How unconstructive is the Cantor-Bernstein theorem?

Cécilia Pradic
$\supseteq$ j.w.w. Chad E. Brown

Continuity, Computability, Constructivity 2025 – Swansea

Constructivity (1/3)

Theorem

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

Constructivity (1/3)

Theorem

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

- we do not know whether $\pi + e$ is transcendental or not...
- nor do we know that for $e \cdot \pi$

Constructivity (1/3)

Theorem

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

- we do not know whether $\pi + e$ is transcendental or not...
- nor do we know that for $e \cdot \pi$

Morality

$\rightsquigarrow$ Not all mathematical arguments are equally informative.

Constructivity (2/2)

In broad strokes

Reject excluded middle and reductio ad absurdum.

$$A \lor \neg A \qquad\qquad \neg\neg A \Rightarrow A$$

- Large amounts of mathematics can still be formalized

  (abstract nonsense, finitary combinatorics, $(\mathbb{Q}, <)$)

- Some stuff breaks down

  (analysis, infinitary combinatorics, ordinals, $(\mathbb{R}, <)$)

- Still expressive: classical logic through $\neg\neg$-translation

  (caveat: sets and function spaces not necessarily left untouched)

Some non-constructive axioms

The limited principle of omniscience (LPO)

"For every $p \in 2^{\mathbb{N}}$, either $p = 0^{\omega}$ or $\exists n \in \mathbb{N}.\ p(n) = 1$."

$\sim$ excluded middle for $\boldsymbol{\Sigma^0_1}$ formulas

The lesser limited principle of omniscience (LLPO)

"For every $p \in 2^{\mathbb{N}}$ s.t. $\exists^{\leq 1} k.\ p(k) = 1$,
either $\quad p(2\mathbb{N}) = \{0\} \quad$ or $\quad p(2\mathbb{N}+1) = \{0\}$."

Equivalent statements in analysis:

LPO	$\forall x, y \in \mathbb{R}.$ either $x = y$ or $\lvert x - y \rvert \geq 2^{-n}$ for some $n \in \mathbb{N}$
LLPO	$\leq$ is a total order over $\mathbb{R}$: $\forall x, y \in \mathbb{R}.\ x \leq y \vee x \geq y$

A more constructive axiom

Markov's principle (MP)

"For every $p \in 2^{\mathbb{N}}$ such that $p \neq 0^\omega$, $\exists n \in \mathbb{N}.\ p(n) = 1$."

- **Postulated** by **some** constructivists
- Corresponds to unbounded search in realizability models
- LPO $\Rightarrow$ LLPO $\wedge$ MP, separations otherwise

In analysis:

LPO	$\forall x, y \in \mathbb{R}.$ either $x = y$ or $\lvert x - y\rvert \geq 2^{-n}$ for some $n \in \mathbb{N}$
LLPO	$\leq$ is a total order over $\mathbb{R}$: $\forall x, y \in \mathbb{R}.\ x \leq y \vee x \geq y$
MP	$\forall x, y \in \mathbb{R}.\ \neg\neg(x = y) \Rightarrow x = y$

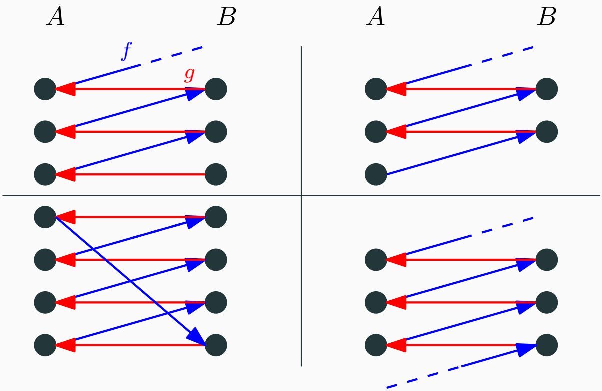
Some non-classical consistent statements

- All functions $\mathbb{N} \to \mathbb{N}$ are computable.
- All functions $\mathbb{N}^{\mathbb{N}} \to 2$ are continuous.
- All functions $\mathbb{N}^{\mathbb{N}} \to 2$ are Borel and LPO.

