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SFI GEC PALAKKAD

MAT 102

Vector Calculus, Differential Equations and Transforms

Text Books :-

Module I & II → 'CALCULUS' by Anton, Biven and Davis

Module III, IV, V → Advanced Engineering Mathematics by Erwin Kreyszig

Module I

Calculus of vector functions

Vector valued function of a single variable

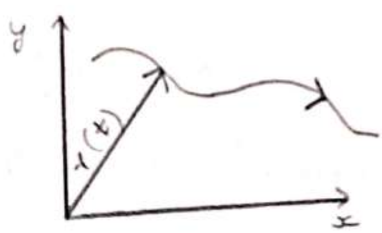
If, for each value of t (real number), there exists a unique vector $\vec{r}(t)$ represented as $\vec{r}(t) = \vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then $\vec{r}(t)$ is called a vector valued function of t in a 3-dimensional space.

$x(t)$, $y(t)$, $z(t)$ are called the components of $\vec{r}(t)$. $\vec{r}(t)$ is also denoted by $\langle x(t), y(t), z(t) \rangle$

Vector valued functions are a basic tool for analysing the motion of particles along curved paths.

eg: a moving car (2-dimension)

a flying bird
roller coaster } (3 dimension)



Graph of a vector valued function $\vec{r}(t)$: is the parametric curve obtained by joining the terminal points of $\vec{r}(t)$. [Parametric curves are curves having a direction (or orientation), the direction being the direction of the increasing parameter t.

Radius vector or position vector

Let the curve C be the graph of $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. The vector $\vec{r}(t)$ with its initial point at the origin and terminal point on C is called radius vector or position vector.

1) The graph of $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$ where $0 \le t \le 2\pi$ represents a circle in anti clockwise direction.

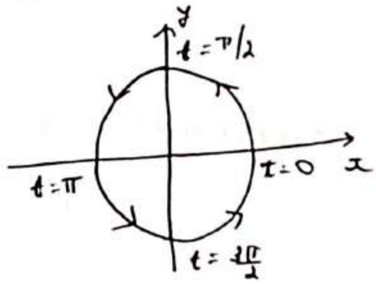
Here $x(t) = \cos t$ & $y(t) = \sin t$

when $t = 0$, $(x, y) = (1, 0)$

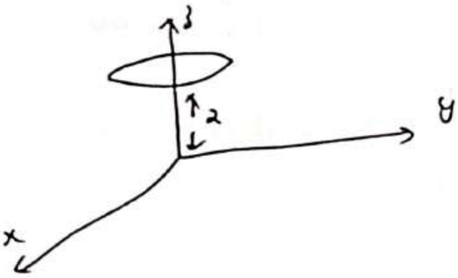
$t = \frac{\pi}{2}$, $(x, y) = (0, 1)$

$t = \pi$, $(x, y) = (-1, 0)$

$t = \frac{3\pi}{2}$, $(x, y) = (0, -1)$



2) The graph of $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2\hat{k}$ is



$t = 0 \Rightarrow (x, y, z) = (1, 0, 2)$

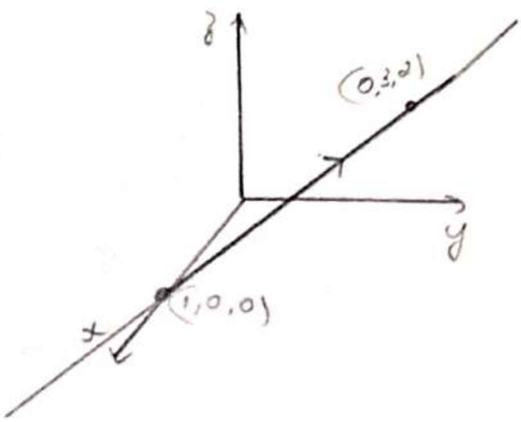
$t = \frac{\pi}{2} \Rightarrow (x, y, z) = (0, 1, 2)$

etc.

3) The graph of $\vec{r}(t) = (1-t)\hat{i} + 3t\hat{j} + 2t\hat{k}$ where $0 \le t \le 1$ represent a line segment in 3-space.

$t = 0 \Rightarrow \vec{r}(t) = \hat{i} = (1, 0, 0)$

$t = 1 \Rightarrow \vec{r}(t) = 3\hat{j} + 2\hat{k} = (0, 3, 2)$



Domain of a vector valued function

If $r(t)$ is defined in terms of component functions, then the natural domain of $\vec{r}(t)$ is the intersection of the natural domain of component functions.

1. Find the domain of $\vec{r}(t) = \sin 2t\,\hat{i} - 4t\,\hat{j}$ and the value of $r(\pi)$.

Soln :- Domain of $\sin 2t = (-\infty, \infty)$

Domain of $4t = (-\infty, \infty)$

$\therefore$ Domain of $\vec{r}(t) = \underline{(-\infty, \infty)}$

$r(\pi) = 0\,\hat{i} - 4\pi\,\hat{j} = \underline{-4\pi\,\hat{j}}$

2) Find the domain of $\vec{r}(t) = \cos \pi t\,\hat{i} - \ln t\,\hat{j} + \sqrt{t-4}\,\hat{k}$

Soln : Domain of $\cos \pi t = (-\infty, \infty)$

Domain of $\ln t = (0, \infty)$

Domain of $\sqrt{t-4} = [4, \infty)$ $\qquad t-4 \geq 0 \Rightarrow t \geq 4$

$\therefore$ Domain of $\vec{r}(t)$ = Intersection of the above domains

$= [4, \infty)$

3) Find the domain of $\vec{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle$

Soln :- Domain of $\ln|t-1| = R - \{1\}$

Domain of $e^t = (-\infty, \infty)$

Domain of $\sqrt{t} = [0, \infty)$

$\therefore$ Domain of $\vec{r}(t) = [0,1) \cup (1, \infty)$

Problems

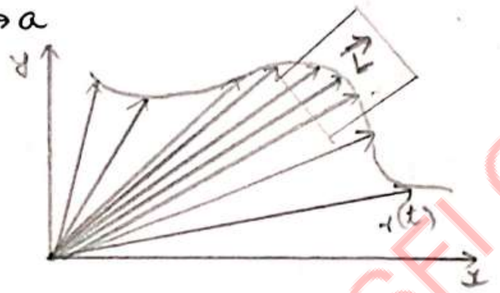
1. Find the domain of $\vec{r}(t) = \langle \sqrt{5t+1}, t^2 \rangle$

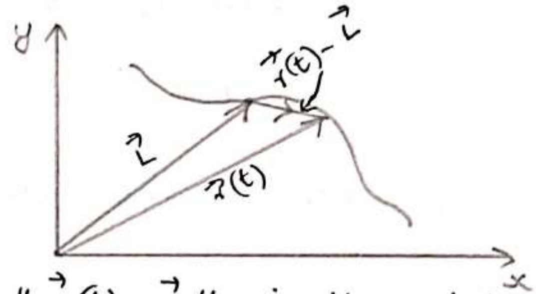
2. Find the domain of $\vec{r}(t) = \langle 2e^{-t}, \cos^{-1} t, \ln(1-t) \rangle$

Calculus of vector-valued functions

Limits: Let $\vec{r}(t)$ be a vector valued function that is defined for all t in some open interval containing a, then

$\lim_{t \to a} \vec{r}(t) = \vec{L}$ if and only if $\lim_{t \to a} \| \vec{r}(t) - \vec{L} \| = 0$





$\| \vec{r}(t) - \vec{L} \|$ is the distance between the terminal points of $\vec{r}(t)$ and $\vec{L}$.

Theorem: If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then

$$\lim_{t \to a} \vec{r}(t) = \left(\lim_{t \to a} x(t) \right)\hat{i} + \left(\lim_{t \to a} y(t) \right)\hat{j} + \left(\lim_{t \to a} z(t) \right)\hat{k}$$

1. Find the limit $\lim_{t \to 0} \vec{r}(t)$ where

$$\vec{r}(t) = t^2 \hat{i} + e^t \hat{j} - (2\cos \pi t)\hat{k}$$

Soln: $\lim_{t \to 0} \vec{r}(t) = \left(\lim_{t \to 0} t^2 \right)\hat{i} + \left(\lim_{t \to 0} e^t \right)\hat{j} + \lim_{t \to 0} (-2\cos \pi t)\hat{k}$

$$= 0\hat{i} + \hat{j} - 2\hat{k}$$

2) Find $\underset{t \to \infty}{Lt} \left\langle \frac{t^3+1}{4t^3+2}, \frac{1}{t} \right\rangle$

<u>Soln</u> :- $\underset{t \to \infty}{Lt} \left\langle \frac{t^3+1}{4t^3+2}, \frac{1}{t} \right\rangle = \left(\underset{t \to \infty}{Lt} \frac{\cancel{t^3}\left(1+\frac{1}{t^3}\right)}{\cancel{t^3}\left(4+\frac{2}{t^3}\right)} \right)\hat{i} + \left(\underset{t \to \infty}{Lt} \frac{1}{t} \right)\hat{j}$

$= \frac{1}{4}\hat{i} + 0\hat{j} = \frac{1}{4}\hat{i}$

<u>Continuity</u>

A vector valued function $\vec{r}(t)$ is said to be continuous at $t = a$ if $\underset{t \to a}{Lt}\, \vec{r}(t) = \vec{r}(a)$

1. Determine whether $\vec{r}(t) = 3\sin t\, \hat{i} + 3t\, \hat{j}$ is continuous at $t = 0$.

<u>Soln</u> :- $\underset{t \to 0}{Lt}\, \vec{r}(t) = \left(\underset{t \to 0}{Lt}\, 3\sin t \right)\hat{i} + \left(\underset{t \to 0}{Lt}\, 3t \right)\hat{j}$

$= 0\hat{i} + 0\hat{j} = \vec{r}(0)$

$\therefore$ $\vec{r}(t)$ is continuous at $t = 0$.

