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Module V

Fourier Integral and Fourier Transforms

Fourier series are powerful tools for problems involving periodic functions and are of interest on a finite interval only. But many problems involve functions that are non-periodic and are of interest on the whole x-axis. So extending the method of Fourier series to such functions will lead to Fourier integral.

Fourier Integral representation

If $f(x)$ is defined in $(-\infty, \infty)$, then

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \quad \text{where}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

If x is a point of discontinuity, the above integral is equal to $\frac{1}{2}[f(x^+) + f(x^-)]$

1. Find the Fourier integral representation of the function $f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega$

Soln :- $f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1}\cos\omega x\,dx = \frac{1}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{1}{\pi}\int_{-1}^{1}\sin\omega x\,dx = \frac{1}{\pi}\left[-\frac{\cos\omega x}{\omega}\right]_{-1}^{1} = \underline{\underline{0}}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\,d\omega$$

~~At x=1, the function is dis~~

From the above, $\displaystyle\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \frac{\pi}{2}f(x) = \begin{cases}\frac{\pi}{2} & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$

At $x=1$, the function is discontinuous. Therefore the integral has the value $\dfrac{f(1^+)+f(1^-)}{2} = \dfrac{0+\frac{\pi}{2}}{2} = \underline{\underline{\dfrac{\pi}{4}}}$

2) Using Fourier integral, show that

$$\int_0^{\infty}\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega = \begin{cases}0 & \text{if } x<0\\ \frac{\pi}{2} & \text{if } x=0\\ \pi e^{-x} & \text{if } x>0\end{cases}$$

Soln:- Let $f(x) = \begin{cases}e^{-x} & \text{if } x>0\\ 0 & \text{if } x<0\end{cases}$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\cos\omega x\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+\omega^2}(-\cos\omega x+\omega\sin\omega x)\right]_0^{\infty}$$

$$\left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)\right]$$

$$= \underline{\underline{\frac{1}{\pi(1+\omega^2)}}}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\sin\omega x\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+\omega^2}(-\sin\omega x-\omega\cos\omega x)\right]_0^{\infty}$$

$$\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)\right]$$

$$= \frac{\omega}{\pi(1+\omega^2)}$$

$$\therefore f(x) = \int_0^{\infty}\left[\frac{1}{\pi(1+\omega^2)}\cos\omega x + \frac{\omega}{\pi(1+\omega^2)}\sin\omega x\right]d\omega$$

From the above $\int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2} d\omega = \pi f(x) = \begin{cases} \pi e^{-x} & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$

At $x=0$, the function is discontinuous, therefore the integral has the value $\frac{f(0^+)+f(0^-)}{2} = \frac{\pi+0}{2} = \frac{\pi}{2}$

Fourier integral representation of even and odd functions

If $f(x)$ is even in $(-\infty,\infty)$, then $B(\omega)=0$

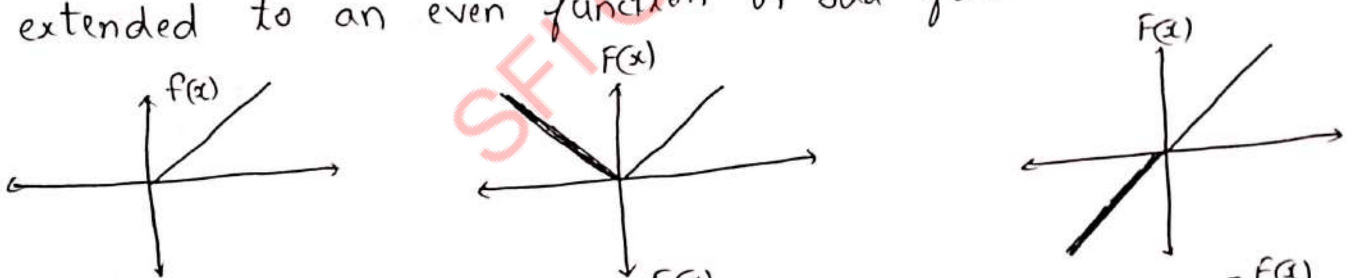
Therefore $f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega$ where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x \, dx$