Cantor-Bernstein

The CB theorem

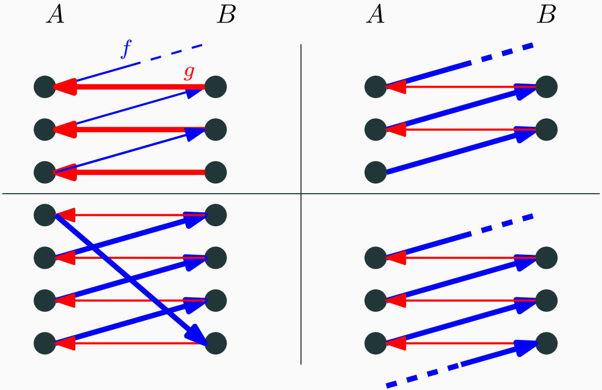
If there exists injection $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



Cantor-Bernstein

The CB theorem

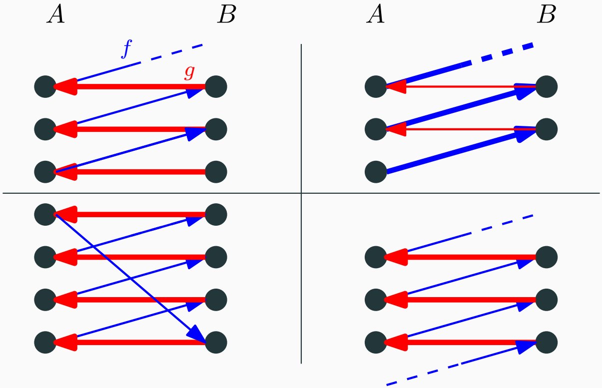
If there exists injection $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



Cantor-Bernstein

The CB theorem

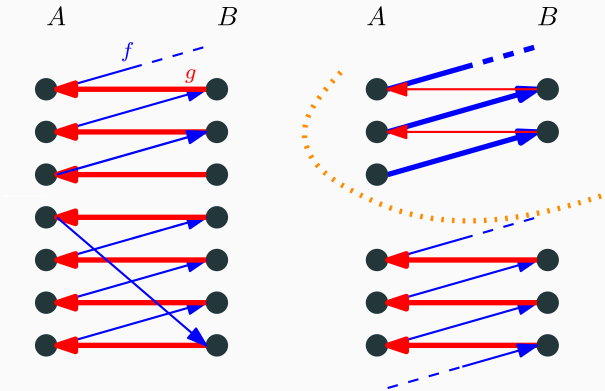
If there exists injection $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



Cantor-Bernstein

The theorem

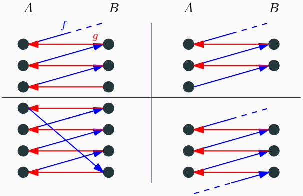
If there exists injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



$\longrightarrow$ excluded middle used to define h by cases

Why isn't this constructive

- We can ask for the successor of a node in the graph
  - given some $x \in A$, apply f; vice-versa for B and g.
- ... but not predecessor



Main question our function cannot ask

Does my input have a finite and odd number of predecessors?

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0, 1]$ and $(0, 1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

  but they are not computably isomorphic!

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0, 1]$ and $(0, 1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

  but they are not computably isomorphic!

Consequence: Cantor-Bernstein fails in a number of models

Failures of Cantor-Bernstein

Idea: adding structure to the map makes CB fail:

Topological and recursion-theoretic failures

- $[0, 1]$ and $(0, 1)$ inject continuously into one another, but aren't homeomorphic!
- $\mathbb{N}$ and the following set computably inject into one another

$$\{e \in \mathbb{N} \mid \text{the } e\text{th Turing machine doesn't halt}\}$$

but they are not computably isomorphic!

Consequence: Cantor-Bernstein fails in a number of models

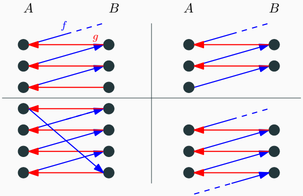
How bad it is?

