സഹായി

SFI GEC PALAKKAD

Module 3

Ordinary Differential Equations (ODE)

Homogeneous linear ODE with constant coefficients

The general form of a second order homogeneous linear ODE with constant coefficients is $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ or $y'' + ay' + by = 0$

eg:- $y'' + 3y' + 2y = 0$

Superposition principle of homogeneous linear ODE

If y_1 and y_2 are two solutions of a second order homogeneous linear ODE, then their linear combination, $y = c_1y_1 + c_2y_2$ will also be a solution.

eg:- Consider the equation $y'' + y = 0$

Here $y_1 = \cos x$ is a solution [Since $y_1' = -\sin x$
$y_1'' = -\cos x$
$y_1'' + y = -\cos x + \cos x = 0$]

Similarly $y_2 = \sin x$ is a solution [Since $y_2' = \cos x$
$y_2'' = -\sin x$
$y_2'' + y_2 = -\sin x + \sin x = 0$]

Then $y = 2\cos x + 3\sin x$ is also a solution.

[Since $y' = -2\sin x + 3\cos x$; $y'' = -2\cos x - 3\sin x$

$y'' + y = -2\cos x - 3\sin x + 2\cos x + 3\sin x = 0$]

Note:- This principle does not hold for a non homogeneous linear or non linear ODE.

Initial value problem (i.v.p)

For a second order homogeneous linear ODE, i.v.p consists of two initial conditions, $y(x_0) = k_0$ and $y'(x_0) = k_1$

These conditions are used to determine the arbitrary constants c_1 and c_2 in the general solution $y = c_1 y_1 + c_2 y_2$. The solution thus obtained is called particular solution.

eg:- Consider $y'' + y = 0$, $y(0) = 3$, $y'(0) = -0.5$
Solve the initial value problem.

Soln :- General solution is $y = c_1 \cos x + c_2 \sin x$

$$y(0) = 3 \implies c_1 \cos 0 + c_2 \sin 0 = 3 \implies c_1 = 3$$

$$y'(0) = -0.5 \implies \left[-c_1 \sin x + c_2 \cos x\right]_{x=0} = -0.5$$

$$\implies c_2 = -0.5$$

$\therefore y = 3\cos x - 0.5 \sin x$ is the particular solution

General Solution and Basis

The general solution of a second order homogeneous linear ODE will be in the form $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are linearly independent solutions of the equation.

The set $\{y_1, y_2\}$ of linearly independent solutions is called a basis.

To check linear dependence and independence

Two solutions y_1 and y_2 are said to be linearly dependent if $y_1 = ky_2$ or $\frac{y_1}{y_2} = k$, constant. Otherwise, linearly independent.

eg:- Consider $y'' + y = 0$

Here $y_1 = \cos x$ and $y_2 = \sin x$ are two solutions and they are linearly independent because

$$\frac{y_1}{y_2} = \frac{\cos x}{\sin x} = \cot x \neq \text{constant}.$$

Hence $y = c_1 \cos x + c_2 \sin x$ is the general solution and $\{\cos x, \sin x\}$ forms a basis of the equation $y'' + y = 0$

1. Verify by substitution that $y_1 = e^{x}$ and $y_2 = e^{-x}$ are solutions of the equation $y'' - y = 0$ and are linearly independent. Then solve the i.v.p, $y'' - y = 0$, $y(0) = 6$, $y'(0) = -2$.

Soln :- $y_1 = e^{x}$; $y_1' = e^{x}$; $y_1'' = e^{x}$

$$y_1'' - y_1 = e^{x} - e^{x} = \underline{0}$$

$y_2 = e^{-x}$, $y_2' = -e^{-x}$, $y_2'' = e^{-x}$

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0$$

Therefore $y_1 = e^{+x}$ and $y_2 = e^{-x}$ are solutions.

Also they are linearly independent because

$\frac{y_1}{y_2} = \frac{e^{x}}{e^{-x}} = e^{2x} \neq$ constant.

Therefore general solution is $y = c_1 e^{x} + c_2 e^{-x}$

Given $y(0) = 6 \Rightarrow c_1 e^{0} + c_2 e^{0} = 6 \Rightarrow c_1 + c_2 = 6$ — ①

$y'(0) = -2 \Rightarrow \left[c_1 e^{x} - c_2 e^{-x}\right]_{x=0} = -2 \Rightarrow c_1 - c_2 = -2$ — ②

Solving ① and ② $c_1 = 2$; $c_2 = 4$

$\therefore y = 2e^{x} + 4e^{-x}$ is the particular solution.

To check linear dependence and independence using Wronskian

Two solutions y_1 and y_2 are linearly dependent if and only if their Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

1. Show that $y_1 = e^{-0.5x}$ and $y_2 = e^{-2.5x}$ are linearly independent using Wronskian.

Soln :-
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-0.5x} & e^{-2.5x} \\ -0.5e^{-0.5x} & -2.5e^{-2.5x} \end{vmatrix}$$

$$= -2.5e^{-3x} + 0.5e^{-3x} = -2e^{-3x} \neq 0$$

Hence y_1 and y_2 are linearly independent.

