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Module 1

OSCILLATIONS AND WAVES

A motion is repeated at regular interval of time is called periodic motion.

Eg. Oscillation of simple pendulum.
Motion of earth around the sun

Motion described functions of sine and cosine is called Harmonic oscillation.
A motion in which a particle moves to and fro about a fixed point and repeat the motion after regular interval of time is called oscillatory.

Free Oscillations

If no forces are acting on the particle, the oscillation is continued for indefinite period without change in amplitude.

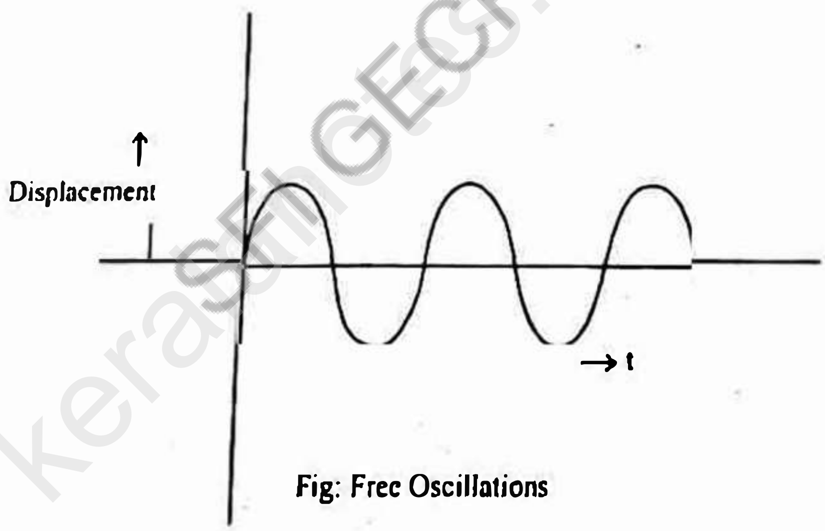


Fig: Free Oscillations

Damped Oscillation

A harmonic oscillator in which the motion is damped by the action of additional force is called damped harmonic oscillator.

When a body oscillates in a medium, it will experience a resistive force depending upon the nature of the medium. A part of energy utilized in overcoming these forces. Hence, amplitude goes on decreasing. These oscillations are called damped oscillation

For small amplitude, the damping force is proportional to velocity $\frac{-dx}{dt}$

Differential equation of damped harmonic oscillation

Consider a particle, executing damped harmonic oscillations

Restoring force = -kx

Damping force = $-b\frac{dx}{dt}$

$\therefore m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{b}{m}\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \text{ ----------------(1)}$$

Where $\frac{k}{m} = \omega_0^2$

$$\frac{b}{m} = 2\gamma$$

Where γ is the damping coefficient

Solution

Suppose the solution of the form $x = Ae^{\alpha t}$

$$\frac{dx}{dt} = Ae^{\alpha t}.\alpha$$

$= \alpha x$

$$\frac{d^2x}{dt^2} = \alpha.Ae^{\alpha t}.\alpha$$

$= \alpha^2 x$

(1)=>

$\alpha^2 x + 2\gamma\,\alpha x + \omega_0^2 x = 0$

$\alpha^2 + 2\gamma\,\alpha + \omega_0^2 = 0$

$$\therefore \alpha = \frac{-2\gamma + \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$= \frac{-2\gamma \pm 2\sqrt{\gamma^2 - \omega_0^2}}{2}$$

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$\therefore$ Solution is $Ae^{\alpha t}$

$$x = Ae^{\left(-\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\right)t}$$

General solution is

$$\boxed{x = A_1 e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t}} \quad \text{——— (a)}$$