2. Determine whether $\vec{r}(t) = t^2\, \hat{i} + \frac{\cos t}{t}\, \hat{j} + t\, \hat{k}$ is continuous at $t = 0$

<u>Soln</u> :- $\underset{t \to 0}{Lt}\, \vec{r}(t) = \left(\underset{t \to 0}{Lt}\, t^2 \right)\hat{i} + \left(\underset{t \to 0}{Lt}\, \frac{\cos t}{t} \right)\hat{j} + \left(\underset{t \to 0}{Lt}\, t \right)\hat{k}$

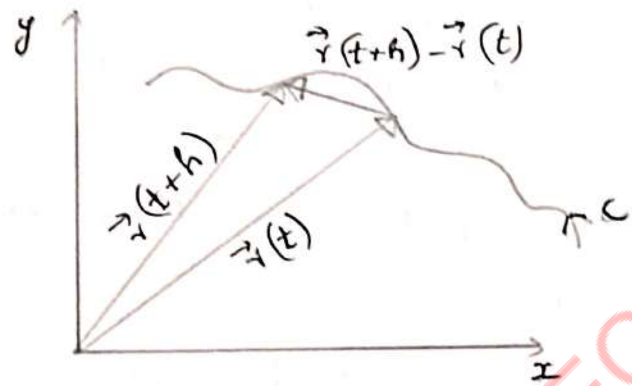
$= 0\hat{i} + \infty\hat{j} + 0\hat{k}$

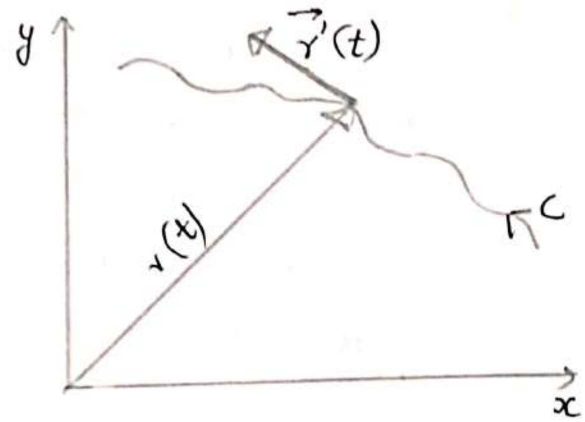
Limit does not exist. Hence $\vec{r}(t)$ is not continuous at $t = 0$.

Derivatives

If $\vec{r}(t)$ is a vector valued function, we define the derivative of $\vec{r}$ with respect to t to be the vector valued function $r'(t)$ given by $$\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Geometrical interpretation





$\vec{r}'(t)$ is the tangent vector drawn at the terminal point of $\vec{r}(t)$ in the direction of the increasing parameter of t.

Theorem :- If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then
$$\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

1. Find $\vec{r}'(t)$ where $\vec{r}(t) = t^2\hat{i} + e^t\hat{j} - (2\cos\pi t)\hat{k}$

Soln :- $\vec{r}'(t) = (2t)\hat{i} + (e^t)\hat{j} + (2\pi\sin\pi t)\hat{k}$

2. Find $\vec{r}'(t)$ where $\vec{r}(t) = 4\hat{i} - \cos t\,\hat{j}$

Soln :- $\vec{r}'(t) = (\sin t)\hat{j}$

3. Let $\vec{r}_1(t) = (\tan^{-1} t)\hat{i} + (\sin t)\hat{j} + (t^2)\hat{k}$
$\vec{r}_2(t) = (t^2 - t)\hat{i} + (2t-2)\hat{j} + (\ln t)\hat{k}$

Show that the graphs of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ intersect at the origin. Find the acute angle between the tangent vectors to the graph of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ at the origin.

Solution:- Consider $\vec{r}_1(t)$. At the origin, $\tan^{-1} t = 0$
$\sin t = 0$
$t^2 = 0$

which implies $t = 0$

Consider $\vec{r}_2(t)$. At the origin, $t^2 - t = 0$ $\quad 2t-2=0 \quad \ln t = 0$
$\Rightarrow t = 0, 1 \quad \Rightarrow t = 1 \quad \Rightarrow t = 1$

$\therefore \; t = 1.$

$\therefore$ when $t=0$, $\vec{r}_1(t)$ and $t=1$, $\vec{r}_2(t)$ intersect at the origin.

Tangent vector of $\vec{r}_1(t)$ at the origin $= \vec{r}_1'(0)$

$$= \left[\left(\frac{1}{1+t^2}\right)\hat{i} + (\cos t)\hat{j} + (2t)\hat{k}\right]_{t=0}$$

$$= \underline{\underline{\hat{i} + \hat{j}}}$$

Tangent vector of $\vec{r}_2(t)$ at the origin $= \vec{r}_2'(1)$

$$= \left[(2t-1)\hat{i} + 2\hat{j} + \frac{1}{t}\hat{k}\right]_{t=1}$$

$$= \underline{\underline{\hat{i} + 2\hat{j} + \hat{k}}}$$

$\therefore$ Acute angle between the tangent vectors,

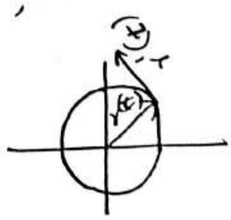
$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|}}$$

$$\cos\theta = \frac{(\hat{i}+\hat{j})\cdot(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{2}\cdot\sqrt{6}}$$

$$= \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \underline{\underline{\frac{\pi}{6}}}$$

<u>Theorem</u> :- If $\vec{r}(t)$ is a differentiable vector-valued function and $\|\vec{r}(t)\|$ is a constant for all t, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$. That is, $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal vectors.



Definite integral of vector-valued function

If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ then,

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b x(t)\,dt\right]\hat{i} + \left[\int_a^b y(t)\,dt\right]\hat{j} + \left(\int_a^b z(t)\,dt\right)\hat{k}$$

1. Evaluate $\int_0^1 \vec{r}(t)\,dt$ where $\vec{r}(t) = t^2\hat{i} + e^t\hat{j} - (2\cos\pi t)\hat{k}$

<u>Soln</u> :- $\int_0^1 \vec{r}(t)\,dt = \left(\int_0^1 t^2 dt\right)\hat{i} + \left(\int_0^1 e^t dt\right)\hat{j} - \left(\int_0^1 2\cos\pi t\,dt\right)\hat{k}$

$$= \left[\frac{t^3}{3}\right]_0^1 \hat{i} + \left[e^t\right]_0^1 \hat{j} - \left[\frac{2\sin\pi t}{\pi}\right]_0^1 \hat{k}$$

$$= \left(\frac{1}{3}\right)\hat{i} + (e-1)\hat{j}$$

2. Evaluate $\int \left[(2t)\hat{i} + (3t^2)\hat{j}\right] dt$

<u>Soln</u> :- $\int \left[(2t)\hat{i} + (3t^2)\hat{j}\right] dt = (t^2 + c_1)\hat{i} + (t^3 + c_2)\hat{j}$

$$= t^2\hat{i} + t^3\hat{j} + c_1\hat{i} + c_2\hat{j}$$

$$= t^2\hat{i} + t^3\hat{j} + C$$ where C is an arbitrary vector constant.

3. Find $\vec{r}(t)$. Given that $\vec{r}\,'(t) = \langle 3, 2t \rangle$ and $\vec{r}(1) = \langle 2, 5 \rangle$

Soln :- $\vec{r}(t) = \int \vec{r}'(t)\,dt = \int (3\hat{i} + 2t\hat{j})\,dt$

$= (3t)\hat{i} + (t^2)\hat{j} + C$

Given $\vec{r}(1) = \langle 2, 5 \rangle \Rightarrow 3\hat{i} + \hat{j} + C = 2\hat{i} + 5\hat{j}$

$\Rightarrow C = -\hat{i} + 4\hat{j}$

$\therefore \vec{r}(t) = (3t - 1)\hat{i} + (t^2 + 4)\hat{j}$

Note :- ① $\frac{d}{dt}\left[\vec{r}_1(t) \cdot \vec{r}_2(t)\right] = \vec{r}_1(t) \cdot \vec{r}_2'(t) + \vec{r}_1'(t) \cdot \vec{r}_2(t)$

② $\frac{d}{dt}\left[\vec{r}_1(t) \times \vec{r}_2(t)\right] = \vec{r}_1(t) \times \vec{r}_2'(t) + \vec{r}_1'(t) \times \vec{r}_2(t)$

Problems

1. Find the limit i) $\lim_{t \to 0}\left(2\sqrt{t}\,\hat{i} + \frac{\sin t}{t}\hat{j}\right)$

ii) $\lim_{t \to 2}\left(5t\hat{i} - 3\hat{j} + t^2\hat{k}\right)$ iii) $\lim_{t \to 1}\left\langle \frac{3}{t^2}, \frac{\ln t}{t^2 - 1}, \cos 3t \right\rangle$

2. Determine whether $\vec{r}(t)$ is continuous at $t = 0$.

i) $\vec{r}(t) = e^t\hat{i} + \hat{j} + \operatorname{cosec} t\,\hat{k}$

ii) $\vec{r}(t) = 5\hat{i} - \sqrt{6t+1}\,\hat{j} + e^{3t}\hat{k}$

3. Find $\vec{r}'(t)$ where i) $\vec{r}(t) = 6\hat{i} - \sin t\,\hat{j}$

ii) $\vec{r}(t) = (\tan^{-1} t)\hat{i} + t\cot t\,\hat{j} - 2\sqrt{t}\,\hat{k}$

iii) $\vec{r}(t) = \langle t, t^2 \rangle$ at $t_0 = 2$

iv) $\vec{r}(t) = \sec t\,\hat{i} + \tan t\,\hat{j}$ at $t_0 = 0$

4. Calculate $\frac{d}{dt}\left[\vec{r}_1(t) \cdot \vec{r}_2(t)\right]$ and $\frac{d}{dt}\left[\vec{r}_1(t) \times \vec{r}_2(t)\right]$ first by differentiating the product directly and then by applying the product rule.

i) $\vec{r}_1(t) = 2t\,\hat{i} + 3t^2\hat{j} + t^3\hat{k}$, $\vec{r}_2(t) = t^4\hat{k}$

ii) $\vec{r}_1(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$, $\vec{r}_2(t) = \hat{i} + t\,\hat{k}$

5) Evaluate the indefinite integral

i) $\int (5\hat{i} + 4t\hat{j})\,dt$ ii) $\int \left(t^2\hat{i} - 4t\hat{j} + \frac{1}{t}\hat{k}\right)dt$

iii) $\int \langle \cos t, \sin t \rangle\, dt$

6) Evaluate the definite integral

i) $\int_0^{\pi/2} \langle \cos 2t, \sin 2t \rangle\, dt$ ii) $\int_{-3}^{3} \langle (3-t)^{3/2}, (3+t)^{3/2}, 1 \rangle\, dt$

iii) $\int_1^9 \left(t^{1/2}\hat{i} + t^{-1/2}\hat{j}\right)dt$

7) Find $\vec{r}(t)$. Given that

i) $\vec{r}\,'(t) = 2t\hat{i} + 3t^2\hat{j}$, $\vec{r}(0) = \hat{i} - 2\hat{j}$

ii) $\vec{r}\,''(t) = \hat{i} + e^t\hat{j}$; $\vec{r}(0) = 2\hat{i}$; $\vec{r}\,'(0) = \hat{j}$

