സഹായി

SFI GEC PALAKKAD

(1)

Module 2

Vector Integral Theorem

Green's Theorem

Let R be a simply connected plane region whose boundary is a simple closed, piecewise smooth curve C oriented anti-clockwise. If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first order partial derivatives on some open set containing R, then

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

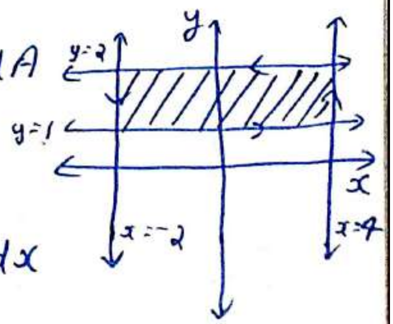
1) Use Green's theorem to evaluate $\oint_C 4xy\,dx + 2xy\,dy$ where C is the rectangle bounded by $x=-2$, $x=4$, $y=1$ and $y=2$

Soln :- Here $f(x,y) = 4xy$ and $g(x,y) = 2xy$

$$\Rightarrow \frac{\partial f}{\partial y} = 4x \qquad \Rightarrow \frac{\partial g}{\partial x} = 2y$$

By Green's theorem,

$$\oint_C 4xy\,dx + 2xy\,dy = \iint_R (2y-4x)\,dA$$



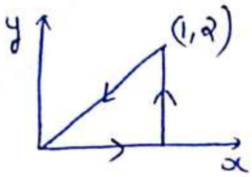
$$= \int_{x=-2}^{4}\int_{y=1}^{2} (2y-4x)\,dy\,dx$$

$$= \int_{x=-2}^{4} \left[y^2 - 4xy\right]_1^2 dx = \int_{x=-2}^{4} (3-4x)\,dx$$

$$= \left[3x - 2x^2\right]_{-2}^{4}$$

$$= \underline{-6}$$

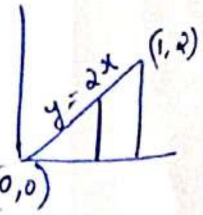
2) Use Green's theorem to evaluate $\int_C x^2 y\,dx + x\,dy$ along the triangular path given by



Soln : Here $f(x,y) = x^2 y$ and $g(x,y) = x$

$$\Longrightarrow \frac{\partial f}{\partial y} = x^2 \qquad \frac{\partial g}{\partial x} = 1$$

$$\oint_C x^2 y\,dx + x\,dy = \iint_R (1-x^2)\,dA$$



$$= \int_{x=0}^{1} \int_{y=0}^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 \left[(1-x^2)y\right]_0^{2x} dx = \int_0^1 (2x - 2x^3)\,dx = \underline{\frac{1}{2}}$$

3) Using Green's theorem, find the work done by the force field $\vec{F}(x,y) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counter clockwise direction.

(3)

<u>Soln</u>: $W = \oint_C \vec{F}\cdot d\vec{r} = \oint_C (e^x - y^3)dx + (\cos y + x^3)dy$

Here $f(x,y) = e^x - y^3$ and $g(x,y) = \cos y + x^3$

$\Rightarrow \frac{\partial f}{\partial y} = -3y^2$ $\qquad \frac{\partial g}{\partial x} = 3x^2$

By Green's theorem, $W = \iint_R (3x^2 + 3y^2)\, dA$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r^2)\, r\, dr\, d\theta$$

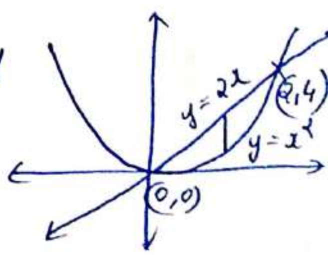
$$= 3\int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^1 d\theta = \frac{3}{4}\left[\theta\right]_0^{2\pi} = \underline{\underline{\frac{3\pi}{2}}}$$

4) Using Green's theorem evaluate $\oint_C (e^x + y^2)dx + (e^y + x^2)dy$ where C is the boundary of the region between $y = x^2$ and $y = 2x$.

<u>Soln</u>:- $f(x,y) = e^x + y^2$ and $g(x,y) = e^y + x^2$

$\Rightarrow \frac{\partial f}{\partial y} = 2y$ $\qquad \frac{\partial g}{\partial x} = 2x$

By Green's theorem, $\oint_C (e^x + y^2)dx + (e^y + x^2)dy$



$$= \iint_R (2x - 2y)\, dA$$

$$= 2\int_{x=0}^{2}\int_{y=x^2}^{2x} (x-y)\, dy\, dx = 2\int_0^2 \left[xy - \frac{y^2}{2}\right]_{x^2}^{2x} dx$$

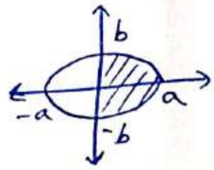
$$= 2\int_0^2 \left(-x^3 + \frac{x^4}{2}\right) dx = \underline{\underline{\frac{-8}{5}}}$$

To find area using Green's theorem

Area of the region, $R = \iint_R dA$

$$= \oint_C x\,dy \quad \text{or} \quad \oint_C (-y)\,dx$$

1) Use line integral to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Soln :- Area $= \oint_C x\,dy = 2\int_{-b}^{b} \frac{a}{b}\sqrt{b^2-y^2}\,dy$

$$= \frac{2a}{b}\int_{-\pi/2}^{\pi/2} \sqrt{b^2 - b^2\sin^2\theta}\; b\cos\theta\, d\theta$$

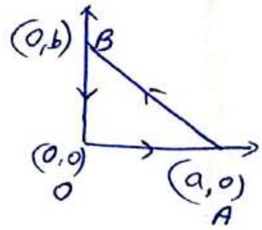
Put $y = b\sin\theta$
$dy = b\cos\theta\, d\theta$

$$= \frac{2a}{b}\int_{-\pi/2}^{\pi/2} b^2\cos^2\theta\, d\theta$$

$$= \frac{2a}{b}\int_{-\pi/2}^{\pi/2} b^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta = ab\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \underline{\underline{\pi ab}}$$

2) Use line integral, to find the area of the triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$ where $a>0$, $b>0$.