Banaschewski and Brümmer's reversal (1/2)

A strengthening of Cantor-Bernstein (CBBB)

If there exists injection $f : A \to B$ and $g : B \to A$, then there exists $h : A \cong B$ *with* $h \subseteq f \cup g^{-1}$

In pictures: we force the bijection to be a subgraph



Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

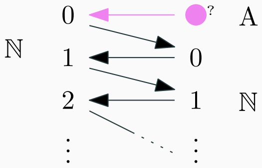
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZ, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n + 1$$



Is A inhabited?
$\to$ is $h(0) = \bullet$ or 0?

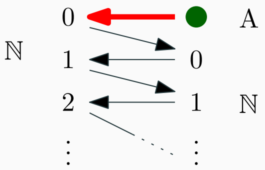
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZF, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n + 1$$



Is A inhabited?

$\to$ is $h(0) = \bullet$ or 0?

Yes!

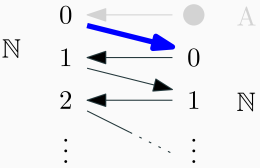
Banaschewski and Brümmer's reversal (2/2)

Theorem (Banaschewski and Brümmer 1986)

Over IZF, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n+1$$

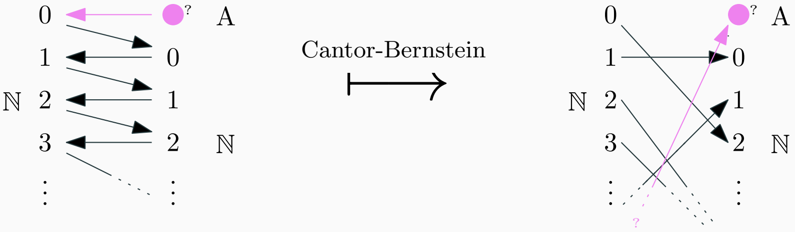


Is A inhabited?

$\to$ is $h(0) = \bullet$ or 0?

No!

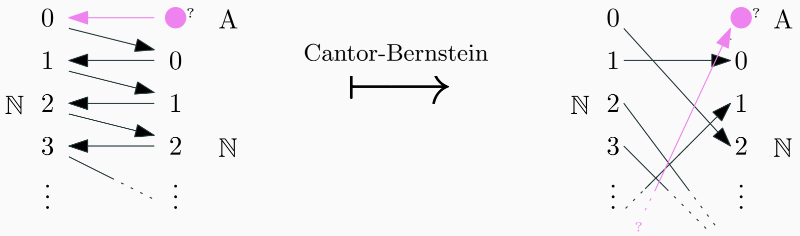
For general Cantor-Bernstein



- $h(0)$ might be uninformative
- But asking "Is $\bullet \in h(\mathbb{N})$?" would be enough

(trivial corollary: CB $\wedge$ LPO $\Rightarrow$ EM)

For general Cantor-Bernstein



- $h(0)$ might be uninformative
- But asking "Is $\bullet \in h(\mathbb{N})$?" would be enough

(trivial corollary: CB $\wedge$ LPO $\Rightarrow$ EM)

Idea

Find some other set $\mathbb{N}_\infty$ for which we can ask our question

"For any $h : \mathbb{N}_\infty \to A \cup \mathbb{N}_\infty$, is $\bullet \in h(\mathbb{N}_\infty)$?"

The conatural numbers $\mathbb{N}_\infty$

Definition as a subset of $2^{\mathbb{N}}$

$$\mathbb{N}_\infty \quad := \quad \{p \in 2^{\mathbb{N}} \mid \exists^{\leq 1} n \in \mathbb{N}.\ p(n) = 1\}$$

- Universal property: final coalgebra for $X \mapsto 1 + X$
- Call ∞ the sequence $n \mapsto 0$
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.