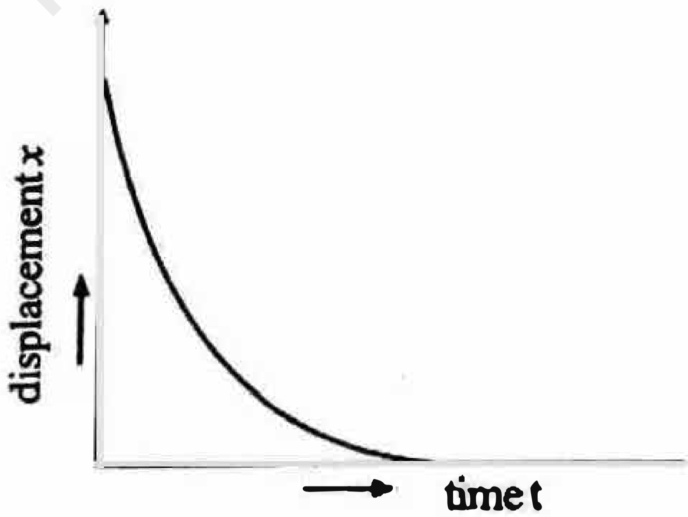
Case 1 – Over damped

If the damping is high, $\gamma > \omega_0$ then, $\sqrt{\gamma^2 - \omega_0^2}$ is real but it is less than r

$\therefore \left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right) t$ and $\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right) t$, both are negative,

So, the displacement x decays exponentially to zero without any oscillation. This oscillation is called over damped or dead beat.

Eg. Dead beat galvanometer, door closing mechanism.



Case 2 Critically Damped

$\omega_0 = \gamma$, then $x = A_1 e^{-\gamma t} + A_2 e^{-\gamma t}$

x= $(A_1 + A_2)e^{-\gamma t}$

=$ce^{-\gamma t}$ where $A_1 + A_2 = c$

The above equation contains only one constant. Hence, this solution is not possible. Letus consider a case that,

$\sqrt{\gamma^2 - \omega_0^2}$ doesn't vanish but equal to very small quantity h

$$\sqrt{\gamma^2 - \omega_0^2} \cong h > 0$$

$$x = A_1 e^{-\gamma t} e^{ht} + A_2 e^{-\gamma t} e^{-ht}$$

$$= e^{-\gamma t}[A_1 e^{+ht} + A_2 e^{-ht}]$$

$$= e^{-\gamma t}[A_1(1 + ht) + A_2(1 - ht)]$$

Neglecting the higher power of h due to small amplitude.

$$= e^{-\gamma t}[(A_1 + A_2)ht + (A_1 - A_2)ht]$$

$$x = e^{-\gamma t}[P + Qt]$$

Where, $A_1 + A_2 = P$ and $(A_1 - A_2)h = Q$

As t increases, (P+Qt) increases. But the exponential term increases more than P+Qt.

∴Displacement decreases from maximum value to zero quickly. This motion is neither damped nor oscillatory. This motion is called critically damped. Here the particle suddenly acquires the equilibrium position.

Applications

1. Automobile shock absorber
2. Door closer mechanism
3. Recoil mechanism in guns.

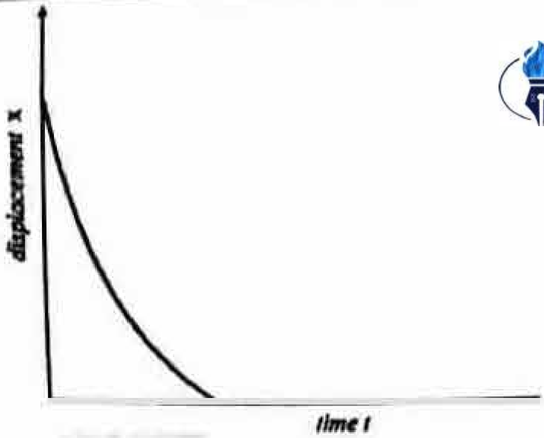


Fig. Critically damped.