2. Show that $y_1 = x^2$, $y_2 = 5x$ $y_3 = 2x$ are linearly dependent.

<u>Soln</u> :- $W = \begin{vmatrix} x^2 & 5x & 2x \\ 2x & 5 & 2 \\ 2 & 0 & 0 \end{vmatrix} = x^2(0) - 5x(-4) + 2x(-10)$

$= 20x - 20x = 0$

Hence y_1, y_2, y_3 are linearly dependent.

<u>Problems</u>

I) Verify that given functions are linearly independent and form a basis of solution of the given ODE. Solve the IVP.

1) $y'' + 9y = 0$; $y(0) = 2$; $y'(0) = -1$

$\cos 3x$, $\sin 3x$

2) $y'' + 2y' + y = 0$; $y(0) = 2$; $y'(0) = -1$

e^{-x}, xe^{-x}

II) Check whether the given solutions are linearly dependent or not.

1) $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$

2) $1, x, x^2, x^3$

To solve a homogeneous linear ODE with constant coefficients

An n^{th} order homogeneous linear ODE with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \quad \text{--- (1)}$$

Usually we write $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2, \dots\dots$

Therefore eq (1) can be written as

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots + a_{n-1} D + a_n) y = 0$$

$$\Rightarrow f(D) y = 0$$

Auxiliary equation or characteristic equation

The equation $f(D) = 0$ is called auxiliary equation or characteristic equation.

Let $D = m_1, m_2, \dots\dots, m_n$ be the roots of A.E

Case I :- If all the roots are real and distinct then the general soln, $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots\dots + c_n e^{m_n x}$

Case II :- If two roots are equal, $m_1 = m_2$, then

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots\dots + c_n e^{m_n x}.$$

If three roots are equal, $m_1 = m_2 = m_3$, then

$$y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots\dots + c_n e^{m_n x}$$

Case III :- If the roots are imaginary, $m_1, m_2 = \alpha \pm i\beta$

then $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots\dots + c_n e^{m_n x}$

Case IV : If two pairs of imaginary roots are equal, $m_1, m_2 = \alpha \pm i\beta$ then m_3, m_4

$$y = e^{\alpha x}\left[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \dots\dots + c_n e^{m_n x}.$$

1. Solve $y'' + 5y' + 6y = 0$

Soln :- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \Rightarrow (D^2 + 5D + 6)y = 0$

Auxiliary eqn (A.E) is $D^2 + 5D + 6 = 0$

$\Rightarrow (D+3)(D+2) = 0$

$\Rightarrow D = -3, -2$ (roots are real & distinct)

$\therefore$ General solution is $y = c_1 e^{-3x} + c_2 e^{-2x}$

2. Find the general solution of

$$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$$

Soln :- The given equation can be written as

$(D^3 - 7D - 6)y = 0$

A.E is $D^3 - 7D - 6 = 0$

$\Rightarrow D = -1, -2, 3$

Roots are real and distinct

$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

$D = -1$ is one root

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$\Rightarrow D^2 - D - 6 = 0$

$\Rightarrow (D-3)(D+2) = 0$

$\Rightarrow D = 3, -2$

3. Solve $y''' - 4y'' + 4y' = 0$

Soln :- $(D^3 - 4D^2 + 4D)y = 0$

A.E is $D^3 - 4D^2 + 4D = 0 \Rightarrow D(D^2 - 4D + 4) = 0$

$\Rightarrow D(D-2)^2 = 0$

$\Rightarrow D = 0, 2, 2$

One root is real & distinct, two roots are real & equal.

$\therefore y = c_1 e^{0x} + (c_2 x + c_3) e^{2x} = c_1 + (c_2 x + c_3) e^{2x}$

4. Solve $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

Soln :- The given equation is $(D^4 + 13D^2 + 36)y = 0$

A.E is $D^4 + 13D^2 + 36 = 0$

$\Rightarrow D = \pm 3i, \pm 2i$

Roots are imaginary

$\therefore y = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$+ e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$

Put $D^2 = u$

$\therefore u^2 + 13u + 36 = 0$

$(u+9)(u+4) = 0$

$u = -9 \quad u = -4$

$\Rightarrow D^2 = -9 \quad D^2 = -4$

$\Rightarrow D = \pm 3i \quad D = \pm 2i$

5. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Soln :- A.E is $D^2 - 4D + 1 = 0$

$\Rightarrow D = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

Roots are real & distinct

$\therefore y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

6. Find the general solution of

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$$

<u>Soln</u> :- The given equation is

$$(D^2 + 6D + 9)x = 0 \quad \text{where } D = \frac{d}{dt}$$

A.E is $D^2 + 6D + 9 = 0 \Rightarrow (D+3)(D+3) = 0$

$\Rightarrow D = -3, -3$

Roots are real & equal.