If $f(x)$ is odd in $(-\infty,\infty)$, then $A(\omega)=0$

Therefore $f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$ where $B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x \, dx$

Fourier Sine Integral and Cosine Integral

If $f(x)$ is a function defined in $(0,\infty)$, it can be extended to an even function or odd function in $(-\infty,\infty)$



$F(-x) = F(x)$
$F(x) \rightarrow$ even extension of $f(x)$

$F(-x) = -F(x)$
$F(x) \rightarrow$ odd extension of $f(x)$

Hence in $(0,\infty)$, $f(x)$ can be written as a cosine integral or sine integral.

Cosine Integral

$f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega$ where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x \, dx$

Sine Integral

$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$ where $B(\omega) =$

1. Represent $f(x) = e^{-kx}$ where $x>0$, $k>0$ as a fourier cosine & sine integral.

<u>Soln</u> :- $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x) \cos\omega x \, dx$

$$= \frac{2}{\pi}\int_0^\infty e^{-kx} \cos\omega x \, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-kx}}{k^2+\omega^2}\left(-k\cos\omega x + \omega \sin\omega x\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\,\frac{k}{k^2+\omega^2}$$

$$\therefore f(x) = \frac{2k}{\pi}\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}\, d\omega$$

<u>Note</u> :- From the above $\boxed{\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2k} e^{-kx}.}$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kx} \sin\omega x \, dx$$

$$= \frac{2}{\pi}\left\{\frac{e^{-kx}}{k^2+\omega^2}\left[-k\sin\omega x - \omega\cos\omega x\right]\right\}_0^\infty$$

$$= \frac{2}{\pi}\,\frac{\omega}{k^2+\omega^2}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin\omega x}{k^2+\omega^2}\, d\omega.$$

<u>Note</u> :- From the above $\boxed{\int_0^\infty \frac{\omega\sin\omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2} e^{-kx}}$

2) Represent $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$ as a Fourier cosine integral.

Soln :- $A(\omega) = \frac{2}{\pi}\int_0^1 \cos\omega x\, dx = \frac{2}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_0^1 = \frac{2}{\pi}\frac{\sin\omega}{\omega}$

$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x\, d\omega$

3) Represent $f(x) = \frac{1}{1+x^2}$, $x>0$ as a fourier Cosine integral.

Soln : $A(\omega) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+x^2}\cos\omega x\, dx$

$= \frac{2}{\pi}\times\frac{\pi}{2}e^{-\omega}$ $\left[\because \int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}d\omega = \frac{\pi}{2k}e^{-kx}\right]$

$= e^{-\omega}$

$\therefore f(x) = \int_0^\infty e^{-\omega}\cos\omega x\, d\omega$

4) Represent $f(x) = \begin{cases}\sin x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi\end{cases}$ as fourier Cosine integral.

Soln :- $A(\omega) = \frac{2}{\pi}\int_0^\pi \sin x\cos\omega x\, dx = \frac{2}{\pi}\int_0^\pi\left[\frac{\sin(1+\omega)x + \sin(1-\omega)x}{2}\right]dx$

$= \frac{1}{\pi}\left[\frac{-\cos(1+\omega)x}{1+\omega} - \frac{\cos(1-\omega)x}{1-\omega}\right]_0^\pi$

$= \frac{1}{\pi}\left[\frac{-\cos(1+\omega)\pi}{1+\omega} - \frac{\cos(1-\omega)\pi}{1-\omega} + \frac{1}{1+\omega} + \frac{1}{1-\omega}\right]$

$= \frac{1}{\pi}\left[\frac{\cos\pi\omega}{1+\omega} + \frac{\cos\pi\omega}{1-\omega} + \frac{2}{1-\omega^2}\right]$

$= \frac{1}{\pi}\left[\frac{2\cos\pi\omega + 2}{1-\omega^2}\right]$

$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\pi\omega + 1}{1-\omega^2}\cos\omega x\, d\omega$

5) Show that $\int_0^\infty \frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\cos\omega x\,d\omega = \begin{cases} \frac{\pi}{2}\cos x & \text{if } 0<|x|<\frac{\pi}{2} \\ 0 & \text{if } |x|\geq\frac{\pi}{2} \end{cases}$

