Equational theories of algebraic operators on Weihrauch problems

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Dagstuhl meeting 25141

Setting for the motivation

Type 2 computability

Turing Machines with

- Input tape containing some $i \in 2^\omega$
- Write-and-go-right-only-output tape

- Natural setting to compute with infinite objects

(the "real" 2^ω is representable)

The category of represented spaces ReprSp

- Objects: (X, δ_X) where δ_X is a **partial** surjection $2^\omega \twoheadrightarrow X$
- Morphisms: maps $X \to X'$ with a type 2-computable witness

- Super nice: extensive, lcc, W/M-types
- ($\cong$ subcategory of the modest sets in the Kleene-Vesley topos)

Weihrauch problems

Definition of Weihrauch problems as *containers*

A Weihrauch problem P is an internal family in ReprSp, i.e.

$$P : \text{positions}(P) \to \text{shape}(P)$$

- $\text{shape}(P)$ is the space of **questions**
- $\text{positions}(P)$ is the space of **answers**
- P links answers with the questions they are answering
- **Notation:** $P_i = P^{-1}(i)$

Examples:

- $\mathsf{C}_{\mathbb{N}}$: "Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p"

$$\{(p, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \mid n \notin \mathsf{range}(p)\} = \text{positions}(\mathsf{C}_{\mathbb{N}}) \xrightarrow{\pi_1} \text{shape}(\mathsf{C}_{\mathbb{N}}) \subseteq \mathbb{N}^{\mathbb{N}}$$

- WKL_0: "given an infinite binary tree, produce an infinite path"

Weihrauch reducibility

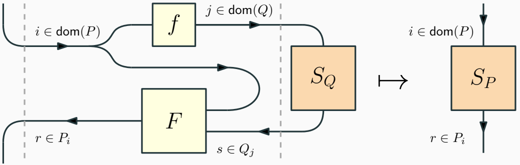
TL;DR: Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_W Q$ if there are **computable**

$$f : \mathrm{shape}(P) \to \mathrm{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{shape}(P)} (Q_{f(i)} \to P_i)$$



Reductions compose + Quotienting by $\equiv_W$ $\rightsquigarrow$ **Weihrauch degrees**

The more general picture: container morphisms

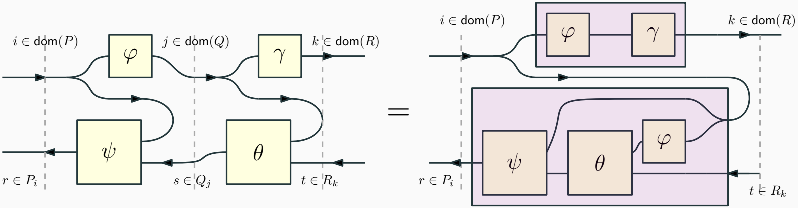
- Fix a category $\mathcal{C}$ with **pullbacks**
- $\mathsf{Cont}(\mathcal{C})$ has internal families in $\mathcal{C}$ as objects

Official definition

A morphism $P \to Q$ in $\mathsf{Cont}(\mathcal{C})$ is a pair (f, F) of

$$f : \mathrm{shape}(P) \to \mathrm{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{shape}(P)} (Q_{f(i)} \to P_i)$$

(To make sense of what F is: requires pullbacks)



A Weihrauch reduction $P \leq_{\mathrm{W}} Q$ = morphism $P \to Q$ in $\mathsf{Cont}(\mathsf{ReprSp})$

Some functors on containers/Weihrauch problems

- Coproducts (joins) $+$:

$$\mathrm{shape}(P+Q) \cong \mathrm{shape}(P) + \mathrm{shape}(Q) \qquad \begin{array}{rcl} (P+Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P+Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Cartesian product $\times$: "given inputs for both, solve one"

$$\mathrm{shape}(P \times Q) \cong \mathrm{shape}(P) \times \mathrm{shape}(Q) \quad (P \times Q)_{i,j} = P_i + Q_j$$

- Parallel product $\otimes$: "solve both problems"

$$\mathrm{shape}(P \otimes Q) \cong \mathrm{shape}(P) \times \mathrm{shape}(Q) \quad (P \otimes Q)_{i,j} = P_i \times Q_j$$

- $\mathbf{I}$: $\mathrm{shape}(\mathbf{I}) = \mathrm{positions}(\mathbf{I}) = 1$

Composition, iterated composition

Sequential composition $Q \triangleright P$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\begin{aligned} \mathrm{shape}(Q \triangleright P) &\cong \sum_{i \in \mathrm{shape}(Q)} (Q_i \to \mathrm{shape}(P)) \\ (Q \triangleright P)_{i,f} &\cong \sum_{r \in Q_i} P_{f(r)} \end{aligned}$$

Iterated composition $P^{\triangleright}$

- Explicitly: computed as the least fixpoint of $X \mapsto \mathbf{I} + (P \triangleright X)$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

Fixpoint of operators

least fixpoint	initial algebra	μ
greatest fixpoint	terminal coalgebra	ν

A very plausible conjecture (Folklore?)