8) Show that the graphs of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ intersect at the point P. Find the acute angle between the tangent lines to the graphs of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ at the point P.

i) $\vec{r}_1(t) = t^2\hat{i} + t\hat{j} + 3t^3\hat{k}$

$\vec{r}_2(t) = (t-1)\hat{i} + \frac{1}{4}t^2\hat{j} + (5-t)\hat{k}$; $P(1, 1, 3)$

ii) $\vec{r}_1(t) = 2e^{-t}\hat{i} + \cos t\,\hat{j} + (t^2+3)\hat{k}$

$\vec{r}_2(t) = (1-t)\hat{i} + t^2\hat{j} + (t^3+4)\hat{k}$; $P(2, 1, 3)$

Motion along a curve - velocity, acceleration and speed

If $\vec{r}(t)$ is the position vector of a particle moving along a curve, then instantaneous velocity, acceleration and speed is given by,

Velocity, $v(t) = \dfrac{d\vec{r}}{dt}$

Acceleration, $a(t) = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

Speed $= \|v(t)\|$ or $\dfrac{ds}{dt}$ where s represents the arc length (or distance travelled)

1. Find the velocity, acceleration and speed where $\vec{r}(t) = t\hat{i} + t^2\hat{j}$ at $t=2$

Soln: $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$ $\qquad \vec{v}(2) = \hat{i} + 4\hat{j}$

$\vec{a}(t) = \dfrac{d\vec{v}}{dt} = 2\hat{j}$ $\qquad \vec{a}(2) = 2\hat{j}$

Speed $= \|v(t)\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$

Speed at $t=2$ is $\sqrt{1+4(2)^2} = \underline{\underline{\sqrt{17}}}$

2. A particle moves along a circular path in such a way that its x and y coordinates at time t are $x = 2\cos t$ $\quad y = 2\sin t$

a) Find velocity and speed

b) Show that at each instant the acceleration vector is perpendicular to velocity vector.

Soln :- Here $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$

a) $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$

Speed $= \|\vec{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \underline{\underline{2}}$

b) $\vec{a}(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j}$

$\vec{v}(t)\cdot\vec{a}(t) = \left(-2\sin t\,\hat{i} + 2\cos t\,\hat{j}\right)\cdot\left(-2\cos t\,\hat{i} - 2\sin t\,\hat{j}\right)$

$= 4\sin t\cos t - 4\sin t\cos t = 0$

$\therefore$ $\vec{v}(t)$ is perpendicular to $\vec{a}(t)$.

3) A particle moves through 3-space in such a way that its velocity is $\vec{v}(t) = \hat{i} + t\hat{j} + t^2\hat{k}$. Find the coordinates of the particle at time $t=1$. Given that the particle is at the point $(-1, 2, 4)$ at $t=0$.

Soln :- $\vec{r}(t) = \int \vec{v}(t)\,dt = \int(\hat{i} + t\hat{j} + t^2\hat{k})\,dt$

$= t\hat{i} + \dfrac{t^2}{2}\hat{j} + \dfrac{t^3}{3}\hat{k} + C$

$\vec{r}(0) = -\hat{i} + 2\hat{j} + 4\hat{k} \Rightarrow 0\hat{i} + 0\hat{j} + 0\hat{k} + C = -\hat{i} + 2\hat{j} + 4\hat{k}$

$\Rightarrow C = -\hat{i} + 2\hat{j} + 4\hat{k}$

$\therefore \vec{r}(t) = (t-1)\hat{i} + \left(\dfrac{t^2}{2} + 2\right)\hat{j} + \left(\dfrac{t^3}{3} + 4\right)\hat{k}$

$\vec{r}(1) = 0\hat{i} + \dfrac{5}{2}\hat{j} + \dfrac{13}{3}\hat{k}$

Displacement and Distance travelled

over the interval $t_1 \le t \le t_2$

i) Displacement of the particle is $\Delta r = \vec{r}(t_2) - \vec{r}(t_1)$ or $\int_{t_1}^{t_2} v(t)\,dt$

ii) Distance travelled = arc length, $s = \int_{t_1}^{t_2} \|v(t)\|\,dt$

1) The position vector of a particle moving along a circular helix in 3-space is given by $\vec{r}(t) = (4\cos\pi t)\hat{i} + (4\sin\pi t)\hat{j} + t\hat{k}$. Find the displacement and distance travelled by the particle during the time interval $1 \le t \le 5$.

Soln :- Displacement, $\Delta r = \vec{r}(t_2) - \vec{r}(t_1)$

$$= \vec{r}(5) - \vec{r}(1)$$

$$= \left[(4\cos 5\pi)\hat{i} + (4\sin 5\pi)\hat{j} + 5\hat{k}\right] - \left[(4\cos\pi)\hat{i} + (4\sin\pi)\hat{j} + \hat{k}\right]$$

$$= \left[-4\hat{i} + 5\hat{k}\right] - \left[-4\hat{i} + \hat{k}\right]$$

$$= \underline{\underline{4\hat{k}}}$$

Distance travelled, $s = \int_1^5 \|\vec{v}(t)\| \, dt$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-4\pi\sin\pi t)\hat{i} + (4\pi\cos\pi t)\hat{j} + \hat{k}$$

$$= \int_1^5 \sqrt{(-4\pi\sin\pi t)^2 + (4\pi\cos\pi t)^2 + 1^2} \, dt$$

$$= \int_1^5 \sqrt{16\pi^2 + 1} \, dt = \sqrt{16\pi^2+1}\,\left[t\right]_1^5$$

$$= \underline{\underline{4\sqrt{16\pi^2+1}}}$$

Problems

1) Find the velocity, acceleration and speed where

i) $\vec{r}(t) = 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$ at $t = \frac{\pi}{2}$

ii) $\vec{r}(t) = e^{t}\hat{i} + e^{-t}\hat{j}$ at $t = 2$

iii) $\vec{r}(t) = (2+4t)\hat{i} + (1-t)\hat{j}$ at $t = 1$

2) Use the given information to find the position and velocity vectors of the particle.

i) $\vec{a}(t) = -\cos t\,\hat{i} - \sin t\,\hat{j}$; $\vec{v}(0) = \hat{i}$; $\vec{r}(0) = 2\hat{j}$

ii) $\vec{a}(t) = \hat{i} + e^{-t}\hat{j}$; $\vec{v}(0) = 2\hat{i} + \hat{j}$; $\vec{r}(0) = \hat{j} - \hat{i}$

iii) $\vec{a}(t) = (t+1)^{-2}\hat{j} - e^{-2t}\hat{k}$; $\vec{v}(0) = 3\hat{i} - \hat{j}$; $\vec{r}(0) = \hat{k}$

3) Find the angle between $\vec{v}$ and $\vec{a}$ for $\vec{r} = t^3\hat{i} + t^2\hat{j}$ when $t = 2$

4) Find the displacement and the distance travelled over the indicated time interval.

i) $\vec{r}(t) = t^2\hat{i} + \frac{t^3}{3}\hat{j}$; $2 \leq t \leq 4$

ii) $\vec{r}(t) = (1 - 3\sin t)\hat{i} + 3\cos t\,\hat{j}$; $0 \leq t \leq \pi$

iii) $\vec{r}(t) = e^{t}\hat{i} + e^{-t}\hat{j} + \sqrt{2}t\,\hat{k}$, $0 \leq t \leq \ln 4$

iv) $\vec{r}(t) = \cos 2t\,\hat{i} + (1 - \cos 2t)\hat{j} + (3 + \frac{1}{2}\cos 2t)\hat{k}$

$0 \leq t \leq \frac{\pi}{2}$.

Directional derivative

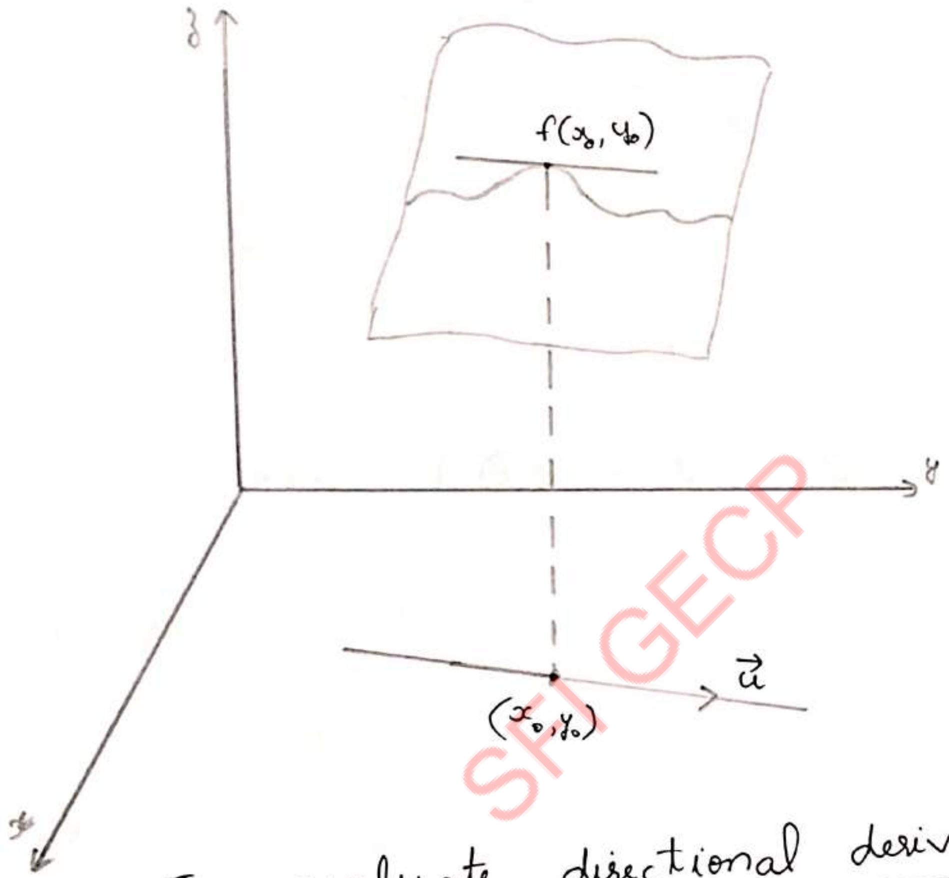
Here we extend the concept of a partial derivative to the more general notion of a directional derivative. We know that partial derivatives of a function give the instantaneous rate of change of that function in directions parallel to the coordinate axes. Directional derivative allow us to compute the instantaneous rate of change of a function in any direction.