$\therefore$ General solution is $\underline{\underline{x = (c_1 t + c_2)e^{-3t}}}$

7. Solve $y''' - 3y'' + 3y' - y = 0$

<u>Soln</u> :- $(D^3 - 3D^2 + 3D - 1)y = 0$

A.E is $D^3 - 3D^2 + 3D - 1 = 0$

$\Rightarrow (D-1)^3 = 0 \Rightarrow D = 1, 1, 1$ (equal roots)

$\therefore y = \underline{\underline{(c_1 x^2 + c_2 x + c_3)e^x}}$

8. Solve the <u>I.V.P</u> $y'' + 4y' + 29y = 0$; $y(0) = 0$; $y'(0) = 15$

<u>Soln</u> :- A.E is $D^2 + 4D + 29 = 0$

$$D = \frac{-4 \pm \sqrt{16 - 116}}{2} = -2 \pm 5i$$

$$\therefore y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$$

$y(0) = 0 \Rightarrow e^0(c_1 + 0) = 0 \Rightarrow \underline{\underline{c_1 = 0}}$

$y'(0) = 15 \Rightarrow \left[e^{-2x}(-5c_1 \sin 5x + 5c_2 \cos 5x) - 2e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)\right]_{x=0} = 15$

$\Rightarrow 5c_2 - 2c_1 = 15 \Rightarrow 5c_2 = 15 \Rightarrow \underline{\underline{c_2 = 3}}$

$\therefore\ y = 3e^{-2x} \sin 5x$

9. Solve $\frac{d^4x}{dt^4} + 4x = 0$

Soln: The given equation is $(D^4+4)x = 0$ where $D = \frac{d}{dt}$

A.E is $D^4 + 4 = 0$

$(D^2+2)^2 - (2D)^2 = 0$

$(D^2+2D+2)(D^2-2D+2) = 0$ $\quad [\because a^2-b^2 = (a+b)(a-b)$

$\Rightarrow D = \frac{-2 \pm \sqrt{-4}}{2} \qquad D = \frac{2 \pm \sqrt{-4}}{2}$

$\Rightarrow D = -1 \pm i \qquad D = 1 \pm i$

$\therefore\ x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{t}(c_3 \cos t + c_4 \sin t)$

10. Solve $[(D^2+1)^3 (D^2+D+1)^2]y = 0$

Soln: A.E is $(D^2+1)^3 (D^2+D+1)^2 = 0$

$\Rightarrow (D^2+1)^3 = 0$ and $(D^2+D+1)^2 = 0$

$\Rightarrow (D^2+1)(D^2+1)(D^2+1) = 0 \qquad (D^2+D+1)(D^2+D+1) = 0$

$\Rightarrow D = \pm i, \pm i, \pm i \qquad D = \frac{-1 \pm \sqrt{-3}}{2}, \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore\ y = (c_1x^2 + c_2x + c_3)\cos x + (c_4x^2 + c_5x + c_6)\sin x$

$+ e^{-\frac{1}{2}x}\left[(c_7x + c_8)\cos\left(\frac{\sqrt{3}}{2}\right)x + (c_9x + c_{10})\sin\frac{\sqrt{3}x}{2}\right]$

To find the homogeneous linear ODE if the solutions are given

I) Find a second order homogeneous linear ODE for which the given functions are the solutions.

i) e^{-x} , e^{-2x}

Soln :- General solution is $y = c_1 e^{-x} + c_2 e^{-2x}$

$\therefore$ Roots of the A.E are -1 and -2.

$\therefore$ A.E is $(D+1)(D+2) = 0 \Rightarrow D^2 + 3D + 2 = 0$

$\therefore$ The equation is $(D^2 + 3D + 2)y = 0$ or

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

ii) $\cos 5x$, $\sin 5x$

Soln :- General solution is $y = c_1 \cos 5x + c_2 \sin 5x$

Here $\alpha = 0$ and $\beta = 5$

$\therefore$ Roots of the A.E are $\alpha \pm i\beta = \pm 5i$

A.E is $(D - 5i)(D + 5i) = 0 \Rightarrow D^2 + 25 = 0$

$\therefore$ The equation is $(D^2 + 25)y = 0$ or

$$\frac{d^2y}{dx^2} + 25y = 0$$

iii) $e^{-2.5x} \cos 0.5x$, $e^{-2.5x} \sin 0.5x$

Soln :- General solution is $y = e^{-2.5x}(c_1 \cos 0.5x + c_2 \sin 0.5x)$

Here $\alpha = -2.5$ and $\beta = 0.5$

$\therefore$ Roots of the A.E are $-2.5 \pm i(0.5)$

$\therefore$ A.E is $\left[D - (-2.5 + i\,0.5)\right]\left[D - (-2.5 - i\,0.5)\right] = 0$

$$\Longrightarrow \left[(D + 2.5) - i\,0.5\right]\left[(D + 2.5) + i\,0.5\right] = 0$$

$$\Longrightarrow (D + 2.5)^2 + (0.5)^2 = 0$$

$$\Longrightarrow D^2 + 5D + 6.5 = 0$$

$\therefore$ The eqn is $(D^2 + 5D + 6.5)\,y = 0$ or

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6.5y = 0$$

Problems

Solve the following differential equations

1) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$

2) $y'' + (a+b)y' + aby = 0$

3) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

4) $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$

5) $y^{iv} - 5y'' + 4y = 0$

6) $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

7) $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$; $x(0) = 0$; $x'(0) = 0$