<u>Soln</u>:- Let $f(x) = \begin{cases} \cos x & \text{if } 0<|x|<\frac{\pi}{2} \\ 0 & \text{if } |x|\geq\frac{\pi}{2} \end{cases}$

$$A(\omega) = \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos\omega x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos(1+\omega)x + \cos(1-\omega)x}{2}\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+\omega)x}{1+\omega} + \frac{\sin(1-\omega)x}{1-\omega}\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega} + \frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{\omega\pi}{2}}{1+\omega} + \frac{\cos\frac{\omega\pi}{2}}{1-\omega}\right]$$

$$= \frac{2}{\pi}\,\frac{\cos\frac{\omega\pi}{2}}{1-\omega^2}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\frac{\omega\pi}{2}}{1-\omega^2}\cos\omega x\,d\omega$$

$$\Longrightarrow \int_0^\infty \frac{\cos\frac{\omega\pi}{2}}{1-\omega^2}\cos\omega x\,d\omega = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2}\cos x & \text{if } 0<|x|<\frac{\pi}{2} \\ 0 & \text{if } |x|\geq\frac{\pi}{2} \end{cases}$$

6) Represent $f(x)$ as a Fourier Sine integral.

$$f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$$

Soln :- $B(\omega) = \frac{2}{\pi}\int_0^a x \sin \omega x \, dx$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos\omega x}{\omega}\right) + \int \frac{\cos\omega x}{\omega} d\omega \right]_0^a$$

$$= \frac{2}{\pi}\left[\frac{-x\cos\omega x}{\omega} + \frac{\sin\omega x}{\omega^2}\right]_0^a$$

$$= \frac{2}{\pi}\left[\frac{-a\cos\omega a}{\omega} + \frac{\sin\omega a}{\omega^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{-a\cos\omega a}{\omega} + \frac{\sin\omega a}{\omega^2}\right] \sin\omega x \, d\omega$$

7) Represent $f(x) = \begin{cases} e^x & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$ as Fourier sine integral.

Soln :- $B(\omega) = \frac{2}{\pi}\int_0^1 e^x \sin\omega x \, dx = \frac{2}{\pi}\left[\frac{e^x}{1+\omega^2}\left(\sin\omega x - \omega\cos\omega x\right)\right]_0^1$

$$= \frac{2}{\pi}\left[\frac{e}{1+\omega^2}\left(\sin\omega - \omega\cos\omega\right) + \frac{\omega}{1+\omega^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{e(\sin\omega - \omega\cos\omega) + \omega}{1+\omega^2}\right]\sin\omega x \, d\omega$$

8) Show that $\displaystyle\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x \, d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

Soln :- $\cancel{B(\omega) = 2}$ Let $f(x) = \begin{cases} \frac{\pi}{2} & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

$$B(\omega) = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2} \sin \omega x \, dx = \left[-\frac{\cos \omega x}{\omega}\right]_0^{\pi} = -\frac{\cos \pi\omega}{\omega} + \frac{1}{\omega}$$

$$\therefore f(x) = \int_0^{\infty} \left(\frac{1-\cos \pi\omega}{\omega}\right) \sin \omega x \, d\omega$$

$$\Longrightarrow \int_0^{\infty}\left(\frac{1-\cos \pi \omega}{\omega}\right) \sin \omega x \, d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

9) Show that $\displaystyle\int_0^{\infty} \frac{\sin \pi\omega \sin \omega x}{1-\omega^2} \, d\omega = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases}$

<u>Soln</u>:- Let $f(x) = \begin{cases} \sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases}$

$$B(\omega) = \frac{2}{\pi}\int_0^{\pi} \sin x \sin \omega x \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{\cos(1-\omega)x - \cos(1+\omega)x}{2} \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-\omega)x}{1-\omega} - \frac{\sin(1+\omega)x}{1+\omega}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-\omega)\pi}{1-\omega} - \frac{\sin(1+\omega)\pi}{1+\omega}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin \pi\omega}{1-\omega} + \frac{\sin \pi\omega}{1+\omega}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin \pi\omega}{1-\omega^2}\right]$$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{2\sin \pi\omega}{1-\omega^2}\right)\sin \omega x \, d\omega$$

$$\Longrightarrow \int_0^{\infty} \frac{\sin \pi\omega \sin \omega x}{1-\omega^2} \, d\omega = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2}\sin x & , 0 \le x \le \pi \\ 0 & , x > \pi \end{cases}$$