The conatural numbers $\mathbb{N}_\infty$

Definition as a subset of $2^{\mathbb{N}}$

$$\mathbb{N}_\infty \quad := \quad \{p \in 2^{\mathbb{N}} \mid \exists^{\leq 1} n \in \mathbb{N}.\ p(n) = 1\}$$

- Universal property: final coalgebra for $X \mapsto 1 + X$
- Call ∞ the sequence $n \mapsto 0$
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.
- LPO $\iff$ $\mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\}$.
- Can constructively define addition, but not subtraction or an equality map $\mathbb{N}_\infty^2 \to 2$

$\mathbb{N}_\infty$ is searchable

Constructive theorem (Escardó 2013)

There is a map $\varepsilon : 2^{\mathbb{N}_\infty} \to \mathbb{N}_\infty$ that picks witnesses

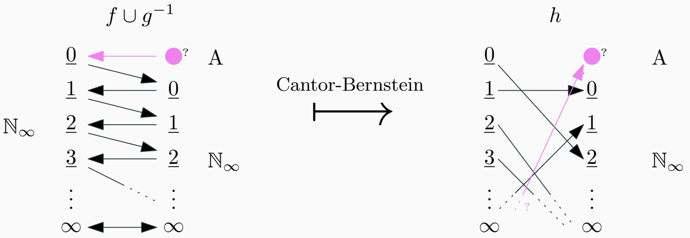
$$\forall p \in 2^{\mathbb{N}_\infty}.\ \ (\exists n \in \mathbb{N}_\infty.\ p(n) = 1) \Longrightarrow p(\varepsilon(p)) = 1$$

Idea: $\varepsilon(p)$ outputs 0s until it finds some $n \in \mathbb{N}$ s.t. $p(\underline{n}) = 1$.

Definition by co-recursion:

$$\varepsilon(p) = \begin{cases} \underline{0} & \text{if } p(\underline{0}) = 1 \\ \underline{\mathsf{Succ}}(\varepsilon(p \circ \underline{\mathsf{Succ}})) & \text{otherwise} \end{cases}$$

Cantor-Bernstein implies excluded middle



- Define $p \in 2^{\mathbb{N}_\infty}$ by $p(n) :=$ "$h(n) = \bullet$"
- Conclude using $p(\varepsilon(p)) = 1 \iff \bullet \in A$

Corollary (Brown, P. 2017)

Cantor-Bernstein implies excluded middle.

Is this actually informative?

The argument relies one making one of the set horrible dependent on some arbitrary proposition we want to decide.

- Gives only lousy concrete counter-examples in non 2-valued models (afaik)
- Does not speak to what we could know if we limit the complexity of A, B, f and g...

The Myhill isomorphism theorem

A sort of ambiantal version of Cantor-Bernstein

Reduction

$A \subseteq \mathbb{N}$ **reduces** to $B \subseteq \mathbb{N}$ **via** $f : \mathbb{N} \to \mathbb{N}$ iff $f^{-1}(B) = A$.

Constructive theorem (Myhill 1955)

If $A, B \subseteq \mathbb{N}$ are inter-reducible via injections $\mathbb{N} \to \mathbb{N}$, then there exists a bijection $h : \mathbb{N} \to \mathbb{N}$ with $h(A) = B$.

- Official original version: insert two “computable” above
- A and B could be **arbitrarily horrible**

The Myhill isomorphism theorem

A sort of ambiantal version of Cantor-Bernstein

Reduction

$A \subseteq \mathbb{N}$ **reduces** to $B \subseteq \mathbb{N}$ **via** $f : \mathbb{N} \to \mathbb{N}$ iff $f^{-1}(B) = A$.

Constructive theorem (Myhill 1955)

If $A, B \subseteq \mathbb{N}$ are inter-reducible via injections $\mathbb{N} \to \mathbb{N}$, then there exists a bijection $h : \mathbb{N} \to \mathbb{N}$ with $h(A) = B$.