8) If $\dfrac{d^4x}{dt^4} = m^4x$, show that

$$x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$$

To solve a non homogeneous linear ODE with constant coefficients

Consider the non homogeneous linear ODE

$f(D)y = X$ where X is either a constant or any function of x

If $y = y_h$ is the general solution of $f(D)y = 0$

and $y = y_p$ is a particular solution of $f(D)y = X$,

then the general solution of $f(D)y = X$ is

$$y = y_h + y_p.$$

y_h is called complementary function and

y_p is called particular integral.

To find the particular integral

1) Method of undetermined coefficients
2) Method of variation of parameters

1) Method of undetermined coefficients

This method is suitable for linear ODE with constant coefficients $y'' + ay' + by = X$ where X is an exponential function, a Cosine or Sine function, a power of x or sum or product of such functions. (Because, these functions have derivatives similar to X itself)

To find the particular integral, we choose a trial solution containing unknown constants which are determined by substitution in the given equation.

$\underline{X}$	Trial Solution
ke^{ax}	Ae^{ax}
kx^n	$A + Bx + Cx^2 + \dots\dots + Nx^n$
$k\cos ax$	$A\cos ax + B\sin ax$
$k\sin ax$	"
$ke^{ax}\cos bx$	$e^{ax}(A\cos bx + B\sin bx)$
$ke^{ax}\sin bx$	"

1. Solve $y'' + 5y' + 6y = 2e^{-x}$

Soln :- A.E is $(D^2 + 5D + 6) = 0$

$\Rightarrow (D+3)(D+2) = 0 \Rightarrow D = -3, -2$ (real & distinct)

Complementary function, $y_h = c_1 e^{-3x} + c_2 e^{-2x}$

Let the particular integral be $y_p = Ae^{-x}$

$\therefore y_p' = -Ae^{-x}$; $y_p'' = Ae^{-x}$

Substituting in the given equation,

$Ae^{-x} + 5(-Ae^{-x}) + 6Ae^{-x} = 2e^{-x}$

$\Rightarrow 2Ae^{-x} = 2e^{-x} \Rightarrow 2A = 2 \Rightarrow A = 1$

$\therefore y_p = e^{-x}$

$\therefore$ General solution is $y = y_h + y_p = \underline{\underline{c_1 e^{-3x} + c_2 e^{-2x} + e^{-x}}}$

2. Solve $\frac{d^2y}{dx^2} - y = 3\cos x$

<u>Soln</u> :- A.E is $D^2 - 1 = 0 \implies D = \pm 1$ (real + distinct)

$\therefore \quad y_h = c_1 e^{x} + c_2 e^{-x}$

Let $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x \;;\; y_p'' = -A\cos x - B\sin x$

Substituting in the given equation,

$-A\cos x - B\sin x - A\cos x - B\sin x = 3\cos x$

$\implies -2A\cos x - 2B\sin x = 3\cos x$

Comparing the coefficients,

$-2A = 3$ and $-2B = 0$

$\implies A = \frac{-3}{2}$ $\quad B = 0$

$\therefore \quad y_p = \frac{-3}{2}\cos x$

$\therefore$ General solution is $y = y_h + y_p = c_1 e^{x} + c_2 e^{-x} - \frac{3}{2}\cos x$

3. Solve $y'' + 3y' + 2y = 12x^2$

<u>Solution</u> :- A.E is $D^2 + 3D + 2 = 0$

$\implies (D+2)(D+1) = 0 \implies D = -2, -1$

$\therefore \quad y_h = c_1 e^{-2x} + c_2 e^{-x}$

Let $y_p = A + Bx + Cx^2$

$y_p' = B + 2Cx \;;\; y_p'' = 2C$

Substituting in the given equation,

$2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) = 12x^2$

$\implies (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = 12x^2$

Comparing the coefficients,

$2C = 12 \qquad\qquad 6C + 2B = 0 \qquad\qquad 2C + 3B + 2A = 0$

$\Rightarrow C = 6 \qquad\qquad 6(6) + 2B = 0 \qquad\qquad 2(6) + 3(-18) + 2A = 0$

