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Module IV

Laplace Transform

Let $f(t)$ be a given function that is defined for all $t \geq 0$. Then

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$ is called the Laplace transform of $f(t)$.

Note :- Here the original function depends on t and its transform on s. That is, a function in one space is transformed to a function in another space. or time domain is changed to frequency domain.

The original function $f(t)$ is called inverse of $F(s)$ denoted by $f(t) = L^{-1}[F(s)]$

Laplace transform of some important functions

	$f(t)$	$F(s) = L[f(t)]$
1)	0	0
2)	1	$\frac{1}{s}$
3)	t	$\frac{1}{s^2}$
4)	e^{at}	$\frac{1}{s-a}$
5)	$t^n \,,\; n \geq 0$	$\frac{n!}{s^{n+1}}$
6)	$\cos at$	$\frac{s}{s^2+a^2}$
7)	$\sin at$	$\frac{a}{s^2+a^2}$
8)	$\cosh at$	$\frac{s}{s^2-a^2}$
9)	$\sinh at$	$\frac{a}{s^2-a^2}$
10)	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
11)	$e^{at} \cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$
12)	$e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$

1) $L(0) = \int_0^\infty e^{-st}(0)\,dt = \underline{0}$

2) $L(1) = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \underline{\underline{\frac{1}{s}}}$

3) $L(t) = \int_0^\infty e^{-st}(t)\,dt = \left[t\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{s^2}\right]_0^\infty = \underline{\underline{\frac{1}{s^2}}}$

4) $L(e^{at}) = \int_0^\infty e^{-st}\,e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$

$= \underline{\underline{\frac{1}{s-a}}}$

Linearity Property

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

8 & 9) $L(\cosh at)$ & $L(\sinh at)$

$\cosh at = \frac{e^{at} + e^{-at}}{2}$ $\qquad \sinh at = \frac{e^{at} - e^{-at}}{2}$

$L(\cosh at) = L\left[\frac{e^{at} + e^{-at}}{2}\right] = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \underline{\underline{\frac{s}{s^2-a^2}}}$

$L(\sinh at) = L\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \underline{\underline{\frac{a}{s^2-a^2}}}$

6 & 7) $L(\cos at)$ & $L(\sin at)$

$L(e^{iat}) = L(\cos at) + iL(\sin at)$ $\qquad [\because e^{iat} = \cos at + i\sin at]$

$\Longrightarrow \frac{1}{s-ia} = L(\cos at) + iL(\sin at)$

$$\Rightarrow \frac{s+ia}{s^2+a^2} = L(\cos at) + iL(\sin at)$$

Comparing real & imaginary parts

$$L(\cos at) = \frac{s}{s^2+a^2} \quad \& \quad L(\sin at) = \frac{a}{s^2+a^2}$$

5) $$L(t^n) = \int_0^\infty e^{-st}\, t^n \, dt = \left(t^n\left(\frac{e^{-st}}{-s}\right)\right)_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\, n t^{n-1}\, dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt$$

$$= \frac{n}{s} L(t^{n-1}) \quad \text{—(1)}$$

Similarly,

$$L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$$

$$L(t^{n-2}) = \frac{n-2}{s} L(t^{n-3})$$

$$\vdots$$

$$L(t^2) = \frac{2}{s} L(t)$$

$$L(t) = \frac{1}{s} L(1)$$

$$L(1) = \frac{1}{s}$$

Substituting in (1)

$$L(t^n) = \frac{n}{s}\left(\frac{n-1}{s}\right)\left(\frac{n-2}{s}\right) \cdots\cdots \left(\frac{2}{s}\right)\left(\frac{1}{s}\right)\left(\frac{1}{s}\right)$$

$$= \underline{\underline{\frac{n!}{s^{n+1}}}}$$

First shifting theorem

If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] = F(s-a)$

10) $L(e^{at} \sin bt)$

Let $f(t) = \sin bt \Longrightarrow F(s) = L(\sin bt) = \frac{b}{s^2+b^2}$

$\therefore L(e^{at} \sin bt) = F(s-a) = \frac{b}{(s-a)^2+b^2}$

11) $L(e^{at} \cos bt)$

$f(t) = \cos bt \Longrightarrow F(s) = L(\cos bt) = \frac{s}{s^2+b^2}$

$L(e^{at} \cos bt) = F(s-a) = \frac{(s-a)}{(s-a)^2+b^2}$

12) $L(e^{at} t^n)$

$f(t) = t^n \Longrightarrow F(s) = L(t^n) = \frac{n!}{s^{n+1}}$

$\therefore L(e^{at} t^n) = F(s-a) = \frac{n!}{(s-a)^{n+1}}$

Problems

1) $L(2t+8) = 2L(t) + 8L(1) = \frac{2}{s^2} + \frac{8}{s}$

2) $L(a-bt)^2 = L(a^2 + b^2t^2 - 2abt) = \frac{a^2}{s} + b^2\left(\frac{2!}{s^3}\right) - \frac{2ab}{s^2}$

3) $L(\cos 2\pi t) = \frac{s}{s^2+4\pi^2}$

4) $L(\cos^2 2t) = L\left[\frac{1+\cos 4t}{2}\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$

5) $L(\sin 3t \cos 2t) = L\left[\frac{\sin 5t + \sin t}{2}\right]$

$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

6) $L(e^{3t}\sinh 4t)$

Let $f(t) = \sinh 4t \Longrightarrow F(s) = L(\sinh 4t)$

$= \frac{4}{s^2-16}$

$\therefore L(e^{3t}\sinh 4t) = F(s-3) = \frac{4}{(s-3)^2-16}$

7) $L[\cos(3t+\theta)] = L[\cos 3t \cos\theta - \sin 3t \sin\theta]$

$= \cos\theta \, L(\cos 3t) - \sin\theta \, L(\sin 3t)$

$= \cos\theta\left(\frac{s}{s^2+9}\right) - \sin\theta\left(\frac{3}{s^2+9}\right)$

8) $L\left[2\sin\left(3t-\frac{\pi}{2}\right)\right] = 2L\left[-\sin\left(\frac{\pi}{2}-3t\right)\right] = -2L(\cos 3t)$

$= \frac{-2s}{s^2+9}$

9) $L(t^3 e^{-2t})$

Let $f(t) = t^3 \Longrightarrow F(s) = L(t^3) = \frac{3!}{s^4}$

$\therefore L(e^{-2t}t^3) = F(s+2) = \frac{3!}{(s+2)^4}$

10) $L\left(2e^{-\frac{t}{2}}\sin 4\pi t\right)$

$f(t) = \sin 4\pi t \implies F(s) = L(\sin 4\pi t) = \dfrac{4\pi}{s^2+16\pi^2}$

$\therefore L\left(2e^{-\frac{t}{2}}\sin 4\pi t\right) = 2F\left(s+\frac{1}{2}\right) = 2\left[\dfrac{4\pi}{\left(s+\frac{1}{2}\right)^2+16\pi^2}\right]$

11) $L(\sinh t\cos t) = L\left\{\left(\dfrac{e^t - e^{-t}}{2}\right)\cos t\right\}$