Case 3: Under Damped

If damping is low, $\gamma < \omega_0$ (Oscillatory condition)

$$\sqrt{\gamma^2 - \omega_0^2} = \sqrt{-(\omega_0^2 - \gamma^2)} = i\omega$$

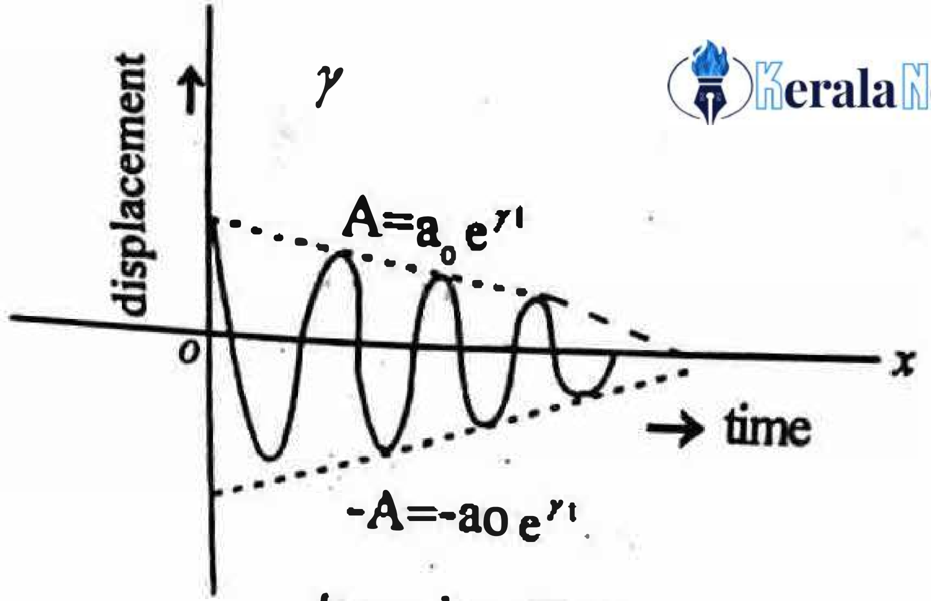
Where w=$\sqrt{\omega_0^2 - \gamma^2}$

$$x = A_1 e^{(-\gamma + i\omega)t} + A_2 e^{(-\gamma - i\omega)t}$$

$x = e^{-\gamma t}\,(A_1 e^{i\omega t} + A_2 e^{-i\omega t})$
$x = e^{-\gamma t}[A_1(Cos\omega t + i\,Sin\omega t) + A_2(Cos\omega t - i\,Sin\omega t)]$
$x = e^{-\gamma t}\,[Cos\omega t(A_1 + A_2) + i\,Sin\omega t(A_1 - A_2)]$
$x = e^{-\gamma t}\,[Cos\omega t\; a_0\, sin\Phi + Sin\omega t a_0\, \cos\Phi]$

Where$A_1 + A_2 = a_0\, sin\Phi$ and $(A_1 - A_2)i = a_0 \cos\Phi$
$x = e^{-\gamma t}\,[a_0\, sin\Phi Cos\omega t + a_0 \cos\Phi\, Sin\omega t]$
$x = a_0 e^{-\gamma t}[Sin(\omega t + \Phi)]$

The amplitude $a_0 e^{-\gamma t}$ is not a constant. But decreases with time. The period of oscillation $T = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$



damped oscillations

Effect of damping

1. Amplitude of oscillation decreases exponentially with time
2. The frequency of oscillation of damped oscillator is less than frequency of undamped oscillation.

Forced Harmonic Oscillator

If an external force acts on a damped oscillator, the oscillating system is called forced harmonic oscillator.

The energy of a damped oscillation decreases on a passage of time. It is possible to compensate the energy loss by applying suitable external force. This supplies energy to the oscillator. When a particle is executing oscillation under an external forces there are 3 forces acting on it.

1. Restoring force ∝ displacement
2. Damping force ∝ velocity
3. External periodic force, which is represented by $F_0 Sin\omega_f t$

∴ Total force $F = F_1 + F_2 + F_3$

$$\frac{md^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 Sin\omega_f t$$

$$\frac{md^2x}{dt^2} + kx + b\frac{dx}{dt} = F_0 Sin\omega_f t$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{b}{m}\frac{dx}{dt} = \frac{F_0}{m} Sin\omega_f t$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x + 2\gamma \frac{dx}{dt} = F_0 Sin\omega_f t_0 \quad -(1)$$

Where $\frac{k}{m} = \omega_0^2, \frac{b}{m} = 2\gamma, \frac{F_0}{m} = f_0$

This is the differential equation for forced harmonic oscillatior.