$\Rightarrow B = -18 \qquad\qquad \Rightarrow A = 21$

$$\therefore y_p = 21 - 18x + 6x^2$$

$$\therefore y = y_h + y_p = c_1 e^{-2x} + c_2 e^{-x} + 6x^2 - 18x + 21$$

4. Solve $y'' + 4y' + 4y = e^{-x} \cos x$

<u>Soln</u> :- A.E is $D^2 + 4D + 4 = 0$

$\Rightarrow (D+2)(D+2) = 0 \Rightarrow D = -2, -2$ (equal roots)

$$\therefore y_h = (c_1 x + c_2) e^{-2x}$$

Let $y_p = e^{-x}(A\cos x + B\sin x) = Ae^{-x}\cos x + Be^{-x}\sin x$

$$y_p' = -Ae^{-x}\sin x - Ae^{-x}\cos x + Be^{-x}\cos x - Be^{-x}\sin x$$

$$y_p'' = \cancel{-Ae^{-x}\cos x} + Ae^{-x}\sin x + Ae^{-x}\sin x + \cancel{Ae^{-x}\cos x}$$
$$- \cancel{Be^{-x}\sin x} - Be^{-x}\cos x - Be^{-x}\cos x + \cancel{Be^{-x}\sin x}$$

$$= 2Ae^{-x}\sin x - 2Be^{-x}\cos x$$

Substituting in the given equation,

$$2Ae^{-x}\sin x - 2Be^{-x}\cos x + 4(-Ae^{-x}\sin x - Ae^{-x}\cos x + Be^{-x}\cos x - Be^{-x}\sin x)$$
$$+ 4(Ae^{-x}\cos x + Be^{-x}\sin x) = e^{-x}\cos x$$

$$\Rightarrow (2A - 4A - 4B + 4B)e^{-x}\sin x + (-2B - 4A + 4B + 4A)e^{-x}\cos x = e^{-x}\cos x$$

$$\Rightarrow (-2A)e^{-x}\sin x + (2B)e^{-x}\cos x = e^{-x}\cos x$$

Comparing the coefficients,

$-2A = 0$ and $2B = 1$

$\Rightarrow A = 0$ and $B = \frac{1}{2}$

$\therefore y_p = \frac{1}{2} e^{-x} \sin x$

$\therefore y = y_h + y_p = (c_1 x + c_2) e^{-2x} + \frac{1}{2} e^{-x} \sin x$

5. Solve $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$

Soln: A.E is $D^2 + 2D + 4 = 0$

$\Rightarrow D = \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm i2\sqrt{3}}{2}$

$= -1 \pm i\sqrt{3}$ (complex roots)

$\therefore y_h = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

Let $y_p = (A + Bx + Cx^2) + De^{-x}$

$y_p' = B + 2Cx - De^{-x}$; $y_p'' = 2C + De^{-x}$

Substituting in the given equation,

$2C + De^{-x} + 2(B + 2Cx - De^{-x}) + 4(A + Bx + Cx^2 + De^{-x})$
$= 2x^2 + 3e^{-x}$

$\Rightarrow (2C + 2B + 4A) + (4C + 4B)x + (4C)x^2 + (D - 2D + 4D)e^{-x}$
$= 2x^2 + 3e^{-x}$

Comparing the coefficients,

$2C + 2B + 4A = 0$; $4C + 4B = 0$; $4C = 2$; $3D = 3$

$\Rightarrow C = \frac{1}{2}$; $\Rightarrow D = 1$

$\Rightarrow 4(\frac{1}{2}) + 4B = 0$

$\Rightarrow B = \frac{-1}{2}$

$$2C + 2B + 4A = 0 \Rightarrow 2\left(\frac{1}{2}\right) + 2\left(\frac{-1}{2}\right) + 4A = 0$$
$$\Rightarrow 4A = 0 \Rightarrow A = 0$$

$$\therefore y_p = \frac{-1}{2}x + \frac{1}{2}x^2 + e^{-x}$$

$$\therefore y = y_h + y_p = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \left(\frac{-1}{2}x + \frac{1}{2}x^2 + e^{-x}\right)$$

Modification rule

If any term in the trial solution appears in the complementary function, we multiply this trial solution by x. If this solution corresponds to a double root of A.E, multiply by x^2 and so on.

6. Solve $(D^2+1)y = \sin x$

Soln :- A.E is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$\therefore y_h = c_1 \cos x + c_2 \sin x$$

Let $y_p = (A\cos x + B\sin x)x = Ax\cos x + Bx\sin x$

$$y_p' = -Ax\sin x + A\cos x + Bx\cos x + B\sin x$$

$$y_p'' = -Ax\cos x - A\sin x - A\sin x - Bx\sin x + B\cos x + B\cos x$$
$$= -Ax\cos x - 2A\sin x - Bx\sin x + 2B\cos x$$

Substituting in the given equation,

$$-\cancel{Ax\cos x} - 2A\sin x - \cancel{Bx\sin x} + 2B\cos x + \cancel{Ax\cos x} + \cancel{Bx\sin x} = \sin x$$

$$\Rightarrow -2A\sin x + 2B\cos x = \sin x$$

comparing the coefficients,

$-2A = 1$ and $2B = 0$

$\Rightarrow A = -\frac{1}{2}$ $\quad B = 0$

$\therefore \quad y_p = -\frac{1}{2} x \cos x$

$\therefore \quad y = y_h + y_p = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$

7. Solve $y'' + 2y' + y = e^{-x}$

Soln: A.E is $(D^2 + 2D + 1) = 0 \Rightarrow (D+1)^2 = 0 \Rightarrow D = -1, -1$ (equal roots)