Definition :- If $z = f(x, y)$ is a function of x and y, and if $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector, then the directional derivative of f in the direction of u at (x_0, y_0) is denoted by $D_u f(x_0, y_0)$ and is defined by

$$D_u f(x_0, y_0) = \frac{dz}{dt} = \frac{d}{dt}\left[f(x_0 + tu_1, y_0 + tu_2)\right]_{t=0}$$

Geometrical meaning :-

$D_u f(x_0, y_0)$ can be interpreted as the slope of the tangent of the surface $z = f(x, y)$ in the direction of u at the point $(x_0, y_0, f(x_0, y_0))$.



To evaluate directional derivative

$$D_u f(x_0, y_0) = \frac{dz}{dt} \quad \text{where } z = f(x, y) = f(x_0 + tu_1, y_0 + tu_2)$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\Longrightarrow \boxed{D_u f(x_0, y_0) = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2}$$

1. Let $f(x,y) = xy$. Find $D_u f(1, 2)$ for the unit vector $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

Soln :- $f_x = y$ $\quad$ $f_x(1, 2) = 2$; $u_1 = \frac{\sqrt{3}}{2}$

$f_y = x$ $\quad$ $f_y(1, 2) = 1$; $u_2 = \frac{1}{2}$

$\therefore D_u f(1,2) = f_x(1,2)u_1 + f_y(1,2)u_2$

$= \left(2 \times \frac{\sqrt{3}}{2}\right) + \left(1 \times \frac{1}{2}\right) = \underline{\underline{\sqrt{3} + \frac{1}{2}}}$

2) Find $D_u f(5,0)$. Given $f(x,y) = e^{2xy}$ and $\vec{u} = \frac{-3}{5}\hat{i} + \frac{4}{5}\hat{j}$

Soln :- $f_x(5,0) = \left(2y e^{2xy}\right)_{(5,0)} = \underline{\underline{0}}$

$f_y(5,0) = \left(2x e^{2xy}\right)_{(5,0)} = \underline{\underline{10}}$

$u_1 = \frac{-3}{5}$ and $u_2 = \frac{4}{5}$.

$\therefore D_u f(5,0) = f_x(5,0)u_1 + f_y(5,0)u_2 = \left(0 \times \frac{-3}{5}\right) + \left(10 \times \frac{4}{5}\right) = \underline{\underline{8}}$

3) Find $D_u f(0,0)$. Given $f(x,y) = \ln(1 + x^2 + y^2)$ and $\vec{u} = \frac{-1}{\sqrt{10}}\hat{i} - \frac{3}{\sqrt{10}}\hat{j}$.

Soln :- $f_x(0,0) = \left(\frac{2x}{1+x^2+y^2}\right)_{(0,0)} = \underline{\underline{0}}$

$f_y(0,0) = \left[\frac{2y}{1+x^2+y^2}\right]_{(0,0)} = \underline{\underline{0}}$

$D_u f(0,0) = f_x(0,0)u_1 + f_y(0,0)u_2 = \left(0 \times \frac{-1}{\sqrt{10}}\right) + \left(0 \times \frac{-3}{\sqrt{10}}\right) = \underline{\underline{0}}$

4) Find the directional derivative of $f(x,y,z) = x^2y - yz^3 + z$ at the point $(1,-2,0)$ in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$

Soln :- Here $\vec{a}$ is not a unit vector

$\vec{u} = \frac{\vec{a}}{||a||} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

$$f_x(1,-2,0) = (2xy)_{(1,-2,0)} = \underline{\underline{-4}}$$

$$f_y(1,-2,0) = (x^2 - z^3)_{(1,-2,0)} = \underline{\underline{1}}$$

$$f_z(1,-2,0) = (-3yz^2 + 1)_{(1,-2,0)} = \underline{\underline{1}}$$

$$\therefore D_u f(1,-2,0) = f_x(1,-2,0)u_1 + f_y(1,-2,0)u_2 + f_z(1,-2,0)u_3$$

$$= \left(-4 \times \frac{2}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(1 \times \frac{-2}{3}\right) = \underline{\underline{-3}}$$

5) Find the directional derivative of $f(x,y) = y^2 \ln x$ at $(1,5)$ in the direction of $\vec{a} = -3\hat{i} + 3\hat{j}$

Soln :- $\vec{u} = \dfrac{\vec{a}}{\|a\|} = \dfrac{-3\hat{i} + 3\hat{j}}{\sqrt{18}} = \dfrac{-\hat{i}}{\sqrt{2}} + \dfrac{\hat{j}}{\sqrt{2}}$

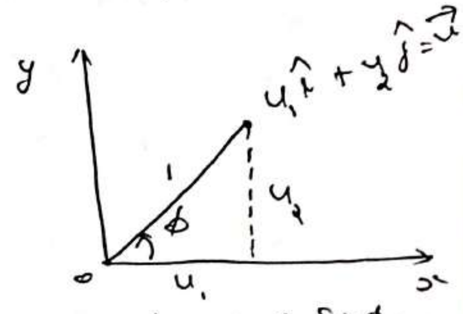
$$f_x(1,5) = \left(\frac{y^2}{x}\right)_{(1,5)} = \underline{\underline{25}} \qquad f_y(1,5) = (2y\ln x)_{(1,5)} = \underline{\underline{0}}$$

$$\therefore D_u f(1,5) = f_x(1,5)u_1 + f_y(1,5)u_2 = \left(25 \times \frac{-1}{\sqrt{2}}\right) + \left(0 \times \frac{1}{\sqrt{2}}\right)$$

$$= \underline{\underline{\frac{-25}{\sqrt{2}}}}$$

Another formula for directional derivative

A unit vector $\vec{u}$ in the xy-plane can be expressed as $\vec{u} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ where ϕ is the angle from +ve x-axis to u.



$\cos\phi = u_1$ & $\sin\phi = u_2$

$$\therefore D_u f(x_0, y_0) = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi$$

1. Find the directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of the unit vector that makes an angle of $\frac{\pi}{3}$ with the +ve x-axis.

<u>Soln</u> :- $f_x(-2,0) = \left(y e^{xy}\right)_{(-2,0)} = \underline{\underline{0}}$

$f_y(-2,0) = \left(x e^{xy}\right)_{(-2,0)} = \underline{\underline{-2}}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\therefore D_u f(-2,0) = f_x(-2,0)\cos\left(\frac{\pi}{3}\right) + f_y(-2,0)\sin\left(\frac{\pi}{3}\right)$

$= \left(0 \times \frac{1}{2}\right) + \left(-2 \times \frac{\sqrt{3}}{2}\right) = \underline{\underline{-\sqrt{3}}}$

2) Find the directional derivative of $f(x,y) = \sqrt{xy}$ at $P(1,4)$ in the direction of a vector making an angle of $\theta = \frac{\pi}{6}$ with +ve x-axis

<u>Soln</u> :- $D_u f(1,4) = f_x(1,4)\cos\frac{\pi}{6} + f_y(1,4)\sin\left(\frac{\pi}{6}\right)$

$f_x(1,4) = \left(\frac{y}{2\sqrt{xy}}\right)_{(1,4)} = \frac{4}{4} = \underline{\underline{1}}$; $f_y(1,4) = \left(\frac{x}{2\sqrt{xy}}\right)_{(1,4)} = \underline{\underline{\frac{1}{4}}}$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$; $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

$\therefore D_u f(1,4) = \left(1 \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{\sqrt{3}}{2} + \frac{1}{8} = \underline{\underline{\frac{4\sqrt{3}+1}{8}}}$

Problems

1) Find $D_u f$ at the point P.

i) $f(x,y) = (1+xy)^{3/2}$; $P(8,1)$; $u = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

ii) $f(x,y,z) = 4x^5y^2z^2$; $P(2,-1,1)$; $u = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$

iii) $f(x,y,z) = \ln(x^2+2y^2+4z^2)$; $P(-1,2,4)$; $u = \frac{-3}{13}\hat{i} - \frac{4}{13}\hat{j} - \frac{12}{13}\hat{k}$

2) Find the directional derivative of f at P in the direction of $\vec{a}$.

i) $f(x,y) = 4x^3y^2$; $P(2,1)$; $\vec{a} = 4\hat{i} - 3\hat{j}$

ii) $f(x,y) = e^x \cos y$; $P(0, \frac{\pi}{2})$; $\vec{a} = 5\hat{i} - 2\hat{j}$

iii) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$; $P(-1,1)$; $\vec{a} = -\hat{i} - \hat{j}$

iv) $f(x,y,z) = \frac{z-x}{z+y}$; $P(1,0,-3)$; $\vec{a} = -6\hat{i} + 3\hat{j} - 2\hat{k}$

3) Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

i) $f(x,y) = \tan(2x+y)$; $P\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$; $\theta = \frac{7\pi}{4}$

ii) $f(x,y) = \frac{x-y}{x+y}$; $P(-1,-2)$; $\theta = \frac{\pi}{2}$

4) Find the directional derivative of $f(x,y) = \frac{x}{x+y}$ at $P(1,0)$ in the direction of $Q(-1,-1)$.

[Hint : $\vec{PQ} = -2\hat{i} - \hat{j} \Rightarrow \vec{u} = \frac{-2\hat{i}-\hat{j}}{\sqrt{5}}$]

5) Find the directional derivative of $f(x,y) = e^{-x}\sec y$ at $P(0, \frac{\pi}{4})$ in the direction of the origin.

6) Find the directional derivative of $f(x,y) = \sqrt{xy}\, e^y$ at $P(1,1)$ in the direction of the negative y-axis

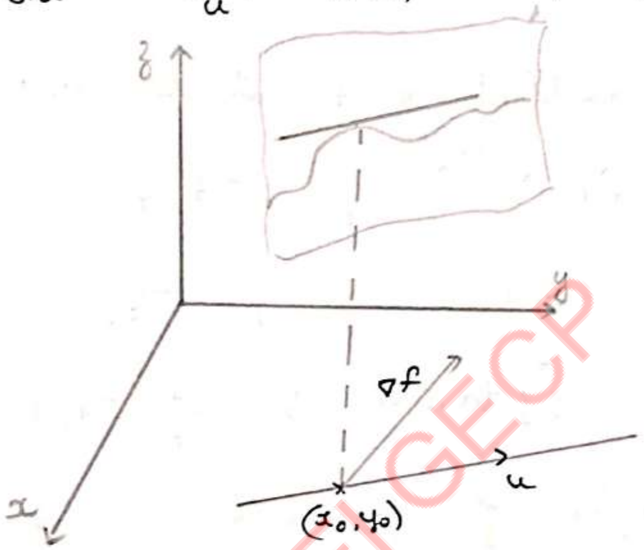
Gradient

Definition:

If f is a function of x and y, then the gradient of f, denoted by ∇f, is given by,

$$\nabla f(x,y) = f_x(x,y)\,\hat{i} + f_y(x,y)\,\hat{j}.$$

Therefore $D_u f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}$



1. Find the gradient of f at the indicated point

i) $f(x,y) = (x^2+xy)^3$ at $(-2,-1)$

Soln:
$$f_x(-2,-1) = \left[3(x^2+xy)^2(2x+y)\right]_{(-2,-1)} = -540$$

$$f_y(-2,-1) = \left[3(x^2+xy)^2\,x\right]_{(-2,-1)} = -216$$

$$\therefore \nabla f(-2,-1) = f_x(-2,-1)\,\hat{i} + f_y(-2,-1)\,\hat{j} = -540\,\hat{i} - 216\,\hat{j}$$

2. Find ∇w where $w = \ln\sqrt{x^2+y^2+z^2}$

Soln:
$$w_x = \left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\left(\frac{2x}{2\sqrt{x^2+y^2+z^2}}\right) = \frac{x}{x^2+y^2+z^2}$$

Similarly $w_y = \dfrac{y}{x^2+y^2+z^2}$ and $w_z = \dfrac{z}{x^2+y^2+z^2}$

$$\therefore \nabla w = \frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}$$

Problems

1) Find ∇z when i) $z = 4x - 10y$

ii) $z = e^{-3y}\cos 5x$

2) Find ∇w when $w = e^{-4x}\sec x^2 yz$

3) Find the gradient of f at the indicated point.

i) $f(x,y) = (x^2+y^2)^{-1/2}$; $(-3,4)$

ii) $f(x,y,z) = y\ln(x+y+z)$; $(-4,5,0)$

iii) $f(x,y,z) = y^2 z\tan^3 x$; $\left(\frac{\pi}{4}, -2, 1\right)$

Properties of gradient

Theorem: Let f be a function of two or three variables and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$ respectively. Assume that f is differentiable at P.

a) If $\nabla f = 0$ at P, then all directional derivative of f at P are zero. (That is, the surface $z = f(x,y)$ has relative maxima or minima)

b) If $\nabla f \neq 0$ at P, then among all possible directional derivative of f at P, the derivative in the direction of ∇f at P has the largest value and value is $\|\nabla f\|$ at P.

c) If $\nabla f \neq 0$, then among all possible directional derivative of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value and the value is $-\|\nabla f\|$ at P.

1. Let $f(x,y) = x^2 e^y$. Find the maximum value of directional derivative at $(-2,0)$ and find the unit vector in the direction in which the maximum value occurs.

Soln :
$$\nabla f(-2,0) = f_x(-2,0)\hat{i} + f_y(-2,0)\hat{j}$$
$$= (2xe^y)_{(-2,0)}\hat{i} + (x^2e^y)_{(-2,0)}\hat{j}$$
$$= -4\hat{i} + 4\hat{j}$$

Maximum value of the directional derivative
$$= \|\nabla f(-2,0)\| = \sqrt{(-4)^2 + (4)^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2,0)$

The unit vector in this direction is
$$\vec{u} = \frac{\nabla f(-2,0)}{\|\nabla f(-2,0)\|} = \frac{-4\hat{i}+4\hat{j}}{4\sqrt{2}} = \frac{-\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$$

2) Find a unit vector in the direction in which $f(x,y) = 4x^3y$ increase most rapidly at $P(-1,1)$ and find the rate of change of f at P in that direction.

Soln : ~~The~~ $f(x,y)$ increases most rapidly in the direction of $\nabla f(-1,1)$.

$$\nabla f(-1,1) = f_x(-1,1)\hat{i} + f_y(-1,1)\hat{j}$$
$$= (12x^2y)_{(-1,1)}\hat{i} + (4x^3)_{(-1,1)}\hat{j}$$
$$= 12\hat{i} - 4\hat{j}.$$

The unit vector in this direction is

$$\vec{u} = \frac{\nabla f(-1,1)}{\|\nabla f(-1,1)\|} = \frac{12\hat{i} - 4\hat{j}}{\sqrt{12^2 + (-4)^2}} = \frac{3}{\sqrt{10}}\hat{i} - \frac{1}{\sqrt{10}}\hat{j}$$

Rate of change of f at P in that direction

= Max. value of the directional derivative

$$= \|\nabla f(-1,1)\| = 4\sqrt{10}$$

3. Find a unit vector in the direction in which $f(x,y) = 20 - x^2 - y^2$ decreases most rapidly at $P(-1,3)$ and find the rate of change of f at P in that direction.

Soln: $f(x,y)$ decreases most rapidly in the direction of $-\nabla f(-1,3)$

$$-[\nabla f(-1,3)] = -[f_x(-1,3)\hat{i} + f_y(-1,3)\hat{j}]$$
$$= -[(-2x)_{(-1,3)}\hat{i} + (-2y)_{(-1,3)}\hat{j}]$$
$$= -[2\hat{i} - 6\hat{j}] = -2\hat{i} + 6\hat{j}$$

The unit vector in this direction is

$$\vec{u} = \frac{-\nabla f(-1,3)}{\|-\nabla f(-1,3)\|} = \frac{-2\hat{i} + 6\hat{j}}{\sqrt{(-2)^2 + (6)^2}} = \frac{-\hat{i}}{\sqrt{10}} + \frac{3\hat{j}}{\sqrt{10}}$$

Rate of change of f at P in that direction

= Min. value of the directional derivative

$$= -\|\nabla f(-1,3)\| = -2\sqrt{10}$$

Problems

1. Find a unit vector in the direction in which f increases most rapidly at P and find the rate of change of f at P in that direction

i) $f(x, y) = \sqrt{x^2+y^2}$; $P(4, 3)$

ii) $f(x, y) = \dfrac{x}{x+y}$; $P(0, 4)$

iii) $f(x, y, z) = x^3 z^2 + y^3 z + z - 1$; $P(1, 1, -1)$

iv) $f(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$; $P(1, 2, -2)$

2. Find a unit vector in the direction in which f decreases most rapidly at P and find the rate of change of f at P in that direction.

i) $f(x, y) = e^{xy}$; $P(1, 3)$

ii) $f(x, y) = \cos(3x - y)$; $P\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

iii) $f(x, y, z) = \dfrac{x+z}{z-y}$; $P(5, 7, 6)$

iv) $f(x, y, z) = 4e^{xy}\cos z$; $P\left(0, 1, \frac{\pi}{4}\right)$

Vector fields

Definition :- A vector field in a plane is a function that associates with each point (x,y) in the plane, a unique vector $\vec{F}(x,y)$ parallel to the plane.

$$\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$$

Similarly, a vector field in 3-space is a function that associates with each point (x,y,z), a unique vector $\vec{F}(x,y,z)$ in 3-space.

$$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

eg:- gravitational field, hurricanes, flowing river.

Note :- Vector field is the mathematical description of a flow. It is used for analysing the flow of fluid, flow of electricity etc.

Another notation :- If (x,y) can be represented by a radius vector, $\vec{r} = x\hat{i} + y\hat{j}$, then vector field can be written as $\vec{F}(\vec{r})$ or $\vec{F}$.

Gradient field or conservative field

If ϕ is a function of three variables, then the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

This formula defines a vector field $\vec{F}$ in 3-space called the gradient field of ϕ or conservative field of ϕ. Here ϕ is called the ~~poten~~ potential function.

1. The function $\phi(x, y, z) = xy + yz + xz$ is a potential function for the vector field $\vec{F}$. Find $\vec{F}$.

<u>Soln</u> :- $\vec{F} = \nabla\phi = \phi_x \hat{i} + \phi_y \hat{j} + \phi_z \hat{k} = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$

2. Confirm that $\phi(x, y, z) = x^2 - 3y^2 + 4z^3$ is a potential function for $\vec{F}(x, y, z) = 2x\hat{i} - 6y\hat{j} + 12z^2\hat{k}$

<u>Soln</u> :- $\nabla\phi = 2x\hat{i} - 6y\hat{j} + 12z^2\hat{k} = \vec{F}(x, y, z)$

Hence the result.

<u>Divergence and Curl</u>

There are two important operations on vector fields - the divergence and curl. These names originate in the study of fluid flow, in which case, the divergence relates to the way in which fluid flows towards or away from a point and the curl relates to the rotational properties of the fluid at a point.

<u>Divergence</u>

If $\vec{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}$, then

$$\text{div}\,\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \quad \text{or} \quad \nabla\cdot\vec{F}$$

$\left[\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f\hat{i} + g\hat{j} + h\hat{k}\right)\right]$

div $\vec{F}$ is a scalar quantity.

Curl

$$\text{Curl } \vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

Curl $\vec{F}$ is a vector.

It can be expressed in determinant form.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \nabla \times \vec{F}$$

1. Find the divergence and curl of the vector field $\vec{F}(x,y,z) = x^2y\,\hat{i} + 2y^3z\,\hat{j} + 3z\,\hat{k}$

Soln :- $\text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$$

$$= 2xy + 6y^2z + 3$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = \hat{i}(0-2y^3) - \hat{j}(0-0) + \hat{k}(0-x^2)$$

$$= (-2y^3)\hat{i} - (x^2)\hat{k}$$

Physical meaning of divergence and curl

The divergence of a vector function representing any physical quantity gives at each point, the rate per unit volume at which physical quantity is issuing from that point.

eg:- If V represents heat flux, div V is the rate at which heat is issuing from a point per unit volume

Curl of any vector function gives the measure of the angular velocity at any point of the vector field.

eg:- hurricane.

2. Find div $\vec{F}$ and curl $\vec{F}$ where

$$\vec{F} = e^{xy}\hat{i} - 2\cos y\,\hat{j} + \sin^2 z\,\hat{k}$$

Soln :- $\text{div}\,\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$$= ye^{xy} + 2\sin y + 2\sin z\cos z$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & -2\cos y & \sin^2 z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(e^{xy}\cdot x)$$

$$= (xe^{xy})\hat{k}$$

3. Find $\nabla\cdot(\vec{F}\times\vec{G})$ where $\vec{F}(x,y,z) = 2x\hat{i} + \hat{j} + 5y\hat{k}$

$$\vec{G}(x,y,z) = x\hat{i} + y\hat{j} - z\hat{k}.$$

Soln:- $\vec{F}\times\vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 1 & 5y \\ x & y & -z \end{vmatrix} = \hat{i}(-z-5y^2) - \hat{j}(-2xz-5xy) + \hat{k}(2xy-x)$

$$\nabla\cdot(\vec{F}\times\vec{G}) = \frac{\partial}{\partial x}(-z-5y^2) + \frac{\partial}{\partial y}(+2xz+5xy) + \frac{\partial}{\partial z}(2xy-x)$$

$$= 0 + 5x + 0 = 5x$$

4) Find $\nabla\cdot(\nabla\times\vec{F})$ where $\vec{F}(x,y,z) = \sin x\,\hat{i} + \sin(x-y)\hat{j} + z\hat{k}$

Soln: $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \sin(x-y) & z \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}\cos(x-y)$

$\therefore \nabla\cdot(\nabla\times\vec{F}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}[\cos(x-y)] = \underline{\underline{0}}$

Sources and sinks

The points at which $\operatorname{div} F > 0$ are called sources and the points at which $\operatorname{div} F < 0$ are called sinks. That is, fluid enters the flow at a source and drains out at a sink.

