

Weihrauch problems are containers. The equational theory of slightly extended Weihrauch degrees with composition.

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Dagstuhl meeting 25131

Weihrauch problems

Definition

A Weihrauch problem P is given

- a set of instances $\text{dom}(P) \subseteq \mathbb{N}^{\mathbb{N}}$
- for each $i \in \text{dom}(P)$ a non-empty set of solutions $P_i \subseteq \mathbb{N}^{\mathbb{N}}$

Examples:

- $C_{\mathbb{N}}$: “Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p ”

$$\text{dom}(C_{\mathbb{N}}) = \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n \, n \notin \text{range}(p)\} \quad C_{\mathbb{N}}(p) = \{1^n 0^\omega \mid n \notin \text{range}(p)\}$$

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Comparing the hardness of problems \rightsquigarrow via a notion of reducibility

Weihrauch reducibility

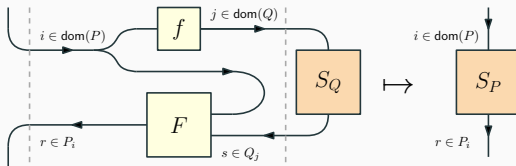
TL;DR: Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_W Q$ if there are **computable**

$$f : \text{dom}(P) \rightarrow \text{dom}(Q) \quad \text{and} \quad F : \prod_{i \in \text{dom}(P)} (Q_{f(i)} \rightarrow P_i)$$



Reductions compose + Quotienting by $\equiv_W \rightsquigarrow$ **Weihrauch degrees**

Containers

Fix a category \mathcal{C} with **pullbacks**

- minimal assumption to talk about “families of sets in \mathcal{C} ”
- formally: morphisms $f : A \rightarrow B$ represents $(f^{-1}(a))_{a \in A}$

Definition

A container P is given by

- an object of shapes $\text{shape}(P)$
- a family of solutions $(P_i)_{i \in \text{shape}(P)}$
(formally a morphism $\text{positions}(P) \rightarrow \text{shape}(P)$)

Example (Weihrauch problems)

Call $\text{pMod}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2)$ the category of subspaces of $\mathbb{N}^{\mathbb{N}}$ and computable maps between them.

All Weihrauch problems are containers.

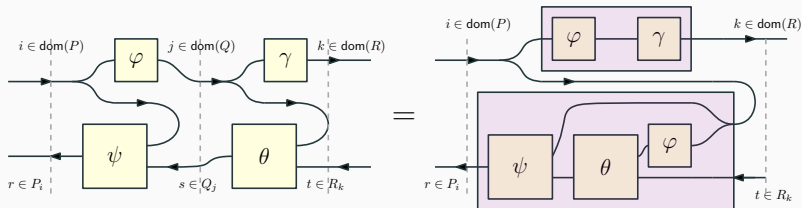
Container morphisms

Official definition

A morphism $P \rightarrow Q$ in $\text{Cont}(\mathcal{C})$ is a pair (f, F) of

$$f : \text{shape}(P) \rightarrow \text{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \text{shape}(P)} (Q_{f(i)} \rightarrow P_i)$$

(To make sense of what F is: requires pullbacks)



Containers over $\mathbf{pMod}(\mathcal{K}_2, \mathcal{K}_2^{\text{rec}}) \approx$ Weihrauch problems

Not all containers in $\mathbf{pMod}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2)$ are Weihrauch problems

$$\text{dom}(\top) = \{\bullet\} \quad \top_{\bullet} = \emptyset$$

Call those containers where $P_i \not\equiv 0$ *answerable*

Contention/Theorem (P., Price)

Weihrauch problems/reducibility



the fullsubcategory of answerable containers over $\mathbf{pMod}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2)$.

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(Theorem: the degree structures are isomorphic)

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(Theorem: the degree structures are isomorphic)

- For structural stuff, answerability is annoying
- ~~answerable~~ = **slightly extended Weihrauch problems**

(terminology suggestions welcome)

Extended Weihrauch problems

- Assume AC for this slide
- $\text{pAsm}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2) = \text{multirepresented subspaces of some } \nabla(X) \times \mathbb{N}^{\mathbb{N}}$

Theorem (P., Price)

The degree structure of containers over $\text{pAsm}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2)$ is the same as extended Weihrauch degrees.

- This says nothing about instance reducibility in general.

