INDIAN INSTITUTE OF INFORMATION TECHNOLOGY

Approximation, Probability, and Optimization

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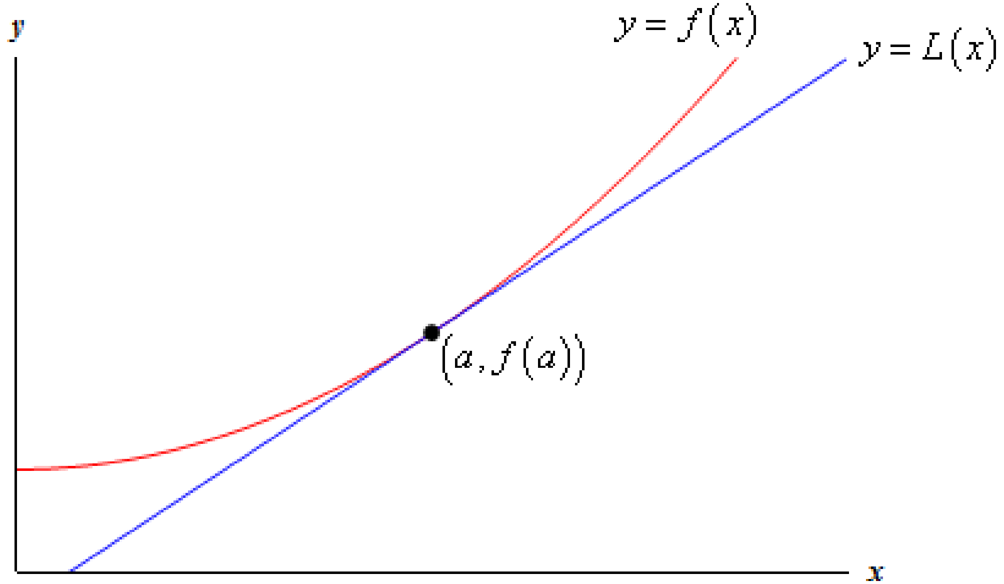
Approximation

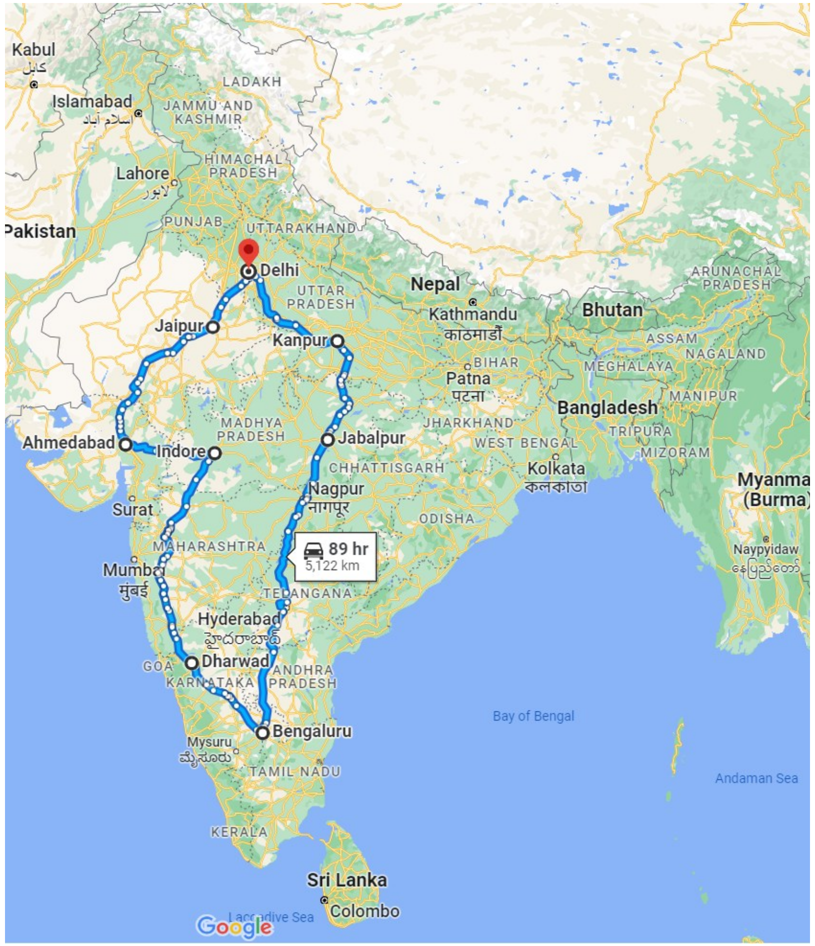
- Efficient algorithms that find **approximate solutions** to NP optimization problems.
- Solution with provable guarantees such that the solution is near-exact solution.
- A tangent to a curve is a straight line that best approximates the curve at a specific point.
- Linear approximation for the curve near the point of tangency. For a value of x very close to x_0, the function's value f(x) can be approximated by the value on the tangent line at a point (x_0,y_0) is given by $y - y_0 = m(x - x_0)$ ➔ $f(x) \approx f(x_0) + f'(x_0)(x-x_0)$.
- Example:
  - Vertex Cover Problem,
  - Traveling Salesman Problem,
  - Set-covering problem (resource-selection problems)
  - Subset-sum problem



https://en.wikipedia.org/wiki/Approximation_algorithm

Approximation

- Karp (1972) proved the TSP to be NP-hard, but effective heuristic approximation methods were developed (Lin and Kernighan, 1973).
- The traveling-salesperson problem (TSP) is a standard combinatorial problem in theoretical computer science (Lawler et al., 1992).
- Arora (1998) devised a fully polynomial approximation scheme for Euclidean TSPs.



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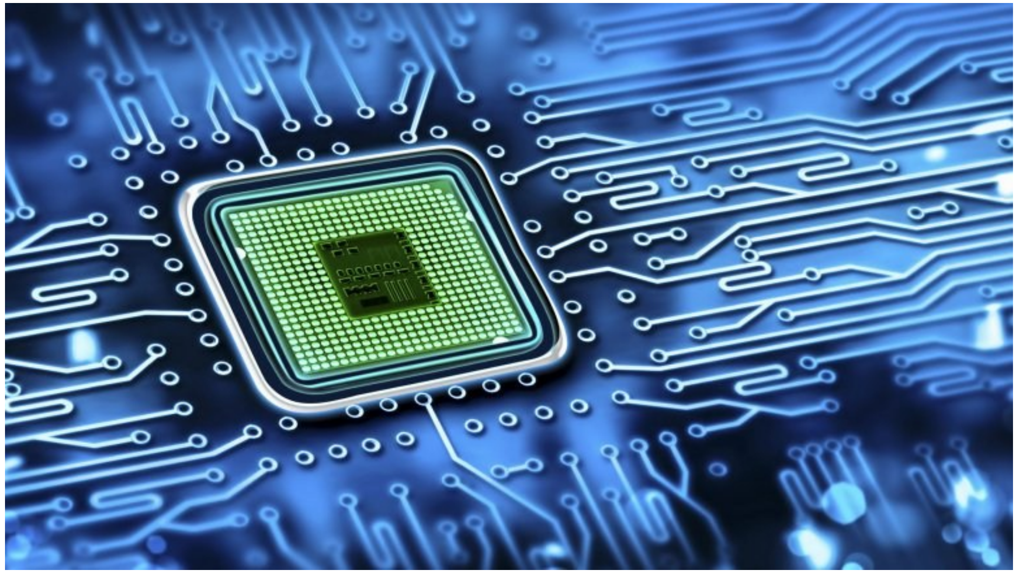
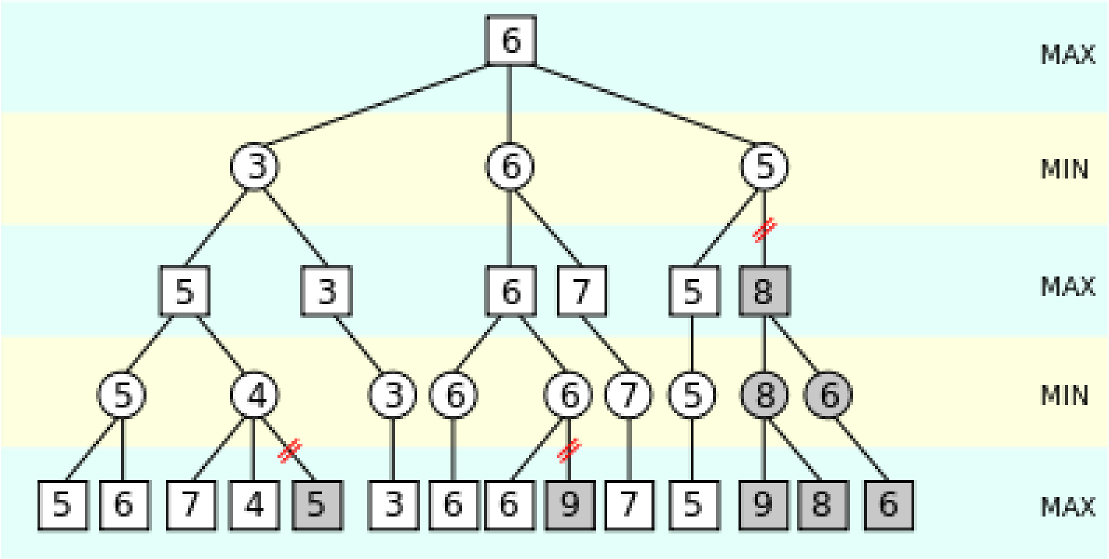
Stuart Russel, and Peter Norvig. "Artificial intelligence: A modern approach. Third edit." Upper Saddle River, New Jersey 7458 (2015). http://aima.cs.berkeley.edu/

Approximation

- VLSI layout methods are surveyed by Shahookar and Mazumder (1991), and many layout optimization papers appear in VLSI journals.
- Approximation in Mini-Max is to cut the search off at some point





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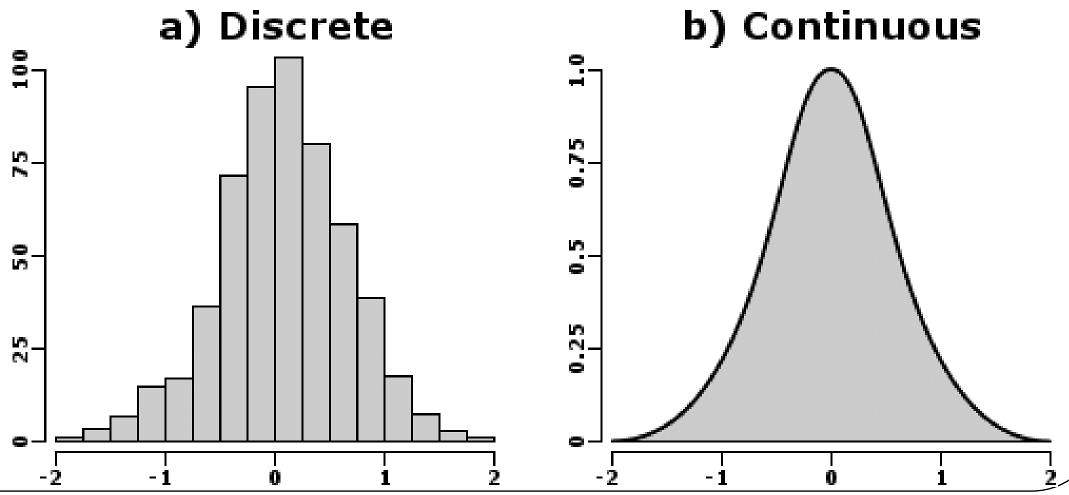
AI with Approximation

- Used when NP-Hard problems are difficult to solve exactly
- Approximation provides near-optimal solutions efficiently
- When a problem is **NP-Complete or NP-Hard**, exact polynomial-time solutions likely don't exist. So, **Approximation** becomes important.
- Examples:
  - **Traveling Salesman Problem (TSP)** → NP-Hard
    ✓ Polynomial approximation with guarantees
  - **Vertex Cover** → NP-Complete
    ✓ 2-approximation algorithm exists
  - **Knapsack** → NP-Complete
    ✓ Fully Polynomial Time Approximation Scheme (FPTAS)
- Approximation is a way to handle NP-Hard problems **efficiently**, trading accuracy for speed.

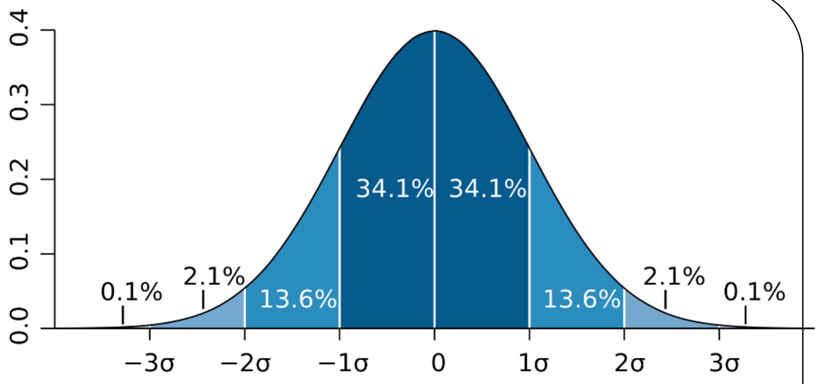
Probability and AI

Probabilistic Vs Stochastic

- Deterministic system are predictable.
  - The state of the system can be forecasted for given input, constraints, and mathematical model.
  - States of a deterministic system are pre-determined.
- Non-Deterministic system are unpredictable
  - Stochastic and Probabilistic are usually used interchangeably.
  - Both represent the randomness present in the system.
  - Probabilistic is superset concept of stochastic.



Probabilistic Vs Stochastic

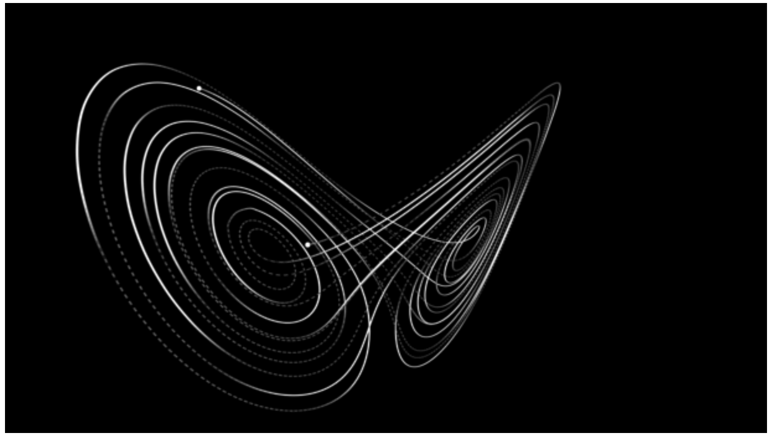


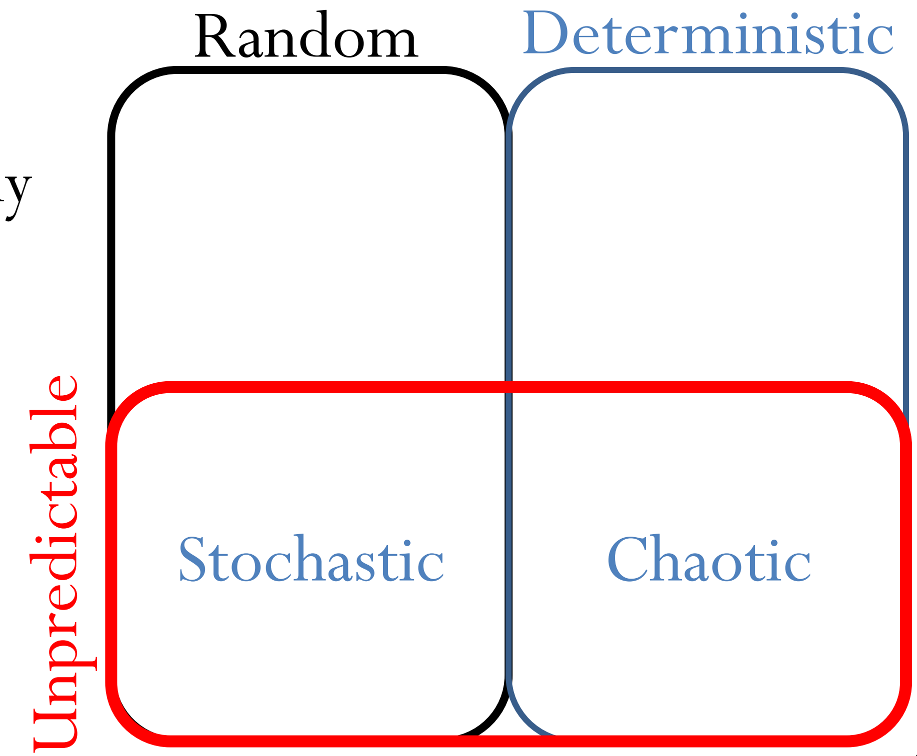
- Probabilistic models are independent of time,
  - which describes system with numerical chances or likelihood of an event to occur.
  - E.g., Lottery numbers are independent of each other.
  - Each instance is determined by the same probability distributions, but with no memory of older instance.
- Stochastic models are time dependent systems,
  - whose changes are described by its past and probabilities for successive changes.
  - E.g., Price of a stock is its old price and an uncertain change.
  - The uncertainty are small, which is semi-predictable.
  - If the stock was closed at 100,
    - then its opening value is predictable around 90 or 110.

https://qr.ae/pvU18E
https://qr.ae/pvU11n

Stochastic and Chaotic

- A **chaotic** system is deterministic in theory.
  - It responds drastically to infinitesimal changes in initial and boundary conditions, making it in practice unpredictable and unstable.
- Deterministic chaos, or simply Chaos.
- **Chaos:** “When the present determines the future, but the approximate present does not approximately determine the future.”
- A **stochastic** system is a random phenomenon.
- Stochastic and Chaotic two terms interchangeably.
- It is hard to distinguish between chaotic and stochastic systems.





https://en.wikipedia.org/wiki/Chaos_theory
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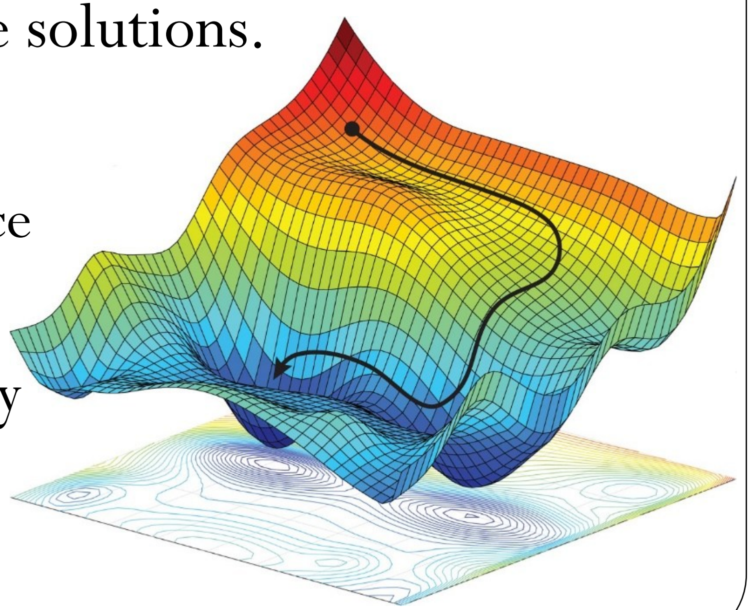
AI with Probability

- Randomized algorithms handle NP-hardness
- Randomization helps solve **hard problems faster on average**, even if worst-case is NP-Hard.
- Examples:
  - **Randomized SAT solvers (WalkSAT)** → Practical AI search
  - **Monte Carlo Tree Search** → Used in reinforcement learning (e.g., AlphaGo)
  - **Probabilistic Graphical Model Inference** → NP-Hard, solved via sampling
- Some NP problems become tractable **on average** using randomness.
- **Probability softens NP-hardness** in practice, especially in AI.

Optimization and AI

Optimization

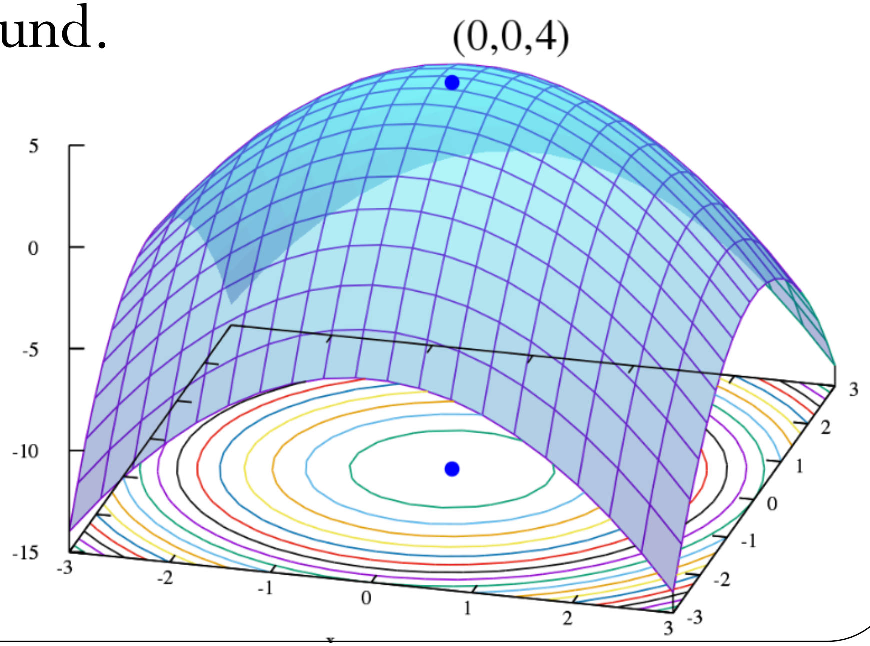
- An optimization problem can be represented in the following way:
  - Given: a function $f : A \longrightarrow \mathbb{R}$ from some set A to the real numbers
  - Goal: an element $x_0 \in A$ such that $f(x_0) \leq f(x)$ for all $x \in A$ ("minimization") or such that $f(x_0) \geq f(x)$ for all $x \in A$ ("maximization").
- The domain A of f is called the search space or the choice set
- The elements of A are called candidate solutions or feasible solutions.
- A is some subset of the
  - Euclidean space $\mathbb{R}^n$, Geometry represent N-dimensional space
  - Specified by a set of constraints,
  - Equalities or inequalities that the members of A have to satisfy



https://en.wikipedia.org/wiki/Mathematical_optimization

Optimization

- Problem of finding the best solution from all feasible solutions.
- **Discrete optimization:** A problem with discrete variables in which an object must be found from a countable set like integer, permutation or graph
  - **Combinatorial optimization**
- **Continuous optimization:** A problem with continuous variables in which an optimal value from a continuous function must be found.
  - **Constrained problems**
  - **Multimodal problems**



https://en.wikipedia.org/wiki/Optimization_problem

Combinatorial optimization

- Finds an optimal object from a finite set of objects, where the set of feasible solutions is discrete or can be reduced to a discrete set.
  - Exhaustive search uses algorithms that quickly rule out large parts of the search space,
  - Or use **Approximation or Probabilistic algorithms**.
- A combinatorial optimization problem A is a quadruple (I, f, m, g), where
  - I is a set of instances; given an instance $x \in I$, $f(x)$ is the set of feasible solutions;
  - Given an instance x and a feasible solution y of x, $m(x, y)$ denotes the measure of y, which is usually a positive real.
  - g is the goal function, and is either min or max.
  - Goal is to find for some instance x an optimal solution, that is, a feasible solution of y

$$m(x,y) = g\{m(x,y') \mid y' \in f(x)\}.$$

https://en.wikipedia.org/wiki/Combinatorial_optimization

Constrained optimization

- Process of optimizing an objective function with respect to some variables in the presence of constraints on those variables.
  - primarily equality constraints, inequality constraints, and integer constraints
  - set of candidate solutions that satisfy all constraints is called the feasible set
- A general constrained minimization problem may be written as follows:

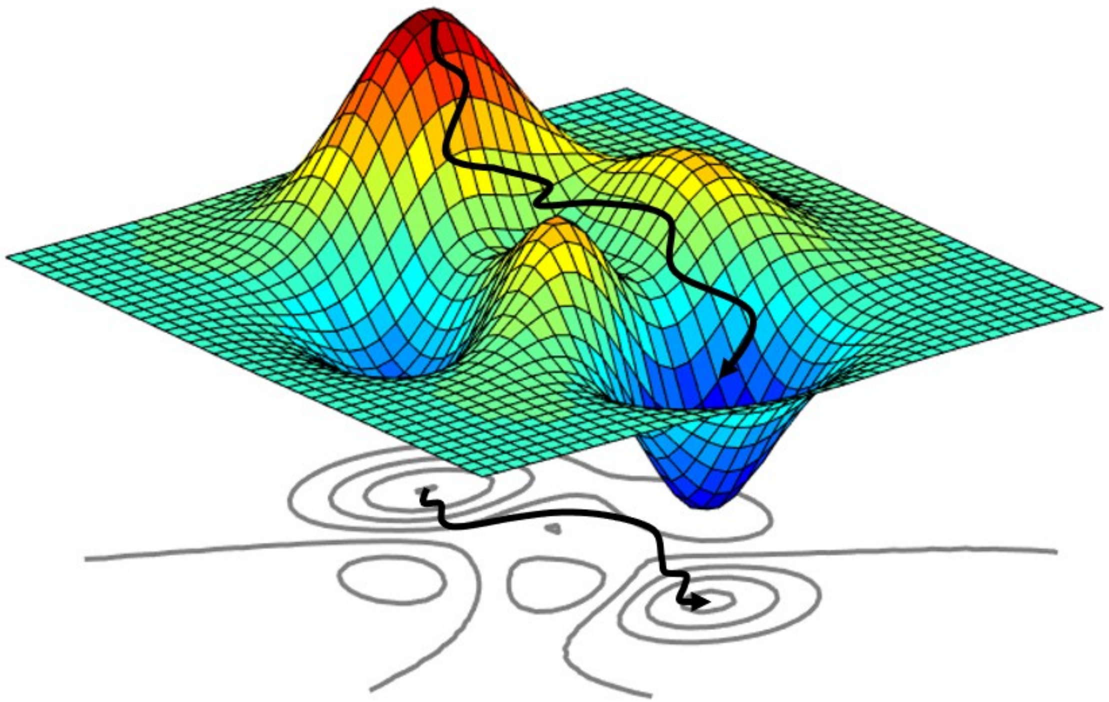
$$
\begin{array}{rlll}
\min & f(\mathbf{x}) & & \\
\text{subject to} & g_i(\mathbf{x}) = c_i & \text{for } i = 1, \ldots, n & \text{Equality constraints} \\
& h_j(\mathbf{x}) \geqq d_j & \text{for } j = 1, \ldots, m & \text{Inequality constraints}
\end{array}
$$

where $g_i(x)$ and $h_j(x)$ are constraints that are required to be satisfied, and $f(x)$ is the objective function that needs to be optimized subject to the constraints

https://en.wikipedia.org/wiki/Constrained_optimization

Multimodal optimization

- Finds all or most of the multiple (at least locally optimal) solutions of a problem, as opposed to a single best solution.
  - Evolutionary Multimodal Optimization is a branch of **Evolutionary Computation**
- **Evolutionary algorithms:**
  - Genetic Algorithms (GAs),
  - Evolution Strategy (ES),
  - Differential Evolution (DE),
  - Particle Swarm Optimization (PSO) etc.



https://en.wikipedia.org/wiki/Evolutionary_multimodal_optimization

AI with Optimization

- Many Optimization problems are NP-Hard, Examples:
  - Maximum Clique → NP-Complete
  - Set Cover → NP-Complete
  - Max-Cut → NP-Hard
  - Integer Programming → NP-Hard
- Optimization appears everywhere:
  - Operations research
  - Machine learning (training a model = optimization)
  - Robotics path planning
- Many real-world optimization problems are **NP-Hard**, which forces us to use:

  ✓ heuristics,

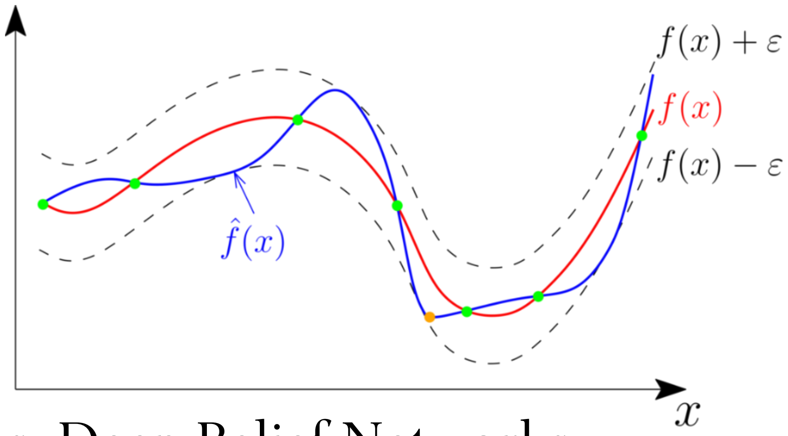
  ✓ greedy strategies

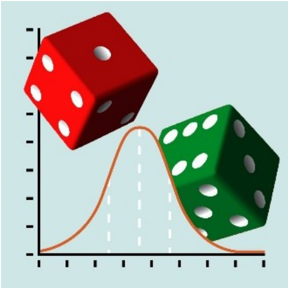
  ✓ meta-heuristics (GA, PSO, Simulated Annealing)

  ✓ relaxations (LP/SDP)

Approximation, Probability, and Optimization

- Deterministic nature of a system does not make it predictable.
- Exhaustive search is intractable for NP-hard problems
- Used Approximations, Probabilistic, and Optimization algorithms in
  - Travelling Salesman Problem (TSP), Minimum Spanning Tree (MST), and Knapsack
  - Minimax, Alpha–Beta pruning,
  - Monte Carlo, Markov chain Monte Carlo,
  - Constraint Satisfaction Problems,
  - Likelihood weighting, Maximum-Likelihood,
  - Stochastic Differential Equations (SDEs),
  - Belief states, Belief propagation, Bayesian Networks, Deep Belief Networks,
  - Various Machine Learning, Deep Learning, Reinforcement Learning algorithms,
  - and several other problems





https://mattiheino.com/2016/12/27/deterministic-not-predictable/

AI with Approximation, Probability, