Equational theories of algebraic operators on Weihrauch problems

Cécilia Pradic
⊇ **lies** and *revisionism*
and j.w.w. Eike Neumann, Arno Pauly & Ian Price
Dagstuhl meeting 25141

Setting for the motivation

Type 2 computability

Turing Machines with

- Input tape containing some $i \in 2^\omega$

- Write-and-go-right-only-output tape

- Natural setting to compute with infinite objects

(the "real" 2^ω is representable)

The category of represented spaces ReprSp

- Objects: (X, δ_X) where δ_X is a **partial** surjection $2^\omega \twoheadrightarrow X$
- Morphisms: maps $X \to X'$ with a type 2-computable witness

- Super nice: extensive, lcc, W/M-types
- ($\cong$ subcategory of the modest sets in the Kleene-Vesley topos)

Definition of Weihrauch problems as *containers*

A Weihrauch problem P is an internal family in ReprSp, i.e.

$$P : \mathrm{positions}(P) \to \mathrm{shape}(P)$$

- $\mathrm{shape}(P)$ is the space of **questions**
- $\mathrm{positions}(P)$ is the space of **answers**
- P links answers with the questions they are answering
- **Notation:** $P_i = P^{-1}(i)$

Examples:

- $\mathsf{C}_\mathbb{N}$: "Given $p \in \mathbb{N}^\mathbb{N}$, find something not enumerated by p"

$$\{(p, 1^n 0^\omega) \in \mathbb{N}^\mathbb{N} \mid n \notin \mathsf{range}(p)\} = \mathrm{positions}(\mathsf{C}_\mathbb{N}) \xrightarrow{\pi_1} \mathrm{shape}(\mathsf{C}_\mathbb{N}) \subseteq \mathbb{N}^\mathbb{N}$$

- WKL_0: "given an infinite binary tree, produce an infinite path"
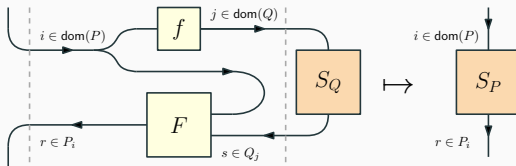
Weihrauch reducibility

TL;DR: Turing reducibility, but
- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_{\mathrm{W}} Q$ if there are **computable**

$$f : \mathrm{shape}(P) \to \mathrm{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{shape}(P)} (Q_{f(i)} \to P_i)$$

Reductions compose + Quotienting by $\equiv_{\mathrm{W}} \leadsto$ **Weihrauch degrees**

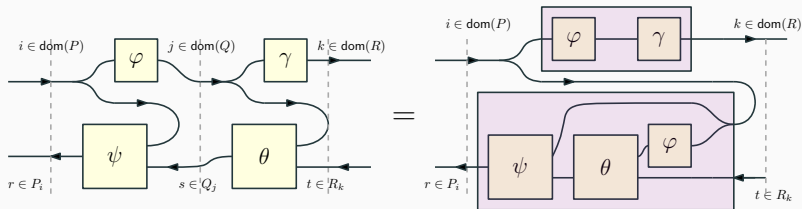# The more general picture: container morphisms

- Fix a category $\mathcal{C}$ with **pullbacks**
- $\mathsf{Cont}(\mathcal{C})$ has internal families in $\mathcal{C}$ as objects

Official definition

A morphism $P \to Q$ in $\mathsf{Cont}(\mathcal{C})$ is <u>**a**</u> pair (f, F) of

$$f : \mathrm{shape}(P) \to \mathrm{shape}(Q) \quad \text{and} \quad F : \prod_{i \in \mathrm{shape}(P)} (Q_{f(i)} \to P_i)$$

(To make sense of what F is: requires pullbacks)

A Weihrauch reduction $P \leq_{\mathrm{W}} Q = $ morphism $P \to Q$ in $\mathsf{Cont}(\mathsf{ReprSp})$

Some functors on containers/Weihrauch problems

- Coproducts (joins) +:

$$\text{shape}(P + Q) \cong \text{shape}(P) + \text{shape}(Q) \qquad \begin{array}{rcl} (P + Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P + Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Cartesian product ×: "given inputs for both, solve one"

$$\text{shape}(P \times Q) \cong \text{shape}(P) \times \text{shape}(Q) \qquad (P \times Q)_{i,j} = P_i + Q_j$$

- Parallel product ⊗: "solve both problems"

$$\text{shape}(P \otimes Q) \cong \text{shape}(P) \times \text{shape}(Q) \qquad (P \otimes Q)_{i,j} = P_i \times Q_j$$

