

# Introduction to Bayesian Statistics with R

3: Exercises

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## Exercise 3.1 - Monte Carlo integration

Computing expectations can be applied to any continuous function

$$E[f(x)] = \int f(x)p(x)dx$$

so that integrals where we recognise  $p(x)$  as (proportional to) a probability distribution may be estimated with Monte Carlo methods since

$$E[f(x)] \approx \frac{1}{N} \sum_{i=1}^N f(x_i)$$

for  $N$  random samples  $x_i$  sampled according to  $p(x)$ . Use samples from a Gaussian to estimate the following three integrals:

$$\int |x|e^{-x^2}dx, \quad \int \sin(x)e^{-x^2}dx, \quad \int \cos(x)e^{-x^2}dx$$

Compare the estimated values to the exact values of the integrals. How does the accuracy depend on the sample size?

## Exercise 3.2 - MCMC

For MCMC we can walk randomly and accept according to the MH ratio to eventually sample proportionally to any target distribution  $p(x)$

```

# simple MCMC function
# n_its is the number of iterations
# start_x the initial position
# rw_sd is the sd of the Gaussian random walk
basicMCMC <- function(n_its = 1e3, start_x = 0, rw_sd = 1, ...) {
  xs <- rep(NA, n_its) # to store all the sampled values
  x <- start_x # starting point
  xs[1] <- x # first value
  p_x <- target_density(x, ...) # probability density at current value of x
  for (ii in 2:n_its) { # MCMC iterations
    x_prop <- x + rnorm(1, mean = 0, sd = rw_sd) # Gaussian random walk to propose next x
    p_x_prop <- target_density(x_prop, ...) # probability density at proposed x
  }
}

```

```

    if (runif(1) < p_x_prop/p_x) { # MH acceptance probability
      x <- x_prop # accept move
      p_x <- p_x_prop # update density
    }
    xs[iii] <- x # store current position, even when move rejected
  }
  return(xs)
}

```

For example, if we want to sample from a Student- $t$  distribution we can use the following target

```

target_density <- function(x, nu) {
  dt(x, nu) # student-t density
}

```

and run a short chain with  $\nu = 5$

```

basicMCMC(nu = 5)

```

Examine the output MCMC chain for different lengths. How many samples would we need to get close to the Student- $t$  distribution?

Use the samples to estimate

$$\int \cos(t) f_5(t) dt, \quad f_\nu(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$