10) Show that $\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^2}\sin\omega x\,d\omega = \begin{cases} \frac{\pi}{2}x, & 0<x<1 \\ \frac{\pi}{4}, & x=1 \\ 0, & x>1 \end{cases}$

Soln :- Let $f(x) = \begin{cases} x & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

$$B(\omega) = \frac{2}{\pi}\int_0^1 x\sin\omega x\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos\omega x}{\omega}\right) + \int\frac{\cos\omega x\,dx}{\omega}\right]_0^1$$

$$= \frac{2}{\pi}\left[-\frac{x\cos\omega x}{\omega} + \frac{\sin\omega x}{\omega^2}\right]_0^1$$

$$= \frac{2}{\pi}\left[-\frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\omega\cos\omega + \sin\omega}{\omega^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty\left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)\sin\omega x\,d\omega$$

$$\Longrightarrow \int_0^\infty\left(\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right)\sin\omega x\,d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2}x & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$$

At $x=1$, the integral must be equal to $\frac{f(1^+)+f(1^-)}{2}$

$$= \frac{0+\frac{\pi}{2}}{2} = \underline{\underline{\frac{\pi}{4}}}$$

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I Represent $f(x)$ as a Fourier Cosine Integral

1) $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

2) $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

3) $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

II Represent $f(x)$ as a Fourier Sine Integral

1) $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

2) $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

3) $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

III

1) Show that $\displaystyle\int_0^\infty \frac{\cos wx}{1+w^2}\, dw = \frac{\pi}{2} e^{-x}$ if $x \geq 0$

2) Show that $\displaystyle\int_0^\infty \frac{w^3 \sin wx}{w^4 + 4}\, dw = \frac{\pi}{2} e^{-x} \cos x$ if $x > 0$

Fourier Transform

Fourier transform of $f(x)$ defined in $(-\infty, \infty)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \hat{f}(\omega)$$

The original function $f(x)$ is called the inverse fourier transform of $\hat{f}(\omega)$ and is given by $f(x) = F^{-1}[\hat{f}(\omega)]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \cdot d\omega.$$

1. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Soln : $F[f(x)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1}$$

$$= \frac{1}{-i\omega\sqrt{2\pi}} \left[e^{-i\omega} - e^{i\omega} \right]$$

$e^{i\omega} = \cos\omega + i\sin\omega$ and $e^{-i\omega} = \cos\omega - i\sin\omega$

$\cancel{e^{i\omega} - e^{-i\omega}}$ $e^{-i\omega} - e^{i\omega} = -2i\sin\omega$

$$\therefore \hat{f}(\omega) = \frac{1}{-i\omega\sqrt{2\pi}} (-2i\sin\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

2. Find the Fourier transform of $f(x) = \begin{cases} e^{-ax} & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$

$a>0$

Soln:- $F\left(e^{-ax}\right) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cdot e^{-i\omega x} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_0^\infty = \frac{1}{\sqrt{2\pi}\,(a+i\omega)}$$

3. Find the fourier transform of $f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{otherwise} \end{cases}$

Soln:- $F[f(x)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a x \cdot e^{-i\omega x} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{-i\omega x}}{(-i\omega)} + \int \frac{e^{-i\omega x}}{i\omega} dx \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{x \cdot e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{a e^{-i\omega a}}{-i\omega} + \frac{e^{-i\omega a}}{\omega^2} - \frac{1}{\omega^2} \right]$$

$$= \frac{1}{\omega^2 \sqrt{2\pi}} \left[(ia\omega + 1) e^{-i\omega a} - 1 \right]$$

4) Find the Fourier transform of $f(x) = \begin{cases} |x| & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$

