Represented spaces of represented spaces

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Motivation

Let $X \in \{\text{Polish, coPolish, quasi-Polish, compact Polish}, \ldots\}$.

We want spaces of X spaces to ask, for some X space $\mathbf{S}$:

- when is $\mathbf{S}$ **uniformly** computably categorical?
- when is $\mathbf{S}$ generic?
- and many other questions...

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Theorem **(for X = compact Polish)**

The Weihrauch degree of uniform categoricity of $\mathcal{S}_1$ is lim.

What is a represented space of represented spaces?

(fighting with definitions)

The category of represented spaces ReprSp

Definition

A represented space $\mathbf{X}$ is a partial surjection $\delta_{\mathbf{X}} \subseteq: \mathbb{N}^{\mathbb{N}} \twoheadrightarrow S$

Idea: $c \in \mathrm{dom}(\delta_{\mathbf{X}})$ is a *name* for $\delta_{\mathbf{X}}(c) \in S$

Computable maps $f : \mathbf{X} \to \mathbf{Y}$

Type 2 computable maps $\ulcorner f \urcorner :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\begin{tikzcd}
\mathbb{N}^{\mathbb{N}} \arrow[r, "\ulcorner f \urcorner"] \arrow[d, "\delta_{\mathbf{X}}"'] & \mathbb{N}^{\mathbb{N}} \arrow[d, "\delta_{\mathbf{Y}}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

- Standard coding of $\mathbb{R}$, $\mathbb{S}$, subspaces, function spaces...
- Includes (quasi-/co) Polish spaces

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- Standard coding of $\mathbb{R}$, $\mathbb{S}$, subspaces, function spaces...
- Includes (quasi-/co) Polish spaces
- Nice (lcc) category: pullbacks, enough regular projectives

What to put in a space of spaces?

Simpler motivating example (†)

In Polish spaces, $(X, Y) \mapsto X \times Y$ is uniformly computable.

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2. For each point $a \in \mathbf{A}$, an interpretation $[\![a]\!] \in \mathsf{ReprSp}_0$
   - (†) …s.t. $\forall c\, d$ points of $\mathbf{A}$, $[\![e(c, d)]\!] \cong [\![c]\!] \times [\![d]\!]$ in ReprSp

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3. Additional coherence data for **uniformity**??
   - (†) compute the iso *computably in c and d*

Dealing with the additional coherence data (1/2)

Taking a leaf from category theorists/type theorists:

- Spaces of spaces are *uniform families* $(\llbracket c \rrbracket)_{c \in \mathbf{A}}$
- **internal** families to a category $\mathcal{C}$ are simply morphisms
  - I-indexed $\cong$ with codomain I

External/internal families in Set

$$\begin{array}{ccc} (I\text{-indexed) families} & \longleftrightarrow & \text{functions (to } I) \\ \\ (A_i)_{i \in I} & \longmapsto & \sum_{i \in I} A_i \xrightarrow{\text{projection}} I \\ \left(f^{-1}(i)\right)_{i \in I} & \longleftarrow\!\shortmid & f : X \to I \end{array}$$

Dealing with the additional coherence data (2/2)

Official definition

A repr. space of repr. spaces is a morphism in ReprSp

Conventions for spaces spaces $\mathrm{El}_{\mathbf{A}} : \mathbf{A}_\bullet \to \mathbf{A}$

- Call $\mathrm{El}_{\mathbf{A}}$ a **bundle**
- Write $[\![a]\!]_{\mathbf{A}}$ for $\mathrm{El}^{-1}(a)$
- $\mathbf{A}$ is the base of the bundle
- $\mathbf{A}_\bullet$ is the total space

Application (†) continued (uniform cartesian products)

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In Polish spaces, $(X, Y) \mapsto X \times Y$ is uniformly computable.

Assuming a space of Polish spaces $\mathrm{El}_{\mathbf{P}} : \mathbf{P}_{\bullet} \to \mathbf{P}$, we want

- a morphism $e : \mathbf{P}^2 \to \mathbf{P}$ and a uniform family

$$\prod_{c,d \in \mathbf{P}} \mathrm{El}_{\mathbf{P}}^{-1}(c) \times \mathrm{El}_{\mathbf{P}}^{-1}(d) \;\cong\; \mathrm{El}_{\mathbf{P}}^{-1}(e(c,d))$$

  - makes sense b/c ReprSp is locally cartesian closed
  - intuition: that's a subspace of $\mathbf{P}^2 \to (\mathbf{P}_{\bullet} \rightharpoonup \mathbf{P}_{\bullet})^2$