When external periodic force is applied both damping and forced terms contribute for the motion of oscillator and try to balance between the contribution of both. After sometime the system reaches a steady state. The resultant oscillations are called forced oscillations.

Let us consider the solution of above equation when steady state is reached.

$$x = A \sin(\omega_f t - \theta)$$

$$\frac{dx}{dt} = A\cos(\omega_f t - \theta)\,\omega_f$$

$$\frac{d^2x}{dt^2} = A\,\omega_f - sin(\omega_f t - \theta)\,\omega_f$$

$$= -A\,\omega_f^2\, Sin(\omega_f t - \theta)$$

Substitutingin (1)

$$-A\,\omega_f^2\, Sin(\omega_f t - \theta) + \omega_0^2 A\, Sin(\omega_f t - \theta) + 2\gamma A\omega_f \,Cos\,(\omega_f t - \theta)$$

$$= F_0 \sin(\omega_f t - \theta + \theta)$$

$$-A\,\omega_f^2\, Sin(\omega_f t - \theta) + \omega_0^2 A\, Sin(\omega_f t - \theta) + 2\gamma A\omega_f \,Cos\,(\omega_f t - \theta)$$

$$= F_0 [\sin(\omega_f t - \theta)cos\theta + cos(\omega_f t - \theta)\sin\theta]$$

$$[-A\,\omega_f^2 + \omega_0^2 A - F_0 Cos\theta]Sin(\omega_f t - \theta) + [2\gamma A\omega_f - F_0 Sin\theta]\,Cos\,(\omega_f t - \theta) = 0\text{------}(2)$$

To find A,
The above relation satisfy for all value of t. The coefficient of terms

$Sin(\omega_f t - \theta)$ and $Cos(\omega_f t - \theta)$ must be zero separately

$-A\,\omega_f^2 + \omega_0^2 A - F_0\,Cos\theta = 0$

$A[\omega_0^2 - \omega_f^2] = F_0\,Cos\theta$ -------------------(3)

$2\gamma\, A\omega_f - F_0\,\text{Sin}\,\theta = 0$

$2\gamma\, A\omega_f = F_0\,\text{Sin}\,\theta$ ------------------(4)

Squaring and adding (3) and (4)

$A^2[\omega_0^2 - \omega_f^2]^2 + 4\gamma^2 A^2\omega_f^2 = F_0{}^2 Cos^2\theta + F_0{}^2 Sin^2\theta$

$A^2[(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2] = F_0{}^2\ (\because Cos^2\theta + Sin^2\theta = 1)$

$$\boxed{A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2}}}$$

Which is the amplitude of forced oscillation,

dividing (4) by (3)

$\Rightarrow \tan\theta = \frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2}$

This gives the phase difference between forced oscillation and applied force,

$\therefore x = A\,Sin(\omega_f t - \theta)$

$$x = \frac{F_0}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2\omega_f^2}}\,Sin\left[\omega_f t - \left(tan^{-1}\frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2}\right)\right]$$

Amplitude Resonance

$If\ \omega_0 = \omega_f$

If ω_f is increased the amplitude increase and reaches a maximum value. The particular frequency of applied force at which the amplitude of oscillation is maximum called resonant frequency.

