

17/8/2017

LINEAR ALGEBRA

MODULE-1

Systems OF Linear Equations

A linear system of 'm' equations in 'n' unknowns x_1, \dots, x_n is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad \rightarrow \textcircled{1}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

If all the b_j are zero, then $\textcircled{1}$ is called a homogeneous system. If at least one b_j is not zero, then $\textcircled{1}$ is called a non-homogeneous system.

Matrix form of the linear system $\textcircled{1}$ is

$$Ax = b \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented Matrix (\tilde{A}):

$$\text{The matrix } \tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

is called the augmented matrix of the system $\textcircled{1}$.

Gauss' Elimination and Back Substitution

- Convert augmented matrix to a triangular matrix using elementary row transformation.

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Q: Solve the system of equations by Gauss elimination.

$$2x + 3y + 2z = 5$$

$$3x - 5y + 5z = 2$$

$$3x + 9y - z = 4$$

$$x + 2y + z = 3$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{array} \right]$$

$$Ax = b$$

$$\tilde{A} = \left[\begin{array}{cccc} 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{array} \right]$$

$$\tilde{A} \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \\ 2 & 3 & 2 & 5 \end{array} \right]$$

$$R_1 \leftrightarrow R_4$$

$$\tilde{A} \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \\ 0 & -1 & 0 & -1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - 2R_1$$

$$\tilde{A} \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \\ 0 & 1 & 0 & -1 \end{array} \right]$$

$$R_4 \rightarrow -R_4$$

$$\sim \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & -4 & -5 \\ 0 & -11 & 2 & -7 \end{array} \right] \quad R_2 \leftrightarrow R_4$$

$$\sim \left[\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 2 & 4 \end{array} \right] \quad R_3 \rightarrow R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad R_4 \rightarrow 2R_4 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -8 \\ 0 \end{bmatrix}$$

$$\text{From } R_1: x + 2y + z = 3$$

$$y = 1$$

$$-4z = -8$$

$$z = 2$$

$$\therefore x = -1$$

$\therefore x = -1, y = 1, z = 2$, is the solution of given system.

Rank of a Matrix

Rank of a matrix is the number of linearly independent rows in a matrix.

(Technique to determine rank : no. of non-zero rows in Echelon form) is the rank of that matrix.

ECHELON FORM:

'Echelon' word meaning is 'step like'.

conditions of Echelon Form

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1. All non-zero rows are lying above all zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All the entries below the leading entry are zeroes

Leading Entry:

First non-zero number in a row is called leading entry.

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Q: By reducing into row echelon form find the rank of the matrix

$$\left(\begin{array}{ccccc} 3 & 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{array} \right)$$

$$A = \begin{bmatrix} 3 & 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{bmatrix} \quad 4 \times 5$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix} \quad R_1 \leftrightarrow R_2$$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{pmatrix} \quad R_2 \rightarrow R_2 - 3R_1$$

$R_3 \rightarrow R_3 - 4R_1$
 $R_4 \rightarrow R_4 - 9R_1$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$R_4 \rightarrow R_4 - 2R_2$

$$\approx \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_4 \rightarrow R_4 - R_3$$

\therefore Rank of the matrix = 3

(This is in echelon form. No. of non-zero rows are three. Therefore, rank of the matrix is 3)

$$\underline{\underline{F_r(A) = 3}}$$

$$\underline{\underline{r(A) = 3}}]$$

CONSISTENCY

Non-Homogeneous System

- If $\text{f}(\tilde{A}) = \text{f}(A)$, then the system is consistent.
 - * If $\text{f}(\tilde{A}) = \text{f}(A) = n$, the system has unique solution (n - no. of unknowns).
 - * If $\text{f}(\tilde{A}) = \text{f}(A) < n$, the system has infinitely many solutions.

Homogeneous System

- If $\text{f}(A) = n$, the system has only trivial solutions.
- If $\text{f}(A) < n$, the system has infinitely many non-trivial solutions.

Q: Test the consistency and hence solve the system of equations:

$$1. \quad 5x + 6y + 3z = 2$$

$$3x + 2y + z = 2$$

$$x + 2y + z = 0$$

$$2x - y + 2z = 5$$

$$x + 3y - z = -3$$

The given system of equation is written in the form $A\mathbf{x} = \mathbf{B}$.

$$\text{i.e., } \begin{bmatrix} 5 & 6 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 5 \\ -3 \end{bmatrix}$$

We know that the system $A\mathbf{x}=\mathbf{B}$ is consistent if $\text{f}(\tilde{A}) = \text{f}(A)$.