Soln:- $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |x| e^{-i\omega x} dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} -x e^{-i\omega x} dx + \int_0^1 x e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[-x\left(\frac{e^{-i\omega x}}{-i\omega}\right) - \frac{e^{-i\omega x}}{\omega^2}\right]_{-1}^{0} + \left[x\left(\frac{e^{-i\omega x}}{-i\omega}\right) + \frac{e^{-i\omega x}}{\omega^2}\right]_{0}^{1}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-1}{\omega^2} + \frac{e^{i\omega}}{i\omega} + \frac{e^{i\omega}}{\omega^2} - \frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega}}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{\omega^2} + \frac{2\cos\omega}{\omega^2} + \frac{2i\sin\omega}{i\omega}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\omega\sin\omega + \cos\omega - 1}{\omega^2}\right]$$

5) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$

Soln :- $F[f(x)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\left(\frac{e^{-i\omega x}}{-i\omega}\right) - (-2x)\frac{e^{-i\omega x}}{(-i\omega)^2} + (-2)\frac{e^{-i\omega x}}{(-i\omega)^3}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{2e^{-i\omega}}{\omega^2} - \frac{2e^{-i\omega}}{i\omega^3} - \frac{2e^{i\omega}}{\omega^2} + \frac{2e^{i\omega}}{i\omega^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{4\cos\omega}{\omega^2} + \frac{4\sin\omega}{\omega^3}\right]$$

$$= \frac{4}{\omega^3\sqrt{2\pi}}\left[\sin\omega - \omega\cos\omega\right]$$

Using inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\omega^3\sqrt{2\pi}} \left[\sin\omega - \omega\cos\omega\right] e^{i\omega x}\, d\omega$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right](\cos\omega x + i\sin\omega x)\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)(\cos\omega x + i\sin\omega x)\, d\omega = \frac{\pi}{2} f(x)$$

Using even & odd fn property,

$$2\int_{0}^{\infty} \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega x\, d\omega = \frac{\pi}{2} f(x)$$

$$\Longrightarrow \int_{0}^{\infty} \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega x\, d\omega = \frac{\pi}{4} f(x) = \frac{\pi}{4}(1-x^2)$$

Put $x = \frac{1}{2}$.

$$\int_{0}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\frac{\omega}{2}\, d\omega = \frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \underline{\underline{\frac{3\pi}{16}}}$$

Changing the variable $\omega = x$

$$\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx = \underline{\underline{\frac{-3\pi}{16}}}$$

<u>H.W</u> Find the Fourier transform

1) $f(x) = \begin{cases} e^{2ix} & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$

2) $f(x) = \begin{cases} xe^{-x} & \text{if } -1<x<0 \\ 0 & \text{otherwise} \end{cases}$

3) $f(x) = e^{-|x|} \quad -\infty < x < \infty$

Fourier Cosine and Sine tranform

Fourier Cosine transform of f(x) defined in $(0,\infty)$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx = \hat{f}_c(\omega)$$

The original function f(x) is called inverse fourier cosine transform and is given by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega$

Fourier Sine transform of f(x) defined in $(0,\infty)$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx = \hat{f}_s(\omega)$$

The original function f(x) is called inverse fourier sine transform and is given by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega$.

1. Find the Fourier cosine and sine tranform of $f(x) = \begin{cases} k & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

Soln :- $F_c[f(x)] = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx$

$$= \sqrt{\frac{2}{\pi}} \, k \left[\frac{\sin \omega x}{\omega}\right]_0^a = \sqrt{\frac{2}{\pi}} \, k \, \frac{\sin a\omega}{\omega}$$

$$F_s[f(x)] = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \, k \left[\frac{-\cos \omega x}{\omega}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \, k \left[\frac{1 - \cos a\omega}{\omega}\right]$$

2) Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Soln :- $F_c[f(x)] = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^2 x \cos \omega x \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin \omega x}{\omega} + \frac{\cos \omega x}{\omega^2} \right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin 2\omega}{\omega} + \frac{\cos 2\omega}{\omega^2} - \frac{1}{\omega^2} \right]$$