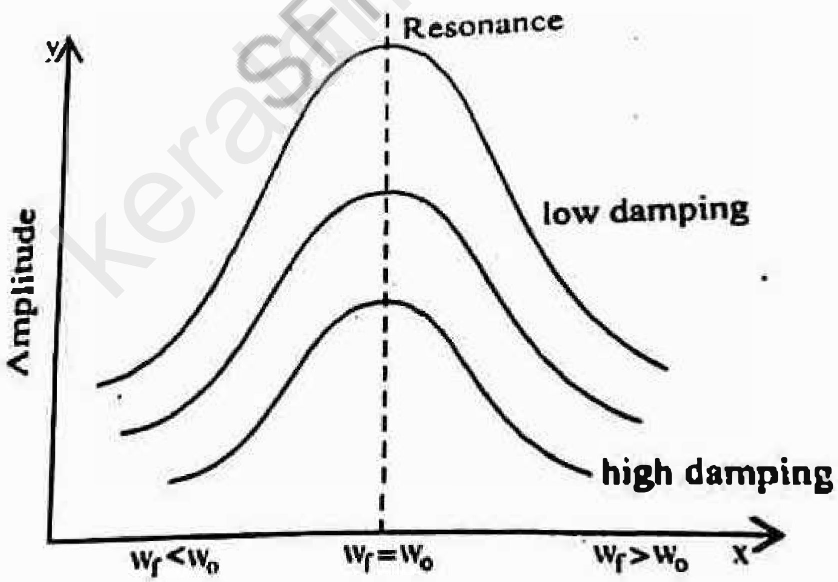
$$\boxed{A_{max} = \frac{F_0}{2\gamma\omega_f}}$$

$\tan\theta = \frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2} = \infty$

$\tan\theta = \infty$

$\theta = \frac{\pi}{2}$

Variation of amplitude A with frequency ω_f of applied force



Sharpness of Resonance

It is the rate of decrease of amplitude with change in frequency of applied periodic force on either side of resonance frequency.

Quality factor
It represents the efficiency of oscillator. It is 2π times the energy stored to the energy dissipated per the cycle.

$$Q = 2\pi \, x \frac{Energy\ stored\ per\ cycle}{Energy\ dissipated\ per\ cycle}$$

$$2\pi \, x \frac{Energy\ stored\ per\ cycle}{Energy\ dissipated\ per\ second\ x\ period\ T}$$

$$= \frac{2\pi E}{\left(\frac{-dE}{dt}\right) x t} \qquad = \frac{2\pi E}{PT}$$

$$\therefore \frac{-dE}{dt} = Power\ P$$

P= γE

$$Q = \frac{2\pi E}{\gamma E\ x\ T}$$

$$= \frac{2\pi}{\gamma T}$$

$$= \frac{2\pi}{\gamma x \frac{2\pi}{\omega}} \qquad = \frac{\omega}{\gamma}$$

$Q = \frac{\omega}{\gamma} = \frac{2m\omega}{b}$ Where $\gamma = \frac{b}{2m}$

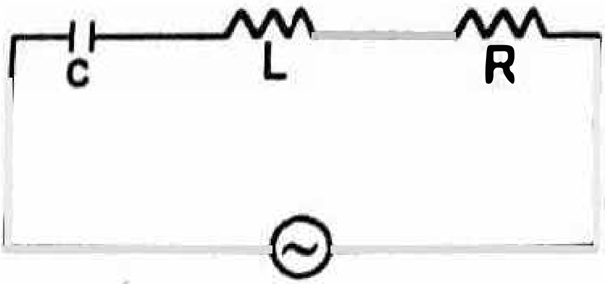
Q will be large if damping coefficient, γ is low ie, efficiency is high.

LCR CIRCUIT
It is an electrical analogue of mechanical oscillator.

Oscillations in L-C Circuit
A pure L-C circuit is electrical analogue of simple pendulum. In simple pendulam energy alternate between potential energy and kinetic energy. In L-C circuit the energy is alternately stored in the capacitor as electric field and in the inductor as magnetic field.
The frequency of oscillation in the L-C circuit

$$n = \frac{1}{2\pi\sqrt{LC}}$$

Forced Oscillations in a LCR Circuit



$v = v_0 \sin \omega t$

Applying Kirchhoff's law to the circuit

$$V_L + V_C + IR = V_0 \text{Sin}\, \omega t$$

$$L\frac{dI}{dt} + \frac{q}{C} + IR = V_0 \text{Sin}\, \omega t$$

$$L\frac{dI}{dt} + \frac{q}{C} + \frac{Rdq}{dt} = V_0 \text{Sin}\, \omega t$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} + \frac{dq}{dt}R = V_0 \text{Sin}\, \omega t \qquad \left[\because I = \frac{dq}{dt},\quad \frac{dI}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2}\right]$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{V_0}{L}\text{Sin}\, \omega t$$

This is the differential equation in electrical oscillator

Physical quantities in mechanical oscillator and electrical oscillator.