Soln :- Area $= \oint_C x\,dy = \int_{OA} x\,dy + \int_{AB} x\,dy + \int_{BO} x\,dy$

OA: $y=0$, $dy=0$; AB: $x = a - \frac{a}{b}y$; BO: $x=0$, $dx=0$

Eqn of AB
$\frac{x-a}{0-a} = \frac{y-0}{b-0}$
$\Rightarrow x = a - \frac{a}{b}y$

$$= 0 + \int_{y=0}^{b}\left(a - \frac{a}{b}y\right)dy + 0$$

$$= \left[ay - \frac{a}{b}\frac{y^2}{2}\right]_0^b = \underline{\underline{\frac{ab}{2}}}$$

H.W Ex 15.4, Question Nos: 5, 7, 10, 11, 12, 13, 29

Excercise 15.4

5) Using Green's theorem, evaluate
$\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$.

7) Using Green's theorem, evaluate
$\oint_C (x^2-y)dx + x\,dy$, where C is the circle
$x^2+y^2=4$.

10) Using Green's theorem, evaluate
$\oint_C x^2y\,dx - y^2x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle
$x^2+y^2=16$.

11) Using Green's theorem, evaluate
$\oint_C \tan^{-1}y\,dx - \frac{y^2x}{1+y^2}\,dy$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

12) Using Green's theorem, evaluate
$\oint_C \cos x\sin y\,dx + \sin x\cos y\,dy$, where C is the triangle with vertices $(0,0)$, $(3,3)$ and $(0,3)$

13) Using Green's theorem, evaluate
$\oint_C x^2y\,dx + (y+xy^2)dy$, where C is the boundary of the region enclosed by $y=x^2$ and $x=y^2$

29) Use Green's theorem to find the work done by the force field $\vec{F}(x,y) = xy\,\hat{i} + \left(\frac{1}{2}x^2 + xy\right)\hat{j}$ on a particle that starts at $(5,0)$, traverses the upper semi circle $x^2+y^2=25$ and returns to its starting point along the x-axis

Surface Integral

Let $f(x,y,z)$ be a continuous function defined on a smooth surface σ. Divide σ into n subsections $\sigma_1, \sigma_2, \ldots\ldots, \sigma_n$ with areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$ respectively.

Let (x_i, y_i, z_i) be any point on the i^{th} section.

Then the surface integral of $f(x,y,z)$ over σ is

$$\iint_\sigma f(x,y,z)\,dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i$$

where $\Delta S_i \to 0$

To evaluate surface integral

Let σ be a surface with equation $z = g(x,y)$ and the region R be its projection on the xy-plane. Then

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA$$

For $y = g(x,z)$ and R its projection on the xz-plane

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(x, g(x,z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2+1}\;dA$$

For $x = g(y,z)$ and R its projection on the yz-plane

$$\iint_\sigma f(x,y,z)\,dS = \iint_R f(g(y,z), y, z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\;dA$$

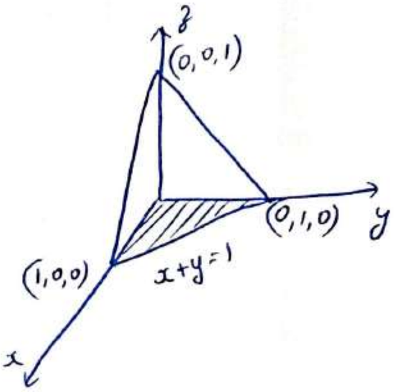
1) Evaluate the surface integral $\iint_{\sigma} xz \, dS$ where σ is the part of the plane $x+y+z=1$ that lies in the first octant.

<u>Soln</u> :- $\sigma : z = 1-x-y$

$$z = 1-x-y, \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$$

$$\iint_{\sigma} xz \, dS = \iint_{R} x(1-x-y)\sqrt{(-1)^2+(-1)^2+1} \; dA$$

$$= \sqrt{3} \int_{x=0}^{1} \int_{y=0}^{(1-x)} \left(x - x^2 - xy\right) dy \, dx$$

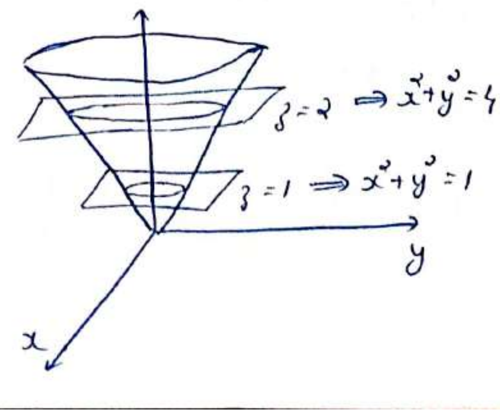


$$= \sqrt{3} \int_{0}^{1} \left[xy - x^2 y - \frac{xy^2}{2}\right]_{0}^{1-x} dx$$

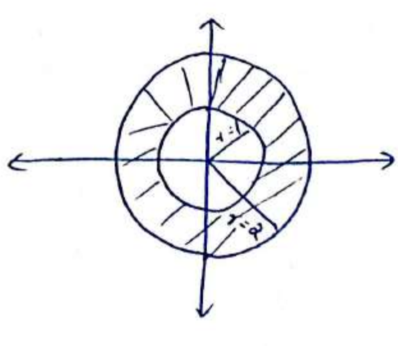
$$= \sqrt{3} \int_{0}^{1} \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx = \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_{0}^{1} = \frac{\sqrt{3}}{24}$$

2) Evaluate the surface integral $\iint_{\sigma} y^2 z^2 \, dS$ where σ is the part of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z=1$ and $z=2$.

<u>Soln</u> :-



Projection R



$$\sigma : z = \sqrt{x^2+y^2} \implies \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \iint_\sigma y^2 z^2 \, dS = \iint_R y^2 (x^2+y^2) \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \; dA$$

$$= \sqrt{2} \iint_R y^2 (x^2+y^2) \, dA$$

$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=1}^{2} (r\sin\theta)^2 (r^2) \, r \, dr \, d\theta$$

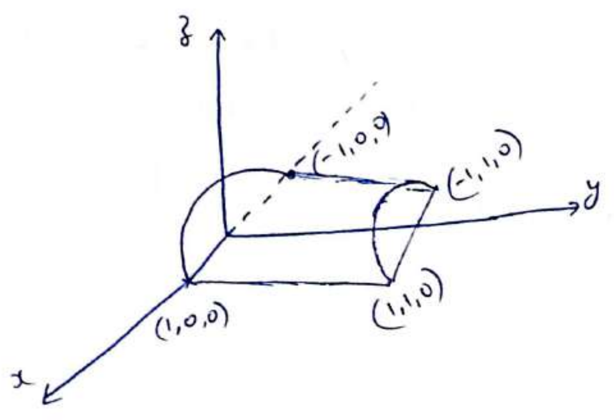
$$= \sqrt{2} \int_0^{2\pi} (\sin^2\theta) \left(\frac{r^6}{6}\right)\Bigg|_1^2 d\theta$$

$$= \sqrt{2} \left(\frac{63}{6}\right) \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\sqrt{2}}{2} \left(\frac{63}{6}\right) \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \underline{\underline{\frac{21\pi}{\sqrt{2}}}}$$

3) Evaluate $\iint_\sigma x^2 y \, dS$ where σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$, $y = 1$ and above the xy-plane.

<u>Soln</u>:-



$$\sigma: \quad x^2+z^2=1 \Rightarrow z=\sqrt{1-x^2}$$

$$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{\partial z}{\partial y}=0$$

$$\therefore \iint_\sigma x^2y\,dS = \iint_R x^2y\sqrt{\left(\frac{-x}{\sqrt{1-x^2}}\right)^2+0+1}\;dA$$

$$=\iint_R x^2y\sqrt{\frac{1}{(1-x^2)}}\;dA$$

$$=\int_{x=-1}^{1}\int_{y=0}^{1}\left(\frac{x^2}{\sqrt{1-x^2}}\,y\right)dy\,dx$$

$$=\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\left(\frac{y^2}{2}\right)_0^1 dx$$

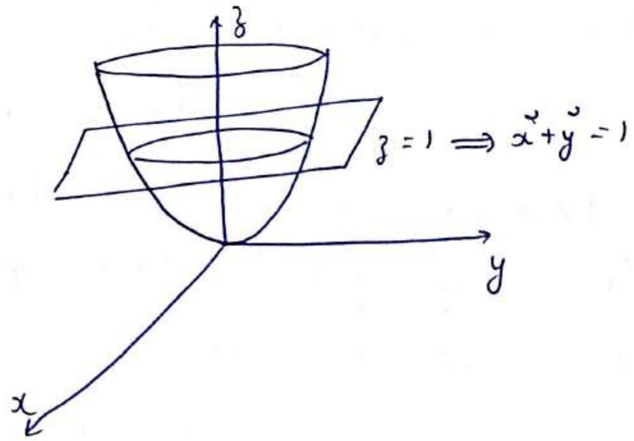
$$=\frac{1}{2}\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx \qquad \text{Put } x=\sin\theta, \quad dx=\cos\theta\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\frac{1-\cos 2\theta}{2}\,d\theta=\frac{1}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}=\frac{\pi}{4}$$

4) Find the mass of the lamina σ that is the portion of the paraboloid $z=x^2+y^2$ below the plane $z=1$ with constant mass density δ_0.

Soln :-



$z = x^2+y^2$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial z}{\partial y} = 2y$

$$\text{Mass} = \iint_\sigma \delta_0 \, dS = \iint_R \delta_0 \sqrt{(2x)^2+(2y)^2+1} \, dA$$

$$= \delta_0 \iint_R \sqrt{4(x^2+y^2)+1} \, dA$$

$$= \delta_0 \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{4r^2+1} \; r \, dr \, d\theta$$

Put

$4r^2+1 = t$

$8r\,dr = dt$

$r\,dr = \frac{dt}{8}$

$$= \delta_0 \int_0^{2\pi} \int_{t=1}^{5} \left(\sqrt{t} \, \frac{dt}{8}\right) d\theta$$

$$= \frac{\delta_0}{8} \int_0^{2\pi} \left(\frac{2}{3} t^{3/2}\right)_1^5 d\theta$$

$$= \frac{\delta_0}{12} \int_0^{2\pi} (5^{3/2}-1) \, d\theta = \frac{\delta_0}{12} (5^{3/2}-1) [\theta]_0^{2\pi}$$

$$= \frac{\delta_0 (5^{3/2}-1)\pi}{6}$$

H.W

Ex 15.5 , Question Nos: 1, 2, 4, 6, 27, 29, 30, Exercise 15.5

I) Evaluate the surface integral $\iint_{\sigma} f(x, y, z)\, dS$

1) $f(x, y, z) = z^2$, σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

2) $f(x, y, z) = xy$, σ is the portion of the plane $x + y + z = 1$ lying in the first octant.

4) $f(x, y, z) = (x^2 + y^2)z$, σ is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.

6) $f(x, y, z) = x + y$, σ is the portion of the plane $z = 6 - 2x - 3y$ in the first octant.

II) Find the mass of the lamina with constant density δ_0.

27) The lamina that is the portion of the circular cylinder $x^2 + z^2 = 4$ that lies directly above the rectangle $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 4\}$ in the xy-plane.

29) Find the mass of the lamina that is the portion of the surface $y^2 = 4 - z$ between the planes $x = 0$, $x = 3$, $y = 0$ and $y = 3$ if the density is $\delta(x, y, z) = y$.