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  - intuition: that's a subspace of $\mathbf{P}^2 \to (\mathbf{P}_\bullet \rightharpoonup \mathbf{P}_\bullet)^2$
- low-level version: $\exists e_\bullet : \mathbf{P}_\bullet^2 \to \mathbf{P}_\bullet$ s.t. we have a pullback

$$\begin{tikzcd}
\mathbf{P}_\bullet^2 \arrow[r, "e_\bullet"] \arrow[d, "\mathrm{El}_{\mathbf{P}}^2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{P}_\bullet \arrow[d, "\mathrm{El}_{\mathbf{P}}"] \\
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A represented space of represented Polish spaces

Polish spaces = completely metrisable + dense sequence

The bundle $\mathsf{PM}_\bullet \to \mathsf{PM}$

- $\mathsf{PM} \subseteq \mathbb{R}^{\mathbb{N}^2}$ consists of the pseudometrics over $\mathbb{N}$

  $$d(x,x) = 0 \qquad d(x,y) = d(y,x) \qquad d(x,y) \leq (x,y) + d(y,z)$$

- $\mathsf{PM}_\bullet \subseteq \mathbb{R}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}}$ consists of pairs (d, s) where s is a fast converging Cauchy sequence for d
- the map is the first projection

Bundles from hyperspaces

Hyperspaces $\mathcal{H}$ over $\mathbf{X} \in \mathsf{ReprSp}_0$

A map $\partial_{\mathcal{H}} : R \to \mathcal{P}(\mathbf{X})$ for some $\mathbf{R} = (R, \delta_{\mathbf{R}}) \in \mathsf{ReprSp}_0$

Given such an hyperspace, build the bundle

- whose base is $\mathbf{R}$
- whose total space $\mathcal{H}_\bullet$ is the subspace of $\mathbf{R} \times \mathbf{X}$ with

$$(r, x) \in \mathcal{H}_\bullet \qquad \text{iff} \qquad x \in \partial_{\mathcal{H}}(r)$$

- which projects onto the $\mathbf{R}$ component

Examples of hyperspaces

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Examples:

- the hyperspace $\mathcal{O}(\mathbf{X})$ of closed subsets of $\mathbf{X}$

$$\partial_{\mathcal{A}(\mathbf{X})} : \qquad p : \mathbf{X} \to \mathbb{S} \quad \longmapsto \quad p^{-1}(\bot)$$

- similarly: opens, $\mathbf{\Pi}^0_2$-subsets, ...
- the hyperspace $\mathcal{V}(\mathbf{X})$ of *overt* subsets of $\mathbf{X}$

  (interprets maps $\exists_{\mathbf{X}} : \mathbf{X}^{\mathbb{S}} \to \mathbb{S}$)

- combined hyperspace $\mathcal{H}_1 \wedge \mathcal{H}_2$

$$\partial_{\mathcal{H}_1 \wedge \mathcal{H}_2}(r_1, r_2)) = A \quad \text{iff} \quad \partial_{\mathcal{H}_i}(r_i) = A \ \text{ for } i \in \{1, 2\}$$

Polish spaces as hyperspaces

Convention: $\mathcal{H}_+$ = restrict to non-empty subspaces

Characterizations and matching hyperspaces

Polish spaces are

- G_δ subsets of the Hilbert cube $\rightsquigarrow$ $\left(\mathbf{\Pi}^0_2 \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_+$
- closed subsets of $\mathbb{R}^{\mathbb{N}}$ $\rightsquigarrow$ $(\mathcal{A} \wedge \mathcal{V})\left(\mathbb{R}^{\mathbb{N}}\right)_+$

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Recall that $\mathsf{PM}_\bullet \to \mathsf{PM}$ is another Polish bundle

Three different definitions

Are they **equivalent**? In which sense?

Embedding and equivalence of spaces of spaces

An embedding of $\mathbf{A}_\bullet \xrightarrow{\mathrm{El}_\mathbf{A}} \mathbf{A}$ into $\mathbf{B}_\bullet \xrightarrow{\mathrm{El}_\mathbf{B}} \mathbf{B}$ is a pair

$e \quad : \quad \mathbf{A} \to \mathbf{B}$		translates $\mathbf{A}$-codes into $\mathbf{B}$-codes…
$E \quad : \quad \prod_{a \in \mathbf{A}} \mathrm{El}_\mathbf{A}^{-1}(a) \cong \mathrm{El}_\mathbf{B}^{-1}(b)$		…w/o modifying the spaces

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Elementary characterization: $\exists$ a pullback

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Equivalence

When there are embedding both ways

Pseudometrics $\subseteq \mathbf{\Pi}^0_2$ (overt) subsets of Hilbert cube

Turn $d \in \mathsf{PM}$ with $d \leq 1$ (wlog) into

$$X_d = \left\{ x \in [0,1]^{\mathbb{N}} \mid \forall k\ \exists m_k\ \forall i \leq k\ |x_i - d(i,m)| < 2^{-k} \right\}$$

Obvious $\mathbf{\Pi}^0_2$, easy to see it is overt $\rightsquigarrow$ code $\ulcorner X_d \urcorner$

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Map between total spaces

$$\begin{array}{ccc}
\mathbb{R}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}} & & \left(\mathbb{S}^{[0,1]^{\mathbb{N}}}_{\mathbf{\Pi^0_2}} \times \mathbb{S}^{\mathbb{S}^{[0,1]^{\mathbb{N}}}}\right) \times [0,1]^{\mathbb{N}} \\
\cup\!| & & \cup\!| \\
\mathsf{PM}_\bullet & \longrightarrow & \left(\mathbf{\Pi}^0_2 \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_{+,\bullet} \\
(d,s) & \longmapsto & \left(\ulcorner X_d \urcorner, \left(k \longmapsto \lim\limits_{n \to +\infty} d(k, s_n)\right)\right)
\end{array}$$

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For the other way around, $\left(\mathbf{\Pi}^0_2 \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_+ \to \mathsf{PM}$

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- **Problem**: this only gives us a multivalued map $\left(\mathbf{\Pi}^0_2 \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_+ \rightrightarrows \mathsf{PM}$
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Solution

Considering bundles $\mathbf{A}_\bullet \to \mathbf{A}$ up to reindexing along

$$\left(\mathbb{N}^{\mathbb{N}} \supseteq\right) \qquad \mathrm{dom}(\delta_{\mathbf{A}}) \quad \longrightarrow \quad \mathbf{A}$$

Intensional equivalence: equivalence up to that reindexing

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(abstract nonsense: reindexing along regular projective covers)

TL;DR natural options are intensionally equivalent

Bundles for Polish spaces

$\mathsf{PM}_\bullet \to \mathsf{PM}$ is intensionally equivalent to bundles given by

$$\left(\mathbf{\Pi}^0_2 \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_+ \qquad (\mathcal{A} \wedge \mathcal{V})\left(\mathbb{R}^{\mathbb{N}}\right)_+ \quad \text{and} \quad \mathcal{V}\left(\mathbb{R}^{\mathbb{N}}\right)_+$$

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Call $\mathsf{TBPM}_\bullet \to \mathsf{TBPM}$ for the variant of $\mathsf{PM}_\bullet \to \mathsf{PM}$ where we add a witness of total boundedness $\in \mathbb{N}^{\mathbb{N}}$.

Bundles for compact Polish spaces

$\mathsf{TBPM}_\bullet \to \mathsf{TBPM}$ is intensionally equivalent to the bundles given by $\left(\mathcal{K} \wedge \mathcal{V}\right)\left([0,1]^{\mathbb{N}}\right)_+$

Uniform computable categoricity

(fun!)

The degree of (uniform) computable categoricity

Let $\mathrm{El} : \mathbf{A}_\bullet \to \mathbf{A}$ be a space of spaces.

Computable categoricity of $S \in \mathsf{ReprSp}_0$ as a problem

- **Input:** $a, b \in \mathbf{A}$ such that $\mathrm{El}^{-1}(a) \cong \mathrm{El}^{-1}(b) \cong S$

  (non-necessarily computably so)
- **Output:** an homeomorphism $\mathrm{El}^{-1}(a) \cong \mathrm{El}^{-1}(b)$

We talk about the **degree** of computable categoricity $\mathrm{CCat}(S)$

S is computably categorical when $\mathrm{CCat}(S, \mathrm{El}) \leq \mathrm{id}$

The degree of (uniform) computable categoricity

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Sanity check: notion stable under intensional equivalence ✓

Cantor space among compact Polish spaces

Theorem

$2^{\mathbb{N}}$ is uniformly computably categorical in $(\mathcal{K} \wedge \mathcal{V})\left([0,1]^{\mathbb{N}}\right)$.

Idea: given a code for $\mathbf{X} \in (\mathcal{K} \wedge \mathcal{V})\left([0,1]^{\mathbb{N}}\right)$:

- look for a cover of $\mathbf{X}$ by two opens U_0, U_1 (compactness)
- such that both are non-empty (overtness)
- and $\overline{U_0} \cap \overline{U_1} = \emptyset$ (compactness)

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For the iso $h : \mathbf{X} \to 2^{\mathbb{N}}$, then set

$$h(x)_0 = i \qquad \iff \qquad x \in U_i$$

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Iterate for the other bits

(find a cover $U_{00} \cup U_{01} \supseteq U_0$ such that…)

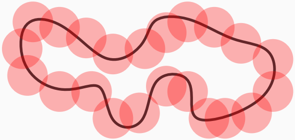
The circle (still among compact Polish spaces)

Theorem

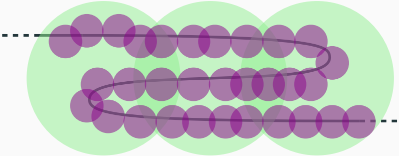
The Weihrauch degree of uniform categoricity of $\mathcal{S}_1$ is lim.

lim computes an iso $\mathbf{X} \to \mathcal{S}_1$ (assuming $\mathbf{X} \cong \mathcal{S}_1$ non-effectively):

- We attempt to cover $\mathbf{X}$ by finer and finer circular chains



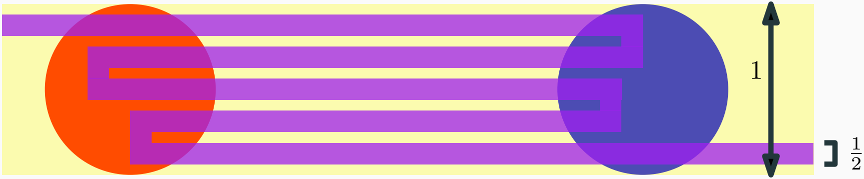
- Use lim to pick refinements without backtrackings



The circle (lim-hardness)

For the converse, first note $\lim \equiv_W \widehat{\textsc{UpperBound}}$.

- For each input $p \in \mathbb{N}^{\mathbb{N}}$ to $\textsc{Upperbound}$, fix two balls and a tube approximating our would-be circle locally
- When asked for a better precision 2^{-k-1}, shrink the tube and add $\max(0, p_{k+1} - \max_{i \le k} p_i)$ backtracks



$\max_{i<0} p_i = 0$ $\quad$ $p_0 = 2$

By the iso $\mathcal{S}_1 \to \mathbf{X} \subseteq [0,1]^{\mathbb{N}}$, bound the # of backtracks

Genericity

The big question

Let $\mathrm{El} : \mathbf{A}_\bullet \to \mathbf{A}$ be a space of spaces.

We need some definitions

What does it mean for a space to be generic?

Here: adapt notions from computability/topology

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Let $\mathrm{El} : \mathbf{A}_\bullet \to \mathbf{A}$ be a space of spaces.

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What does it mean for a space to be generic?

Here: adapt notions from computability/topology

- Concern #1: stability under bundle equivalence
- Concern #2: homeomorphism invariance
- Concern #3: the right notion of pointclass

Concerns #1 and #2

Let $\mathrm{El} : \mathbf{A}_\bullet \to \mathbf{A}$ be a space of spaces.

Equivalence and density: one issue

If $a \in \mathbf{A}$ is computable, the following is equivalent to El:

$$!_X + \mathrm{El} \quad : \quad \mathrm{El}^{-1}(a) + \mathbf{A}_\bullet \longrightarrow 1 + \mathbf{A}$$

But now every dense set in $1 + \mathbf{A}$ contains a copy of $\mathrm{El}^{-1}(a)$!!

Solution:

- Consider the quotient $\pi_\cong : \mathbf{A} \twoheadrightarrow \mathbf{A}_\cong$ that identify codes for isomorphic spaces
- $\pi_\cong \circ \mathrm{El}$ is intensionally equivalent to El

Concern #3: pointclasses

- Recall we have $\mathrm{El} : \mathbf{A}_\bullet \to \mathbf{A}$ and $\pi_{\cong} : \mathbf{A} \twoheadrightarrow \mathbf{A}_{\cong}$ around
- Let $\widetilde{\mathbf{A}} \subseteq \mathbb{N}^{\mathbb{N}}$ the space of names of $\mathbf{A}$ and $\delta_{\mathbf{A}} : \widetilde{\mathbf{A}} \twoheadrightarrow \mathbf{A}$

Standard convention for non-Polish spaces for pointclasses

[Pauly & de Brecht 15, Callard & Hoyrup 20, Hoyrup 20]