Problems

1. Confirm that ϕ is a potential function for $\vec{F}$

i) $\phi(x,y) = \tan^{-1} xy$

$\vec{F}(x,y) = \frac{y}{1+x^2y^2}\hat{i} + \frac{x}{1+x^2y^2}\hat{j}$

ii) $\phi(x,y) = 2y^2 + 3x^2y - xy^4$

$\vec{F}(x,y) = (6xy - y^4)\hat{i} + (4y + 3x^2 - 4xy^3)\hat{j}$

2. Find $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}$.

i) $\vec{F}(x,y,z) = xz^3\hat{i} + 2y^4x^2\hat{j} + 5z^3y\hat{k}$.

ii) $\vec{F}(x,y,z) = \ln y\,\hat{i} + e^{xyz}\hat{j} + \tan^{-1}\left(\frac{z}{x}\right)\hat{k}$

3. Find $\nabla\cdot(\vec{F}\times\vec{G})$ where $\vec{F}(x,y,z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$

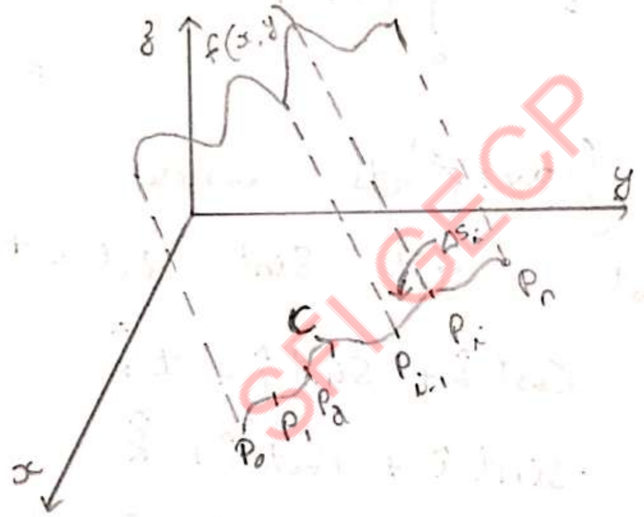
$\vec{G}(x,y,z) = xy\hat{i} + xyz\hat{j}$

4. Find $\nabla\times(\nabla\times\vec{F})$ where $\vec{F}(x,y,z) = y^2z\hat{i} - 3yz\hat{j} + xz\hat{k}$

Line Integral

Let $f(x,y)$ be a function that is continuous on a curve C. Divide C into n small sections at $P_0, P_1, P_2 \ldots\ldots P_n$. Let Δs_i be the length of the arc between P_{i-1} and P_i. Then the line integral of $f(x,y)$ along the curve C is defined as

$$\int_C f(x,y)\,ds = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} f(x,y)\,\Delta s_i \quad \text{where } \Delta s_i \to 0$$



Geometrical meaning

$\int_C f(x,y)\,ds$ represents the area of the sheet that is swept out by a vertical line segment that extends upward from the point (x,y) to a height of $f(x,y)$

Evaluation of the integral

Let C be the curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ where $a \le t \le b$

Then $$\int_C f(x,y)\,ds = \int_{t=a}^{b} f\,(x(t), y(t))\,\|r'(t)\|\,dt$$

1. Evaluate $\int_C (1+xy^2)\, ds$ where

$C: \vec{r}(t) = t\hat{i} + 2t\hat{j} \quad (0 \le t \le 1)$

Soln:
$$\vec{r}'(t) = \hat{i} + 2\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
$$\therefore \int_C (1+xy^2)\, ds = \int_{t=0}^{1} \left[1 + (t)(2t)^2\right] \|r'(t)\| dt$$
$$= \int_0^1 (1+4t^3)\sqrt{5}\, dt = \sqrt{5}\left[t + t^4\right]_0^1$$
$$= \underline{\underline{2\sqrt{5}}}$$

2. Evaluate $\int_C (xy + z^3)\, ds$ where

$C: x(t) = \cos t \quad y(t) = \sin t \quad z(t) = t \quad (0 \le t \le \pi)$

Soln:
$$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$
$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$
$$\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$
$$\therefore \int_C (xy + z^3)\, ds = \int_{t=0}^{\pi} \left[(\cos t)(\sin t) + t^3\right] \sqrt{2}\, dt$$
$$= \sqrt{2} \int_0^{\pi} \left(\frac{\sin 2t}{2} + t^3\right) dt$$
$$= \sqrt{2}\left[-\frac{\cos 2t}{4} + \frac{t^4}{4}\right]_0^{\pi} = \underline{\underline{\frac{\sqrt{2}\,\pi^4}{4}}}$$

3. Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ in the xy-plane to the parabolic cylinder $z = 1 - x^2$.

Soln :- Area of the surface $= \int_C (1-x^2)\,ds$

where $C : x^2+y^2=1$

$\Rightarrow x = \cos t \; ; \; y = \sin t \; (0 \le t \le 2\pi)$

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$\therefore \int_C (1-x^2)\,ds = \int_{t=0}^{2\pi} (1-\cos^2 t)\,dt = \int_0^{2\pi} \sin^2 t\,dt$

$= \int_0^{2\pi} \frac{1-\cos 2t}{2} = \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$

$= \underline{\underline{\pi}}$

4. Suppose that a semicircular wire has the equation $y = \sqrt{25-x^2}$ and that its mass density is $\delta(x,y) = 15-y$. Find the mass of the wire.

Soln :- Mass of the wire $= \int_C \delta(x,y)\,ds = \int_C (15-y)\,ds$

where C is $y = \sqrt{25-x^2} \Rightarrow x^2+y^2 = 25$

$\Rightarrow x = 5\cos t \; ; \; y = 5\sin t \; (0 \le t \le \pi)$

$\vec{r}(t) = 5\cos t\,\hat{i} + 5\sin t\,\hat{j}$

$(\vec{r}\,'(t)) = -5\sin t\,\hat{i} + 5\cos t\,\hat{j}$

$\|\vec{r}\,'(t)\| = 5$

$\therefore$ Mass $= \int_C (15-y)\,ds = \int_0^{\pi} (15-5\sin t)\,5\,dt$

$= 25\left[3t + \cos t\right]_0^{\pi}$

$= 75\pi - 25 - 25 = \underline{\underline{75\pi - 50}}$

Problems :

1. Evaluate $\int_C (x-y)\,ds$ where $C: x = 2t,\ y = 3t^2$ $(0 \le t \le 1)$

2. Evaluate $\int_C xyz^2\,ds$ where $C: x = t,\ y = 3t^2;\ z = 6t^3$ $(0 \le t \le 1)$

3. Evaluate $\int_C \frac{1}{1+x}\,ds$ where $C: \vec{r}(t) = t\,\hat{i} + \frac{2}{3}t^{3/2}\,\hat{j}$ $(0 \le t \le 3)$

4. Evaluate $\int_C 3x^2yz\,ds$ where $C: x = t;\ y = t^2;\ z = \frac{2}{3}t^3$ $(0 \le t \le 1)$

5. Evaluate $\int_C \frac{e^{-z}}{x^2+y^2}\,ds$ when $C: \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t\,\hat{k}$ $(0 \le t \le 2\pi)$

6. Evaluate $\int_C (x^3+y^3)\,ds$ where $C: \vec{r}(t) = e^{t}\,\hat{i} + e^{-t}\,\hat{j}$ $(0 \le t \le \ln 2)$

7. Find the area of the surface that extends upward from the parabola $y = x^2$ $(0 \le x \le 2)$ in the xy-plane to the plane $z = 3x$.

8. Find the area of the surface that extends upward from the semi circle $y = \sqrt{4-x^2}$ in the xy-plane to the surface $z = x^2y$.

9. Find the mass of a thin wire shaped in the form of the circular arc $y = \sqrt{9-x^2}$ $(0 \le x \le 3)$ if the density function is $\delta(x,y) = x\sqrt{y}$.

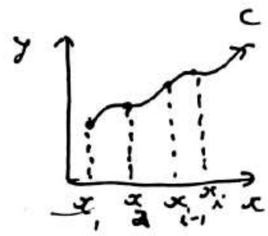
Line integral w.r.t x and y

We now describe a second type of line integral in which we replace the 'ds' in the integral by dx or dy.

Suppose that f is a function defined on a smooth curve C in the xy plane and the partition points of C are denoted by $P_i(x_i, y_i)$. Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$

We define $$\int_C f(x,y)\,dx = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i$$

and $$\int_C f(x,y)\,dy = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta y_i$$



Note :- Unlike Δs_i, the values of Δx_i and Δy_i change sign if the order of partion points along C is reversed. That is, reversing the orientation of the curve will change the sign of the line integral. $\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$.