$y_h = (c_1 x + c_2) e^{-x}$

Let $y_p = A e^{-x} (x^2)$

$y_p' = -Ax^2 e^{-x} + 2Axe^{-x}$

$y_p'' = Ax^2 e^{-x} - 2Axe^{-x} - 2Axe^{-x} + 2Ae^{-x} = Ax^2 e^{-x} - 4Axe^{-x} + 2Ae^{-x}$

Substituting in the given equation,

$(\cancel{Ax^2 e^{-x}} - \cancel{4Axe^{-x}} + 2Ae^{-x}) + 2(-\cancel{Ax^2 e^{-x}} + \cancel{2Axe^{-x}}) + \cancel{Ax^2 e^{-x}} = e^{-x}$

$\Rightarrow 2Ae^{-x} = e^{-x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$\therefore \quad y_p = \frac{1}{2} e^{-x} x^2$

$\therefore \quad y = y_h + y_p = (c_1 x + c_2) e^{-x} + \frac{1}{2} e^{-x} x^2$

8. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

Soln: A.E is $D^2 + D = 0 \Rightarrow D(D+1) = 0 \Rightarrow D = 0, -1$

$$y_h = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$$

Let $y_p = (A + Bx + Cx^2)x = Ax + Bx^2 + Cx^3$

$y_p' = A + 2Bx + 3Cx^2$; $y_p'' = 2B + 6Cx$

Substituting in the given equation,

$$(2B + 6Cx) + (A + 2Bx + 3Cx^2) = x^2 + 2x + 4$$

$$3Cx^2 + (6C + 2B)x + (A + 2B) = x^2 + 2x + 4$$

Comparing the coefficients,

$3C = 1$ $\Rightarrow C = \frac{1}{3}$

$6C + 2B = 2$, $6\left(\frac{1}{3}\right) + 2B = 2$ $\Rightarrow B = 0$

$A + 2B = 4$, $A + 2(0) = 4$ $\Rightarrow A = 4$

$$\therefore y_p = 4x + \frac{1}{3}x^3$$

$$\therefore y = c_1 + c_2 e^{-x} + 4x + \frac{x^3}{3}$$

9. Solve $\frac{d^2y}{dx^2} + 4y = x \sin x$

<u>Soln</u>: A.E is $D^2 + 4 = 0 \Rightarrow D^2 = -4 \Rightarrow D = \pm 2i$

$$\therefore y_h = c_1 \cos 2x + c_2 \sin 2x$$

Let $y_p = (A + Bx)(C \cos x + D \sin x)$

$= AC \cos x + AD \sin x + BCx \cos x + BDx \sin x$

$y_p' = -AC \sin x + AD \cos x - BCx \sin x + BC \cos x + BDx \cos x + BD \sin x$

$y_p'' = -AC \cos x - AD \sin x - BCx \cos x - BC \sin x - BC \sin x - BDx \sin x + BD \cos x + BD \cos x$

$= (-AC + 2BD) \cos x + (-AD - 2BC) \sin x - BCx \cos x - BDx \sin x$

Substituting in the given equation,

$$(-AC + 2BD)\cos x + (-AD - 2BC)\sin x - BCx\cos x - BDx\sin x$$
$$+ 4(AC\cos x + AD\sin x + BCx\cos x + BDx\sin x)$$
$$= x\sin x$$

$$\Rightarrow (3AC + 2BD)\cos x + (3AD - 2BC)\sin x + 3BCx\cos x + 3BDx\sin x$$
$$= x\sin x$$

Comparing the coefficients,

$3AC + 2BD = 0$; $3AD - 2BC = 0$; $3BC = 0$; $3BD = 1$

$\Rightarrow BC = 0$ $\Rightarrow BD = \frac{1}{3}$

$\Rightarrow 3AC + 2\left(\frac{1}{3}\right) = 0$ $\Rightarrow 3AD - 2(0) = 0$

$\Rightarrow AC = \frac{-2}{9}$ $\Rightarrow AD = 0$

$$\therefore y_p = \frac{-2}{9}\cos x + \frac{1}{3}x\sin x$$

$$\therefore \underline{y = y_h + y_p} = c_1\cos 2x + c_2\sin 2x - \frac{2}{9}\cos x + \frac{x\sin x}{3}$$

Problems :-

Solve the following differential equations

1) $\frac{d^3y}{dx^3} + y = 3 + 5e^x$

2) $y'' - 2y' + 5y = \sin 3x$

3) $y'' - 3y' + 2y = 6e^{-3x} + \sin 2x$

4) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$

5) $\frac{d^4y}{dx^4} - y = e^x\cos x$

6) $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2$

7) $(D^2 - 4D + 3)y = \sin 3x\cos 2x$

8) $\frac{d^2y}{dx^2} - 4y = x^2$

9) $y'' + y = 2\cos x$ 10) $y'' + y = x\sin x$

11) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

12) $y'' + 4y = e^x + \sin 2x$

13) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

2) Method of variation of parameters to find the particular integral

Consider a second order linear ODE with constant coefficients,

$$y'' + ay' + by = X$$

Let the complementary function be,

$$y_h = c_1 y_1 + c_2 y_2$$

Then P.I $y_p = u(x)y_1 + v(x)y_2$ where

$$u(x) = -\int \frac{y_2 X}{W}\,dx \quad \text{and} \quad v(x) = \int \frac{y_1 X}{W}\,dx$$

where W is the Wronskian of y_1 and y_2.