	Mechanical Oscillator	Electrical Oscillator
1	Displacement, x	Charge q
2	Velocity $\frac{dx}{dt}$	Current $\frac{dq}{dt}$
3	Mass m	Inductance L
4	Damping Co-efficient γ	Resistance R
5	Force amplitude f_0	Voltage amplitude V_0
6	Driving frequency ω_f	Oscillator frequency ω_0

WAVES

Wave motion is a kind of disturbance or mode of momentum or energy transfer which is due to repeated periodic vibrations of particles of medium about equilibrium position

Mechanical Wave

It requires a material medium for their propagation. They are characterized by transport of energy through matter. Eg. Sound wave

Electromagnetic wave

It requires no medium for their propagation. They travel through vacuum. They are produced by oscillating electric charges. Electric field and magnetic field acts perpendicular to direction of propagation. Eg. Radio wave, micro wave.

Transverse wave	Longitudinal wave
1. Particles of medium vibrate in a direction perpendicular to the direction of propagation 2. Consist of crests and troughs Eg. Light wave.	1. Particles of the medium vibrate parallel to the direction of propagation 2. Consists of compressions and rare faction. Eg.Sound wave.

One dimensional wave equation.

The equation of wave motion is given by

$$u(x,t) = f(x - vt)$$

Differentiating equation (1) with respect to x and t twice

$$\frac{\partial u}{\partial x} = f^{1}(x - vt) \qquad \text{--(2)}$$

$$\frac{\partial^2 u}{\partial x^2} = f^{11}(x - vt)$$

Then,

$\frac{\partial u}{\partial t} = f^{1}(x - vt) - v$ --(4)

$\frac{\partial^2 u}{\partial t^2} = f^{11}(x - vt)v^2$ --(5)

From the eqn (3)

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$ This is one dimensional differential equation of wave motion

Eqn (4) =>

$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}$ (From equn 2)

3-Dimensional wave equation

One dimensional wave equation is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

In 3D,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

$\nabla^2 u = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$

Where $\nabla^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, It is called Laplacian operator

Solution of one dimensional equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}.\frac{\partial^2 u}{\partial t^2} \qquad \text{-------------------(1)}$$

Suppose, the solution of the form

$u(x,t) = X(x)\, T(t)$ -------------------(2)

X(x) is a function of x and T(t) is a function of t.

Differentiating (2) with respect to x and t twice

$$\frac{d^2 u}{dx^2} = T\frac{d^2 X}{dx^2}$$

$$\frac{d^2 u}{dt^2} = X\frac{d^2 T}{dt^2}$$

Eqn (1) =>

$$T\frac{d^2 X}{dx^2} = \frac{1}{v^2} X \frac{d^2 T}{dt^2}$$

Dividing by X T.

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

LHS and RHS contains only one variable.

Each side separately equate to common constant $-k^2$

$$\therefore \frac{1}{X}\frac{d^2 X}{dx^2} = -k^2$$

$$\frac{d^2 X}{dx^2} = -k^2 X$$

$$\frac{d^2 X}{dx^2} + k^2 X = 0 \qquad \text{-----------------------------(3)}$$

Then,

$$\frac{1}{v^2}.\frac{1}{T}.\frac{d^2 T}{dt^2} = -k^2$$

$$\frac{d^2 T}{dt^2} + k^2 v^2 T = 0$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \quad \text{------------} (4)$$

Where $k^2v^2=\omega^2$ $(\omega = kv)$

$X(x) = Ce^{\pm ikx}$

$T(t) = Ce^{\pm i\omega t}$

$u(x,t) = X(x).T(t)$

$u = Ce^{i(\pm kx \pm \omega t)}$

Solution of 3 dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} \quad \text{------------} (1)$$