$= \dfrac{1}{2}\left[L(e^t\cos t) - L(e^{-t}\cos t)\right]$

$= \dfrac{1}{2}\left[\dfrac{s-1}{(s-1)^2+1} - \dfrac{s+1}{(s+1)^2+1}\right]$

12) $L(\cos^3 2t) = L\left(\dfrac{\cos 6t + 3\cos 2t}{4}\right)$

$= \dfrac{1}{4}\left[\dfrac{s}{s^2+36} + \dfrac{3s}{s^2+4}\right]$

Result :- If $L[f(t)] = F(s)$, then

$L[t^n f(t)] = (-1)^n F^n(s)$

1) $L(t\sin 5t)$

Soln : Let $f(t) = \sin 5t \implies F(s) = L(\sin 5t) = \dfrac{5}{s^2+25}$

$\therefore L(t\sin 5t) = (-1)^1 F'(s) = \dfrac{10s}{(s^2+25)^2}$

2) $L(t\cos 3t)$

Soln :- Let $f(t) = \cos 3t \Longrightarrow F(s) = L(\cos 3t) = \frac{s}{s^2+9}$

$$\therefore L(t\cos 3t) = (-1)^1 F'(s) = \frac{s^2-9}{(s^2+9)^2}$$

3) $L(t^2 \sinh 2t)$

Soln :- Let $f(t) = \sinh 2t \Longrightarrow F(s) = L(\sinh 2t) = \frac{2}{s^2-4}$

$$\therefore L(t^2 \sinh 2t) = (-1)^2 F''(s)$$

$$= F'\left[\frac{-4s}{(s^2-4)^2}\right]$$

$$= \frac{(s^2-4)^2(-4) + 4s(2)(s^2-4)(2s)}{(s^2-4)^4}$$

$$= \frac{12s^2+16}{(s^2-4)^3}$$

Result :- If $L[f(t)] = F(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

1. $L\left(\frac{\sin t}{t}\right)$

Soln :- Let $f(t) = \sin t \Longrightarrow F(s) = L(\sin t) = \frac{1}{s^2+1}$

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty \left(\frac{1}{s^2+1}\right) ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

2) $L\left(\frac{\sin^2 t}{t}\right) = L\left[\frac{1-\cos 2t}{2t}\right]$

Let $f(t) = 1-\cos 2t \Rightarrow F(s) = L(1-\cos 2t)$

$= \frac{1}{s} - \frac{s}{s^2+4}$

$\therefore L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log 1 - \log\sqrt{1+\frac{4}{s^2}}\right]_s^\infty$

$= \frac{1}{2}\log\sqrt{1+\frac{4}{s^2}} = \underline{\underline{\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)}}$

3) $L\left[\frac{\cos 3t - \cos 2t}{t}\right]$

<u>Soln</u>: Let $f(t) = \cos 3t - \cos 2t \Rightarrow F(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$\therefore L\left[\frac{\cos 3t - \cos 2t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right) ds$

$= \frac{1}{2}\left[\log(s^2+9) - \log(s^2+4)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+9}{s^2+4}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{1+\frac{9}{s^2}}{1+\frac{4}{s^2}}\right)\right]_s^\infty$

$= \underline{\underline{\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)}}$

Problems

Find the Laplace transform of

1) $\sin 2t \cos 3t$

2) $\sin^3 2t$

3) $e^{-3t}(\cos 4t + 3\sin 4t)$

4) $t e^{-4t} \sin 3t$

5) $\cos(at+b)$

6) $\sin at \sin bt$

7) $\sin^2 3t$

8) $e^{-2t} \sin 4t$

9) $t \sin^2 3t$

10) $t \sin 3t \cos 2t$

11) $t^2 e^{-2t} \cos t$

12) $\dfrac{e^{-t} \sin t}{t}$

13) $\dfrac{1-\cos 2t}{t}$

Inverse Laplace Transforms

	$F(s)$	$f(t) = L^{-1}[F(s)]$
1)	0	0
2)	$\frac{1}{s}$	1
3)	$\frac{1}{s^2}$	t
4)	$\frac{1}{s-a}$	e^{at}
5)	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
6)	$\frac{1}{s^2+a^2}$	$\frac{1}{a} \sin at$
7)	$\frac{s}{s^2+a^2}$	$\cos at$
8)	$\frac{1}{s^2-a^2}$	$\frac{1}{a} \sinh at$
9)	$\frac{s}{s^2-a^2}$	$\cosh at$
10)	$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b} e^{at} \sin bt$
11)	$\frac{(s-a)}{(s-a)^2+b^2}$	$e^{at} \cos bt$
12)	$\frac{1}{(s-a)^{n+1}}$	$e^{at} \frac{t^n}{n!}$

1) Find $L^{-1}\left[\frac{2s+14}{s^2+196}\right]$

<u>Soln</u>: $L^{-1}\left[\frac{2s+14}{s^2+196}\right] = 2L^{-1}\left[\frac{s}{s^2+(14)^2}\right] + L^{-1}\left[\frac{14}{s^2+(14)^2}\right]$

$= 2\cos 14t + \sin 14t$

2) $L^{-1}\left(\frac{s}{l^2s^2+\frac{\pi^2}{4}}\right)$

<u>Soln</u>: $L^{-1}\left[\frac{s}{l^2s^2+\frac{\pi^2}{4}}\right] = \frac{1}{l^2}L^{-1}\left[\frac{s}{s^2+\frac{\pi^2}{4l^2}}\right] = \frac{1}{l^2}\cos\left(\frac{\pi}{2l}t\right)$

3) Find $L^{-1}\left[\frac{2}{s^4}-\frac{48}{s^6}\right]$

<u>Soln</u>: $L^{-1}\left[\frac{2}{s^4}-\frac{48}{s^6}\right] = 2L^{-1}\left[\frac{1}{s^4}\right] - 48L^{-1}\left(\frac{1}{s^6}\right)$

$= 2\frac{t^3}{3!} - 48\frac{t^5}{5!}$

$= \frac{t^3}{3} - \frac{2t^5}{5}$

4) Find $L^{-1}\left[\frac{1}{(s(s+1)s+2)}\right]$

<u>Soln</u>: $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

$\Longrightarrow 1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

Put $s=0 \Longrightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$; Put $s=-1 \Rightarrow 1 = -B \Rightarrow B=-1$

Put $s=-2 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

5) Find $L^{-1}\left[\frac{1}{s^2(s+1)}\right]$

Soln :- $$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\Longrightarrow 1 = As(s+1) + B(s+1) + Cs^2$$

Put $s=0 \Rightarrow 1 = B$

Put $s=-1 \Rightarrow 1 = C$

Compare the coefficient of s^2

$$0 = A+C \Rightarrow 0 = A+1 \Rightarrow A = -1$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s+1)}\right] = L^{-1}\left(\frac{-1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -1 + t + e^{-t}$$

6) Find $L^{-1}\left[\frac{-s+11}{s^2-2s-3}\right]$

Soln :- $$\frac{-s+11}{(s-3)(s+1)} = \frac{2}{s-3} + \frac{(-3)}{s+1}$$ (Using partial fractions)

$$\therefore L^{-1}\left[\frac{-s+11}{s^2-2s-3}\right] = 2L^{-1}\left[\frac{1}{s-3}\right] - 3L^{-1}\left[\frac{1}{s+1}\right] = 2e^{3t} - 3e^{-t}$$