1. Solve $y'' + y = \operatorname{Cosec} x$

Soln :- A.E is $D^2 + 1 = 0 \Rightarrow D = \pm i$

C.F is $y_h = c_1 \cos x + c_2 \sin x$

Here $y_1 = \cos x$; $y_2 = \sin x$ and $X = \operatorname{Cosec} x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$\therefore u(x) = -\int \frac{y_2 X}{W} dx = -\int \sin x \operatorname{cosec} x \, dx = -\int dx = \underline{\underline{-x}}$

$v(x) = \int \frac{y_1 X}{W} dx = \int \cos x \operatorname{cosec} x \, dx = \int \frac{\cos x}{\sin x} dx$

$= \int \frac{1}{u} du$ | Put $\sin x = u$, $\cos x dx = du$

$= \log u = \log(\sin x)$

$\therefore$ P.I, $y_p = u(x) y_1 + v(x) y_2 = \underline{\underline{-x \cos x + \sin x \log(\sin x)}}$

$\therefore y = y_h + y_p = \underline{\underline{c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log(\sin x)}}$

2. Solve $y'' + 4y = 4\sec^2 2x$

<u>Soln</u> :- A.E is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

C.F, $y_h = c_1 \cos 2x + c_2 \sin 2x$

Here $y_1 = \cos 2x$; $y_2 = \sin 2x$; $X = 4\sec^2 2x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$

$= \underline{\underline{2}}$

$u(x) = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin 2x \, 4\sec^2 2x}{2} dx$

$= -2 \int \frac{\sin 2x}{\cos^2 2x} dx = -2 \int \sec 2x \tan 2x \, dx$

$= -2 \frac{\sec 2x}{2} = -\sec 2x$

$$v(x) = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos 2x \, 4\sec^2 2x}{2}\,dx$$

$$= 2\int \sec 2x\,dx$$

$$= 2\,\frac{\log(\sec 2x + \tan 2x)}{2}$$

$$= \log(\sec 2x + \tan 2x)$$

$$\left[\begin{aligned} &\int \sec x\,dx \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \\ &= \log(\sec x + \tan x) \end{aligned}\right.$$

$$\therefore y_p = u(x)y_1 + v(x)y_2 = -\sec 2x \cos 2x + \sin 2x \log(\sec 2x + \tan 2x)$$

$$= -1 + \sin 2x \log(\sec 2x + \tan 2x)$$

$$\therefore y = y_h + y_p = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$$

3. Solve $y'' + 4y = \tan 2x$

Soln :- A.E is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

C.F, $y_h = c_1 \cos 2x + c_2 \sin 2x$

Here, $y_1 = \cos 2x$, $y_2 = \sin 2x$ $X = \tan 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u(x) = -\int \frac{y_2 X}{W}\,dx = -\int \frac{\sin 2x \tan 2x}{2}\,dx = \frac{-1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= \frac{-1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$v(x) = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos 2x \tan 2x}{2}\,dx = \frac{1}{2}\int \sin 2x\,dx$$

$$= -\frac{\cos 2x}{4}$$

$$\therefore\ y_p = u(x)y_1 + v(x)y_2 = -\frac{\cos 2x \log(\sec 2x + \tan 2x)}{4} - \frac{\sin 2x \cos 2x}{4}$$

$$\therefore\ y = y_h + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{\cos 2x \log(\sec 2x + \tan 2x)}{4} - \frac{\sin 2x \cos 2x}{4}$$

4. Solve $y'' - 2y' + 2y = e^{x} \cdot \tan x$

Soln :- A.E is $D^2 - 2D + 2 = 0 \Rightarrow D = 1 \pm i$

$$\therefore\ y_h = e^{x}(c_1 \cos x + c_2 \sin x)$$

Here $y_1 = e^{x}\cos x$; $y_2 = e^{x}\sin x$; $X = e^{x}\tan x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ -e^{x}\sin x + e^{x}\cos x & e^{x}\cos x + e^{x}\sin x \end{vmatrix}$$

$$= e^{2x}\cos^2 x + e^{2x}\cos x \sin x + e^{2x}\sin^2 x - e^{2x}\sin x \cos x$$

$$= e^{2x}$$

$$u(x) = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{x}\sin x\, e^{x}\tan x}{e^{2x}}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$v(x) = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{x}\cos x\, e^{x}\tan x}{e^{2x}}\,dx = \int \sin x\,dx = -\cos x$$

$\therefore y_p = u(x)y_1 + v(x)y_2 = -e^x \cos x \left[\log(\sec x + \tan x) - \sin x\right] - e^x \sin x \cos x$

$\therefore y = y_h + y_p = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \left[\log(\sec x + \tan x) - \sin x\right] - e^x \sin x \cos x$

Problems

Solve using method of variation of parameters

i) $\dfrac{d^2y}{dx^2} + y = \sec x$

ii) $\dfrac{d^2y}{dx^2} + y = x \sin x$

Euler- Cauchy Equation

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

where a_i's are constants and X is a function of x, is called Euler Cauchy's equation.

