Numerical computation of wave propagation in cosmological spacetimes

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1 Motivation and Introduction

1.1 Motivation

- Gravitational waves give us profound insight into astrophysical phenomena in our universe that were hidden to observation via the electromagnetic spectrum.
- Modeling of gravitational waves and their numerical calculations are generally performed in asymptotically flat spacetimes. However, astronomical evidence indicates that our universe undergoes accelerated expansion [1, 2]. It is therefore important to quantitatively understand the impact of the cosmological constant on the propagation of gravitational waves in accelerated universes.
- Given the size of the cosmological constant, most calculations estimate the impact of the accelerated expansion on gravitational waves to be small. However, future generations of gravitational wave detectors such as the Big Bang Observer may need to incorporate cosmological effects into the gravitational wave templates.

1.2 Introduction

- We develop numerical tools adopted to compute wave propagation accurately in asymptotically de Sitter spacetimes by adopting the hyperboloidal framework.
- Result 1: 1a) Compare numerical efficiency of different choices of foliation; 1b) Study impact of parameters on accuracy; 1c) Suggest an improvement that adjusts foliation to location of sources (small black holes represented as particles or Green functions).
- Result 2: Demonstrate the impact of the cosmological constant on certain features of a) scalar; b) gravitational wave propagation in an idealized setting. The code and all computations are publicly available on GitHub.

2 Hyperboloidal foliations in spherical symmetry

We restrict our discussion to 4 dimensional, spherically symmetric spacetimes. The metric can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\omega^2 \,, \tag{1}$$

where $d\omega^2$ is the standard metric on the unit sphere. We are interested in comparing different coordinate choices for the above metric that are best suited for numerical computations of wave propagation problems. We introduce a new time coordinate that preserves the timelike Killing field

$$\tau = t + h(r),$$

where $h(r)$ is the height function. As a consequence, the time direction is a Killing vector. The metric becomes

$$ds^2 = -f d\tau^2 + 2fH d\tau dr + \frac{1 - f^2H^2}{f} dr^2 + r^2 d\omega^2,$$

where $H := dh/dr$. And the scalar wave equation then becomes (see appendix)

$$\frac{1 - f^2H^2}{f}\partial_\tau^2 u = 2fH\partial_r\partial_\tau u + f\partial_r^2 u + (fH)'\partial_\tau u + f'\partial_r u - \frac{1}{r^2}(rf' + \ell^2)u. \tag{2}$$

For the regularity of the metric, we require $1 - f^2H^2 \sim f$ near the zero points of f. Enforcing the radial components of the metric to vanish, $g_{rr} = 0$ (or the non-diagonal components to be unity, $g_{\tau r} = \pm 1$), gives in- and outgoing null coordinates by $fH = \pm 1$, or $h(r) = \pm dr/f$. It is convenient to define the tortoise coordinate as $r_* := \int dr/f$ and write the in- and outgoing null coordinates as

$$u = t - r_*, \qquad v = t + r_*.$$

2.1 Examples

Maybe we talk about the historical examples here, giving some context for the Eddington and Painlevé choices below.

3 de Sitter

The de Sitter (dS) universe is the simplest solution of the Einstein equations with cosmological constant, $\Lambda > 0$. The de Sitter metric on its static patch has the form (1) with

$$f = 1 - \frac{r^2}{L^2}.$$

The only length scale, L, in this universe is related to the cosmological constant via $\Lambda = 3/L^2$. The tortoise coordinate is

$$r_* = \int \frac{dr}{f} = L \operatorname{arctanh} \frac{r}{L} = \frac{1}{2}L\left[\ln\left(1 + \frac{r}{L}\right) - ln\left(1 - \frac{r}{L}\right)\right].$$

In- and outgoing null rays are given by

$$u = t - r_* = t - L\,\mathrm{arctanh}\,\frac{r}{L}, \qquad v = t + r_* = t + L\,\mathrm{arctanh}\,\frac{r}{L}.$$

We want to construct a time slicing that approaches the cosmological horizon to the future, so that the time coordinate resembles the outgoing null coordinate, u, as we approach the cosmological horizon, $r \to L$.

To adapt the foliation to a single horizon, we can use well-known horizon-penetrating coordinates from history. Painlevé coordinates are obtained by requiring that the time slices are Euclidean with $g_{rr} = 1$. In terms of the metric (1), we get the condition

$$fH_P = \pm\sqrt{1-f},$$

with metric

$$ds_P^2 = -f d\tau^2 \pm 2\sqrt{1-f}d\tau dr + dr^2 + r^2 d\omega^2.$$

Chosing the negative sign in de Sitter space, we get

$$fH_P = -\frac{r}{L} \implies h_P(r) = \frac{L}{2}\ln f + L\ln L,$$

with metric

$$ds_P^2 = -f d\tau^2 - \frac{2r}{L}d\tau dr + dr^2 + r^2 d\omega^2.$$

The equation (2) takes the form

$$\partial_\tau^2 u = \frac{2r}{L}\partial_\tau\partial_u + \left(1 - \frac{r^2}{L^2}\right)\partial_r^2 u - \frac{1}{L}\partial_\tau u - \frac{2r}{L^2}\partial_r u - \left(\frac{2}{L^2} - \frac{\ell^2}{r^2}\right)u \qquad (3)$$

These coordinates were first discussed by Parikh [3] (see also [4, 5]). The main advantage of this form of the de Sitter metric is that the constant-time slices are flat.

