1 Wave equation on de Sitter spacetime

1.1 Treatment of the origin

We want to solve the following equation for the unknown $u(r,t)$ using finite differences

$$\partial_t u = \partial_r u + \frac{2}{r} u, \tag{1}$$

on the domain $r \in [0, R]$. Spherical symmetry implies the following boundary condition at the origin

$$\partial_r u(0,t) = 0. \tag{2}$$

One might be inclined to directly discretize Eq. (1) and then impose the boundary condition Eq. (2) in a one-sided derivative operator. It turns out that this approach is numerically unstable. Instead, we use the following identity

$$\frac{u}{r} = \partial_r u - r\,\partial_r\left(\frac{u}{r}\right),$$

to rewrite Eq. (1) as

$$\partial_t u = 3\partial_r u - 2r\,\partial_r\left(\frac{u}{r}\right).$$

Of course, the division by r is still there, but now the discretization with the boundary condition (2) is stable.

2 Wave equation on Schwarzschild-de Sitter spacetime

The Schwarzschild-de Sitter (SdS) metric on the static patch

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\omega^2, \qquad f(r) = 1 - \frac{r^2}{L^2} - \frac{2M}{r}. \tag{3}$$

The metric is singular at the roots of f. Assuming $0 < r_e < r_c$ and setting $r_0 = -(r_e + r_c)$, we write

$$f = \frac{1}{L^2 r}(r - r_e)(r_c - r)(r - r_0), \quad L^2 = r_e^2 + r_e r_c + r_c^2. \tag{4}$$

Hyperboloidal coordinates

Introduce hyperboloidal time τ as usual with the height function $h(r)$ and boost $H(r)$

$$\tau = t - h(r), \qquad H(r) := \frac{dh}{dr}.$$

The hyperboloidal SdS metric reads

$$ds^2 = -f d\tau^2 - 2fH d\tau dr + \frac{1}{f}\left(1 - f^2H^2\right) dr^2 + r^2 d\omega^2. \tag{5}$$

We use the freedom in H to remove the singularity of the metric by requiring $1 - f^2H^2 \sim f$ near the roots of f. There are many choices available to achieve this. We need a choice that has good numerical properties. For example, the characteristic speeds should be reasonable.

The characteristic speeds of spherical light rays can be obtained via setting $ds^2 = 0$ and solving for $c_\pm = dr/d\tau$ which satisfies

$$\frac{1}{f}(1 - f^2H^2)c_\pm^2 - 2fHc_\pm - f = 0.$$

We get

$$c_\pm = -\beta \pm \frac{\alpha}{\gamma} = \alpha^2(fH \pm 1) = \frac{f}{\mp 1 + fH}.$$

By construction, c_+ vanishes at the left boundary and c_- vanishes at the right boundary. We also need $c_\pm$ to have "reasonable" finite values at their respective boundaries when they do not vanish.

Choice 1:

$$fH = 2\frac{r - r_e}{r_c - r_e} - 1. \tag{6}$$

We get the metric

$$ds^2 = -fdt^2 - 2\left(2\frac{r - r_e}{r_c - r_e} - 1\right)d\tau dr + \frac{4L^2r}{(r_c - r_e)^2(r - r_0)}dr^2 + r^2d\omega^2.$$

Now using (4) and (6)

$$c_+ = \frac{1}{2L^2r}(r - r_e)(r - r_0)(r_c - r_e), \qquad c_- = \frac{1}{2L^2r}(r_c - r)(r - r_0)(r_c - r_e)$$

As expected, $c_+(r_e) = 0 = c_-(r_c)$. When they don't vanish at the boundaries, we have

$$c_+(r_c) = \frac{(r_c - r_e)^2(r_e + 2r_c)}{2L^2r_c}, \qquad c_-(r_e) = \frac{(r_c - r_e)^2(r_c + 2r_e)}{2L^2r_e}.$$

We are interested in the large r_c case. We see that $c_(r_e) \sim r_c$. This choice is not good because the ingoing characteristic near the black hole horizon increases with large r_c, overly restricting our CFL condition.

Choice 2:

We write our previous choice as

$$fH = \frac{r_c - r}{r_c - r_e} - \frac{r - r_e}{r_c - r_e},$$

and modify it slightly as

$$fH = \frac{r_e}{r}\frac{r_c - r}{r_c - r_e} - \frac{r - r_e}{r_c - r_e}.$$

The characteristics read now

$$c_+ = \frac{1}{L^2(r + r_e)}(r - r_e)(r - r_0)(r_c - r_e), \qquad c_- = \frac{1}{L^2(r + r_c)}(r_c - r)(r - r_0)(r_c - r_e)$$

The non-vanishing boundary speeds are

$$c_+(r_c) = \frac{(r_c - r_e)^2(r_e + 2r_c)}{L^2(r_c + r_e)}, \qquad c_-(r_e) = \frac{(r_c - r_e)^2(r_c + 2r_e)}{L^2(r_c + r_e)}.$$

Both speeds are on the order of unity for large r_c. The small modification fixes the behavior of the characteristic speeds.