OR (another method)

$$L^{-1}\left[\frac{-s+11}{s^2-2s-3}\right] = L^{-1}\left[\frac{-s+11}{(s-1)^2-4}\right] = L^{-1}\left[\frac{-(s-1)+10}{(s-1)^2-4}\right]$$

$$= -L^{-1}\left[\frac{(s-1)}{(s-1)^2-2^2}\right] + 10\,L^{-1}\left[\frac{1}{(s-1)^2-2^2}\right]$$

$$= -e^{t}\cosh 2t + \frac{10}{2} e^{t}\sinh 2t$$

7) Find $L^{-1}\left[\frac{2\pi}{(s+\pi)^3}\right]$

Soln :- $L^{-1}\left[\frac{2\pi}{(s+\pi)^3}\right] = 2\pi\, e^{-\pi t}\frac{t^2}{2!} = \pi e^{-\pi t} t^2$

8) Find $L^{-1}\left[\frac{2}{s^2+8s+12}\right]$

Soln :- $L^{-1}\left[\frac{2}{s^2+8s+12}\right] = L^{-1}\left[\frac{2}{(s+4)^2-4}\right]$

$$= e^{-4t}\sinh 2t$$

9) Find $L^{-1}\left[\frac{6s+7}{2s^2+4s+10}\right]$

Soln :- $L^{-1}\left[\frac{6s+7}{2s^2+4s+10}\right] = L^{-1}\left[\frac{6s+7}{2(s^2+2s+5)}\right]$

$$= \frac{1}{2}L^{-1}\left[\frac{6s+7}{(s+1)^2+4}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{6(s+1)+1}{(s+1)^2+4}\right]$$

$$= \frac{1}{2}\left\{L^{-1}\left[\frac{6(s+1)}{(s+1)^2+4}\right] + L^{-1}\left[\frac{1}{(s+1)^2+4}\right]\right\}$$

$$= \frac{1}{2}\left[6e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right]$$

10) Find $L^{-1}\left[\dfrac{3s-137}{s^2+2s+401}\right]$

<u>Soln:</u> $L^{-1}\left[\dfrac{3s-137}{s^2+2s+401}\right] = L^{-1}\left[\dfrac{3s-137}{(s+1)^2+400}\right]$

$$= L^{-1}\left[\dfrac{3(s+1)-140}{(s+1)^2+400}\right]$$

$$= 3e^{-t}\cos 20t - \frac{140}{20}e^{-t}\sin 20t$$

$$= 3e^{-t}\cos 20t - 7e^{-t}\sin 20t$$

<u>H.W</u>

1 Find

i) $L^{-1}\left[\dfrac{5s+1}{s^2-25}\right]$

ii) $L^{-1}\left[\dfrac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right]$

iii) $L^{-1}\left[\dfrac{4s+32}{s^2-16}\right]$

iv) $L^{-1}\left[\dfrac{6}{(s+1)^3}\right]$

v) $L^{-1}\left[\dfrac{90}{(s+\sqrt{3})^6}\right]$

vi) $L^{-1}\left[\dfrac{1}{s^3-2\pi s^2}\right]$

Convolution theorem

Convolution :- (mathematical way of combining two signals & form a third signal)

The convolution of two functions $f(t)$ and $g(t)$ denoted by $f*g(t)$ is defined as $\int_0^t f(u)\,g(t-u)\,du$

Convolution theorem

Let $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$.

Then $L(f*g) = L[f(t)]\,L(g(t)) = F(s)\,G(s)$

$$\Rightarrow f*g = L^{-1}[F(s)\,G(s)] = \int_0^t f(u)\,g(t-u)\,du$$

1) Find $1 * -1$

Soln :- $f(t) = 1 \quad g(t) = -1$

$$f*g = 1*(-1) = \int_0^t 1(-1)\,du = -[u]_0^t = \underline{-t}$$

2) Find $e^{-t} * e^{t}$

Soln :- $e^{-t} * e^{t} = \int_0^t f(u)\,g(t-u)\,du = \int_0^t e^{-u}\,e^{t-u}\,du$

$$= \int_0^t e^{t-2u}\,du = \left[\frac{e^{t-2u}}{-2}\right]_0^t$$

$$= \frac{e^{t}-e^{-t}}{2} = \underline{\sinh t}$$

3) Find $\cos wt * 1$

Soln :- $\cos wt * 1 = \int_0^t \cos wu\,du = \left[\frac{\sin wu}{w}\right]_0^t = \underline{\frac{\sin wt}{w}}$

4) Find $t * e^{-t}$

Soln :- $t*e^{-t} = \int_0^t u\,e^{-(t-u)}\,du = \left[u e^{-(t-u)} - e^{-(t-u)}\right]_0^t$

$$= \underline{t - 1 + e^{-t}}$$

H.W Find i) $1 * \sin \omega t$

ii) $\cos \omega t * \cos \omega t$

iii) $e^{at} * e^{bt}$ $(a \neq b)$

To find inverse using convolution theorem

1) Find $L^{-1}\left[\frac{1}{(s-a)s}\right]$

Soln: Let $F(s) = \frac{1}{s-a}$ $\quad G(s) = \frac{1}{s}$

$\Rightarrow f(t) = L^{-1}[F(s)] = L^{-1}\left(\frac{1}{s-a}\right)$

$= e^{at}$

$g(t) = L^{-1}[G(s)]$

$= L^{-1}\left(\frac{1}{s}\right)$

$= 1$

$\therefore L^{-1}\left[\frac{1}{(s-a)s}\right] = \int_0^t f(u) g(t-u)\, du = \int_0^t e^{au}(1)\, du = \left[\frac{e^{au}}{a}\right]_0^t$

$= \frac{e^{at}-1}{a}$

2) Find $L^{-1}\left[\frac{1}{(s+1)(s-4)}\right]$

Soln: Let $F(s) = \frac{1}{s+1}$ $\quad G(s) = \frac{1}{s-4}$

$\Rightarrow f(t) = L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$ $\quad \Rightarrow g(t) = L^{-1}\left(\frac{1}{s-4}\right) = e^{4t}$

$L^{-1}\left[\frac{1}{(s+1)(s-4)}\right] = \int_0^t e^{-u} e^{4(t-u)}\, du = \int_0^t e^{4t-5u}\, du$

$= \left[\frac{e^{4t-5u}}{-5}\right]_0^t = \frac{e^{-t} - e^{4t}}{-5}$

3) Find $L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right]$

Soln :- $F(s) = \frac{1}{s^2+\omega^2}$ $\quad G(s) = \frac{1}{s^2+\omega^2}$

$f(t) = L^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{\sin\omega t}{\omega}$ $\quad g(t) = L^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega t}{\omega}$

$$\therefore L^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \int_0^t \frac{\sin\omega u}{\omega} \cdot \frac{\sin\omega(t-u)}{\omega}\,du$$

$$= \frac{1}{2\omega^2}\int_0^t \left[\cos(2\omega u-\omega t) - \cos\omega t\right] du$$

$$= \frac{1}{2\omega^2}\left[\frac{\sin(2\omega u-\omega t)}{2\omega} - u\cos\omega t\right]_0^t$$

$$= \frac{1}{2\omega^2}\left[\frac{\sin\omega t}{2\omega} - t\cos\omega t + \frac{\sin\omega t}{2\omega}\right]$$

$$= \frac{1}{2\omega^2}\left[\frac{\sin\omega t}{\omega} - t\cos\omega t\right]$$