- $\mathbf{I}$: $\text{shape}(\mathbf{I}) = \text{positions}(\mathbf{I}) = 1$

Composition, iterated composition

Sequential composition $Q \triangleright P$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\text{shape}(Q \triangleright P) \cong \sum_{i \in \text{shape}(Q)} (Q_i \to \text{shape}(P))$$
$$(Q \triangleright P)_{i,f} \cong \sum_{r \in Q_i} P_{f(r)}$$

Iterated composition $P^{\triangleright}$

- Explicitly: computed as the least fixpoint of $X \mapsto \mathbf{I} + (P \triangleright X)$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

Fixpoint of operators

least fixpoint	initial algebra	μ
greatest fixpoint	terminal coalgebra	ν

A very plausible conjecture (Folklore?)

If F is a fibred polynomial endofunctor over containers, the following exists:

- an initial algebra μF for F

- a terminal coalgebra νF for F

- a somewhat canonical bialgebra ζF sitting in-between

Examples:

- $P^{\triangleright} = \mu(X \mapsto \mathbf{I} + P \triangleright X)$
- $P^{\otimes} = \mu(X \mapsto \mathbf{I} + X \otimes P)$
- $P^{\otimes \infty} = \zeta(X \mapsto X \otimes P)$
- $P^{\triangleright \infty} = \zeta(X \mapsto P \triangleright X)$

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \times)$

  (via the Medvedev degrees)

- Thus, $(\mathfrak{W}, +, \times) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Equational theory of the Weihrauch lattice

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into $(\mathfrak{W}, +, \times)$
  (via the Medvedev degrees)
- Thus, $(\mathfrak{W}, +, \times) \models t \leq u$ iff $t \leq u$ is provable from the axioms of distributive lattices. (formulas being implicitly universally quantified)

Driving question
Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

Meta-question
For a given signature, is there anything true in the Weihrauch degree that is not true for **all** (suitable) categories of containers?

Terms with composition and automata
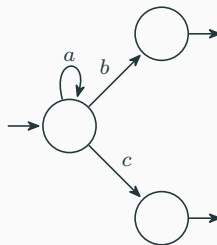
Starting observation

Terms over $0, \mathbf{I}, +, \triangleright, (-)^{\triangleright} =$ can be regarded as regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\times$ = allowing alternating automata
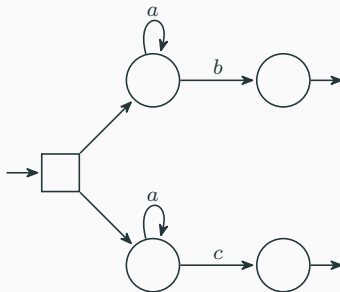
$$(a^{\triangleright} \triangleright b) \times (a^{\triangleright} \triangleright c) \qquad a^{\triangleright} \triangleright (b + c)$$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\partial(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

---

Theorem

$(\mathfrak{W}, +, \times, \triangleright, (-)^{\triangleright}) \models t \leq u$ iff Duplicator wins in $\partial(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\partial(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel
  (using $\mathcal{B}$ exactly once)

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\partial(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

> **Theorem**
>
> $(\mathfrak{W}, +, \times, \rhd, (-)^{\rhd}) \models t \leq u$ iff Duplicator wins in $\partial(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\partial(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

  (using $\mathcal{B}$ exactly once)

$\Rightarrow \mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

> **Theorem**
>
> $(\mathfrak{W}, +, \times, \rhd, (-)^{\rhd}) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

  (using $\mathcal{B}$ exactly once)

$\Rightarrow$ $\mathcal{O}(|\mathcal{B}|2^{|\mathcal{A}|})$ positions

> **Corollary**
>
> "$(\mathfrak{W}, \mathbf{I}, 0, +, \times, \rhd, (-)^{\rhd}) \models t \leq u$?" is decidable.