- Official original version: insert two “computable” above
- A and B could be **arbitrarily horrible**

$\Rightarrow$ h can be built only with info from the injections

Towards a proof of the Myhill isomorphism theorem

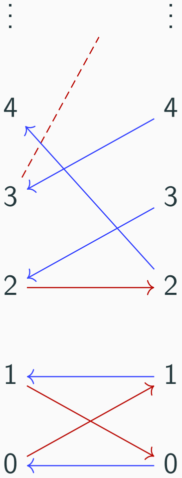
Let's call this the strong Myhill isomorphism theorem

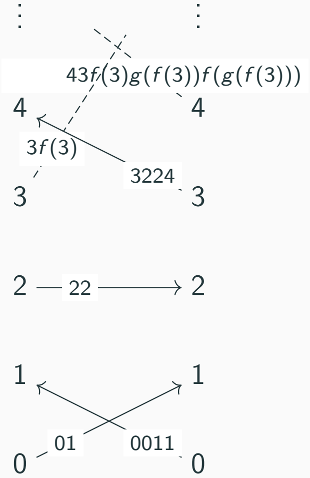
Given two injections $f, g : \mathbb{N} \to \mathbb{N}$, $\exists$ a bijection $h : \mathbb{N} \to \mathbb{N}$ s.t.

$$h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$$

- Compare and contrast with CBBB (when both sets are $\mathbb{N}$):
  - CBBB says $h \subseteq f \cup g^{-1}$ $\quad (m \in \{-1, 0\})$
  - Pictures: we can only use **edges** in the graph given by f and g
  - Relaxation: we can use **paths**
- Implies the Myhill isomorphism theorem
  - If f, g are reductions between A and B, then the connected components are either in $A + B$ or outside.

Proof: a back-and-forth argument





$f : \mathbb{N} \rightleftarrows \mathbb{N} : g$ $\qquad$ $h : \mathbb{N} \longrightarrow \mathbb{N}$

Question: other ambiance than $\mathbb{N}$? (Bauer 2025, fediverse)

Definition

Say that X has the **Myhill property** if:

For all $A, B \subseteq X$ are inter-reducible via injections,

there exists a bijection $h : X \to X$ with $h(A) = B$.

Questions

Is/does the class of sets with the Myhill property

1. closed under $+, \times, \to$?
2. contain $\mathbb{N}_\infty$?

(constructively; classically, that's a corollary of CBBB)

Before we discuss this

Strong Myhill property: defined analogously

Definition

Say that X has the **strong** Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ with $h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$.

- Clearly implies the Myhill property.
- Converse: not clear (to me).

Closure under $+, \times, \to$ is not reasonable

Observation (†)

For $n \in \mathbb{N}$, any $A \subseteq \{0, \ldots, n\}$ has the strong Myhill property.

Proof: $g^{-1} = (f \circ g)^{n!-1} \circ f$

Closure under $+, \times, \to$ is not reasonable

Observation (†)

For $n \in \mathbb{N}$, any $A \subseteq \{0, \dots, n\}$ has the strong Myhill property.

Proof: $g^{-1} = (f \circ g)^{n!-1} \circ f$

Corollary of (†) and the Myhill isomorphism theorem

LPO and the closure of the Myhill property under either $+, \times, \to$ or subsets imply excluded middle.

Proof idea: essentially the same as CBBB $\wedge$ LPO $\quad\Rightarrow\quad$ EM

$\mathbb{N}_\infty$ does not have the Myhill property

- Assume $\mathbb{N}_\infty$ has the strong Myhill property
- Assume $\mathbb{N}_\infty$-choice: every surjection $A \to \mathbb{N}_\infty$ has a section
- (valid in Kleene-Vesley realizability)

Straightforward consequence of all of that

For injections $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$, there is $\iota : \mathbb{N}_\infty \to \mathbb{Z}$ such that

$$h(x) = (f \circ g)^{\iota(x)}(f(x)) \quad \text{is a bijection}$$

(ι tells us how to travel in the graph to define h)

$\mathbb{N}_\infty$ does not have the Myhill property

- Assume $\mathbb{N}_\infty$ has the strong Myhill property
- Assume $\mathbb{N}_\infty$-choice: every surjection $A \to \mathbb{N}_\infty$ has a section
- (valid in Kleene-Vesley realizability)