Other things we know how to do

- Trivially: continuous/generalized W reducibility
- With some work: strong reducibility

Point? (not sure)

Seen in the container literature

\sqcap \sqcup \star \times $(-)^{\diamond}$ \rightarrow $/$ \Rightarrow

- Assuming we are working in a elcc with (co)inductive types
- Sadly not **quite** true for $\mathsf{pAsm}(\mathcal{K}_2^{\text{rec}}, \mathcal{K}_2)$
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Maths is easier when assuming something patently false!
- A lot of work on type theory and containers (von Glehn & Moss 18)
 - Another way to link linear arithmetic/Weihrach reducibility?
(other than (Uftring 21); I'm not optimistic atm)

Some functors on containers (operators on Weihrauch problems)

Many natural operators over Weihrauch problems/degrees:

- Coproducts (joins) \sqcup :

$$\begin{aligned} \text{dom}(P \sqcup Q) &\cong \text{dom}(P) + \text{dom}(Q) & (P \sqcup Q)_{\text{in}_1(i)} &= P_i \\ & & (P \sqcup Q)_{\text{in}_2(j)} &= Q_j \end{aligned}$$

- Meets \sqcap : “given inputs for both, solve one”

$$\text{dom}(P \sqcap Q) \cong \text{dom}(P) \times \text{dom}(Q) \quad (P \sqcap Q)_{i,j} = P_i + Q_j$$

- Products \times : “solve both problems”

$$\text{dom}(P \times Q) \cong \text{dom}(P) \times \text{dom}(Q) \quad (P \times Q)_{i,j} = P_i \times Q_j$$

- 1: “there is a computable instance which has a computable solution”

Fixpoint of operators

least fixpoint	initial algebra	μ
greatest fixpoint	terminal coalgebra	ν

A very plausible conjecture (Folklore?)

If F is a fibred polynomial endofunctor over containers, the following exists:

- an initial algebra μF for F
- a terminal coalgebra νF for F
- a somewhat canonical (co)algebra ζF sitting in-between

Examples:

- $P^\diamond = \mu(X \mapsto 1 \sqcup X \star P)$
- $P^* = \mu(X \mapsto 1 \sqcup X \times P)$
- $\hat{P} = \zeta(X \mapsto X \times P)$
- $P^\infty = \zeta(X \mapsto X \star P)$

Abstract nonsense over!
(Talk 2)

Equational theory of the s.e. Weihrauch lattice

- The (s.e.) Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, \sqcup, \sqcap)$
(via the Medvedev degrees)
- Thus, $(\mathfrak{W}, \sqcup, \sqcap) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

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Meta-question

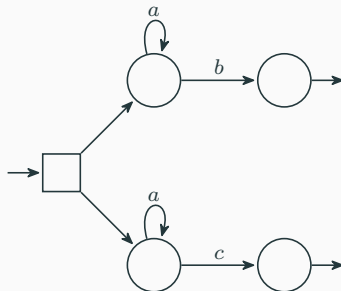
For a given signature, is there anything true in the s.e. Weihrauch degree that is not true for **all** (suitable) categories of containers?

Terms with composition and automata

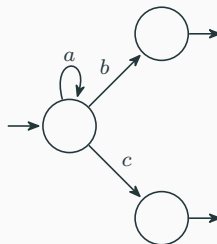
Starting observation

Terms over $0, 1, \sqcup, \star, (-)^\diamond =$ can be regarded as regular expressions.
(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding \sqcap = allowing alternating automata



$(b \star a^\diamond) \sqcap (c \star a^\diamond)$



$(b \sqcup c) \star a^\diamond$

Universal validity and games

Given alternating automata \mathcal{A} and \mathcal{B} , we can define a game $\mathfrak{D}(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Theorem

$(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^\diamond) \models t \leq u$ iff Duplicator wins in $\mathfrak{D}(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\mathfrak{D}(\mathcal{A}, \mathcal{B})$:

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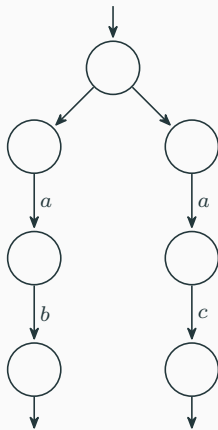
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Corollary

The equational theory of “ $(\mathfrak{W}, 1, 0, \sqcup, \sqcap, \star, (-)^\diamond) \models t \leq u?$ ” is decidable.

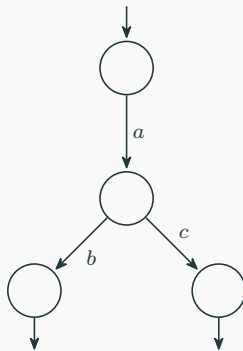
- Conjecture: this is PSPACE-complete.