If F is a fibred polynomial endofunctor over containers, the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical bialgebra ζF sitting in-between

Examples:

- $P^{\triangleright} = \mu(X \mapsto \mathbf{I} + P \triangleright X)$
- $P^{\otimes} = \mu(X \mapsto \mathbf{I} + X \otimes P)$
- $P^{\otimes\infty} = \zeta(X \mapsto X \otimes P)$
- $P^{\triangleright\infty} = \zeta(X \mapsto P \triangleright X)$

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \times)$

  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \times) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

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Meta-question

For a given signature, is there anything true in the Weihrauch degree that is not true for **all** (suitable) categories of containers?

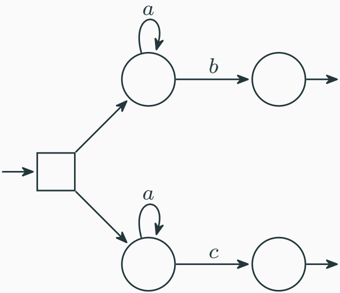
Terms with composition and automata

Starting observation

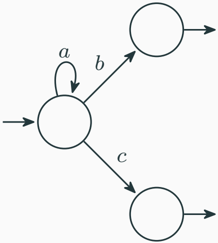
Terms over $0, \mathbf{I}, +, \triangleright, (-)^{\triangleright}$ = can be regarded as regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\times$ = allowing alternating automata



$(a^{\triangleright} \triangleright b) \times (a^{\triangleright} \triangleright c)$



$a^{\triangleright} \triangleright (b + c)$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\partial(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, +, \times, \triangleright, (-)^{\triangleright}) \models t \leq u$ iff Duplicator wins in $\partial(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\partial(\mathcal{A}, \mathcal{B})$:

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(using $\mathcal{B}$ exactly once)

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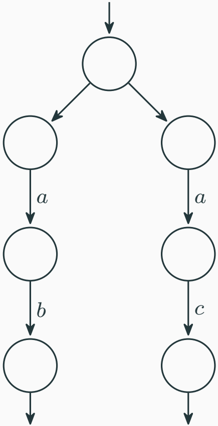
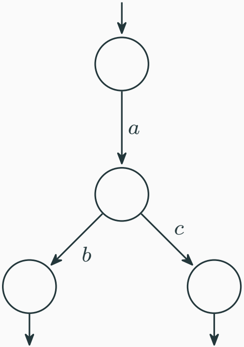
Corollary

"$(\mathfrak{W},\mathbf{I},0,+,\times,\triangleright,(-)^{\triangleright}) \models t \leq u$?" is decidable.

- Conjecture: this is PSPACE-complete.

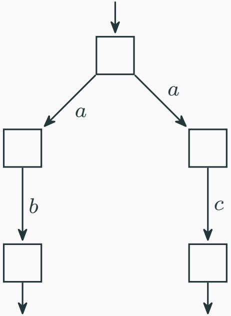
A simple example of simulation and non-simulation

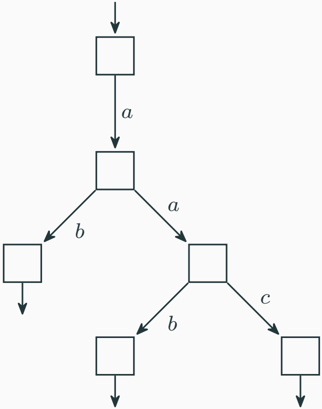




$$(a \triangleright b) + (a \triangleright c) \quad < \quad a \triangleright (b + c)$$

A simulation requiring several concurrent attempts





$$(a \triangleright b) \times (a \triangleright c) \quad < \quad a \triangleright ((a \triangleright (c \times b)) \times b)$$

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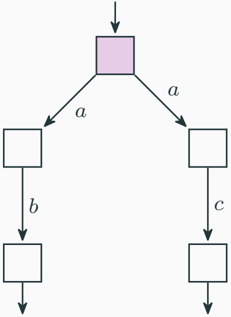

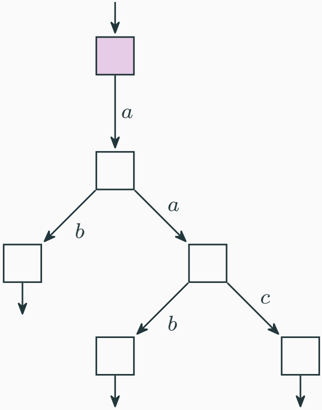

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Let's play ?!

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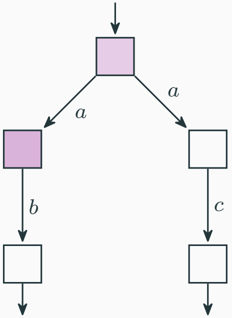

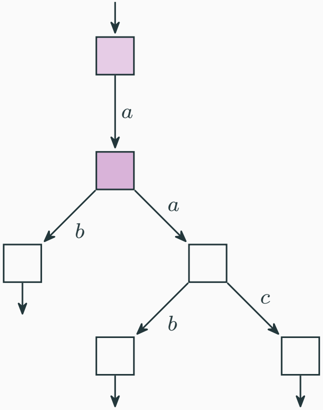

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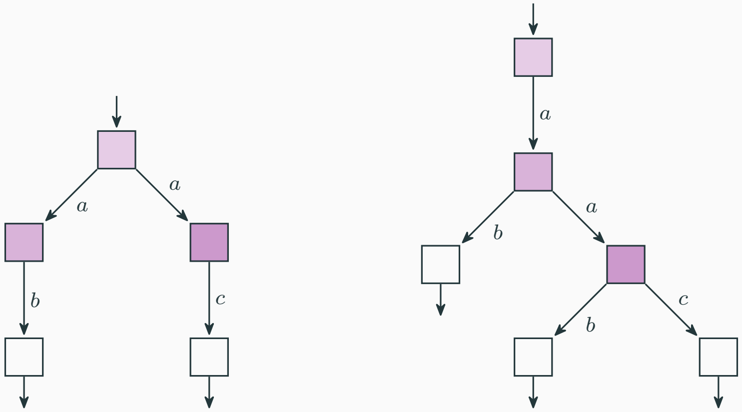

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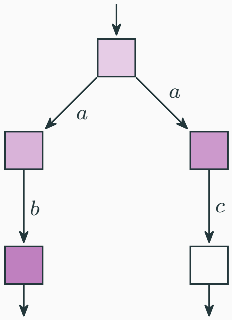

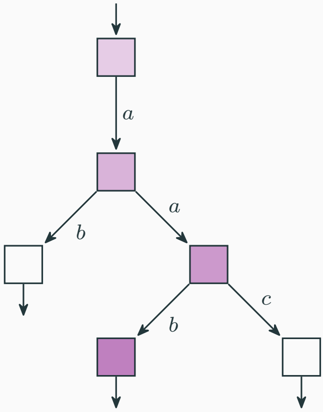

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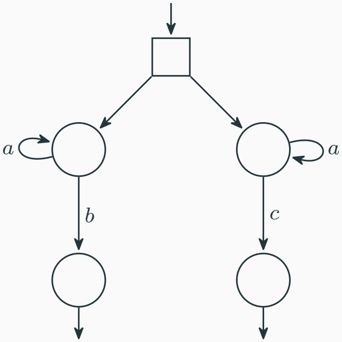

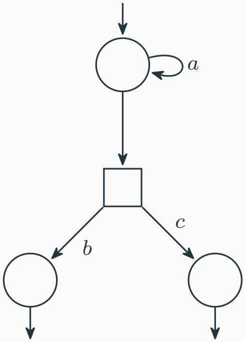

$$(a^{\triangleright} \triangleright b) \times (a^{\triangleright} \triangleright c) \quad \equiv \quad a^{\triangleright} \triangleright (b \times c)$$

Induction principles for $(-)^{\triangleright}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

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Theorem

The above axioms are valid in the extended Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **w/o right-distributivity of $+$ over $\star$**

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Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^{\triangleright})$.

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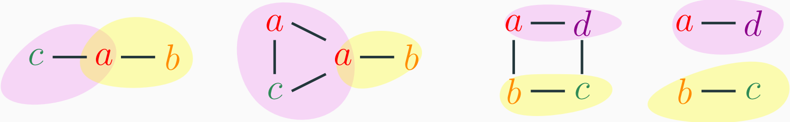
Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^{\triangleright})$.

Proof idea: ∃ positional simulation strategies, induction on the syntax

How this started: the theory of $\otimes, \times$

A notion of combinatorial reduction between graphs

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \to V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0).



$a \otimes (c \times b) \quad < \quad a \otimes ((a \otimes c) \times b) \qquad (a \times c) \otimes (b \times d) \quad < \quad (a \otimes d) \times (b \otimes c)$

Combinatorial characterization (Neumann, Pauly, P.)

$(\mathfrak{W}, \times, \otimes) \models t \leq u$ iff there is a reduction from G_t to G_u.
As a result, deciding $(\mathfrak{W}, \times, \otimes) \models t \leq u$ is Σ_2^p-complete.

- Axiomatizing: harder!
- $+$, $\triangleright$ and $\otimes$: opens the gates of hell (**concurrency theory**)

Conjectures!
(and related mess)

Extending the signature/the simulation game

- Enriching the signature with aforementioned μ = same thing with all finite alternating automata
- Then enriching the signature with ν = parity alternating automata
- Then enriching the signature with ζ (or $(-)^{\triangleright\infty}$) = runs of countable ordinal length
- Enriching with $\otimes$ = going to higher-dimensional automata
  - Dealing with stuff that sounds like concurrency
  - Scarier to me!

Some englobing syntax for all signatures discussed here

$$\frac{x \in \Gamma \cup \Delta}{\Gamma; \Delta \vdash x} \qquad \frac{\Gamma; \Delta \vdash t \qquad \Gamma; \Delta \vdash u \qquad \square \in \{\otimes, \times, +\}}{\Gamma; \Delta \vdash t \mathbin{\square} u}$$

$$\frac{}{\Gamma; \Delta \vdash \mathbf{I}} \qquad \frac{\Gamma; \cdot \vdash t \qquad \Gamma; \Delta \vdash u}{\Gamma; \Delta \vdash t \triangleright u}$$

$$\frac{\Gamma; \Delta \vdash t \qquad \Gamma; \cdot \vdash u}{\Gamma; \Delta \vdash t \blacktriangleright u} \qquad \frac{\Gamma; \cdot \vdash t \qquad \Gamma; \cdot \vdash u \qquad \twoheadrightarrow \in \{\multimap, \Rightarrow\}}{\Gamma; \Delta \vdash t \twoheadrightarrow u}$$

$$\frac{\Gamma; \Delta, x \vdash t \qquad \gamma \in \{\mu, \nu, \zeta\}}{\Gamma; \Delta \vdash \gamma x.t}$$

Another kind of questions

Conjecture(s)

For various signatures, true inequations in the slightly extended Weihrauch degrees are true in **all** categories of containers.

- Proofs of completeness = there exists messy enough problems to not create other true equations in Weihrauch degrees.
- When does that happen in a category $\mathcal{C}$

Conjecture: that's true when

For every $n \in \mathbb{N}$, there is

- an object A in $\mathcal{C}$
- a strong antichain of (regular?) subobjects $(V_i)_{i<n}$ of A
- with all $V_i \cup V_j$ are connected

Meta-question: motivations?

Would anyone care about similar results for other categories of containers?

A vaguer project

Inconvenient truths

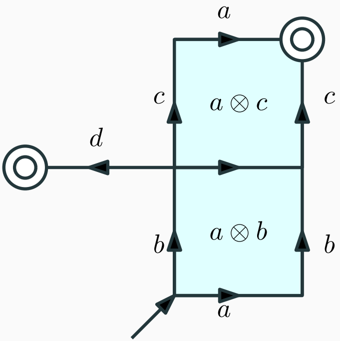
- Weihrauch problems are the containers over **regular projective** represented spaces (subspaces of $\mathbb{N}^{\mathbb{N}}$) for which every question has an answer
- Containers over subspaces of Baire space are only **weakly** locally cartesian closed

- (and also have only weak (co)inductive types)
- It sounds unproblematic in practice because
  - The weak structure is good enough
  - (a systematic way of relating that = this is the category of regular projectives of represented spaces, which is a nice lccc)

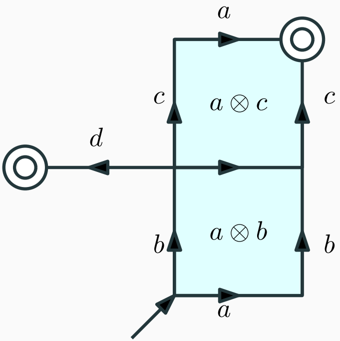
Question(s)

How do we transfer cleanly results about containers on a nice category $\mathcal{C}$ with enough projectives to containers of the full subcategory of projectives?

Example of what's a higher-dimensional automaton



Example of what's a higher-dimensional automaton



(I dislike this HDA, I feel it is not nice enough to interpret in containers)