A| = 8(-12) = -96$

79. Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product K^TJK is

[GATE 2014(CE-Set 1)]

SOLUTION:

$$K^TJK = [1 \quad 2 \quad -1]_{1\times3} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times1}$$

$$= [3+4-1 \quad 2+8-2 \quad 1+4-6]_{1\times3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times1}$$

$$= [6 \quad 8 \quad -1]_{1\times3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{3\times1} = [6+16+1] = [23]$$

80. The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is **[GATE 2014(CE-Set 2)]**

Ans. 88

SOLUTION: Given matrix $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is symmetric

$$\therefore \quad |A| = (-1)\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} \text{[Expanding by } R_2\text{]}$$

$$= (-1)[1(0-1) - 2(6-0) + 3(3-0)] - 3[0 - 1(4-3) + 3(0-9)$$

$$= (-1)[-1 - 12 + 19] - 3(-1 - 27)$$

$$= 4 + 84 = 88$$

81. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is **[GATE 2014(CE-Set 2)]**

Ans. 2

SOLUTION: Given matrix A is a 3×4 matrix. So $\rho(A) \leq 3$.

Consider the minors of order 3.

$$\begin{vmatrix} 0 & 4 & 4 \\ 14 & 8 & 18 \\ -14 & 0 & -10 \end{vmatrix} = 0 - 4(-140 + 252) + 4(0 + 112) = -4(112) + 4(112) = 0$$

Similarly $\begin{vmatrix} 6 & 0 & 4 \\ -2 & 14 & 8 \\ 14 & -14 & 0 \end{vmatrix} = 6(0 + 112) - 0 + 4(28 - 196) = 672 - 672 = 0$

Since $\begin{vmatrix} 6 & 0 \\ -2 & 14 \end{vmatrix} \neq 0$, the rank of A is 2.

82. If the matrix A is such that $A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \quad 9 \quad 5]$ then the determinant of A is equal to

[GATE 2014(CS-Set 2)]

Ans. 0

SOLUTION: $A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}_{3\times 1} [1 \quad 9 \quad 5]_{1\times 3} = \begin{bmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{vmatrix} = \begin{vmatrix} 2 & 18 & 10 \\ 0 & 0 & 0 \\ 7 & 63 & 35 \end{vmatrix} \text{(Applying } R_2 \to R_2 + 2R_1\text{)}$$

$$= 0$$

83. Consider the matrix $J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where α is a non-negative real number. The value of α for which $det(P) = 0$ is **[GATE 2014(EC-Set 1)]**

Ans. 1

SOLUTION: (i) Let $P = I_2 + \alpha J_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$

$\therefore |P| = 1 - \alpha^2$

(ii) Let $P = I_4 + \alpha J_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix}$

$$\therefore |P| = \begin{vmatrix} 1 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \alpha \begin{vmatrix} 0 & 1 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 0 & 0 \end{vmatrix}$$

$$= (1-\alpha^2) - \alpha^2(1-\alpha^2) = (1-\alpha^2)(1-\alpha^2) = (1-\alpha^2)^2$$

(iii) Similarly if $P = I_6 + \alpha J_6$ then

$|P| = (1-\alpha^2)^3$. Given $det(P) = 0 \Rightarrow \alpha = \pm 1 \therefore \alpha = 1$ (since $\alpha > 0$)

84. Which one of the following equations is a correct identity for arbitrary 3×3 real matrices P, Q and R ? **[GATE 2014(ME - Set 4)]**

(A) $P(Q+R) = PQ + RP$ (B) $(P-Q)^2 = P^2 - 2PQ + Q^2$
(C) $det(P+Q) = detP + detQ$ (D) $(P+Q)^2 = P^2 + PQ + QP + Q^2$

Ans. (D)

SOLUTION: Multiplicatication of matrices in general is not commutative.

85. Two matrices A and B are given below: $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$; $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$

If the rank of matrix A is N, then the rank of matrix B is **[GATE 2014(EE - Set 3)]**

(A) $N/2$ (B) $N-1$ (C) N (D) $2N$

Ans. (C)

SOLUTION: We know that rank of a matrix is unaltered by applying the elementary Row (or column) operations. Here the matrix B is obtained from the matrix A by applying the elementary operations. ($C_1 \to C_1 p + C_2 q$ and $C_2 \to C_1 r + C_2 s$). Since the rank of A is N, therefore, the rank of B is also N.

86. If V_1 and V_2 are 4-dimensional subspaces of a 6-dimensional vector space V, then the smallest possible dimension of $V_1 \cap V_2$ is **[GATE 2014(CS - Set 3)]**

Ans. 2

SOLUTION: Let the basis of V be $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

In order for $V_1 \cap V_2$ to have smallest possible dimension, let V_1 and V_2 be respectively $\{e_1, e_2, e_3, e_4\}$ and $\{e_3, e_4, e_5, e_6\}$.

$\therefore$ The basis of $V_1 \cap V_2$ is $\{e_3, e_4\}$.

Hence the smallest possible dimension is 2.