$$\text{Consider } \tilde{A} = \begin{bmatrix} 5 & 6 & 3 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 1 & 0 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 5 & 6 & 3 & 2 \\ 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -2 & 2 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & 1 & -2 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 5R_1 \\ R_4 \rightarrow R_4 - 2R_1 \\ R_5 \rightarrow R_5 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & -4 & -2 & 2 \\ 0 & -5 & 0 & 5 \\ 0 & -4 & -2 & 2 \end{bmatrix} \quad R_2 \leftrightarrow R_5$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & -10 & -10 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 + 4R_2 \\ R_4 \rightarrow R_4 + 5R_2 \\ R_5 \rightarrow R_5 + 4R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_4 \rightarrow R_4 - R_3$$

$$R_5 \rightarrow R_5 - R_3$$

$$f(\tilde{A}) = \text{No. of non-zero rows in } \tilde{A} = 3$$

$$f(A) = \text{No. of non-zero rows in } A = 3$$

$$f(\tilde{A}) = f(A)$$

\therefore The given system of solution is consistent.

$$n = \text{no. of unknown} = 3$$

$$f(\tilde{A}) = f(A) = n$$

Hence, the system has unique solution.

$$x + 2y + z = 0 \Rightarrow x = 1$$

$$y - 2z = -3 \Rightarrow y = -1$$

$$-10z = -10 \Rightarrow z = 1$$

$$\therefore y = -3 + 2 = -1$$

\therefore Solution is $x = 1, y = -1, z = 1$

Q: Solve the system of equations:

$$y - 3z = -1$$

$$x + z = 1$$

$$3x + y = 2$$

$$x + y - 2z = 0$$

The given system of equation is written in the form $AX = B$

$$\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

We know that the system $AX = B$ is consistent if $\mathfrak{f}(\tilde{A}) = \mathfrak{f}(A)$

Consider $\tilde{A} = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_2$$

$$\mathfrak{f}(\tilde{A}) = \text{no. of non-zero rows in } \tilde{A} = 2$$

$$\mathfrak{f}(A) = \text{no. of non-zero rows in } A = 2$$

$$\therefore \mathfrak{f}(X) = \mathfrak{f}(A)$$

\therefore The given system of solution is consistent.

$n = \text{no. of unknowns} = 3$

$$f(A') = f(A) < n$$

\therefore The system has infinite no. of solution.

$$x + z = 1$$

$$y - 3z = -1$$

Let $z = k$ (k is an arbitrary value)

$$y = -1 + 3k$$

$$x = 1 - k$$

$\therefore x = 1 - k, y = -1 + 3k, z = k$ is the solution
of the given system. ($k \in \mathbb{R}$)

Q: Find the values of λ and μ for which the system

of equations $2x + 3y + 5z = 9$

$$-x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

has (i) no solution (ii) unique solution (iii) infinitely many solutions. Find the solutions when they exist.

The given system of equations is written in the form of $Ax = B$.

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 1 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -3/2 & -4/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \quad R_2 \rightarrow R_2 - \frac{1}{2}R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \quad R_2 \rightarrow -2R_2$$

(i) $\lambda=5, \mu \neq 9$

(ii) $\lambda \neq 5, \mu$ can be any value.

(iii) $\lambda=5, \mu=9$

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Fundamental Theorem

Q: Do the equations $x-3y-8z=0, 3x+y=0, 2x+5y+6z=0$ have a non-trivial solution?

The given system is a homogeneous system

and it can be written as $AX=0$

$$\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

We know that, the homogeneous system $AX=0$ has a non-trivial solution, if $f(A) < n$.

Here, $f(A) < 3$

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} & R_2 \rightarrow R_2 - 3R_1 \\ &= \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -2 \end{bmatrix} & R_3 \rightarrow R_3 - \frac{11}{10}R_2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -2 \end{bmatrix} & R_3 \rightarrow R_3 - \frac{11}{10}R_2$$

But $f(A) = 3$

$$f(A) = n = 3$$

\therefore the eqn system has only trivial solution.

$$\therefore x=0, y=0, z=0$$

Hence, the system has no non-trivial solution.

Q: Solve the system of equations:

$$\therefore x+2y-z=0$$

$$3x+y-z=0$$

$$2x-y=0$$

Given system is homogeneous and can be written in the form

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

Hence, the system has infinitely many non-trivial solutions.

$$x + 2y - z = 0 \quad \text{---} ①$$

$$-5y + 2z = 0 \quad \text{---} ②$$

Let $z = t$

$$② \Rightarrow -5y = -2z$$

$$y = \frac{2}{5}z$$

$$y = \frac{2}{5}t$$

$$\therefore \textcircled{1} \Rightarrow x = -2y + z$$

$$= -2 \times \frac{2}{5}t + t$$

$$\therefore x = -\frac{4}{5}t + t = \underline{\underline{\frac{t}{5}}}$$

$$x = \frac{t}{5}$$

\therefore Solution is $x = \frac{t}{5}$, $y = \frac{2}{5}t$, $z = t$,

where, t is any real number.