Choice 3:
Another choice is the one by Hintz and Xie in [6]. They chose the height function as

$$-h(r) = \frac{1}{2\kappa_e} \ln(r - r_e) + \frac{1}{2\kappa_c} \ln(r - r_c).$$

So the boost is then

$$-H = \frac{1}{2\kappa_e(r - r_e)} + \frac{1}{2\kappa_c(r - r_c)}.$$

In particular

$$fH = -\frac{r_e}{r}\frac{r_c - r}{r_c - r_e}\frac{r - r_0}{r_e - r_0} + \frac{r_c}{r}\frac{r - r_e}{r_c - r_e}\frac{r - r_0}{r_c - r_0}.$$

We get the metric

$$ds^2 = -f dt^2 - 2\left(2\frac{r - r_e}{r_c - r_e} - 1\right) d\tau dr + \frac{4\ell^2 r}{(r_c - r_e)^2(r - r_0)} dr^2 + r^2 d\omega^2.$$

Now using (4) and (6)

$$c_+ = \frac{1}{2\ell^2 r}(r - r_e)(r - r_0)(r_c - r_e), \qquad c_- = \frac{1}{2\ell^2 r}(r_c - r)(r - r_0)(r_c - r_e)$$

Choice 4:
Take the tortoise coordinate defined through

$$r_* = \int \frac{1}{f} dr$$

The metric becomes

$$ds^2 = f\left(-dt^2 + dr_*^2\right) + r(r_*)^2 d\omega^2.$$

Define the new time coordinate as

$$\tau = t - \sqrt{1 + r_*^2}.$$

The main advantage of this construction is that it's easy to adapt to the requirements of the numerical computation as follows

$$\tau = t - \sqrt{K^2 + (r_* - p)^2}.$$

For now, we just set $p = 0$ and recompactify space using

$$r_* = \frac{\rho_*}{\Omega} \quad \text{with} \quad \Omega = \frac{1 - \rho_*^2}{2}.$$

This transformation maps the radial coordinate $r_* \in (-\infty, \infty)$ to $\rho \in [-1, 1]$. The metric reads then

$$ds^2 = \frac{1}{\Omega^2}\left\{f\left(-\Omega^2 d\tau^2 - 2\rho_* d\tau d\rho_* + d\rho_*^2\right) + \rho^2 d\omega^2\right\},$$

where we have defined $\rho := \Omega r$. Note that ρ has the same domain and limits as ρ_*. This metric is regular, so the transformed equation will be regular as well.

Scalar wave equation

We consider the scalar wave equation

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

After decomposing in spherical modes and writing out

$$\partial_t^2\psi = f^2\partial_r^2\psi + f\left(\frac{2f}{r} + f'\right)\partial_r\psi - \frac{f\ell^2}{r^2}.$$

We rescale by r via $u := \psi/r$ to get

$$\partial_t^2 u = f^2\partial_r^2 u + ff'\partial_r u - \frac{f}{r^2}(rf' + \ell^2).$$

Transforming into the tortoise coordinate gives us

$$\partial_t^2 u = \partial_{r_*}^2 u - \frac{f}{r^2}(rf' + \ell^2).$$

Now perform the hyperboloidal tranformation

$$\tau = t - \sqrt{1 + r_*^2}$$

in combination with compactification

$$r_* = \frac{\rho_*}{\Omega} \quad \text{with} \quad \Omega = \frac{1-\rho_*^2}{2}$$

The hyperboloidal transformation reads in compactifying coordinates

$$\tau = t - \frac{1+\rho_*^2}{1-\rho_*^2}$$

The derivative operators transform as

$$\partial_\tau = \partial_t, \qquad \partial_{r_*} = \frac{2}{1+\rho_*^2}\left(-\rho_*\partial_\tau + \Omega^2\partial_{\rho_*}\right)$$

The resulting equation reads then

$$-\partial_\tau^2 - 2\rho_*\partial_\tau\partial_{\rho_*} + \Omega^2\partial_{\rho_*}^2 - \frac{\Omega}{1+\rho_*^2}\left(2\partial_\tau + \rho_*(3+\rho_*^2)\partial_{\rho_*}\right) = \frac{f(1+\rho_*^2)}{4\Omega^2 r^2}(rf' + \ell^2).$$

Note that $\Omega^2 r^2$ is regular at both infinities, so we can define $\rho = \Omega^2 r^2$ which lives on the same domain as $\rho_* \in [-1, 1]$.

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