4) Find $L^{-1}\left[\frac{18s}{(s^2+36)^2}\right]$

Soln :- $F(s) = \frac{s}{s^2+36}$ $\quad G(s) = \frac{1}{s^2+36}$

$f(t) = L^{-1}\left[\frac{s}{s^2+36}\right] = \cos 6t$ $\quad g(t) = L^{-1}\left(\frac{1}{s^2+36}\right) = \frac{\sin 6t}{6}$

$$\therefore L^{-1}\left[\frac{18s}{(s^2+36)^2}\right] = 18\int_0^t \cos 6u \cdot \frac{\sin 6(t-u)}{6}\,du$$

$$= \frac{3}{2}\int_0^t \sin 6t + \sin(6t-12u)\,du$$

$$= \frac{3}{2}\left[u\sin 6t + \frac{\cos(6t-12u)}{12}\right]_0^t = \frac{3}{2}t\sin 6t$$

H.W

Find the inverse using convolution theorem.

i) $\frac{1}{(s-a)^2}$

ii) $\frac{2\pi s}{(s^2+\pi^2)^2}$

iii) $\frac{9}{s(s+3)}$

iv) $\frac{\omega}{s^2(s^2-\omega^2)}$

v) $\frac{40}{s(s^2-9)}$

vi) $\frac{1}{(s^2+1)(s^2+25)}$

Laplace transform of derivatives & integrals

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

$$\therefore L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \dots\dots - f^{n-1}(0).$$

1. Using the formula for Laplace transform of derivatives, find $L(\sin at)$.

Soln :- $f(t) = \sin at \quad f'(t) = a\cos at \quad f''(t) = -a^2 \sin at$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$\Rightarrow L[-a^2 \sin at] = s^2 L(\sin at) - a$$

$$\Rightarrow L(\sin at)(s^2 + a^2) = a$$

$$\Rightarrow L(\sin at) = \frac{a}{s^2 + a^2}.$$

2. $L(te^{at})$

$f(t) = te^{at} \quad f'(t) = ate^{at} + e^{at}$

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[ate^{at} + e^{at}] = sL(te^{at})$$

$$\Rightarrow L(te^{at})(s-a) = L(e^{at}) = \frac{1}{s-a}$$

$$\Rightarrow L(te^{at}) = \frac{1}{(s-a)^2}$$

H.W

i) $f(t) = t\sin at$

ii) $f(t) = t\cos 4t$

iii) $f(t) = \cos^2 \omega t$

Laplace transform of integrals

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Thus $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$.

1. Find the inverse using integration.

i) $\frac{1}{s(s^2+\omega^2)}$

Soln:- Let $F(s) = \frac{1}{s^2+\omega^2} \Rightarrow f(t) = L^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega t}{\omega}$

$\therefore L^{-1}\left[\frac{1}{s(s^2+\omega^2)}\right] = \int_0^t \frac{\sin\omega t}{\omega}\,dt = \frac{1}{\omega}\left[\frac{-\cos\omega t}{\omega}\right]_0^t = \frac{1-\cos\omega t}{\omega^2}$

ii) $\frac{2}{s(s+3)}$

Soln:- Let $F(s) = \frac{1}{s+3} \Rightarrow f(t) = L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$

$\therefore L^{-1}\left[\frac{2}{s(s+3)}\right] = 2\int_0^t e^{-3t}\,dt = 2\left[\frac{e^{-3t}}{-3}\right]_0^t = \frac{2}{3}\left[1-e^{-3t}\right]$

H.W

i) Find $L^{-1}\left[\frac{1}{s(s+1)^3}\right]$ using integration

ii) Find $L^{-1}\left[\frac{1}{s^2(s+2)}\right]$ using integration.

Solving differential equations using Laplace transform

1. Solve the initial value problem

$y'' - y = t$. Given $y(0) = 1$ and $y'(0) = 1$

Soln :- Apply Laplace transform on both sides

$$L(y'') - L(y) = L(t)$$

$$\Rightarrow [s^2 L(y) - s\,y(0) - y'(0)] - L(y) = \frac{1}{s^2}$$

$$\Rightarrow (s^2 - 1) L(y) - s - 1 = \frac{1}{s^2}$$

$$\Rightarrow (s^2 - 1) L(y) = \frac{1}{s^2} + s + 1$$

$$\Rightarrow L(y) = \frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s^2(s^2-1)}\right] + L^{-1}\left[\frac{s}{s^2-1}\right] + L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2-1} - \frac{1}{s^2}\right] + L^{-1}\left[\frac{s}{s^2-1}\right] + L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$= \sinh t - t + \cosh t + \sinh t$$

$$= \underline{2\sinh t + \cosh t - t}$$

2. Solve $y'' - 3y' + 2y = 4t - 8$. Given $y(0) = 2$ and $y'(0) = 7$

Soln :- Apply Laplace transform on both sides.

$$L(y'') - 3L(y') + 2L(y) = 4L(t) - 8L(1)$$

$$\Rightarrow [s^2 L(y) - s\,y(0) - y'(0)] - 3[sL(y) - y(0)] + 2L(y) = \frac{4}{s^2} - \frac{8}{s}$$

$$\Rightarrow (s^2 - 3s + 2) L(y) - 2s - 7 + 6 = \frac{4}{s^2} - \frac{8}{s}$$

$$\Rightarrow (s^2 - 3s + 2) L(y) = \frac{4}{s^2} - \frac{8}{s} + 2s + 1$$

$$\Rightarrow L(y) = \frac{4-8s+2s^3+s^2}{s^2(s^2-3s+2)} = \frac{4-8s+2s^3+s^2}{s^2(s-2)(s-1)}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s-1}$$

$$\Rightarrow 4-8s+2s^3+s^2 = As(s-2)(s-1) + B(s-2)(s-1) + Cs^2(s-1) + Ds^2(s-2)$$

Put $s=0$ $\quad 4 = 2B \Rightarrow B = 2$

$s=1$ $\quad -1 = -D \Rightarrow D = 1$

$s=2$ $\quad 8 = 4C \Rightarrow C = 2$

Coefficients of s^3 $\quad 2 = A+C+D \Rightarrow A = 2-C-D = 2-2-1 = -1$

$$\therefore L(y) = \frac{-1}{s} + \frac{2}{s^2} + \frac{2}{s-2} + \frac{1}{s-1}$$

$$\therefore y = \underline{\underline{-1+2t+2e^{2t}+e^{t}}}$$

3. Solve $(D^2+4)y = \sin 2t$. Given $y(0)=3$, $y'(0)=4$

Soln $\quad L(y'') + 4L(y) = L(\sin 2t)$

$$\Rightarrow s^2L(y) - sy(0) - y'(0) + 4L(y) = \frac{2}{s^2+4}$$

$$\Rightarrow (s^2+4)L(y) = \frac{2}{s^2+4} + 3s + 4$$

$$\Rightarrow L(y) = \frac{2}{(s^2+4)^2} + \frac{3s}{s^2+4} + \frac{4}{s^2+4}$$

$$\Rightarrow y = L^{-1}\left[\frac{2}{(s^2+4)^2}\right] + 3L^{-1}\left[\frac{s}{s^2+4}\right] + 4L^{-1}\left[\frac{1}{s^2+4}\right]$$

To find $L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$ use convolution theorem.

Let $F(s) = \frac{1}{s^2+4}$ $\qquad G(s) = \frac{1}{s^2+4}$

$$\Rightarrow f(t) = \frac{1}{2}\sin 2t \qquad g(t) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left[\frac{1}{(s^2+4)^2}\right] = \int_0^t \frac{1}{2}\sin 2u \, \frac{1}{2}\sin 2(t-u)\, du$$

$$\cancel{= \frac{1}{4}\int_0^t \left[\cos 2t + \cos(4u-2t)\right] du} \qquad = \frac{1}{4}\int_0^t \frac{\cos(4u-2t) - \cos 2t}{2}\, du$$

$$\cancel{= \frac{1}{4}\left[u\cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t} \qquad = \frac{1}{8}\left[\frac{\sin(4u-2t)}{4} - u\cos 2t\right]_0^t$$

$$\cancel{= \frac{1}{4}\left[t\cos 2t + \frac{\sin 2t}{4} - \frac{\sin 2t}{4}\right]} \qquad = \frac{1}{8}\left[\frac{\sin 2t}{4} - t\cos 2t + \frac{\sin 2t}{4}\right]$$

$$\cancel{= \frac{t}{4}\cos 2t} \qquad = \frac{1}{8}\left[\frac{\sin 2t}{2} - t\cos 2t\right]$$

$$\therefore y = 2\left(\frac{t}{4}\cos 2t\right) + 3\cos 2t + \sin 2t$$

4) Solve $y' + \frac{2}{3}y = -4\cos 2t$. Given $y(0) = 0$

Soln :-

$$L(y') + \frac{2}{3}L(y) = -4L(\cos 2t)$$

$$sL(y) - y(0) + \frac{2}{3}L(y) = \frac{-4s}{s^2+4}$$

$$\left[s + \frac{2}{3}\right]L(y) = \frac{-4s}{s^2+4}$$

$$\Longrightarrow L(y) = \frac{-4s}{(s^2+4)\left(s+\frac{2}{3}\right)} = \frac{-12s}{(s^2+4)(3s+2)}$$

$$\frac{-12s}{(s^2+4)(3s+2)} = \frac{As+B}{s^2+4} + \frac{C}{3s+2}$$

$$-12s = (As+B)(3s+2) + C(s^2+4)$$

Put $s = \frac{-2}{3}$ $\quad 8 = C\left(\frac{4}{9}+4\right) \Rightarrow C = \frac{9}{5}$

Coefficients of s^2, $\quad 0 = 3A + C \Rightarrow A = \frac{-C}{3} = \frac{-9}{15}$

Coefficient of s, $\quad -12 = 2A + 3B \Rightarrow -12 = \frac{-18}{15} + 3B \Rightarrow B = \frac{-18}{5}$

$$y = \frac{-9}{15} L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{18}{5} L^{-1}\left(\frac{1}{s^2+4}\right) + \frac{9}{15} L^{-1}\left[\frac{1}{\left(s+\frac{2}{3}\right)}\right]$$

$$= \frac{-9}{15}\cos 2t - \frac{18}{10}\sin 2t + \frac{9}{15} e^{-\frac{2t}{3}}$$

5) Solve $y'' + 2y' - 3y = 0$. Given $y(2) = -3$ & $y'(2) = -5$

Ans Here $x_0 = 2$

Introduce $\tilde{x} = x - x_0 = x - 2$

$\therefore \tilde{y}'' + 2\tilde{y}' - 3\tilde{y} = 0$ $\quad \tilde{y}(0) = -3$ $\quad \tilde{y}'(0) = -5$

Apply Laplace transform on both sides,

$$\left[s^2 L(\tilde{y}) - s\tilde{y}(0) - \tilde{y}'(0)\right] + 2\left[s L(\tilde{y}) - \tilde{y}(0)\right] - 3L(\tilde{y}) = 0$$

$$\Longrightarrow (s^2 + 2s - 3) L(\tilde{y}) + 3s + 5 + 6 = 0$$

$$\Longrightarrow L(\tilde{y}) = \frac{-3s-11}{(s+3)(s-1)} = \frac{1}{2(s+3)} - \frac{7}{2(s-1)} \quad \text{(Using partial fraction)}$$

$$\Longrightarrow \tilde{y} = L^{-1}\left[\frac{1}{2(s+3)}\right] - L^{-1}\left(\frac{7}{2(s-1)}\right) = \frac{1}{2}e^{-3\tilde{x}} - \frac{7}{2}e^{\tilde{x}}$$

$$\Longrightarrow y = \frac{1}{2}e^{-3(x-2)} - \frac{7}{2}e^{x-2}.$$

H.W

1) Solve $y' + 2y = 0$. Given $y(0) = 1.5$

2) Solve $y'' + y' - 6y = 0$. Given $y(0) = 1$ & $y'(0) = 1$

3) Solve $y'' + 9y = 10e^{-t}$ $\quad y(0) = 0$ $\quad y'(0) = 0$

4) Solve $y'' - 6y' + 5y = 29\cos 2t$ $\quad y(0) = 3.2$ $\quad y'(0) = 6.2$

5) Solve $y'' + 7y' + 12y = 21e^{3t}$ $\quad y(0) = 3.5$ $\quad y'(0) = -10$.

6) Solve $y' - 6y = 0$. Given $y(-1) = 4$

7) Solve $y'' + y = 2t$. $\quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$ $\quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$

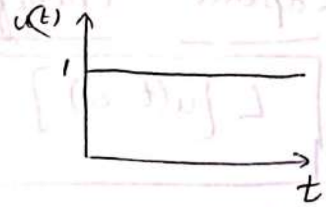
8) Solve $y'' + 2y' + 5y = 50t - 100$ Given $y(2) = -4$ & $y'(2) = 14$

Unit step function and second shifting theorem

The unit step function, also called Heaviside function is defined as

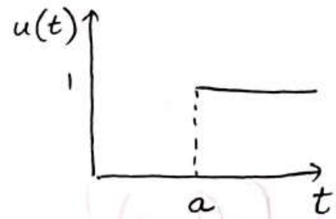
$$u(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t\geq 0 \end{cases}$$

The graph will be :



Similarly $u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t\geq a \end{cases}$