30) Find the mass of the lamina that is the portion of the cone $z = \sqrt{x^2 + y^2}$ between

$z = 1$ and $z = 4$ if the density is
$\delta(x, y, z) = x^2 z$.

Application of surface integral :- Flux

Imagine that a fluid is flowing through a surface. Then flux is the volume of fluid that passes through the surface in one unit of time.

Let $F(x,y,z)$ be the vector field at a point (x,y,z) on the surface σ. Then

flux $\phi = \iint_{\sigma} \vec{F} \cdot \hat{n}\, dS$ where n is the unit normal vector towards the positive side of σ.

Let σ be the surface with equation $z = g(x,y)$ and R be its projection on the xy-plane.

Then $\phi = \iint_{\sigma} \vec{F} \cdot \hat{n}\, dS = \iint_{R} \vec{F} \cdot \left(-\hat{i} \frac{\partial z}{\partial x} - \hat{j} \frac{\partial z}{\partial y} + \hat{k} \right) dA$

(oriented up)

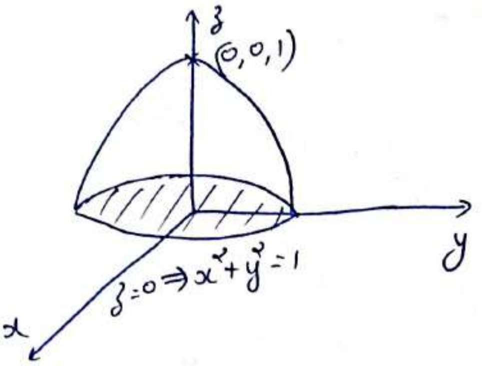
or

$\iint_{R} \vec{F} \cdot \left(\hat{i} \frac{\partial z}{\partial x} + \hat{j} \frac{\partial z}{\partial y} - \hat{k} \right) dA$

(oriented down)

1) Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane and suppose that σ is oriented up. Find the flux of the vector field, $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

Soln:-



$\sigma: z = 1 - x^2 - y^2$

$\frac{\partial z}{\partial x} = -2x$

$\frac{\partial z}{\partial y} = -2y$

$$\phi = \iint_{\sigma} \vec{F}\cdot\hat{n}\,dS = \iint_{R} \vec{F}\cdot\left(-\hat{i}\,\frac{\partial z}{\partial x} - \hat{j}\,\frac{\partial z}{\partial y} + \hat{k}\right) dA$$

$$= \iint_{R} \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot\left(-\hat{i}\,(-2x) - \hat{j}\,(-2y) + \hat{k}\right) dA$$

$$= \iint_{R} \left(2x^2 + 2y^2 + z\right) dA$$

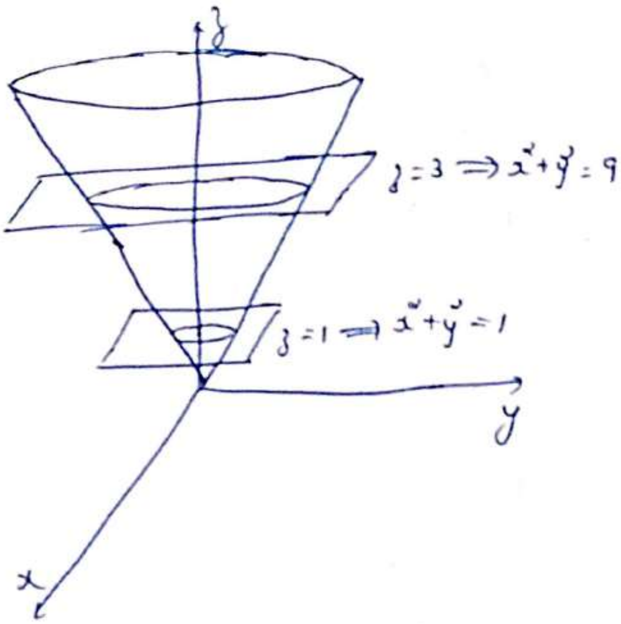
$$= \iint_{R} \left[2x^2 + 2y^2 + (1 - x^2 - y^2)\right] dA$$

$$= \iint_{R} \left(x^2 + y^2 + 1\right) dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r^2+1)\,r\,dr\,d\theta$$

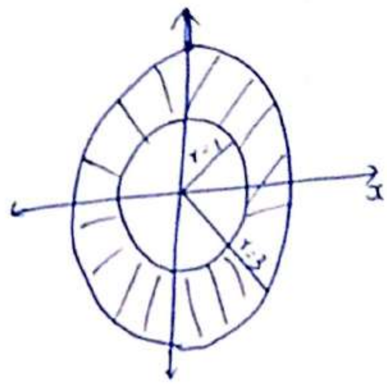
$$= \int_{0}^{2\pi}\left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 d\theta = \frac{3}{4}\left[\theta\right]_0^{2\pi} = \frac{3\pi}{2}$$

2) Let σ be the portion of the cone $z^2 = x^2 + y^2$ between the planes $z=1$ and $z=3$ oriented by upward unit normal. Find the flux of the vector field $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + 2z\hat{k}$.