3) Find the Fourier sine transform of

$f(x) = e^{-ax}$, $a > 0$

Soln :- $F_s[f(x)] = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin \omega x \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\omega^2} \left(-a \sin \omega x - \omega \cos \omega x \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{a^2+\omega^2} \right)$$

4) Find the Fourier sine transform of $f(x) = e^{-|x|}$

Hence evaluate $\int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2} \, d\omega$

Soln :- $F_s[f(x)] = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \omega x \, dx$ $\left[\text{in } (0, \infty) \; e^{-|x|} = e^{-x} \right]$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\omega^2} \left[-\sin \omega x - \omega \cos \omega x \right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right)$$

Using inverse fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2} d\omega$$

$$\Longrightarrow \int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} f(x) = \underline{\underline{\frac{\pi}{2} e^{-|x|}}}$$

<u>H.W</u>

1. Find $F_c(e^{-x})$

2. Find $\hat{f}_s(\omega)$ if $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ -1 & \text{if } 1<x<2 \\ 0 & \text{if } x>2 \end{cases}$

3. Find $\hat{f}_c(\omega)$ if $f(x) = \begin{cases} x^2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

<u>Linearity property</u>

$$F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$$

Similarly $F_c[af(x)+bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

$F_s[af(x)+bg(x)] = aF_s[f(x)] + bF_s[g(x)]$

where a, b are constants.

<u>Fourier transform of derivatives</u>

$$F[f'(x)] = i\omega F[f(x)]$$

$$F[f''(x)] = (i\omega)^2 F[f(x)]$$

Similarly $F[f^n(x)] = (i\omega)^n F[f(x)]$

1. Find the Fourier transform of xe^{-x^2}

Soln :- $F\left[xe^{-x^2}\right] = F\left[-\frac{1}{2}\left(e^{-x^2}\right)'\right]$

$= -\frac{1}{2} F\left[\left(e^{-x^2}\right)'\right]$

$= -\frac{1}{2}\, i\omega\, F\left(e^{-x^2}\right)$

$= -\frac{1}{2}\, i\omega \frac{1}{\sqrt{2}} e^{-\omega^2/4}$ $\left[\because F\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}\right]$

$= -\frac{i\omega}{2\sqrt{2}} e^{-\omega^2/4}$

Fourier Cosine & Sine transform of derivatives

$$\boxed{\begin{aligned} F_c\left[f'(x)\right] &= \omega F_s\left[f(x)\right] - \sqrt{\tfrac{2}{\pi}}\, f(0) \\ F_s\left[f'(x)\right] &= -\omega F_c\left[f(x)\right] \end{aligned}}$$

$F_c\left[f''(x)\right] = \omega F_s\left[f'(x)\right] - \sqrt{\frac{2}{\pi}}\, f'(0)$

$$\boxed{\Rightarrow F_c\left[f''(x)\right] = -\omega^2 F_c\left[f(x)\right] - \sqrt{\tfrac{2}{\pi}}\, f'(0)}$$

Similarly $F_s\left[f''(x)\right] = -\omega^2 F_s\left[f(x)\right] + \sqrt{\frac{2}{\pi}}\, \omega f(0)$

1. Find the Fourier cosine transform of e^{-ax} using derivatives.

Soln : $f(x) = e^{-ax}$ $\quad f'(x) = -ae^{-ax}$ $\quad f''(x) = a^2 e^{-ax}$

$\Rightarrow F_c\left[f''(x)\right] = a^2 F_c\left[e^{-ax}\right]$

$$\Rightarrow -\omega^2 F_c(e^{-ax}) - \sqrt{\frac{2}{\pi}}\, f'(0) = a^2 F_c(e^{-ax})$$

$$\Rightarrow (a^2+\omega^2) F_c(e^{-ax}) = a\sqrt{\frac{2}{\pi}} \qquad [\because f'(0) = -a]$$

$$\Rightarrow F_c(e^{-ax}) = \left(\frac{a}{a^2+\omega^2}\right)\sqrt{\frac{2}{\pi}}$$

Convolution theorem

Convolution:- convolution of two functions $f(x)$ and $g(x)$ denoted by $f*g$ is defined as $f*g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$