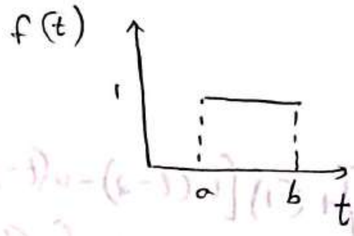
The graph will be

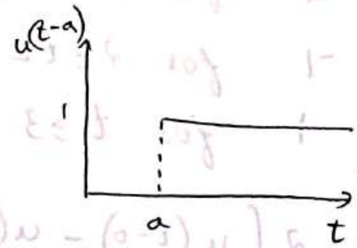


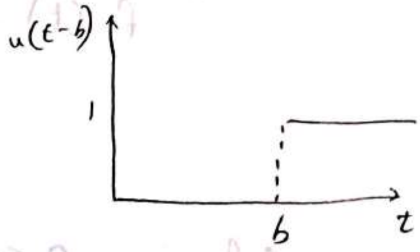
The fn $f(t)$ defined by $f(t) = \begin{cases} 1 & \text{for } a \leq t \leq b \\ 0 & \text{, otherwise} \end{cases}$

can be expressed in terms of unit step fn as

$$f(t) = u(t-a) - u(t-b)$$







Let $f(t)$ be a piecewise continuous fn defined as

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < t_1 \\ f_2(t) & \text{for } t_1 < t < t_2 \end{cases}$$

$$f(t) = f_1(t)\left[u(t-0) - u(t-t_1)\right] + f_2(t)\left[u(t-t_1) - u(t-t_2)\right]$$

Laplace transform of unit step function

$$\boxed{L\left[u(t-a)\right] = \frac{e^{-as}}{s}}$$

Proof:-

$$L\left[u(t-a)\right] = \int_0^\infty e^{-st} u(t-a)\,dt = \int_0^a (e^{-st} \times 0)\,dt + \int_a^\infty (e^{-st} \times 1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

$$L\left[u(t-a)\right] = \frac{e^{-as}}{s} \Longrightarrow \boxed{L^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a)}$$

1. Express the function in terms of unit step function and then find its Laplace transform.

$$f(t) = \begin{cases} 2 & \text{for } 0 < t < 2 \\ -1 & \text{for } 2 \le t < 3 \\ 1 & \text{for } t \ge 3 \end{cases}$$

Soln :- $f(t) = 2\left[u(t-0) - u(t-2)\right] + (-1)\left[u(t-2) - u(t-3)\right] + (1)\left[u(t-3)\right]$

$$\Longrightarrow f(t) = 2 - 3u(t-2) + 2u(t-3).$$

$$L\left[f(t)\right] = L(2) - 3L\left[u(t-2)\right] + 2L\left[u(t-3)\right] = \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-3s}}{s}$$

Second Shifting theorem

If $L[f(t)] = F(s)$, then

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

From the above, $L^{-1}[e^{-as}F(s)] = [f(t-a)u(t-a)]$

1. Express the function in terms of unit step function and find its Laplace transform.

i) $g(t) = \begin{cases} 0 & , t<4 \\ (t-4)^2 & , t \geq 4 \end{cases}$

Soln :- $g(t) = (t-4)^2 u(t-4)$

$L[g(t)] = L[(t-4)^2 u(t-4)]$ Here $f(t-4) = (t-4)^2$

$\Rightarrow f(t) = t^2$

$= e^{-4s}\dfrac{2}{s^3}$ (by second shifting theorem) $\Rightarrow L(f(t)) = \dfrac{2}{s^3} = F(s)$

ii) $g(t) = t-3$ for $t \geq 3$

Soln :- $g(t) = (t-3)u(t-3)$

$L[g(t)] = L[(t-3)u(t-3)]$ Here $f(t-3) = t-3$

$\Rightarrow f(t) = t$

$\Rightarrow L(f(t)) = \dfrac{1}{s^2} = F(s)$

$= \dfrac{e^{-3s}}{s^2}$

iii) $g(t) = t \quad (0<t<2)$

Soln :- $g(t) = t[u(t-0) - u(t-2)]$

$= t\,u(t-0) - t\,u(t-2)$

(4) LT-(25)

$$L[g(t)] = L[(t-0)u(t-0)] - L\{[(t-2)+2]u(t-2)\}$$

$$= L[(t-0)u(t-0)] - L[(t-2)u(t-2)] - 2L[u(t-2)]$$

$$= \frac{e^{-0s}}{s^2} - \frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s} = \frac{1}{s^2} - e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

iv) $g(t) = e^{-t} \quad (0 < t < \pi)$

soln :- $g(t) = e^{-t}[u(t-0) - u(t-\pi)]$

$$L[g(t)] = L[e^{-(t-0)}u(t-0)] - L[e^{-(t-\pi)-\pi}u(t-\pi)]$$

$$= L[e^{-(t-0)}u(t-0)] - e^{-\pi}L[e^{-(t-\pi)}u(t-\pi)]$$

$$= \frac{1}{s+1} - e^{-\pi}\frac{e^{-\pi s}}{s+1}$$

v) $g(t) = e^{-\frac{\pi t}{2}}$ for $(1 < t < 3)$

soln :- $g(t) = e^{-\frac{\pi t}{2}}[u(t-1) - u(t-3)]$

$$L[g(t)] = L[e^{-\frac{\pi t}{2}}u(t-1)] - L[e^{-\frac{\pi t}{2}}u(t-3)]$$

$$= L(e^{-\frac{\pi}{2}(t-1)-\frac{\pi}{2}}u(t-1)) - L(e^{-\frac{\pi}{2}(t-3)-\frac{3\pi}{2}}u(t-3))$$

$$= e^{-\frac{\pi}{2}}\cdot\frac{e^{-s}}{s+\frac{\pi}{2}} - e^{-\frac{3\pi}{2}}\frac{e^{-3s}}{s+\frac{\pi}{2}}$$

vi) $g(t) = \sin t$ for $\frac{\pi}{2} < t < \pi$

soln :- $g(t) = \sin t\,(u(t-\frac{\pi}{2}) - u(t-\pi))$

$$L[g(t)] = L[\sin t\, u(t-\tfrac{\pi}{2})] - L[\sin t\, u(t-\pi)]$$

$$= L\left\{\sin\left[\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right]u\left(t-\frac{\pi}{2}\right)\right\} - L\left\{\sin\left[(t-\pi)+\pi\right]u(t-\pi)\right\}$$

$$= L\left[\cos\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)\right] + L\left[\sin(t-\pi)\,u(t-\pi)\right]$$

$$= e^{-\frac{\pi s}{2}}\frac{s}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

vii) $g(t) = 2t^2$ for $t > \frac{5}{2}$

Soln :- $g(t) = 2t^2\, u\left(t-\frac{5}{2}\right)$

$$L[g(t)] = L\left[2t^2 u\left(t-\frac{5}{2}\right)\right]$$

$$= 2L\left\{\left[\left(t-\frac{5}{2}\right)^2 + 5t - \frac{25}{4}\right]u\left(t-\frac{5}{2}\right)\right\}$$

$$= 2L\left\{\left[\left(t-\frac{5}{2}\right)^2 + 5\left(t-\frac{5}{2}+\frac{5}{2}\right) - \frac{25}{4}\right]u\left(t-\frac{5}{2}\right)\right\}$$

$$= 2L\left[\left(t-\frac{5}{2}\right)^2 u\left(t-\frac{5}{2}\right)\right] + 10L\left[\left(t-\frac{5}{2}\right)u\left(t-\frac{5}{2}\right)\right] + 2L\left[\left(\frac{25}{2}-\frac{25}{4}\right)u\left(t-\frac{5}{2}\right)\right]$$

$$= 2e^{-\frac{5s}{2}}\frac{2}{s^3} + 10\frac{e^{-\frac{5s}{2}}}{s^2} + 2\left(\frac{25}{4}\right)\frac{e^{-\frac{5s}{2}}}{s}$$

$$= 2e^{-\frac{5s}{2}}\left[\frac{2}{s^3} + \frac{5}{s^2} + \frac{25}{4s}\right]$$