Straightforward consequence of all of that

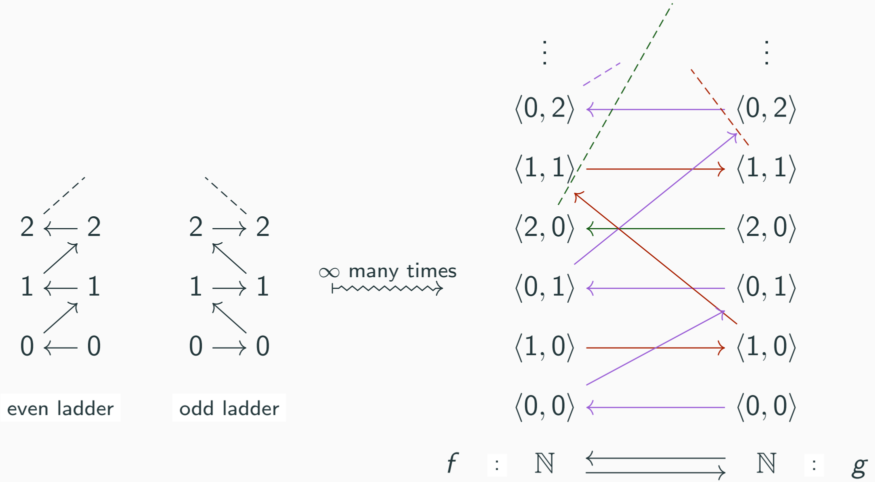
For injections $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$, there is $\iota : \mathbb{N}_\infty \to \mathbb{Z}$ such that

$$h(x) = (f \circ g)^{\iota(x)}(f(x)) \quad \text{is a bijection}$$

(ι tells us how to travel in the graph to define h)

$\iota : \mathbb{N}_\infty \to \mathbb{Z}$ is continuous iff it is eventually constant.

Forcing ι to oscillate between positive and negative (boom)



Formally

Theorem

If $\mathbb{N}_\infty$ has the strong Myhill property, MP holds and $\mathbb{N}_\infty$-choice holds, then LPO holds.

Technical lemma, in Kleene-Vesley realizability

If X is a partitioned modest set and has the Myhill property, then it has the strong Myhill property.

Proof: given f and g, make $A, B \subseteq \mathbb{N}_\infty$ horrible enough.

Theorem

$\mathbb{N}_\infty$ does not have the Myhill property in KV realizability.

But...

- We have *not really* shown that a reasonable bijection is impossible to build from f and g alone.
- Only that it is not induced by a continuous $\iota : \mathbb{N}_\infty \to \mathbb{Z}$

Fix by inserting $\neg\neg$

Say that X has the **strong** $\neg\neg$-Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ such that

$\neg\neg\,(\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$ for every $x \in X$

But...

- We have *not really* shown that a reasonable bijection is impossible to build from f and g alone.
- Only that it is not induced by a continuous $\iota : \mathbb{N}_\infty \to \mathbb{Z}$

Fix by inserting $\neg\neg$

Say that X has the **strong** $\neg\neg$-Myhill property if:

For any injections $f, g : X \to X$

there exists a bijection $h : X \to X$ such that

$\neg\neg\,(\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$ for every $x \in X$

Theorem

If MP holds, $\mathbb{N}_\infty$ has the strong $\neg\neg$-Myhill property.

Very rough proof idea

Assume $f, g : \mathbb{N}_\infty \to \mathbb{N}_\infty$ injective.

Observation

If f, g are continuous, $f(\infty) = g(\infty) = \infty$

Start an optimistic back-and-forth on the elements $< \infty$

- If we need the value of $f(\underline{n})$, actually query $\min(f(\infty), f(\underline{n}))$.
- If $\min(f(\infty), f(\underline{n})) = f(\infty)$, f is discontinuous and LPO holds $\Longrightarrow$ we have $\mathbb{N}_\infty \cong \mathbb{N}$ (all becomes easy)
- Otherwise $f(\underline{n}) < \infty$; we're happy and we carry on.
- (completely analogous for g queries)

Some subtleties, but h can be built from that and the $\neg\neg$ in the correctness criterion allows the use of classical logic there.

The $\neg\neg$-Myhill property beyond $\mathbb{N}_\infty$?