Evaluation of the integral

Let C be the curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ $a \le t \le b$.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

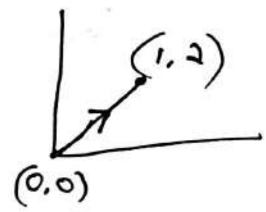
$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

1. Evaluate $\int_C (x+y)\,dx$ where C is the curve $x = 2t$; $y = 3t^2$ $\quad 0 \le t \le 1$

Soln :- $\int_C (x+y)\,dx = \int_0^1 (2t + 3t^2)\,2\,dt$ $\qquad x = 2t$, $dx = 2dt$

$$= 2\left[t^2 + t^3\right]_0^1 = \underline{\underline{4}}$$

2. Evaluate $\int_C 3xy\,dy$ where C is the line segment joining $(0,0)$ and $(1,2)$ in the given direction.



Soln :- Eqn of C is $\frac{x-0}{1-0} = \frac{y-0}{2-0}$

$$\Rightarrow y = 2x$$

$y = 2x$
$\Rightarrow dy = 2\,dx$

$$\therefore \int_C 3xy\,dy = \int_{x=0}^{1} 3x(2x)\,2\,dx$$

$$= \int_0^1 12x^2\,dx = \left[4x^3\right]_0^1 = \underline{\underline{4}}$$

3. Evaluate $\int_C 2xy\,dx + (x^2+y^2)\,dy$ along C given by $x = \cos t$ $\quad y = \sin t$ $\qquad 0 \le t \le \frac{\pi}{2}$

Soln :- $\int_C 2xy\,dx + (x^2+y^2)\,dy = \int_0^{\pi/2} 2(\cos t)(\sin t)(-\sin t)\,dt$

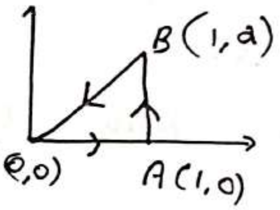
$$+ \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\cos t\,dt$$

$$= -2\int_0^{\pi/2} \cos t \sin^2 t\,dt + \int_0^{\pi/2} \cos t\,dt$$

Put $\sin t = u$

$$= -2\int_0^1 u^2\,du + \int_0^{\pi/2} \cos t\,dt = -2\left[\frac{u^3}{3}\right]_0^1 + \left[\sin t\right]_0^{\pi/2} = \underline{\underline{\frac{1}{3}}}$$

4) Evaluate $\int_C x^2 y\,dx + x\,dy$ where C is the triangular path



Solution:

Along OA, $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{OA} x^2 y\,dx + x\,dy = \int 0\,dx = \underline{\underline{0}}$$

Along AB, $x = 1 \Rightarrow dx = 0$

$$\int_{AB} x^2 y\,dx + x\,dy = \int_{y=0}^{2} dy = [y]_0^2 = \underline{\underline{2}}$$

Along BO, $y = 2x \Rightarrow dy = 2dx$

$$\int_{BO} x^2 y\,dx + x\,dy = \int_{x=1}^{0} 2x^3\,dx + x(2dx) = \left[\frac{x^4}{2} + x^2\right]_1^0$$

$$= \underline{\underline{\frac{-3}{2}}}$$

$$\therefore \int_C x^2 y\,dx + x\,dy = 0 + 2 - \frac{3}{2} = \underline{\underline{\frac{1}{2}}}$$

Problems:

1. Evaluate $\int_C (x-y)\,dx$ and $\int_C (x-y)\,dy$ where $x = 2t$; $y = 3t^2$ $0 \le t \le 1$

2. Evaluate $\int_C xyz^2\,dx$, $\int_C xyz^2\,dy$ and $\int_C xyz^2\,dz$ where C is $x = t$, $y = 3t^2$; $z = 6t^3$ $(0 \le t \le 1)$

3. Evaluate $\int_C (3x + 2y)\,dx + (2x - y)\,dy$ along the curve

a) the line segment from $(0,0)$ to $(1,1)$

b) the parabolic arc $y = x^2$ from $(0,0)$ to $(1,1)$

c) the curve $y = \sin\left(\frac{\pi x}{2}\right)$ from $(0,0)$ to $(1,1)$

d) the curve $x = y^3$ from $(0,0)$ to $(1,1)$.

4 Evaluate the line integral along the curve C.

i) $\int_C (x^2 - y^2)\,dx + x\,dy$ $\quad C: x = t^{2/3} \quad y = t \quad (-1 \le t \le 1)$

ii) $\int_C -y\,dx + x\,dy$ $\quad C: y^2 = 3x$ from $(3,3)$ to $(0,0)$

iii) $\int_C (x^2 + y^2)\,dx - x\,dy$ $\quad C: x^2 + y^2 = 1$ anticlockwise from $(1,0)$ to $(0,1)$

iv) $\int_C yz\,dx - xz\,dy + xy\,dz$ $\quad C: x = e^t;\ y = e^{3t};\ z = e^{-t}$ $\quad 0 \le t \le 1$

Line integral of a vector field along a curve C.

There is an alternative notation for line integrals w.r.t x and y.

Let $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j} \Rightarrow \vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C \left[f(x,y)\hat{i} + g(x,y)\hat{j}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \int_C \vec{F}(x,y)\cdot\vec{dr}$$

1. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \cos x\,\hat{i} + \sin x\,\hat{j}$

when $C: \vec{r}(t) = \frac{-\pi}{2}\hat{i} + t\,\hat{j} \quad (1 \le t \le 2)$

<u>Soln</u> :- Here $x(t) = \frac{-\pi}{2}$ and $y(t) = t$

$\Rightarrow dx = 0 \qquad dy = dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left((\cos x\,\hat{i} + \sin x\,\hat{j})\right)\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{t=1}^{2} \left[\cos\left(\frac{-\pi}{2}\right)\hat{i} + \sin\left(\frac{-\pi}{2}\right)\hat{j}\right]\cdot\left(0\hat{i} + dt\,\hat{j}\right]$$

$$= \int_{t=1}^{2} (-1)\,dt = \left[-t\right]_1^2 = \underline{\underline{-1}}$$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = x^2\hat{i} + xy\,\hat{j}$

and $C: \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} \quad (0 \le t \le \pi)$

<u>Soln</u> :- Here $x(t) = 2\cos t$ and $y(t) = 2\sin t$

$\Rightarrow dx = -2\sin t\,dt \qquad \Rightarrow dy = 2\cos t\,dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2\hat{i} + xy\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C x^2\,dx + xy\,dy$$

$$= \int_{t=0}^{\pi} (2\cos t)^2(-2\sin t)\,dt + (2\cos t)(2\sin t)\,2\cos t\,dt$$

$$= \int_0^{\pi} -8\cos^2 t\sin t\,dt + 8\cos^2 t\sin t\,dt = \underline{\underline{0}}$$

Problems :-

Evaluate $\int_C \vec{F} \cdot \vec{dr}$ along the curve C.

1) $\vec{F}(x, y) = x^2 y \hat{i} + 2\hat{j}$

$C: \vec{r}(t) = e^t \hat{i} + e^{-t} \hat{j} \quad (0 \le t \le 1)$

2) $\vec{F}(x, y) = (x^2 + y^2)^{-3/2} (x\hat{i} + y\hat{j})$

$C: \vec{r}(t) = e^t \sin t \, \hat{i} + e^t \cos t \, \hat{j} \quad (0 \le t \le 1)$

3) $\vec{F}(x, y, z) = z\hat{i} + x\hat{j} + y\hat{k}$

$C: \vec{r}(t) = \sin t \, \hat{i} + 4\sin t \, \hat{j} + \sin^2 t \, \hat{k} \quad (0 \le t \le \frac{\pi}{2})$

Work done as line integral

Suppose that under the influence of a continuous force field $\vec{F}$, a particle moves along a smooth curve C and that C is oriented in the direction of motion of the particle. Then the work performed by the force field on the particle is

$$W = \int_C \vec{F} \cdot \vec{dr}$$

1. Find the work done by the force field $\vec{F}(x, y) = xy\hat{i} + x^2\hat{j}$ on a particle that moves along the curve $C: x = y^2$ from (0,0) to (1,1)

Soln :- $x = y^2 \Rightarrow dx = 2y\,dy$

Work done, $W = \int_C \vec{F} \cdot \vec{dr} = \int_C (xy\hat{i} + x^2\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$

$= \int_C xy\,dx + x^2\,dy$

$$= \int_{y=0}^{1} (y^2)(y)(2y\,dy) + (y^2)^2 dy$$

$$= \int_0^1 2y^4 dy + y^4 dy = \int_0^1 3y^4 dy = \left[\frac{3y^5}{5}\right]_0^1 = \frac{3}{5}$$

2. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along a straight line from $(0,0,0)$ to $(2,1,3)$.

<u>Soln</u> :- $W = \int_C \vec{F}\cdot d\vec{r} = \int \left(3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

$$= \int 3x^2 dx + (2xz - y)dy + z\,dz$$

$$= \int_{y=0}^{1} 3(2y)^2(2dy) + \left[2(2y)(3y) - y\right]dy + 3y(3)dy$$

$$= \int_0^1 24y^2 dy + (12y^2 - y)dy + 9y\,dy$$

$$= \int_0^1 (36y^2 + 8y)dy = \left[12y^3 + 4y^2\right]_0^1$$

$$= \underline{16}$$

Eqn of st. line joining $(0,0,0)$ to $(2,1,3)$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$\Rightarrow x = 2y$
$z = 3y$

(Intersection of two planes is a st. line)

<u>Problems</u> :-

Find the work done by the force field $\vec{F}$ on a particle that moves along the curve C.

1) $\vec{F}(x,y) = (x^2 + xy)\hat{i} + (y - x^2y)\hat{j}$

C: $x = t$, $y = \frac{1}{t}$ $(1 \le t \le 3)$

2) $\vec{F}(x,y,z) = (xy)\hat{i} + (yz)\hat{j} + (xz)\hat{k}$

C: $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ $(0 \le t \le 1)$

3) $\vec{F}(x, y, z) = (x+y)\hat{i} + (xy)\hat{j} - z^2\hat{k}$

C: along line segments from (0,0,0) to (1,3,1) to (2,-1,5)

Independence of path

Consider the line integral $\int_C \vec{F}\cdot d\vec{r}$

The curve C is called the path of integration.

If $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r}$ does not depend on the path of integration.

1. $\vec{F}(x,y) = y\hat{i} + x\hat{j}$ is a conservative field. Show that $\int_C \vec{F}\cdot d\vec{r}$ is same along the paths

a) line segment $y = x$ from (0,0) to (1,1)

b) parabola $y = x^2$ from (0,0) to (1,1)

c) cubic $y = x^3$ from (0,0) to (1,1).