H.W

1) $g(t) = \cos 2t \quad (0 < t < \pi)$

2) $g(t) = \sin \pi t \quad (2 < t < 4)$

3) $g(t) = t^2 \quad (1 < t < 2)$

4) $g(t) = \begin{cases} 2 & 0 < t < 1 \\ \frac{t^2}{2} & 1 \le t < \frac{\pi}{2} \\ \cos t & t \ge \frac{\pi}{2} \end{cases}$

To find the inverse

1) Find the inverse of $\left(\frac{se^{-2s}}{s^2+16}\right)$

Soln :- By second shifting theorum,

$$L^{-1}\left[e^{-as}F(s)\right] = f(t-a)\,u(t-a)$$

Here $a = 2$ $\quad F(s) = \frac{s}{s^2+16} \Rightarrow f(t) = \cos 4t$

$\Rightarrow f(t-2) = \cos 4(t-2)$

$$\therefore L^{-1}\left[\frac{se^{-2s}}{s^2+16}\right] = \cos 4(t-2)\,u(t-2)$$

2) Find the inverse of $\frac{e^{-3s}}{s^6}$

Soln :- Here $a = 3$ $\quad F(s) = \frac{1}{s^6} \Rightarrow f(t) = \frac{t^5}{5!}$

$\Rightarrow f(t-3) = \frac{(t-3)^5}{5!}$

$$L^{-1}\left[\frac{e^{-3s}}{s^6}\right] = \frac{(t-3)^5}{5!}\,u(t-3)$$

3) Find $L^{-1}\left[\frac{e^{-2s}}{(s-1)^3}\right]$

Soln :- Here $a = 2$ $\quad F(s) = \frac{1}{(s-1)^3} \Rightarrow f(t) = \frac{t^2}{2}e^t$ (By first shifting theorem)

$$L^{-1}\left[\frac{e^{-2s}}{(s-1)^3}\right] = f(t-2)\,u(t-2)$$

$$= \frac{(t-2)^2}{2}e^{t-2}\,u(t-2)$$

4) Find $L^{-1}\left[\frac{1-e^{-\pi s}}{s^2+4}\right]$

Soln: $L^{-1}\left[\frac{1-e^{-\pi s}}{s^2+4}\right] = L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{e^{-\pi s}}{s^2+4}\right]$

$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t.$

To find $L^{-1}\left(\frac{e^{-\pi s}}{s^2+4}\right)$. Here $a = \pi$

$F(s) = \frac{1}{s^2+4} \Rightarrow f(t) = \frac{1}{2}\sin 2t$

$\Rightarrow f(t-\pi) = \frac{1}{2}\sin 2(t-\pi)$

$\therefore L^{-1}\left(\frac{e^{-\pi s}}{s^2+4}\right) = f(t-\pi)\,u(t-\pi) = \frac{1}{2}\sin 2(t-\pi)\,u(t-\pi).$

$\therefore L^{-1}\left[\frac{1-e^{-\pi s}}{s^2+4}\right] = \frac{1}{2}\sin 2t - \frac{1}{2}\sin 2(t-\pi)\,u(t-\pi)$

H.W

1) Find $L^{-1}\left[\frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}\right]$

2) Find $L^{-1}\left[\frac{e^{-2s} - 2e^{-5s}}{s}\right]$

3) Find $L^{-1}\left\{2\left[\frac{e^{-s} - e^{-3s}}{s^2-4}\right]\right\}$

To solve differential equations

1) Solve $y'' + 3y' + 2y = 1$ if $0<t<1$ and 0 if $t \geq 1$

Given $y(0) = 0$ $y'(0) = 0.$

Soln :- $y'' + 3y' + 2y = u(t-0) - u(t-1)$

Apply Laplace transform on both sides.

$L(y'') + 3L(y') + 2L(y) = L[u(t-0) - u(t-1)]$

$\Rightarrow [s^2 L(y) - sy(0) - y'(0)] + 3[sL(y) - y(0)] + 2L(y) = \frac{1}{s} - \frac{e^{-s}}{s}$

$\Rightarrow [s^2 + 3s + 2]L(y) = \frac{1}{s} - \frac{e^{-s}}{s}$

$\Rightarrow L(y) = \frac{1}{s(s+2)(s+1)} - \frac{e^{-s}}{s(s+1)(s+2)}$

$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$

$1 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$

Put $s = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$s = -2 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$s = -1 \Rightarrow 1 = -C \Rightarrow C = -1$

$\therefore L(y) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{(s+1)} - \frac{e^{-s}}{2s} - \frac{e^{-s}}{2(s+2)} + \frac{e^{-s}}{s+1}$

$\Rightarrow y = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}u(t-1) - \frac{1}{2}e^{-2(t-1)}u(t-1) + e^{-(t-1)}u(t-1)$

HW

1) Solve $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$

Given $y(0) = 0$ $y'(0) = 0$

2) Solve $y'' + y = 2t$ if $0 < t < 1$ and 2 if $t > 1$

Given $y(0) = 0$ $y'(0) = 0$

3) solve $y'' + y' - 2y = 3\sin t - \cos t$ if $0 < t < 2\pi$ and $3\sin 2t - \cos 2t$ if $t > 2\pi$. Given $y(0) = 0$ & $y'(0) = -1$

2) Solve $y'' + 4y = r(t)$ where $r(t) = 4\cos t$ if $0 < t < \pi$
and 0 if $t > \pi$

Given $y(0) = 0$ & $y'(0) = 0$

<u>Soln</u>:

$$y'' + 4y = 4\cos t\,[u(t) - u(t-\pi)]$$

$$\Rightarrow y'' + 4y = 4\cos t\,u(t) + 4\cos(t-\pi)\,u(t-\pi)$$

Apply Laplace transform on both sides

$$L(y'') + 4L(y) = L[4\cos t\,u(t)] + L[4\cos(t-\pi)\,u(t-\pi)]$$

$$\Rightarrow [s^2 L(y) - s\,y(0) - y'(0)] + 4L(y) = \frac{4s}{s^2+1} + 4\frac{e^{-\pi s}s}{s^2+1}$$

$$\Rightarrow (s^2+4)L(y) = \frac{4s}{s^2+1} + 4\frac{e^{-\pi s}s}{s^2+1}$$

$$\Rightarrow L(y) = \frac{4s}{(s^2+1)(s^2+4)} + 4\frac{e^{-\pi s}s}{(s^2+1)(s^2+4)}$$

$$= 4\left[\frac{1}{3}\left(\frac{s}{s^2+1}\right) - \frac{1}{3}\left(\frac{s}{s^2+4}\right)\right] + 4\left[\frac{e^{-\pi s}s}{3(s^2+1)} - \frac{e^{-\pi s}s}{3(s^2+4)}\right]$$

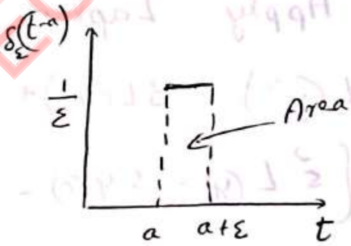
$$\Rightarrow y = \frac{4}{3}\cos t - \frac{4}{3}\cos 2t + \frac{4}{3}\cos(t-\pi)\,u(t-\pi) - \frac{4}{3}\cos 2(t-\pi)\,u(t-\pi)$$

Dirac Delta function or Unit impulse function

An airplane making a "hard" landing, a mechanical system being hit by a hammer blow, a ship being hit by a single high wave, a tennis ball hit by a racket and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces (large) are applied over short interval of time. Such phenomena can be modelled by Dirac delta function.