Strong counter-examples

If MP holds and any of

$$\mathbb{N} + \mathbb{N}_\infty \quad \mathbb{N} \times \mathbb{N}_\infty \quad \mathbb{N}_\infty^2 \quad 2^{\mathbb{N}} \quad \text{or} \quad \mathbb{N}^{\mathbb{N}}$$

have the strong $\neg\neg$-Myhill property, then LPO holds.

Boils down to finding easy injections f, g such that no continuous bijection h can do the job.

Remaining conjecture for converses (easy?)

$2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ have the property $\Longrightarrow$ $\mathbf{\Sigma}_1^1$-excluded middle.

The $\neg\neg$-Myhill property beyond $\mathbb{N}_\infty$?

Strong counter-examples

If MP holds and any of

$$\mathbb{N} + \mathbb{N}_\infty \quad \mathbb{N} \times \mathbb{N}_\infty \quad \mathbb{N}_\infty^2 \quad 2^{\mathbb{N}} \quad \text{or} \quad \mathbb{N}^{\mathbb{N}}$$

have the strong $\neg\neg$-Myhill property, then LPO holds.

Boils down to finding easy injections f, g such that no continuous bijection h can do the job.

Remaining conjecture for converses (easy?)

$2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ have the property $\Longrightarrow$ $\mathbf{\Sigma^1_1}$-excluded middle.

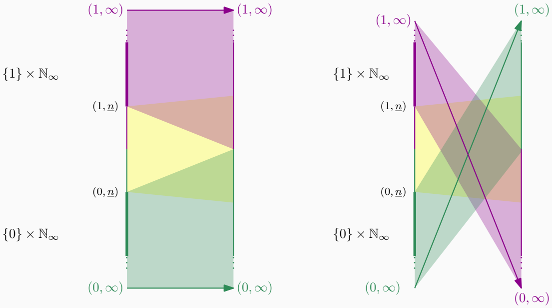
Missing $k \times \mathbb{N}_\infty$ for $k \in \mathbb{N} \setminus \{0, 1\}$?

$2 \times \mathbb{N}_\infty$: h can be continuous

A positive result

Assuming LPO, given uniformly continuous injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there exists a continuous bijection $h : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$ such that $h \subseteq \bigcup_{m \in \mathbb{Z}} (f \circ g)^m \circ f$.

B/c continuous injections $2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$ look like that:



$2 \times \mathbb{N}_\infty$: h cannot be continuously computed from f and g

Theorem

In KV realizability, $2 \times \mathbb{N}_\infty$ does **not** have the $\neg\neg$-Myhill property.

$2 \times \mathbb{N}_\infty$: h cannot be continuously computed from f and g

Theorem

In KV realizability, $2 \times \mathbb{N}_\infty$ does **not** have the $\neg\neg$-Myhill property.

Quantifying the obstruction via modalities

For any two injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there $\mathsf{LLPO}^* \star \mathsf{LPO}^8$-exists a suitable bijection h such that $\forall x \in 2 \times \mathbb{N}_\infty.\ \bigcirc_{\mathsf{LPO}} (\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$.

- LPO^8 can be dropped when f and g are continuous
- Plausible conjecture: then LLPO^* is optimal

So, where do we end up at? (assuming MP)

- For operators:

$$(\text{Closure under } +, \times, \to) \implies \text{excluded middle}$$

- For simple sets:

... having the ¬¬-Myhill property	is equivalent to ...
$\mathbb{N}$ subfinite sets $\mathbb{N}_\infty$	$\top$
$\mathbb{N}_\infty \times 2$ $\mathbb{N}_\infty \times 3$ ...	$? \in [\mathsf{LLPO}, \mathsf{LPO}]$
$\mathbb{N} + \mathbb{N}_\infty$ $\mathbb{N} \times \mathbb{N}_\infty$ $\mathbb{N}_\infty^2$	LPO
$2^{\mathbb{N}}$ $\mathbb{N}^{\mathbb{N}}$	$\mathbf{\Sigma}^1_1 - \mathsf{EM}$?

Some takeaways

- KV realizability useful for intuitions!
- As well as oracle modalities/functors
  - can be used in a model-agnostic way in the logic
  - connecting Weihrauch complexity to higher-order problems
- Frivolous, but reasonably fun??
- Does not speak much to other CB-flavored works out there?