Solution :- a) $\int_C \vec{F}\cdot d\vec{r} = \int_C (y\hat{i} + x\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$

$$= \int_C y\,dx + x\,dy$$

$y = x$
$\Rightarrow dy = dx$

$$= \int_{x=0}^{1} x\,dx + x\,dx$$

$$= \int_{x=0}^{1} 2x\,dx = [x^2]_0^1 = \underline{\underline{1}}$$

b) $\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + x\,dy = \int_{x=0}^{1} x^2\,dx + x(2x\,dx)$

$y = x^2$
$dy = 2x\,dx$

$$= \int_0^1 3x^2\,dx = [x^3]_0^1 = \underline{\underline{1}}$$

c) $\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + x\,dy = \int_{x=0}^{1} x^3 dx + x\,(3x^2 dx)$ $\left| \begin{array}{l} y = x^3 \\ dy = 3x^2 dx \end{array}\right.$

$= \int_0^1 4x^3 dx = [x^4]_0^1 = \underline{\underline{1}}$

Fundamental theorem of line integral

Theorem: Suppose that $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1). If $\vec{F}(x,y) = \nabla\phi(x,y)$ and if C is any curve that starts from (x_0, y_0) and ends at (x_1, y_1), then $\int_C \vec{F}\cdot d\vec{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$

1. a) show that $\vec{F}(x,y) = y\hat{i} + x\hat{j}$ is conservative with potential function $\phi(x,y) = xy$

b) Use fundamental theorem of line integral to evaluate $\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r}$

Soln:- a) $\phi(x,y) = xy$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} = y\hat{i} + x\hat{j} = \vec{F}(x,y)$$

$\therefore \vec{F}(x,y)$ is conservative.

b) $\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \phi(1,1) - \phi(0,0) = 1 - 0 = \underline{\underline{1}}$

Note:- If $\vec{F}(x,y)$ is a conservative vector field and C is any closed path, then

$$\oint_C \vec{F}\cdot d\vec{r} = \phi(x_0, y_0) - \phi(x_0, y_0) = \underline{\underline{0}}$$

A test for conservative vector fields

Theorem :- If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives in some open region D and if $\vec{F}(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$ is conservative on D, then

$$\boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}}.$$

Conversely, if D is simply connected and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then $\vec{F}(x,y)$ is conservative

1. Determine whether $\vec{F}(x,y) = (y+x)\hat{i} + (y-x)\hat{j}$ is conservative.

Soln :- Here $f(x,y) = y+x$ and $g(x,y) = y-x$

$$\frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = -1$$

$$\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}. \quad \text{Hence not conservative.}$$

Conservative vector field in 3-space

Let $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$.

If D is simply connected and

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\ ; \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \quad \text{or}$$

equivalently if curl $\vec{F} = 0$, then $\vec{F}(x,y,z)$ is conservative

1. Show that $\int_C yz\,dx + xz\,dy + yx^2\,dz$ is not independent of path.

Soln : To show that the given integral is not independent of path, it is enough to show that $\vec{F}$ is not conservative.

$$\int_C yz\,dx + xz\,dy + yx^2\,dz = \int_C \left(yz\hat{i} + xz\hat{j} + yx^2\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int \vec{F}\cdot\vec{dr}.$$

Here $\vec{F} = yz\hat{i} + xz\hat{j} + yx^2\hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & yx^2 \end{vmatrix} = \hat{i}(x^2 - x) - \hat{j}(2xy - y) + \hat{k}(z - z)$$

$$\neq 0.$$

$\vec{F}$ is not conservative. Hence the integral is not independent of path.

2. Let $\vec{F}(x,y) = 2xy^3\hat{i} + (1 + 3x^2y^2)\hat{j}$

a) Show that $\vec{F}$ is conservative

b) Find ϕ, by integrating $\frac{\partial \phi}{\partial x}$

c) Find ϕ, by integrating $\frac{\partial \phi}{\partial y}$

d) Using ϕ, evaluate $\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy$

Soln : $f(x,y) = 2xy^3$ $\qquad g(x,y) = 1 + 3x^2y^2$

$$\frac{\partial f}{\partial y} = 6xy^2 \qquad \frac{\partial g}{\partial x} = 6xy^2$$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Hence $\vec{F}$ is conservative.

b) Since $\vec{F}$ is conservative, $\vec{F} = \nabla \phi$

$$\Rightarrow 2xy^3\,\hat{i} + (1+3x^2y^2)\,\hat{j} = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y}$$

Comparing both sides,

$$\frac{\partial \phi}{\partial x} = 2xy^3 \qquad \frac{\partial \phi}{\partial y} = 1 + 3x^2y^2 \text{ —— (1)}$$

Integrating $\frac{\partial \phi}{\partial x}$ w.r.t x (keeping y constant)

$$\phi = x^2y^3 + k(y) \text{ —— (2)}$$

To find $k(y)$, diff. (2) partially w.r.t y

$$\frac{\partial \phi}{\partial y} = 3x^2y^2 + k'(y) \text{ —— (3)}$$

Comparing (1) and (3) $1 + 3x^2y^2 = 3x^2y^2 + k'(y)$

$$\Rightarrow k'(y) = 1 \Rightarrow k(y) = y + c$$

$$\therefore \phi = \underline{\underline{x^2y^3 + y + c}}$$

c) Integrating $\frac{\partial \phi}{\partial y}$ w.r.t y (keeping x constant)

$$\phi = y + x^2y^3 + k(x) \text{ —— (4)}$$

Differentiating (4) partially w.r.t x,

$$\frac{\partial \phi}{\partial x} = 2xy^3 + k'(x) \text{ —— (5)}$$

Comparing (1) and (5) $2xy^3 = 2xy^3 + k'(x)$

$$\Rightarrow k'(x) = 0 \Rightarrow k(x) = c$$

$$\therefore \phi = \underline{\underline{y + x^2y^3}} + c$$

d) $$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^3)\,dy = \int_{(1,4)}^{(3,1)} \vec{F}\cdot d\vec{r}$$

$$= \phi(3,1) - \phi(1,4) = (10 + c) - (68 + c)$$

$$= \underline{-58}$$

3) Determine whether $\vec{F}$ is conservative. If so, find a potential function for it.

a) $\vec{F}(x,y) = 6y^2\hat{i} + 12xy\hat{j}$

Soln :- Here $f(x,y) = 6y^2$ $\quad g(x,y) = 12xy$

$$\frac{\partial f}{\partial y} = 12y \qquad \frac{\partial g}{\partial x} = 12y$$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Hence $\vec{F}$ is conservative.

To find ϕ

$$\vec{F} = \nabla\phi \Rightarrow 6y^2\hat{i} + 12xy\hat{j} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 6y^2 \qquad \frac{\partial\phi}{\partial y} = 12xy \quad \text{---(1)}$$

Integrating $\frac{\partial\phi}{\partial x}$ w.r.t x (keeping y constant)

$$\phi = 6xy^2 + k(y) \quad \text{---(2)}$$

Diff. (2) partially w.r.t y,

$$\frac{\partial\phi}{\partial y} = 12xy + k'(y) \quad \text{---(3)}$$

Comparing (1) and (3), $k'(y) = 0 \Rightarrow k(y) = c$

$$\therefore \phi = 6xy^2 + c$$

4) Show that $\int_{(1,2)}^{(4,0)} 4y\,dx + 4x\,dy$ is independent of path and evaluate the integral using fundamental theorem.

Soln :- Here $\vec{F}(x,y) = 4y\hat{i} + 4x\hat{j}$

$\frac{\partial f}{\partial y} = 4 \qquad \frac{\partial g}{\partial x} = 4$. $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Hence $\vec{F}$ is conservative and hence the integral is independent of path.

To find ϕ

$$\vec{F} = \nabla\phi \Rightarrow 4y\hat{i} + 4x\hat{j} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 4y \qquad \frac{\partial\phi}{\partial y} = 4x \quad \text{—(1)}$$

Integrating $\frac{\partial\phi}{\partial x}$,

$$\phi = 4xy + k(y) \Rightarrow \frac{\partial\phi}{\partial y} = 4x + k'(y) \quad \text{—(2)}$$

Comparing (1) and (2) $k'(y) = 0 \Rightarrow k(y) = c$

$$\therefore \phi = \underline{\underline{4xy}} + c$$

$$\int_{(1,2)}^{(4,0)} 4y\,dx + 4x\,dy = \phi(4,0) - \phi(1,2)$$

$$= c - (8 + c) = \underline{\underline{-8}}$$

5) Let $\vec{F}(x,y) = e^y\hat{i} + xe^y\hat{j}$ denote a force field

a) Verify that $\vec{F}$ is conservative.

b) Find the work done by the field on a particle that moves from $(1,0)$ to $(-1,0)$ along the semi-circular path

Soln:- a) $f(x,y) = e^y$; $g(x,y) = xe^y$

$$\frac{\partial f}{\partial y} = e^y \qquad \frac{\partial g}{\partial x} = e^y$$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Hence $\vec{F}$ is conservative.

b) Work done $= \int_{(1,0)}^{(-1,0)} \vec{F}\cdot d\vec{r} = \phi(-1,0) - \phi(1,0)$

To find ϕ

$$\vec{F} = \nabla\phi \Rightarrow e^y\hat{i} + xe^y\hat{j} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = e^y \qquad \frac{\partial\phi}{\partial y} = xe^y \quad \text{—(1)}$$

Integrating $\frac{\partial \phi}{\partial x}$,

$$\phi = xe^{y} + k(y) \Rightarrow \frac{\partial \phi}{\partial y} = xe^{y} + k'(y) \quad \text{---} \ (2)$$

From (1) and (2), $k'(y) = 0 \Rightarrow k(y) = c$

$$\therefore \phi = xe^{y} + c$$

$$\therefore W = \phi(-1,0) - \phi(1,0) = (-1+c) - (1+c) = \underline{\underline{-2}}$$

<u>Problems</u>

1) Determine whether $\vec{F}$ is conservative. If so, find a potential function.

 i) $\vec{F}(x,y) = 2x\hat{i} + 2y\hat{j}$

 ii) $\vec{F}(x,y) = x^2y\hat{i} + 6xy^2\hat{j}$

 iii) $\vec{F}(x,y) = 2e^{2x}\cos 2y\,\hat{i} - 2e^{2x}\sin 2y\,\hat{j}$

 iv) $\vec{F}(x,y) = (\cos y + y\cos x)\hat{i} + (\sin x - x\sin y)\hat{j}$.

2) Show that the integral is independent of path and evaluate the integral using fundamental theorem

 i) $$\int_{(0,0)}^{(1,\pi/2)} e^{x}\sin y\,dx + e^{x}\cos y\,dy$$

 ii) $$\int_{(0,0)}^{(3,2)} 3x^2e^{y}\,dx + x^3e^{y}\,dy$$

 iii) $$\int_{(-1,2)}^{(0,1)} (3x-y+1)\,dx - (x+6y+2)\,dy$$

3) Determine whether the force field $\vec{F}$ is conservative and find the work done by $\vec{F}$ on a particle moving along a curve from P to Q.

 i) $\vec{F}(x,y) = xy^2\hat{i} + x^2y\hat{j}$; $P(2,2)$, $Q(0,0)$

 ii) $\vec{F}(x,y) = ye^{xy}\hat{i} + xe^{xy}\hat{j}$; $P(-1,1)$, $Q(2,1)$

 iii) $\vec{F}(x,y) = e^{-y}\cos x\,\hat{i} - e^{-y}\sin x\,\hat{j}$; $P(\frac{\pi}{2},1)$, $Q(\pi,0)$