Consider the function $\delta_{\varepsilon}(t-a) = \begin{cases} \frac{1}{\varepsilon} & , a < t < a+\varepsilon \\ 0 & , \text{otherwise} \end{cases}$

The graph will be



Area $= 1 = \int_a^{a+\varepsilon} \delta_{\varepsilon}(t-a)\,dt$

As $\varepsilon \to 0$, the height of the strip increases infinitely

So Dirac delta function is defined as

$$\delta(t-a) = \lim_{\varepsilon \to 0} \delta_{\varepsilon}(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

such that $\int_0^{\infty} \delta(t-a)\,dt = 1$

Laplace transform of $\delta(t-a)$

$$\boxed{L[\delta(t-a)] = e^{-as}}$$

Proof

$$L[\delta_{\varepsilon}(t-a)] = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\varepsilon} = \frac{1}{\varepsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-s(a+\varepsilon)}}{s}\right]$$

Therefore $L[\delta(t-a)] = \lim_{\varepsilon \to 0} L[\delta_\varepsilon(t-a)]$

$$= \lim_{\varepsilon\to 0} \frac{e^{-as}}{s\varepsilon}\left[1-e^{-s\varepsilon}\right]$$

$$= e^{-as} \lim_{\varepsilon\to 0}\left[\frac{1-e^{-s\varepsilon}}{s\varepsilon}\right]$$

$$= e^{-as} \lim_{\varepsilon\to 0} e^{-s\varepsilon} \quad \text{[Using L'Hospital's rule]}$$

$$= \underline{\underline{e^{-as}}}$$

1) Solve $y'' + 3y' + 2y = \delta(t-1)$. Given $y(0)=0$ & $y'(0)=0$

<u>Soln</u>:- Apply Laplace transform on both sides,

$$L(y'') + 3L(y') + 2L(y) = L[\delta(t-1)]$$

$$\Longrightarrow \left[s^2 L(y) - s\,y(0) - y'(0)\right] + 3\left(sL(y) - y(0)\right] + 2L(y) = e^{-s}$$

$$\Longrightarrow (s^2+3s+2)L(y) = e^{-s}$$

$$\Longrightarrow L(y) = \frac{e^{-s}}{(s+1)(s+2)} = \frac{e^{-s}}{s+1} - \frac{e^{-s}}{s+2} \quad \text{(using partial fractions)}$$

$$\Longrightarrow y = L^{-1}\left[\frac{e^{-s}}{s+1}\right] - L^{-1}\left[\frac{e^{-s}}{s+2}\right]$$

$$= e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)$$

$$\therefore = \underline{\underline{\left[e^{-(t-1)} - e^{-2(t-1)}\right]u(t-1)}}$$

2) Solve $y'' + 9y = \delta\left(t-\frac{\pi}{2}\right)$ Given $y(0)=2$ & $y'(0)=0$

<u>Soln</u>:- $L(y'') + 9L(y) = L\left[\delta\left(t-\frac{\pi}{2}\right)\right]$

$$\Longrightarrow \left[s^2 L(y) - s\,y(0) - y'(0)\right] + 9L(y) = e^{-\frac{\pi}{2}s}$$

$$\Longrightarrow (s^2+9)L(y) = e^{-\frac{\pi}{2}s} + 2s$$

$$\Longrightarrow L(y) = \frac{e^{-\frac{\pi}{2}s}}{s^2+9} + \frac{2s}{s^2+9}$$

$$\Longrightarrow y = L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s^2+9}\right] + L^{-1}\left[\frac{2s}{s^2+9}\right]$$

$$= \frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right) + 2\cos 3t$$

3) Solve $y'' + 2y' + 5y = 25t - 100\,\delta(t-\pi)$

Given $y(0) = -2$ & $y'(0) = 5$

Soln :- $L(y'') + 2L(y') + 5L(y) = 25L(t) - 100L[\delta(t-\pi)]$

$$\Longrightarrow \left[s^2 L(y) - s\,y(0) - y'(0)\right] + 2\left[sL(y) - y(0)\right] + 5L(y) = \frac{25}{s^2} - 100e^{-\pi s}$$

$$\Longrightarrow (s^2+2s+5)L(y) = \frac{25}{s^2} - 100e^{-\pi s} - 2s + 1$$

$$\Longrightarrow L(y) = \frac{25}{s^2(s^2+2s+5)} - 100\frac{e^{-\pi s}}{s^2+2s+5} - \frac{2s}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$$= \left[\frac{-2s+5}{s^2} + \frac{2s-1}{s^2+2s+5}\right] - 100\frac{e^{-\pi s}}{s^2+2s+5} - \frac{2s}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$$= \left(\frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1)-3}{(s+1)^2+4}\right] - 100\frac{e^{-\pi s}}{(s+1)^2+4} - \left[\frac{2(s+1)-2}{(s+1)^2+4}\right] + \frac{1}{(s+1)^2+4}$$

$$\Longrightarrow y = L^{-1}\left(\frac{-2}{s}\right) + L^{-1}\left(\frac{5}{s^2}\right) + 2L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right] - 3L^{-1}\left[\frac{1}{(s+1)^2+4}\right] - 100L^{-1}\left(\frac{e^{-\pi s}}{(s+1)^2+4}\right] - 2L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right]$$

$$+ 2L^{-1}\left(\frac{1}{(s+1)^2+4}\right] + L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$\therefore y = -2 + 5t + \cancel{2e^{-t}\cos 2t} - \cancel{\frac{3}{2}e^{-t}\sin 2t} - \frac{100}{2}e^{-(t-\pi)}\sin 2(t-\pi)\,u(t-\pi)$$
$$- \cancel{2e^{-t}\cos 2t} + \cancel{e^{-t}\sin 2t} + \cancel{\frac{1}{2}e^{-t}\sin 2t}$$

$$\Rightarrow y = -2 + 5t \underline{- 50e^{-(t-\pi)}} \sin 2(t-\pi)\,u(t-\pi)$$

H.W

1) Solve $y'' + 16y = 4\delta(t-3\pi)$. Given $y(0) = 2$ & $y'(0) = 0$.

2) Solve $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$. Given $y(0) = 0$ & $y'(0) = 1$

3) Solve $y'' + 3y' + 2y = \sin t + \delta(t-1)$ Given $y(0) = 1$ & $y'(0) = -1$

4) Solve $y'' + 3y' + 2y = u(t-1) + \delta(t-2)$. Given $y(0) = 0$ & $y'(0) = 1$

5) Solve $4y'' + 16y' + 17y = 3e^{-t} + \delta(t - \frac{1}{4})$. Given $y(0) = \frac{3}{5}$ & $y'(0) = \frac{-3}{5}$