Essential caveat

A $\mathbf{\Pi}^2_0$ subset of $\mathbf{A}_{\cong}$ is given

- by a morphism $\mathbf{A}_{\cong} \to \mathbb{S}_{\mathbf{\Pi}^0_2}$
  - $\mathbb{S}_{\mathbf{\Pi}^0_2} = 2^{\mathbb{N}} /$"both finite or not"
  - equivalently: a $\mathbf{\Pi}^2_0$ set of $\widetilde{\mathbf{A}}$ that respects $\pi_{\cong} \circ \delta_{\mathbf{A}}$
- not necessarily by $\bigcap_{n \in \mathbb{N}} U_n$ with $U_n \in \mathcal{O}(\mathbf{A}_{\cong})$!
  - The U_ns do not need to respect the quotients

The definition

Recall we have $\mathrm{El} : \mathbf{A}_{\bullet} \to \mathbf{A}$ and $\pi_{\cong} : \mathbf{A} \twoheadrightarrow \mathbf{A}_{\cong}$ around

Let $\mathcal{C}$ be a pointclass

$\mathcal{C}$-genericity

S is $\mathcal{C}$-generic in El if for every dense set $D \in \mathcal{C}(\mathbf{A}_{\cong})$

$$\exists a \in D.\ \ \mathrm{El}^{-1}(a) \cong S$$

- Note: $\mathrm{El}^{-1}(a) \cong \mathrm{El}^{-1}(a')$ and $a \in D$ imply $a' \in D$

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Sanity check

Intensionally equivalent bundles $\Rightarrow$ same $\mathcal{C}$-generic spaces

Genericity in compact Polish spaces

Proposition

All infinite compact Polish spaces are $\boldsymbol{\Sigma}^0_1$-generic.

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$2^{\mathbb{N}}$ is the only $\boldsymbol{\Pi}^0_2$-generic compact Polish space.

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Proof idea: being isomorphic to $2^{\mathbb{N}}$ is a $\mathbf{\Pi}^0_2$ property

$$
\begin{aligned}
&\forall n \in \mathbb{N}. \forall r \in \mathbb{Q}_{>0}. \exists m \in \mathbb{N}. \exists s \in \mathbb{Q}_{>0}. \exists m' \in \mathbb{N}. \exists s \in \mathbb{Q}_{>0}. \\
&\qquad B(x_n, r) \cap X \neq \emptyset \implies \overline{B}(x_n, r) \cap X \subseteq B(x_m, s) \cup B(x_{m'}, s') \\
&\qquad\qquad\qquad\qquad\qquad\quad \wedge \ B(x_m, s) \cap X \neq \emptyset \\
&\qquad\qquad\qquad\qquad\qquad\quad \wedge \ B(x_{m'}, s') \cap X \neq \emptyset \\
&\qquad\qquad\qquad\qquad\qquad\quad \wedge \ \overline{B}(x_m, s) \cap \overline{B}(x_{m'}, s') \cap X = \emptyset
\end{aligned}
$$

Conclusion

What happened

- The notion of spaces of spaces as bundles
  - As type-theoretic universes
  - (but typically we like them somewhere in the middle between discrete and indiscrete)
- Equivalent presentations for (compact) Polish spaces
  - $\rightsquigarrow$ effectivizes equivalent characterizations
- Notions of uniform computable categoricity and genericity
- Some results for compact Polish spaces

What could happen: coPolish and quasiPolish spaces

Some groundwork for quasiPolish spaces in [dB21]

- A space of spaces based on ideal presentations
- Effective closure under countable products & more…

$$\begin{tikzcd}
\mathbf{QP}_\bullet^{\mathbb{N}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{QP}_\bullet \arrow[d] \\
\mathbf{QP}^{\mathbb{N}} \arrow[r, "\prod_{\mathbb{N}}"'] & \mathbf{QP}
\end{tikzcd}$$

Further things to determine

Different equivalent bundles for different characterization?

Same thing for coPolish spaces?

Effectivize Y coPolish $\wedge$ X quasiPolish $\Longrightarrow$ X^Y quasiPolish?

What could happen: categoricity

Obvious questions

What is the degree of categoricity of $[0, 1[$? $\mathbb{N}^{\mathbb{N}}$? $\mathbb{R}$? …

Beyond that:

- What happens if we restrict isomorphisms to be
  - uniformly continuous
  - isometries
  - …
- Links with other notions of computable categoricity
  - in computable structure theory
  - for Banach spaces
  - …

What could happen: genericity

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What are $\mathbf{\Pi}^0_2$-generic Polish spaces?

Beyond that, what link with genericity in other settings?

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Thanks for listening! Questions?