Convolution theorem

$$F(f*g) = \sqrt{2\pi}\, F[f(x)]\, F[g(x)]$$

1. Verify convolution theorem for $f(x) = e^{-x}$, $x>0$ and $g(x) = x$, $x>0$

Soln:- $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-i\omega x}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\right]_0^{\infty} = \frac{1}{\sqrt{2\pi}}\frac{1}{(1+i\omega)}$$

$$F[g(x)] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x \cdot e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\left[x\left(\frac{e^{-i\omega x}}{-i\omega}\right) + \frac{e^{-i\omega x}}{\omega^2}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{-1}{\omega^2}\right)$$

$$\sqrt{2\pi}\, F[f(x)]\, F[g(x)] = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\omega}\right)\left(\frac{-1}{\omega^2}\right)$$

$$f*g = \int_0^x f(u)\, g(x-u)\,du = \int_0^x e^{-u}(x-u)\,du$$

$$= \left[(x-u)\left(\frac{e^{-u}}{-1}\right) + e^{-u}\right]_0^x$$

$$= e^{-x} + x - 1$$

$$F(f*g) = \frac{1}{\sqrt{2\pi}} \int_0^\infty (e^{-x}+x-1)e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(e^{-(1+i\omega)x} + x e^{-i\omega x} - e^{-i\omega x}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1+i\omega)x}}{-(1+i\omega)} + x\frac{e^{-i\omega x}}{(-i\omega)} + \frac{e^{-i\omega x}}{\omega^2} + \frac{e^{-i\omega x}}{i\omega} \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+i\omega} - \frac{1}{\omega^2} - \frac{1}{i\omega} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{(1+i\omega)\omega^2} \right]$$

Therefore $F(f*g) = \sqrt{2\pi}\; F[f(x)]\, F[g(x)]$

To find inverse fourier transform using convolution theorem

$$F(f*g) = \sqrt{2\pi}\; F[f(x)]\, F[g(x)]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} f*g = F^{-1}\left[F[f(x)]\, F[g(x)]\right] = F^{-1}\left[\hat{f}(\omega)\,\hat{g}(\omega)\right]$$

1. Find $F^{-1}\left[\frac{1}{6+5i\omega-\omega^2}\right]$

Soln :- $F^{-1}\left[\frac{1}{6+5i\omega-\omega^2}\right] = F^{-1}\left[\frac{1}{(2+i\omega)(3+i\omega)}\right]$

$= F^{-1}\left[\hat{f}(\omega)\,\hat{g}(\omega)\right]$ where

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$\hat{f}(\omega) = \dfrac{1}{2+i\omega}$

$\Rightarrow f(x) = F^{-1}[\hat{f}(\omega)] = F^{-1}\left(\dfrac{1}{2+i\omega}\right)$

$= \sqrt{2\pi}\, e^{-2x}, \quad x>0$

$\hat{g}(\omega) = \dfrac{1}{3+i\omega}$

$\Rightarrow g(x) = F^{-1}[\hat{g}(\omega)]$

$= F^{-1}\left[\dfrac{1}{3+i\omega}\right]$

$= \sqrt{2\pi}\, e^{-3x}, \quad x>0$

$\left[\text{Since } F(e^{-ax}, x>0) = \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{(a+i\omega)}\right.$

$\left.\Rightarrow F^{-1}\left(\dfrac{1}{a+i\omega}\right) = \sqrt{2\pi}\, e^{-ax}, \quad x>0\right]$

$\therefore F^{-1}[\hat{f}(\omega)\hat{g}(\omega)] = F^{-1}\left[\dfrac{1}{(2+i\omega)(3+i\omega)}\right]$

$= \dfrac{1}{\sqrt{2\pi}}\, f * g$

$= \dfrac{1}{\sqrt{2\pi}} \int_0^x \sqrt{2\pi}\, e^{-2u} \cdot \sqrt{2\pi}\, e^{-3(x-u)}\, du$

$= \dfrac{2\pi}{\sqrt{2\pi}}\, e^{-3x} \int_0^x e^{u}\, du$

$= \sqrt{2\pi}\, e^{-3x}(e^{x}-1)$