Matrix Eigen Value Problems

[Let $A = [a_{jk}]$ be a given non zero square matrix of dimension $n \times n$. Consider the following vector equation:

$$Ax = \lambda x \rightarrow \textcircled{1}$$

The problem of finding non zero x 's and λ 's that satisfy the equation is called an eigenvalue problem.

The λ 's that satisfy $\textcircled{1}$ are called eigen values of A and the corresponding non zero x 's that also satisfy $\textcircled{1}$ are called eigen vectors of A .

[Value of λ for which (1) has a solution $x \neq 0$ is called an eigenvalue or characteristic value of the matrix A. The corresponding solutions $x \neq 0$ of (1) are called the eigen vectors or characteristic vectors of A corresponding to that eigenvalue λ .]

The set of all the eigenvalues of A is called the spectrum of A. The largest of the absolute values of the eigenvalues of A is called the spectral radius of A.

21/8/2019 Let A be an $n \times n$ square matrix, then the problem of determining λ and x in the system $Ax = \lambda x$, is generally known as Eigen value Problem.

$x=0$ is always a solution. But in the Eigen value problem, we are considering only the non-zero x .

Note:

We can solve the Eigen value problem $Ax = \lambda x$ by considering $Ax - \lambda x = 0$

$$(A - \lambda I)x = 0$$

which is a homogenous system of n equations in n unknowns!

We know that $(A - \lambda I)x = 0$ has a non-trivial solution only if $|A - \lambda I| = 0$.

Note:

$|A - \lambda I| = 0$ is known as the characteristic equation of A , which is a n th degree eqn. in λ . So, let it has n roots, say, $\lambda_1, \lambda_2, \dots, \lambda_n$.

Now the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the characteristic equation $|A - \lambda I| = 0$ are known as the Eigen values of A .

The non-zero solution of the homogenous system $(A - \lambda I)x = 0$ corresponding to each of the Eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ are known as the Eigen vectors of A .

Q:1) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & -2 & 3 \end{bmatrix}$.

Q:2) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

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Answers:

1) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$ can be found out by either
 $\lambda^3 - (\text{trace of } A)\lambda^2 + \left(\begin{array}{l} \text{sum of the} \\ \text{cofactor of} \\ \text{the diagonal} \\ \text{elements of } A \end{array} \right) \lambda - |A| = 0$

OR

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + (4+5+2)\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Put $\lambda=1 \Rightarrow 1-6+11-6 = 0$

$\Rightarrow \lambda=1$ is a root of $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ 1 & \quad & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

(synthetic division)

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)$$

\therefore Roots are 1, 2, 3.

\therefore Eigen values are 1, 2, 3.

Eigen vector of A corresponding to $\lambda=1$

Eigen vector corresponding to λ is

$$(A - \lambda I)x = 0$$

$$(A - I)x = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A-I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_1$$

$$\xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

$$-x_3 = 0 \Rightarrow x_3 = 0$$

$$\text{Put } x_2 = t$$

$$x_1 = -t$$

$$\therefore \underline{x_1 = -t}, \underline{x_2 = t}, \underline{x_3 = 0}, t \in \mathbb{R}$$

$$\text{Put } t=1$$

$$\Rightarrow x_1 = -1, x_2 = 1, x_3 = 0$$

$$(-1, 1, 0)$$

Eigen vector corresponding to $\lambda=2$

$$(A - 2I)x = 0$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{array} \right] \quad R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{array} \right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$2x_2 - x_3 = 0$$

$$\text{Put } x_3 = t$$

$$x_1 = -x_3 = -t$$

$$x_2 = \frac{x_3}{2} = \frac{t}{2}$$

$$\therefore x_1 = -t, x_2 = \underline{\frac{t}{2}}, x_3 = \underline{t}$$

Put $t=2$, (t can be any value)

$$x_1 = -2, x_2 = 1, x_3 = 2$$

$$\underline{(-2, 1, 2)}$$

Eigen value corresponding to $\lambda=3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \quad R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + 2R_2.$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-2x_2 + x_3 = 0$$

Put $x_3 = t$

$$x_2 = \frac{-x_3}{-2} = \frac{t}{2}$$

$$\begin{aligned}
 x_1 &= x_2 - x_3 \\
 &= \frac{t}{2} - t \\
 &= -\frac{t}{2}
 \end{aligned}$$

$$\therefore x_1 = -\frac{t}{2}, x_2 = \frac{t}{2}, x_3 = t$$

Put $t=2$,

$$x_1 = -1, x_2 = 1, x_3 = 2$$

$$(-1, 1, 2)$$

$$2) PA = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\lambda^3 - (\text{trace of } A)\lambda^2 + \left(\begin{array}{l} \text{sum of the} \\ \text{cofactor of} \\ \text{diagonal elements} \\ \text{of } A \end{array} \right) \lambda - |A| = 0$$

$$\lambda^3 + \lambda^2 + (-12 + -3 + -6)\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

Put $\lambda = 5$

$$125 + 25 - 105 - 45 = 0$$

$\Rightarrow \lambda = 5$ is a root of $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r}
 5 \left| \begin{array}{cccc}
 1 & 1 & -21 & -45 \\
 0 & 5 & 30 & 45 \\
 \hline
 1 & 6 & 9 & 0
 \end{array} \right.
 \end{array}$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

$$\lambda = -3, -3$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda - 5)(\lambda + 3)(\lambda + 3)$$

Eig.: Roots are 5, -3, -3.