A simple example of simulation and non-simulation



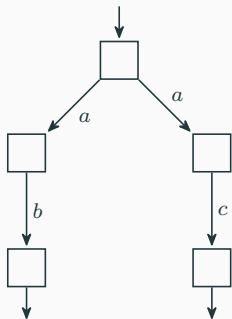
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$<$



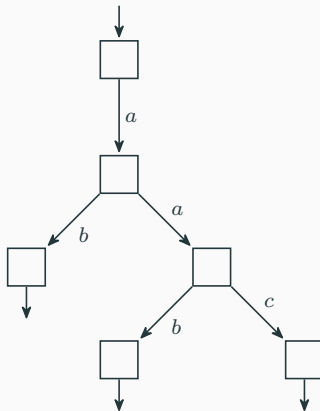
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A simulation requiring several concurrent attempts



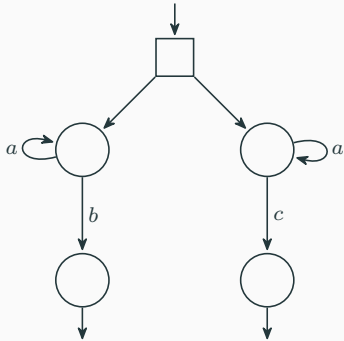
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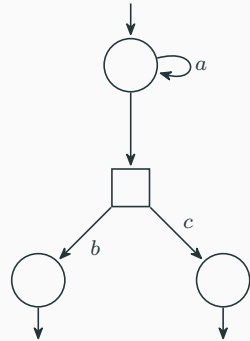
$$(((c \sqcap b) \star a) \sqcap b) \star a$$

Another simulation requiring several concurrent attempts



$$(b \star a^\diamond) \sqcap (c \star a^\diamond)$$

\equiv



$$(b \sqcap c) \star a^\diamond$$

Induction principles for $(-)^{\diamond}$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

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Theorem

The above axioms are valid in the extended Weihrauch degrees.

Completeness

Candidate axiomatization of inequations

- All the axioms of RKA **minus left-distributivity of \sqcup over \star**

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Proof idea: \exists positional simulation strategies, induction on the syntax

Conjectures!
(and related mess)

Extending the signature/the simulation game

- Enriching the signature with aforementioned μ = same thing with all finite alternating automata
- Then enriching the signature with ν = parity alternating automata
- Then enriching the signature with ζ (or $(-)^{\infty}$) = runs of countable ordinal length
- Enriching with \times = going to higher-dimensional automata
 - Dealing with stuff that sounds like concurrency
 - Scarier to me!

Some englobing syntax for all signatures discussed here

$$\frac{x \in \Gamma \cup \Delta}{\Gamma; \Delta \vdash x}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \Delta \vdash u \quad \square \in \{\times, \sqcap, \sqcup\}}{\Gamma; \Delta \vdash t \square u}$$

$$\frac{}{\Gamma; \Delta \vdash 1}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \Delta \vdash u}{\Gamma; \Delta \vdash t \star u}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \cdot \vdash u}{\Gamma; \Delta \vdash u \rightarrow t}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \cdot \vdash u \quad \multimap \in \{\neg, \circ, \Rightarrow\}}{\Gamma; \Delta \vdash t \multimap u}$$

$$\frac{\Gamma; \Delta, x \vdash t \quad \gamma \in \{\mu, \nu, \zeta\}}{\Gamma; \Delta \vdash \gamma x.t}$$

Another kind of questions

Conjecture(s)

For various signatures, true inequations in the slightly extended Weihrauch degrees are true in **all** categories of containers.

- Proofs of completeness = there exists messy enough problems to not create other true equations in Weihrauch degrees.
- When does that happen in a category \mathcal{C}

Conjecture: that's true when

For every $n \in \mathbb{N}$, there is

- an object A in \mathcal{C}
- a strong antichain of (regular?) subobjects $(V_i)_{i < n}$ of A
- with all $V_i \cup V_j$ are connected

A vaguer project

Containers over subspaces of Baire space are only **weakly** locally cartesian closed

- (and also have only weak (co)inductive types)
- It sounds unproblematic in practice because
 - The weak structure is good enough
 - (a systematic way of relating that = this is the category of regular projectives of represented spaces, which is a nice lccc)

Question(s)

How do we transfer cleanly results about containers on a nice category \mathcal{C} with enough projectives to containers of the full subcategory of projectives?

Example of what's a higher-dimensional automaton