The outgoing radial null rays read, up to a constant

$$u = \tau - L\ln\left(1 + \frac{r}{L}\right).$$

Another choice is designed such that outgoing null rays take the particularly simple form of $u = \tau - r$. Imposing this condition on the height function, we obtain

$$fH_E = -\frac{r^2}{L^2} \implies h_E(r) = \int \frac{r^2}{L^2 - r^2}dr = r - L\,\mathrm{arctanh}\,\frac{r}{L},$$

with metric

$$ds_E^2 = -f d\tau^2 - \frac{2r^2}{L^2}d\tau dr + \left(1 + \frac{r^2}{L^2}\right)dr^2 + r^2 d\omega^2.$$

4 Schwarzschild-de Sitter

The case of the Schwarzschild-de Sitter is more complicated due to the presence of two horizons and two scales: M and L. The Schwarzschild-de Sitter metric on its static patch has the form (1) with

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{L^2},$$

where M is the black-hole mass and L is the cosmological length scale related to the cosmological constant Λ as before via $\Lambda = 3/L^2$.

The Schwarzschild-de Sitter metric becomes singular at the three roots of $f(r)$. We denote the two positive roots as r_b and r_c. The third root is negative given by $-(r_b + r_c)$. We think of r_b as the black hole horizon, r_c as the cosmological horizon, and we are interested in the domain $r \in [r_b, r_c]$. With $0 < r_b < r_c$ and $r_0 = -(r_b + r_c)$, we can write the function f in terms of its roots as

$$f = \frac{1}{L^2 r}(r - r_b)(r_c - r)(r - r_0).$$

The SdS metric is a two-parameter family. We can either use M and L as free parameters, or r_b and r_c. The relationship between these parametrizations is

$$L^2 = r_b^2 + r_b\, r_c + r_c^2 M = \frac{r_b r_c (r_b + r_c)}{2L^2}.$$

It is convenient to define the numerical grid through a prescribed r_b and r_c.

To write down the expression for the tortoise coordinate, it is helpful to define the following quantity associated with each root r_i of $f(r)$ as $\kappa_i = \frac{1}{2}\left|\frac{df}{dr}\right|_{r=r_i}$. Then

$$\kappa_b = \frac{(r_c - r_b)\,(r_b - r_0)}{2\,L^2\, r_b}\,, \kappa_c = \frac{(r_c - r_b)\,(r_c - r_0)}{2\,L^2\, r_c}\,, \kappa_0 = -\frac{(r_b - r_0)\,(r_c - r_0)}{2\,L^2\, r_0}\,.$$

We can then compute the tortoise coordinate as

$$r_* = \int \frac{dr}{f(r)} = \frac{1}{2\kappa_b} \ln\left|\frac{r}{r_b} - 1\right| - \frac{1}{2\kappa_c} \ln\left|1 - \frac{r}{r_c}\right| + \frac{1}{2\kappa_0} \ln\left|\frac{r}{r_0} - 1\right|\,.$$

4.1 Slow-roll coordinates

[6, 7, 8, 9]

4.2 QNM coordinates

$$h(r) = \frac{1}{\kappa_b} \ln|r - r_b| + \frac{1}{\kappa_c} \ln|r - r_c|.$$

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A Scalar wave equation

Consider the wave equation

$$\Box\Psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Psi = 0$$

for the metric (1) it takes the form

$$\frac{1}{r^2\sin\theta^2}\partial_\phi^2\Psi + \frac{1}{r^2}\partial_\theta^2\Psi + \frac{\cot\theta}{r^2}\partial_\theta\Psi + \left(\frac{2f}{r} + f'\right)\partial_r\Psi + f\partial_r^2\Psi - \frac{1}{f}\partial_t^2\Psi = 0$$

Decomposing in spherical modes as

$$\Psi(t,r,\theta,\phi) = \sum_{l=0}^{m}\sum_{m=-l}^{l}\psi_{lm}(t,r)Y_{lm}(\theta,\phi)$$

results in the following equation for radial modes $\psi_{lm}(t,r)$ which we will simply call ψ

$$\partial_t^2\psi = f^2\partial_r^2\psi + f\big(\frac{2f}{r} + f'\big)\partial_r\psi - \frac{f\ell^2}{r^2}\psi.$$

where $\ell^2 = l(l+1)$ Rescaling by r as $u := \psi/r$ to get

$$\partial_t^2 u = f^2\partial_r^2 u + ff'\partial_r u - \frac{f}{r^2}\big(f'r - \ell^2\big)u.$$

Now performing time transformation yileds the following wave equation

$$\frac{1-f^2H^2}{f}\partial_\tau^2 u = 2fH\partial_r\partial_\tau u + f\partial_r^2 u + (fH)'\partial_\tau u + f'\partial_r u - \frac{1}{r^2}(rf' + \ell^2)u.$$