Soln:- $\sigma: z^2 = x^2 + y^2 \implies \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$



Projection R



Flux $\phi = \iint_\sigma \vec{F} \cdot \hat{n}\, dS = \iint_R \left(x\hat{i} + y\hat{j} + 2z\hat{k}\right) \cdot \left(-\hat{i}\frac{x}{\sqrt{x^2+y^2}} - \hat{j}\frac{y}{\sqrt{x^2+y^2}} + \hat{k}\right) dA$

$$= \iint_R \left(\frac{-x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z\right) dA$$

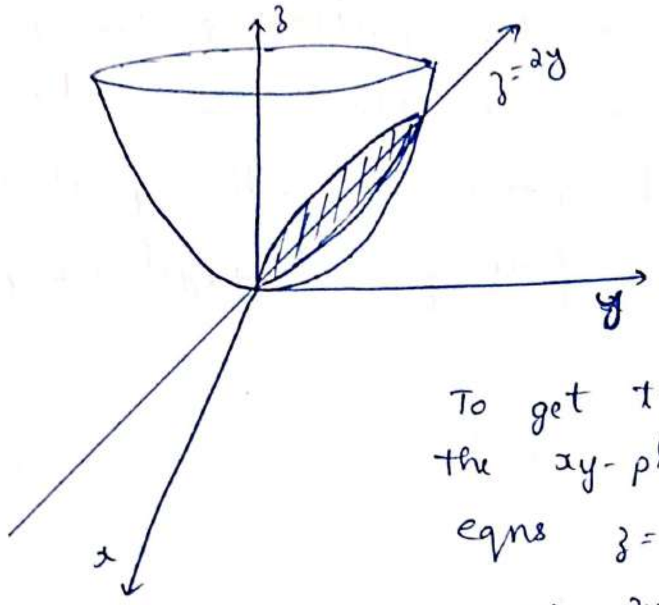
$$= \iint_R \left(\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + 2\sqrt{x^2+y^2}\right) dA$$

$$= \iint_R \left(\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right) dA = \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \left(\frac{r^2}{r}\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_1^3 = \frac{26}{3}\left[\theta\right]_0^{2\pi} = \underline{\underline{\frac{52\pi}{3}}}$$

3) Let σ be the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 2y$ oriented by downward unit normal. Find the flux of the vector field $\vec{F} = x\hat{k}$

Soln :-



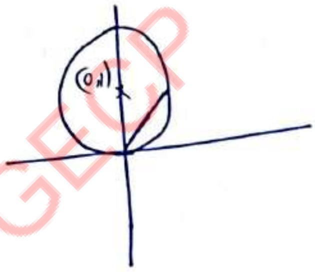
To get the projection R on the xy-plane, solve the eqns $z = x^2+y^2$ and $z = 2y$

$\Rightarrow 2y = x^2+y^2$

$\Rightarrow (x-0)^2 + (y-1)^2 = 1$

Circle → centre (0,1)

Radius = 1



In polar form, eqn is

$x^2+y^2-2y=0$

$r^2 - 2r\sin\theta = 0$

$r(r-2\sin\theta) = 0$

$\Rightarrow r=0 \quad , r = 2\sin\theta$

$$\therefore \text{Flux}, \phi = \iint_\sigma \vec{F}\cdot\hat{n}\,dS = \iint_R x\hat{k}\cdot\left(\hat{i}\frac{\partial z}{\partial x} + \hat{j}\frac{\partial z}{\partial y} - \hat{k}\right)dA$$

$$= \iint_R x\hat{k}\cdot\left(\hat{i}(2x) + \hat{j}(2y) - \hat{k}\right)dA$$

$$= \iint_R (-x)\,dA = \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} (-r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} -\cos\theta\left(\frac{r^3}{3}\right)_0^{2\sin\theta} d\theta = \int_0^{\pi} -\cos\theta\left(\frac{8\sin^3\theta}{3}\right)d\theta$$

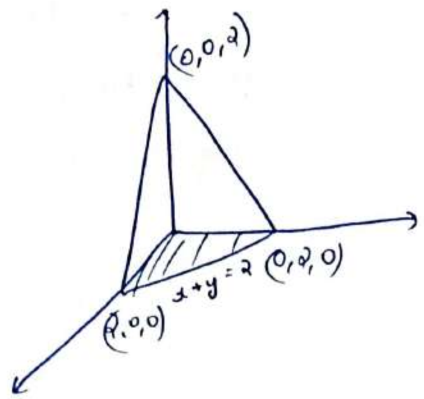
Put $\sin\theta = u$

$\cos\theta\,d\theta = du$

$$= \frac{-8}{3}\int_0^0 u^3\,du = \underline{\underline{0}}$$

4) Let σ be the portion of the plane $x+y+z=2$ in the first octane, oriented by unit normals with positive side. Find the flux of $\vec{F} = (x+y)\hat{i} + (y+z)\hat{j} + (z+x)\hat{k}$

<u>soln</u> :-



$\sigma : z = 2-x-y$

$\frac{\partial z}{\partial x} = -1$

$\frac{\partial z}{\partial y} = -1$

$$\phi = \iint_\sigma \vec{F}\cdot\hat{n}\, dS = \iint_R \left[(x+y)\hat{i} + (y+z)\hat{j} + (z+x)\hat{k}\right]\cdot\left[-\hat{i}(-1) - \hat{j}(-1) + \hat{k}\right] dA$$

$$= \iint_R \left[(x+y) + (y+z) + (z+x)\right] dA$$

$$= \iint_R \left[x+y + y + (2-x-y) + (2-x-y) + x\right] dA$$

$$= \iint_R 4\, dA = \int_{x=0}^{2}\int_{y=0}^{2-x} 4\, dy\, dx = 4\int_0^2 [y]_0^{2-x}\, dx$$

$$= 4\int_0^2 (2-x)\, dx$$

$$= 4\left[2x - \frac{x^2}{2}\right]_0^2 = \underline{\underline{8}}$$

H.W

Ex 15.6, Question Nos: 7, 10, 12, 23, 24

Exercise 15.6

Find the flux of the vector field $\vec{F}$ across σ.

7) $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + 2z\hat{k}$, σ is the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane oriented by upward normals.

10) $\vec{F}(x,y,z) = y\hat{j} + \hat{k}$, σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, oriented by downward unit normals.

12) $\vec{F}(x,y,z) = x^2\hat{i} + yx\hat{j} + zx\hat{k}$, σ is the portion of the plane $6x + 3y + 2z = 6$ in the first octant, oriented by unit normals with positive components.

23) $\vec{F}(x,y,z) = \hat{i} + \hat{j} + \hat{k}$, σ is the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$ oriented by downward unit normals

24) $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$, σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 1$ and $y = -2$, oriented by outward unit normals.