∴ Eigen values are -3, -3, 5.

Eigen vector corresponding to λ is

$$(A - \lambda I)x = 0$$

Eigen vector corresponding to $\lambda = -3$.

$$(A + 3I)x = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$\text{Let } x_2 = t$$

$$x_3 = s$$

$$x_1 = 3x_3 - 2x_2$$

$$= 3s - 2t$$

$$\therefore x_1 = 3s - 2t, x_2 = t, x_3 = s$$

$$X = \begin{bmatrix} -2t + 3s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 3s \\ 0 \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 5$.

$$(A - 5I)x = 0$$

$$A - 5I = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}, R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \quad R_1 \rightarrow (-1)R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 7R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 5x_3 = 0$$

$$-8x_2 - 16x_3 = 0$$

$$\text{Let } x_3 = t$$

$$-8x_2 = 16x_3$$

$$x_2 = -2t$$

$$x_1 = -2x_2 - 5x_3$$

$$= 4t - 5t$$

$$= \underline{\underline{-t}}$$

$$\therefore x_1 = t, \quad x_2 = -2t, \quad x_3 = t$$

Put $t=1$

$$x_1 = 1, \quad x_2 = -2, \quad x_3 = 1$$

$$\underline{x = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}$$

5/9/2019

Q: Find the Eigen values and vectors of

$$A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

Eigen values are given by

$$|A - \lambda I| = 0$$

$$\lambda^3 - 3\lambda^2 + [2+(-1)+2]\lambda - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0$$

\therefore Roots are 1, 1, 1.

Eigen values are 1, 1, 1.

Eigen vector corresponding to $\lambda = 1$

$$(A - \lambda I)x = 0$$

$$(A - I)x = 0$$

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - I = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -5 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2 \quad -5+4$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0$$

$$-x_2 + x_3 = 0$$

$$x_3 = x_2$$

$$\text{Let } x_3 = t$$

$$\therefore x_2 = t$$

$$x_1 = -2x_2 - x_3$$

$$= -2t - t$$

$$= -3t$$

$$\therefore x_1 = -3t, \quad x_2 = t, \quad x_3 = t$$

Put $t = 1$

$$X = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

DIAGONALISATION:

Let \exists A be a square matrix, then, we say that A is diagonalisable, if there exist an invertible matrix, P (modal matrix) such that $P^{-1}AP = D$, where, D is a diagonal matrix whose diagonal elements are eigen values of A .

Q: Determine whether the following matrices are diagonalisable or not.

$$(i) A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$(iii) A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$(i) A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

Eigen vector of A corresponding to $\lambda = 1$

$$(A - I)x = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector of A corresponding to $\lambda = 2$.

$$(A - 2I)x = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -t \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Eigen vector of A corresponding to $\lambda=3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} -t/2 \\ t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -2 & -2 & 0 \\ 3 & 3 & \frac{3}{2} \end{bmatrix}$$

$$\bar{P}^{-1}AP = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -2 & -2 & 0 \\ 3 & 3 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 2 & \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

$$\bar{P}^{-1}AP = D$$

$\therefore A$ is diagonalizable.

$$(ii) A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = -3, -3, 5$$

$$P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\bar{P}^{-1} = \frac{\text{adj} P}{|P|}$$

$$= \frac{1}{-8} \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\bar{P}^{-1} A = \frac{1}{-8} \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \frac{1}{-8} \begin{bmatrix} 5 & 10 & -15 \\ -6 & 12 & 18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$\bar{P}^{-1} A P = \frac{1}{-8} \begin{bmatrix} 5 & 10 & -15 \\ -6 & 12 & 18 \\ 3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{-8} \begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\bar{P}^{-1} A P = D$$

$\therefore A$ is diagonalizable.

$$(iii) A = \begin{bmatrix} -3 & -1 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$(\lambda - 1)^3 = 0$$

\therefore Eigen values are 1, 1, 1.

$$\lambda = 1, 1, 1$$

Eigen vector corresponding to $\lambda = 1$

$$(A - I)x = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} -3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Since, there is only one independent vector, the given matrix is not diagonalizable.