(Gowers 1996, Goodrick 2001, ...)

Some questions

- What is the complexity of $\neg\neg$-CBBB for $\mathbb{N}$? $\mathbb{N}_\infty$? $k \times \mathbb{N}_\infty$?
- Can a univalent universe have the Myhill property?

  (not sure if that was one of the questions of Andrej)
- Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

Some questions

- What is the complexity of $\neg\neg$-CBBB for $\mathbb{N}$? $\mathbb{N}_\infty$? $k \times \mathbb{N}_\infty$?
- Can a univalent universe have the Myhill property?

  (not sure if that was one of the questions of Andrej)
- Can we say something about "set divison" theorems?

$$X \times k \cong Y \times k \implies X \cong Y \qquad (k \in \mathbb{N})$$

Thanks for listening!
Questions? :)

Modalities associated to problems

Definition

Given an $F : I \to \mathcal{P}(O)$, define

$$\begin{array}{rccc} \bigcirc_F : & \Omega & \longrightarrow & \Omega \\ & \varphi & \longmapsto & \exists i \in I.\, \forall o \in F(i).\, \varphi \end{array}$$

- Intuition for proving $\bigcirc_F \varphi$: if someone has an answer to a F-question of my choosing, I can prove φ.
- We always $\varphi \Rightarrow \bigcirc_F \varphi$ if I is inhabited.
- Only one call; $\bigcirc_F \bigcirc_F \varphi \not\Rightarrow \bigcirc_F \varphi$ in general
- number of other sanity checks can be made

$$\bigcirc_F \varphi \wedge (\forall i \in I. \exists o \in F(i)) \Rightarrow \varphi \qquad \forall i \in I. \bigcirc_F (\exists o \in F(i)) \qquad \ldots$$

Endofunctors associated to problems

Definition

Given an $F : I \to \mathcal{P}(O)$, define

$$\begin{array}{rccc} \bigcirc_F : & \mathrm{Set} & \longrightarrow & \mathrm{Set} \\ & X & \longmapsto & \{f : F(i) \to X \mid f \text{ constant}, i \in I\}_{/\sim} \end{array}$$

- Having an $\tilde{x} \in \bigcirc_F X$: should you be able to solve an arbitrary F-challenge, you can get an $x \in X$!
- (any solution $\to$ same result)
- (identify things that ultimately yield the same $x \in X$)
- Modalities: functorial action on injections into 1.

Modalities in action

$\mathsf{LPO}(p) = \{n+1 \mid p(n) = 1\} \cup \{0 \mid p = 0^\omega\} \ \ldots$

Memento $2 \times \mathbb{N}_\infty$

For any two injections $f, g : 2 \times \mathbb{N}_\infty \to 2 \times \mathbb{N}_\infty$, there $\mathsf{LLPO}^* \star \mathsf{LPO}^8$-exists a suitable bijection h such that $\forall x \in 2 \times \mathbb{N}_\infty.\ \bigcirc_{\mathsf{LPO}} (\exists m \in \mathbb{Z}.\ h(x) = (f \circ g)^m(f(x)))$.

An endofunctor in action

The problem $\mathsf{C}_{\omega+1,2}$

- **Input:** a decreasing sequence $s \in (\omega+1)^\omega$
- **Output:** $b \in 2$ equal to the parity of $\min(s)$ if $\min(s) \neq \omega$

Call η the canonical map $2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$

CBBB for $2^{\mathbb{N}}$ and continuous maps (Neumann, Pauly, P.)

In KV-realizability, for any injections $f, g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, there is a "bijection" $h : 2^{\mathbb{N}} \to \bigcirc_{\mathsf{C}_{\omega+1,2}}(2^{\mathbb{N}})$ such that, for every $p \in 2^{\mathbb{N}}$,

$$\bigcirc_{\mathsf{C}_{\omega+1,2}} \left(h(x) = \eta(f(x)) \quad \vee \quad h(x) = \eta(g^{-1}(x)) \right)$$