$\therefore$ *solution is*

$u(x, y, x, t) = X(x). Y(y). Z(z)T(t)$

$X = ce^{\pm ik_x x}$

$Y = ce^{\pm ik_y y}$

$Z = ce^{\pm ik_z z}$

$T = ce^{\pm i\omega t}$

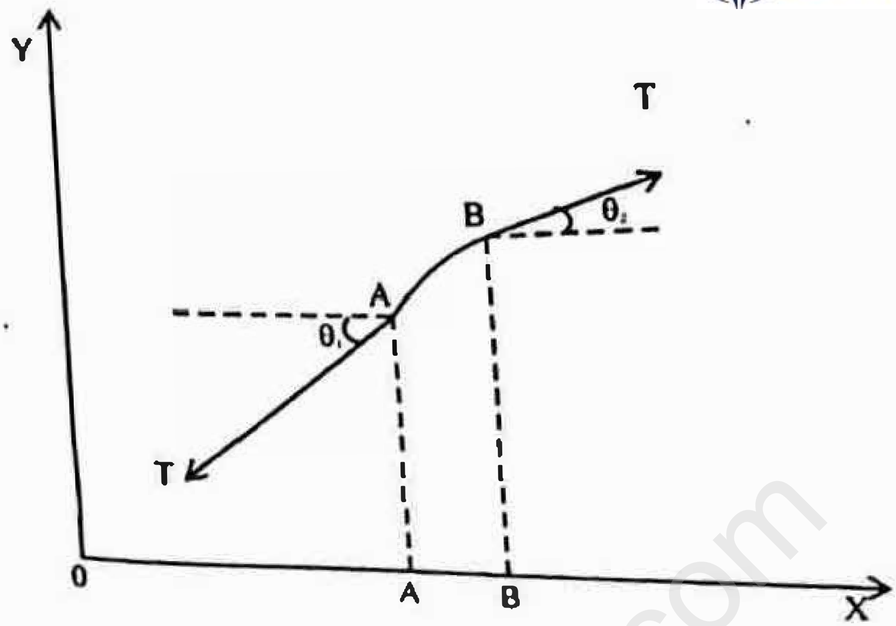
Solution is

$u(x,y,z,t) = X(x), Y(y), Z(z)\, T(t)$

$= ce^{\pm i(K_x x + K_y y + K_z z + \omega t)}$

$u(x,y,z,t) = ce^{\pm i(\vec{K}.\,\vec{r} + \omega t + \phi)}$

Transverse waves in a stretched string.



Consider a perfectly uniform string stretched between two points by a constant tension T. Let μ be mass per unit length of the wire. Let the string be slightly displaced along y-axis and released. Then, transverse vibrations set up in the string. The string is made up of large no.of small elements of length δx, θ_1 and θ_2 are the angles made by the tension T with x-axis at A and B

Then,

$F_x \quad = T\cos\theta_2 \quad - T\cos\theta_1$

$F_y \quad = T\sin\theta_2 \quad - T\sin\theta_1$

For small θ, $F_x = 0$, [$\cos\theta = 1$]

$$F_y = T\tan\theta_2 - T\tan\theta_1 \quad [\sin\theta \cong \tan\theta]$$

Tan θ is the slop of the wave at any point

$$\tan\theta = \frac{\partial y}{\partial x}$$

$$\tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$$

$\text{Tan}\ \theta_1 = \left(\frac{\partial y}{\partial x}\right)_x$

Then,

$$Fy = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$\mu\delta x\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right]$$

We have $Lt\ \delta x \to 0\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] = \frac{\partial^2 y}{\partial x^2}$

$\therefore \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$ ------- (A)

This is the one dimensional differential equation of a transverse waves in a stretched string.

$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$ ------ (B)

Comparing (A) and (B)

$\therefore V = \sqrt{\frac{T}{\mu}}$

Frequency $\upsilon = \frac{v}{\lambda}$

$$\boxed{\upsilon = \frac{1}{\lambda}\sqrt{\frac{T}{\mu}}}$$

Where T is tension and μ is mass per unit length

Law of Vibration

The frequency of transverse waves is directly proportional to square root of tension an inversely proportional to wavelength and square root of linear density of the string.