Such equations can be reduced to linear differential equations with constant coefficients by the substitution, $x = e^z$ or $z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{where } D = \frac{d}{dz}$$

Similarly, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$; $x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)$ etc.

1. Solve $x^2y'' - 5xy' + 9y = 0$

Soln :- The given equation is an Euler-Cauchy equation.

Put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Substituting in the given equation,

$$[D(D-1) - 5D + 9]y = 0$$

$$\Rightarrow [D^2 - 6D + 9]y = 0$$

A·E is $D^2 - 6D + 9 = 0 \Rightarrow D = 3, 3$

$$\therefore \quad y = (c_1 z + c_2)e^{3z} = \underline{\underline{(c_1\log x + c_2)x^3}}$$

2. Solve $4x^2y'' + 5y = 0$

Soln :- Put $x = e^z \Rightarrow z = \log x$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

Substituting in the given equation,

$$[4D(D-1) + 5]y = 0$$

A·E is $4D^2 - 4D + 5 = 0$

$$\Rightarrow D = \frac{4 \pm \sqrt{16 - 80}}{8} = \frac{1}{2} + i$$

$$\therefore \quad y = e^{\frac{z}{2}}(c_1\cos z + c_2\sin z)$$

$$= \underline{\underline{\sqrt{x}\,[c_1\cos(\log x) + c_2\sin(\log x)]}}$$

3. Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Soln :- Put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Substituting in the given equation,

$$[D(D-1) - D + 1]y = z \quad \text{——— (1)}$$

A.E is $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$

$\therefore$ C.F, $y_h = (c_1 z + c_2)e^z = (c_1 \log x + c_2)x$

Let $y_p = A + Bz$

$y_p' = B \; ; \; y_p'' = 0$

Substituting in eq(1)

$0 - 2B + A + Bz = z \Rightarrow B = 1$ and $-2B + A = 0$

$\Rightarrow -2 + A = 0$

$\Rightarrow A = 2$

$\therefore y_p = 2 + z = 2 + \log x$

$\therefore y = y_h + y_p = (c_1 \log x + c_2)x + 2 + \log x$

4. Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Ans Put $x = e^z \Rightarrow z = \log x$

$x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$; $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

Substituting in the given equation,

$$[D(D-1)(D-2)+2D(D-1)+2]y = 10(e^{z}+e^{-z})$$

$$\Rightarrow (D^3 - D^2 + 2)y = 10(e^{z}+e^{-z}) \quad —(1)$$

A.E is $D^3 - D^2 + 2 = 0 \Rightarrow D = -1,\ 1 \pm i$

$\therefore$ C.F, $y_h = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z)$

$$= \frac{c_1}{x} + x\left[c_2 \cos(\log x) + c_3 \sin(\log x)\right]$$

Let $y_p = Ae^{z} + Be^{-z} z$

$$y_p' = Ae^{z} - Bze^{-z} + Be^{-z}$$

$$y_p'' = Ae^{z} + Bze^{-z} - Be^{-z} - Be^{-z} = Ae^{z} + Bze^{-z} - 2Be^{-z}$$

$$y_p''' = Ae^{z} - Bze^{-z} + Be^{-z} + 2Be^{-z} = Ae^{z} - Bze^{-z} + 3Be^{-z}$$

Substituting in eq(1)

$$\cancel{Ae^{z}} - \cancel{Bze^{-z}} + 3Be^{-z} - \cancel{Ae^{z}} - \cancel{Bze^{-z}} + 2Be^{-z} + 2Ae^{z} + \cancel{2Bze^{-z}}$$

$$= 10e^{z} + 10e^{-z}$$

$$\Rightarrow 2Ae^{z} + 5Be^{-z} = 10e^{z} + 10e^{-z}$$

$\Rightarrow 2A = 10$ and $5B = 10$

$\Rightarrow A = 5$ $\qquad B = 2$

$$\therefore y_p = 5e^{z} + 2ze^{-z} = 5x + \frac{2\log x}{x}$$

$$\therefore y = y_h + y_p = \frac{c_1}{x} + x\left[c_2 \cos(\log x) + c_3 \sin(\log x)\right] + 5x + \frac{2\log x}{x}$$

Problems

Solve

1) $$x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 25y = 50$$

2) $$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

3) $$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x+1)^2$$

4) $$x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$$ [Hint:- Multiply throughout by x]

5) $$x^4\frac{d^3y}{dx^3} + 2x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1$$

6) $$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$

7) $$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$$