Gauss Divergence Theorem

Let V be a solid whose surface σ is oriented outward. If $F(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$ where f, g, h have continuous first order partial derivatives on some open set containing V and if n is the outward unit normal on σ, then,

$$\iint_{\sigma} \vec{F}\cdot\hat{n}\, dS = \iiint_{V} (\text{div } F)\, dV = \iiint_{V} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dV$$

1. Use divergence theorem to find the outward flux of the vector field $F(x,y,z) = 2x\hat{i} + 3y\hat{j} + z^2\hat{k}$ across the unit cube.

Soln: Flux $\phi = \iint_{\sigma} \vec{F}\cdot\hat{n}\, dS = \iiint_{V} (\text{div } F)\, dV$

$$= \iiint_{V} \left[\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (5 + 2z)\, dx\, dy\, dz$$

$$= \int_{0}^{1} \int_{0}^{1} (5 + 2z)[x]_0^1\, dy\, dz$$

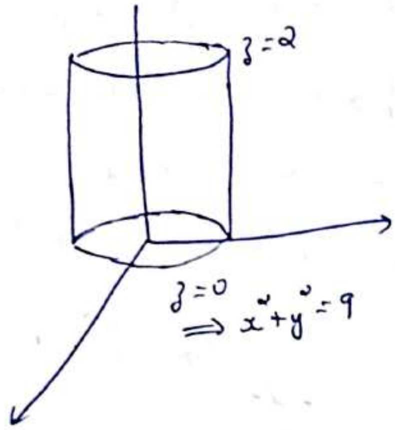
$$= \int_{0}^{1} (5 + 2z)[y]_0^1\, dz = \left[5z + z^2\right]_0^1$$

$$= \underline{\underline{6}}$$

2) Use divergence theorem to find the outward flux of the vector field $F(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^2\hat{k}$

across the surface of the region that is enclosed by the circular cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=2$.

Soln :-



$$\phi = \iiint_V (\operatorname{div} F)\, dV = \iiint_V (3x^2+3y^2+2z)\, dv$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{2} (3r^2+2z)\, r\, dz\, dr\, d\theta \quad \text{(Using cylindrical co-ordinates)}$$

$$= \int_0^{2\pi}\int_0^{3} \left[3r^3 z + z^2 r\right]_0^2 dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} (6r^3+4r)\, dr\, d\theta = \int_0^{2\pi}\left[\frac{6r^4}{4}+\frac{4r^2}{2}\right]_0^3 d\theta$$

$$= \frac{279}{2}\left[\theta\right]_0^{2\pi} = \underline{\underline{279\pi}}$$

3) Use divergence theorem to find the outward flux of the vector field $F(x,y,z) = z\hat{k}$ across the sphere $x^2+y^2+z^2=a^2$.

Soln :- $\phi = \iiint_V (\operatorname{div} F)\, dV = \iiint dV$ = Volume of the sphere

$$= \underline{\underline{\frac{4}{3}\pi a^3}}$$

4) Use divergence theorem to find the outward flux of the vector field $F(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ across the surface of the region that is enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$.

Soln :- $\phi = \iiint_V (\operatorname{div} F)\, dV = \iiint_V (3x^2 + 3y^2 + 3z^2)\, dV$

$$= 3\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{a} (\rho^2)\, \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta \quad \text{(Using spherical coordinates)}$$

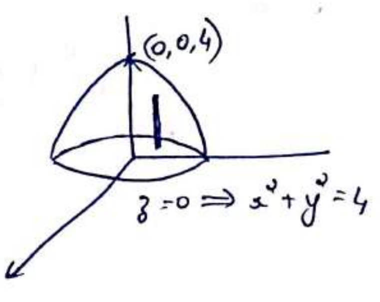
$$= 3\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left[\frac{\rho^5}{5}\right]_0^a d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2} \left(\frac{a^5}{5}\sin\phi\right) d\phi\, d\theta = \frac{3a^5}{5}\int_0^{2\pi} (-\cos\phi)_0^{\pi/2}\, d\theta$$

$$= \frac{3a^5}{5}\left[\theta\right]_0^{2\pi} = \frac{6\pi a^5}{5}$$

5) Use DT to find the outward flux of the vector field $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$ across the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Soln :-



$$\phi = \iiint_V (\operatorname{div} F)\, dV = \iiint_V 3\, dV = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-x^2-y^2} r\, dz\, dr\, d\theta$$

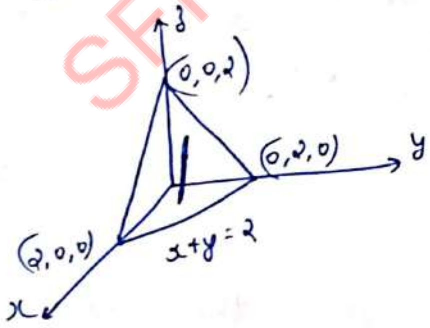
$$= 3\int_0^{2\pi}\int_0^{2} r\,[z]_0^{4-r^2}\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{2} (4r - r^3)\, dr\, d\theta = 3\int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

$$= 3\int_0^{2\pi} 4\, d\theta = 12\,[\theta]_0^{2\pi} = \underline{\underline{24\pi}}$$

6) Use DT to find the outward flux of the vector field $\vec{F}(x,y,z) = (x^2+y)\hat{i} + xy\hat{j} - (2xz+y)\hat{k}$ across the surface of the tetrahedron in the first octant bounded by $x+y+z=2$ and the co-ordinate planes.