- Conjecture: this is Pspace-complete.
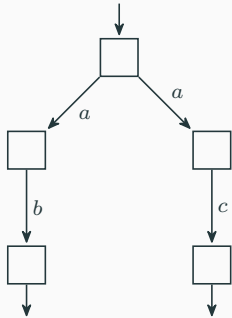
A simple example of simulation and non-simulation
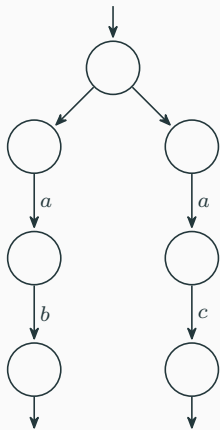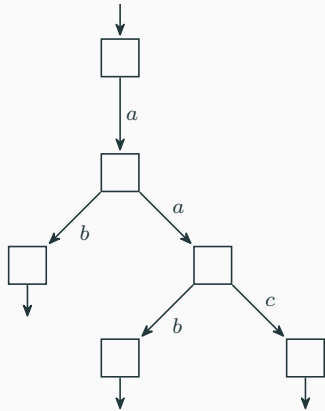

$$(a \triangleright b) + (a \triangleright c) \qquad < \qquad a \triangleright (b + c)$$
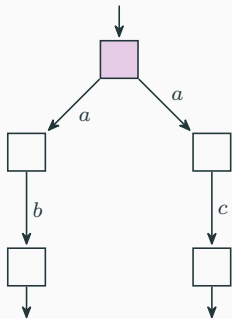
A simulation requiring several concurrent attempts
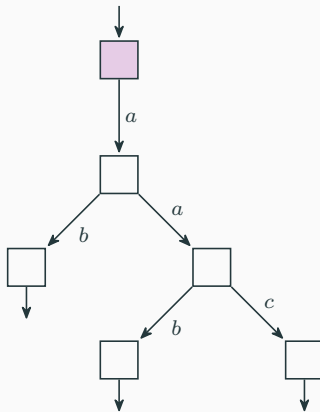

$$(a \triangleright b) \times (a \triangleright c) \qquad < \qquad a \triangleright ((a \triangleright (c \times b)) \times b)$$

$$(a \triangleright b) \times (a \triangleright c) \qquad < \qquad a \triangleright ((a \triangleright (c \times b)) \times b)$$

Let's play ?!

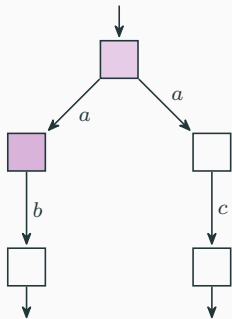$$(a \triangleright b) \times (a \triangleright c) \qquad < \qquad a \triangleright ((a \triangleright (c \times b)) \times b)$$

Let's play ?!

A simulation requiring several concurrent attempts

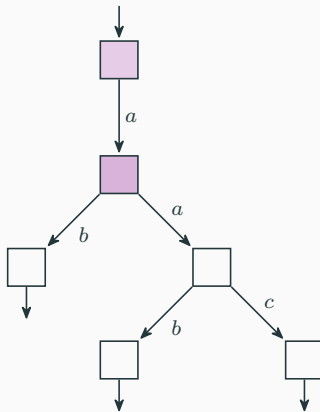$$(a \triangleright b) \times (a \triangleright c) \qquad < \qquad a \triangleright ((a \triangleright (c \times b)) \times b)$$

Let's play ?!

A simulation requiring several concurrent attempts

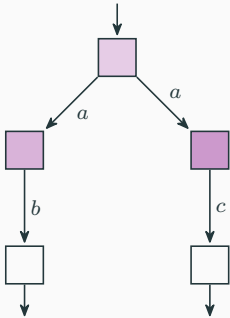$$(a \triangleright b) \times (a \triangleright c) \qquad < \qquad a \triangleright ((a \triangleright (c \times b)) \times b)$$
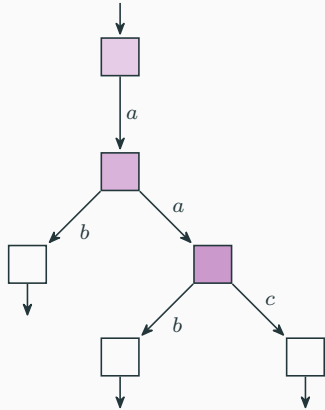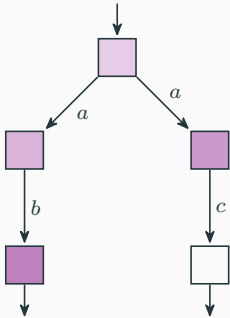
Let's play ?!

$$(a^{\triangleright} \triangleright b) \times (a^{\triangleright} \triangleright c) \qquad \equiv \qquad a^{\triangleright} \triangleright (b \times c)$$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \qquad \Rightarrow \qquad x^\triangleright \leq x$$

Induction principles for $(-)^\triangleright$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \le x \qquad \Rightarrow \qquad x^\triangleright \le x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \le x \qquad \Rightarrow \qquad y^\triangleright \triangleright x \le x$$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \le x \qquad \Rightarrow \qquad x^{\triangleright} \le x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \le x \qquad \Rightarrow \qquad y^{\triangleright} \triangleright x \le x$$

- For $\times$, it seems like we sometimes need

$$(y \triangleright x) \times z \le x \qquad \Rightarrow \qquad (y^{\triangleright} \triangleright x) \times z \le x$$

(key example: $\mathbf{I} \le a \times b$ implies $a^{\triangleright} \times b^{\triangleright} \le (a \times b)^{\triangleright}$)

Non-trivial useful axiom for fixpoints (Westrick, 2021)

The following is valid in the Weihrauch degrees

$$x \triangleright x \leq x \qquad \Rightarrow \qquad x^{\triangleright} \leq x$$

- Similar: an axiom of **left-handed Kleene algebras** (LKA)

$$y \triangleright x \leq x \qquad \Rightarrow \qquad y^{\triangleright} \triangleright x \leq x$$

- For $\times$, it seems like we sometimes need

$$(y \triangleright x) \times z \leq x \qquad \Rightarrow \qquad (y^{\triangleright} \triangleright x) \times z \leq x$$

(key example: $\mathbf{I} \leq a \times b$ implies $a^{\triangleright} \times b^{\triangleright} \leq (a \times b)^{\triangleright}$)

Theorem

The above axioms are valid in the extended Weihrauch degrees.

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over** ⋆

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over ⋆**
  - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_{\mathrm{W}} (R \triangleright P) + (R \triangleright Q)$

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over ⋆**
  - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_{\mathrm{W}} (R \triangleright P) + (R \triangleright Q)$
  - why: LKA = language inclusions, but we want simulations

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over ⋆**
  - i.e. it can be the case that $R \rhd (P + Q) \not\equiv_{\mathrm{W}} (R \rhd P) + (R \rhd Q)$
  - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{aligned}
(y \rhd x) \times (z \rhd x) &\leq (y \times z) \rhd x \\
(y \rhd x) \times z &\leq y \rhd (x \times z) \\
(y \rhd x) \times z \leq x &\Rightarrow (y^{\rhd} \rhd x) \times z \leq x \\
1 \rhd x = 1 \qquad & \quad 0 \rhd x = 0
\end{aligned}
$$

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over ⋆**
  - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_{\mathrm{W}} (R \triangleright P) + (R \triangleright Q)$
  - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{aligned}
(y \triangleright x) \times (z \triangleright x) &\leq (y \times z) \triangleright x \\
(y \triangleright x) \times z &\leq y \triangleright (x \times z) \\
(y \triangleright x) \times z \leq x &\Rightarrow (y^{\triangleright} \triangleright x) \times z \leq x \\
1 \triangleright x = 1 &\qquad 0 \triangleright x = 0
\end{aligned}
$$

Theorem

Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^{\triangleright})$.

Completeness

Candidate axiomatization of inequations

- All the axioms of LKA **minus right-distributivity of + over ⋆**
  - i.e. it can be the case that $R \triangleright (P + Q) \not\equiv_W (R \triangleright P) + (R \triangleright Q)$
  - why: LKA = language inclusions, but we want simulations
- The distributive lattice axioms **with units** +

$$
\begin{aligned}
(y \triangleright x) \times (z \triangleright x) &\leq (y \times z) \triangleright x \\
(y \triangleright x) \times z &\leq y \triangleright (x \times z) \\
(y \triangleright x) \times z \leq x &\Rightarrow (y^\triangleright \triangleright x) \times z \leq x \\
1 \triangleright x = 1 &\qquad 0 \triangleright x = 0
\end{aligned}
$$

Theorem
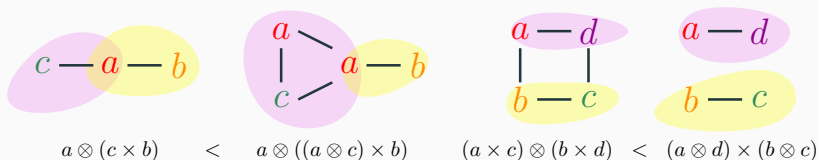
Complete for the equational theory of $(\mathfrak{W}, \mathbf{I}, 0, 1, +, \times, \triangleright, (-)^\triangleright)$.