<u>Soln</u> :-

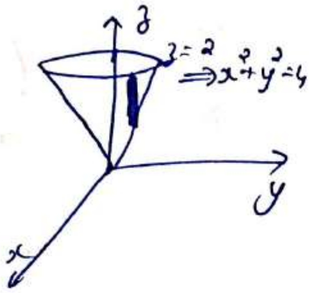


$$\phi = \iiint_V (\operatorname{div} F)\, dV = \iiint_V x\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{2-x-y} x\, dz\, dy\, dx$$

$$= \int_0^2\int_0^{2-x} (2x - x^2 - xy)\, dy\, dx = \int_0^2 \left[2xy - x^2y - x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^2 \left(-2x^2 + \frac{x^3}{2} + 2x\right) dx = \underline{\frac{2}{3}}$$

7) Use DT to find the outward flux of the vector field $\vec{F}(x,y,z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ across the surface of the conical solid bounded by $z = \sqrt{x^2+y^2}$ and $z = 2$



Soln :

$$\phi = \iiint_V (\text{div } F)\, dv$$

$$= 2\iiint_V (x+y+z)\, dv$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=\sqrt{x^2+y^2}}^{2} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{2}\left[r^2(\cos\theta+\sin\theta)z + \frac{rz^2}{2}\right]_r^2 dr\, d\theta$$

$$= 2\int_0^{2\pi} \left[\frac{2r^3}{3}(\cos\theta+\sin\theta) + r^2 - \frac{r^4}{4}(\cos\theta+\sin\theta) - \frac{r^4}{8}\right]_0^2 d\theta$$

$$= 2\int_0^{2\pi}\left[(\cos\theta+\sin\theta)\left(\frac{4}{3}\right) + 2\right] d\theta$$

$$= 2\left[\frac{4}{3}(\sin\theta - \cos\theta) + 2\theta\right]_0^{2\pi} = \underline{\underline{8\pi}}$$

H.W

Ex 15.7 Question Nos: 3, 4, 11, 13, 18, 19

H.W

Use Divergence Theorem to find the outward flux of the vector field

1) $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$; σ is the surface of the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$

2) $\vec{F}(x,y,z) = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region and the bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $z=3$.

3) $\vec{F}(x,y,z) = (x^2+y)\hat{i} + z^2\hat{j} + (e^y - z)\hat{k}$ where σ is the surface of the rectangular solid bounded by the co-ordinate planes and the planes $x=3$, $y=1$ and $z=3$.

Excercise 15.7

3) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = 2x\hat{i} - yz\hat{j} + z^2\hat{k}$, the surface σ is the paraboloid $z = x^2 + y^2$ capped by the disk $x^2 + y^2 \le 1$ in the plane $z = 1$

4) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = xy\hat{i} + yz\hat{j} + xz\hat{k}$, σ is the surface of the cube bounded by the planes $x=0, x=2, y=0, y=2, z=0, z=2$

11) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = (x-z)\hat{i} + (y-x)\hat{j} + (z-y)\hat{k}$; σ is the surface of the cylindrical solid bounded by $x^2+y^2 = a^2$, $z=0$ and $z=1$.

13) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, σ is the surface of the cylindrical solid $x^2+y^2=4$, $z=0$ and $z=3$

18) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = x^2y\hat{i} - xy^2\hat{j} + (z+2)\hat{k}$, σ is the

surface of the solid bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.

19) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = x^3\hat{i} + x^2y\hat{j} + xy\hat{k}$; σ is the surface of the solid bounded by $z = 4-x^2$, $y+z = 5$, $z=0$ and $y=0$

Stoke's theorem

Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C. If the components of the vector field

$$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

are continuous and have continuous first order partial derivatives on some open set containing σ, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_\sigma (\text{Curl } \vec{F})\cdot \hat{n}\, dS$

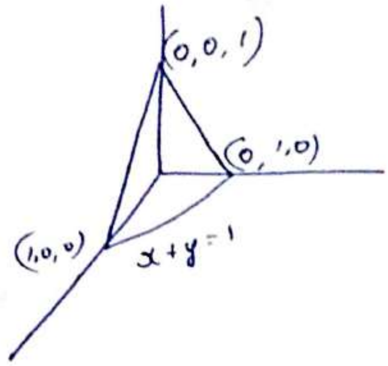
where n is the unit normal towards the positive side of σ.

1. Use Stoke's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x-y)\hat{i} + (y-z)\hat{j} + (z-x)\hat{k}$ where C is the boundary of the portion of the plane $x+y+z=1$ in the first octant. Assume that the surface has an upward direction.

<u>Soln</u> :- $\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y) & (y-z) & (z-x) \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_\sigma \text{Curl}\vec{F}\cdot \hat{n}\, dS = \iint_R (\hat{i}+\hat{j}+\hat{k})\cdot\left(-\hat{i}\frac{\partial z}{\partial x} - \hat{j}\frac{\partial z}{\partial y} + \hat{k}\right) dA$$

$$= \iint_R (\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})\, dA \quad \Bigg| \begin{array}{l} z = 1-x-y \\ \frac{\partial z}{\partial x} = -1 \\ \frac{\partial z}{\partial y} = -1 \end{array}$$



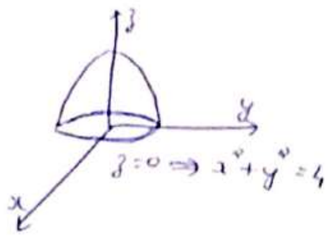
$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} dy\,dx$$

$$= 3\int_0^1 [y]_0^{1-x}\,dx = 3\int_0^1 (1-x)\,dx$$

$$= 3\left[x - \frac{x^2}{2}\right]_0^1 = \underline{\underline{\frac{3}{2}}}$$

2) Use Stoke's Theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $F(x,y,z) = 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$ and σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward direction and C to be positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane

Soln:-



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_\sigma \text{curl}\,\vec{F}\cdot\hat{n}\,dS$$

$$= \iint_R \text{curl}\,\vec{F}\cdot\left(-\hat{i}\frac{\partial z}{\partial x} - \hat{j}\frac{\partial z}{\partial y} + \hat{k}\right)dA$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\hat{i} + 2\hat{j} + 3\hat{k}$$

$$z = 4 - x^2 - y^2 \implies \frac{\partial z}{\partial x} = -2x \quad \text{and} \quad \frac{\partial z}{\partial y} = -2y$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (5\hat{i}+2\hat{j}+3\hat{k})\cdot(2x\hat{i}+2y\hat{j}+\hat{k})\,dA$$

$$= \iint_R (10x+4y+3)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}(10r\cos\theta+4r\sin\theta+3)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{10r^3}{3}\cos\theta+\frac{4r^3}{3}\sin\theta+\frac{3r^2}{2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left(\frac{80}{3}\cos\theta+\frac{32}{3}\sin\theta+6\right)d\theta$$

$$= \left[\frac{80}{3}\sin\theta-\frac{32}{3}\cos\theta+6\theta\right]_0^{2\pi} = \underline{\underline{12\pi}}$$

3) Use Stoke's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = z^2\hat{i}+3x\hat{j}-y^3\hat{k}$ and C is the circle $x^2+y^2=1$ in the xy-plane with counter clockwise direction looking down the +ve z-axis.

Soln :- $\quad \text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 3x & -y^3 \end{vmatrix} = (-3y^2)\hat{i}+(2z)\hat{j}+3\hat{k}$

$z=0 \Rightarrow \frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$.

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{Curl}\,\vec{F}\cdot\left(-\hat{i}\frac{\partial z}{\partial x}-\hat{j}\frac{\partial z}{\partial y}+\hat{k}\right)dA$$

$$= \iint_R \left[(-3y^2)\hat{i}+(2z)\hat{j}+3\hat{k}\right]\cdot\hat{k}\,dA$$

$$= \iint_R 3\,dA = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,dr\,d\theta = 3\int_0^{2\pi}\frac{1}{2}\,d\theta = \underline{\underline{3\pi}}$$

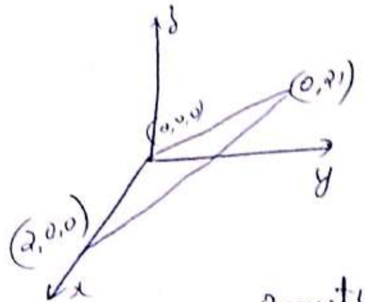
(24)

4) Use Stoke's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = -3y^2\hat{i} + 4z\hat{j} + 6x\hat{k}$ and C is the triangle in the plane $z = \frac{y}{2}$ with vertices $(2,0,0)$, $(0,2,1)$ and $(0,0,0)$ with a counter clockwise direction looking down the +ve z-axis.

Soln :- $\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = -4\hat{i} - 6\hat{j} + 6y\hat{k}$

$z = \frac{y}{2} \implies \frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = \frac{1}{2}$

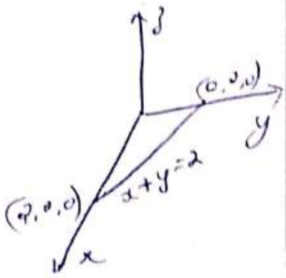
$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(-4\hat{i} - 6\hat{j} + 6y\hat{k}\right)\cdot\left(-\frac{1}{2}\hat{j} + \hat{k}\right) dA$



$= \iint_R (3 + 6y)\, dA$

$= \int_{x=0}^{2} \int_{y=0}^{2-x} (3 + 6y)\, dy\, dx$

Projection



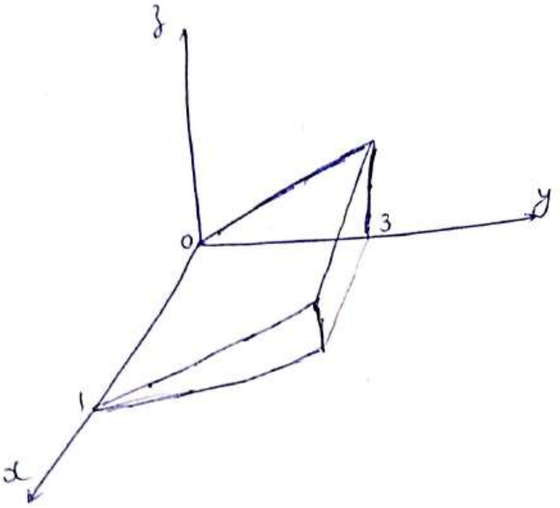
$= \int_0^2 \left(3y + 3y^2\right)_0^{2-x} dx$

$= \int_0^2 \left(3x^2 - 15x + 18\right) dx = \left[x^3 - \frac{15x^2}{2} + 18x\right]_0^2$

$= \underline{\underline{14}}$

5) Use stoke's theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $F(x,y,z) = x^2\hat{i} + 4xy^3\hat{j} + y^2x\hat{k}$ and C is the rectangle $0 \le x \le 1$, $0 \le y \le 3$ in the plane $z = y$ in the upward direction.

Soln:-



$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = (2xy)\hat{i} - (y^2)\hat{j} + (4y^3)\hat{k}$$

$$z = y \implies \frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 1$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_R \left[(2xy)\hat{i} - (y^2)\hat{j} + (4y^3)\hat{k}\right]\cdot(-\hat{j}+\hat{k})\,dA$$

$$= \iint_R (y^2 + 4y^3)\,dA = \int_{y=0}^{3}\int_{x=0}^{1}(y^2+4y^3)\,dx\,dy$$

$$= \int_0^3 (y^2+4y^3)\,[x]_0^1\,dy$$

$$= \left[\frac{y^3}{3} + y^4\right]_0^3 = \underline{\underline{90}}$$

H.W

Ex 15.8

I) Use stoke's theorum to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where

1) $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$; σ is the portion of the cone $z = \sqrt{x^2+y^2}$ below the plane $z=1$

2) $\vec{F} = (z-y)\hat{i} + (z+x)\hat{j} - (x+y)\hat{k}$;

σ is the portion of the paraboloid

$z = 4 - x^2 - y^2$ above the xy-plane.

3) $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$; σ is the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.