Proof idea: ∃ positional simulation strategies, induction on the syntax

How this started: the theory of $\otimes, \times$

A notion of combinatorial reduction between graphs

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \rightarrow V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0).

$$a \otimes (c \times b) \quad < \quad a \otimes ((a \otimes c) \times b) \qquad (a \times c) \otimes (b \times d) \quad < \quad (a \otimes d) \times (b \otimes c)$$

Combinatorial characterization (Neumann, Pauly, P.)

$(\mathfrak{W}, \times, \otimes) \models t \leq u$ iff there is a reduction from G_t to G_u.
As a result, deciding $(\mathfrak{W}, \times, \otimes) \models t \leq u$ is Σ_2^p-complete.

- Axiomatizing: harder!
- $+, \triangleright$ and $\otimes$: opens the gates of hell (**concurrency theory**)

Conjectures!
(and related mess)

Extending the signature/the simulation game

- Enriching the signature with aforementioned μ = same thing with all finite alternating automata

- Then enriching the signature with ν = parity alternating automata

- Then enriching the signature with ζ (or $(-)^{\triangleright\infty}$) = runs of countable ordinal length

- Enriching with $\otimes$ = going to higher-dimensional automata
  - Dealing with stuff that sounds like concurrency
  - Scarier to me!

Some englobing syntax for all signatures discussed here

$$\frac{x \in \Gamma \cup \Delta}{\Gamma; \Delta \vdash x}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \Delta \vdash u \quad \Box \in \{\otimes, \times, +\}}{\Gamma; \Delta \vdash t \Box u}$$

$$\frac{}{\Gamma; \Delta \vdash \mathbf{I}}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \Delta \vdash u}{\Gamma; \Delta \vdash t \rhd u}$$

$$\frac{\Gamma; \Delta \vdash t \quad \Gamma; \cdot \vdash u}{\Gamma; \Delta \vdash t \blacktriangleright u}$$

$$\frac{\Gamma; \cdot \vdash t \quad \Gamma; \cdot \vdash u \quad \multimap\ast \in \{\multimap, \Rightarrow\}}{\Gamma; \Delta \vdash t \multimap\ast u}$$

$$\frac{\Gamma; \Delta, x \vdash t \quad \gamma \in \{\mu, \nu, \zeta\}}{\Gamma; \Delta \vdash \gamma x.t}$$

Another kind of questions

Conjecture(s)

For various signatures, true inequations in the slightly extended Weihrauch degrees are true in **all** categories of containers.

- Proofs of completeness = there exists messy enough problems to not create other true equations in Weihrauch degrees.
- When does that happen in a category $\mathcal{C}$

Conjecture: that's true when

For every $n \in \mathbb{N}$, there is

- an object A in $\mathcal{C}$
- a strong antichain of (regular?) subobjects $(V_i)_{i<n}$ of A
- with all $V_i \cup V_j$ are connected

Would anyone care about similar results for other categories of containers?

A vaguer project

Inconvenient truths

- Weihrauch problems are the containers over **regular projective** represented spaces (subspaces of $\mathbb{N}^{\mathbb{N}}$) for which every question has an answer

- Containers over subspaces of Baire space are only **weakly** locally cartesian closed

- (and also have only weak (co)inductive types)
- It sounds unproblematic in practice because
  - The weak structure is good enough
  - (a systematic way of relating that = this is the category of regular projectives of represented spaces, which is a nice lccc)
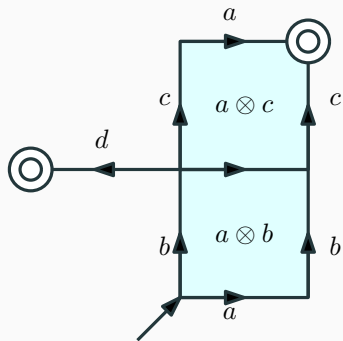
Question(s)

How do we transfer cleanly results about containers on a nice category $\mathcal{C}$ with enough projectives to containers of the full subcategory of projectives?

Example of what's a higher-dimensional automaton
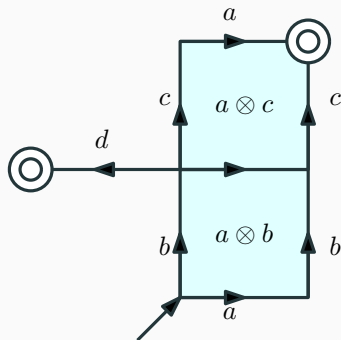
Example of what's a higher-dimensional automaton

(I dislike this HDA, I feel it is not nice enough to interpret in containers)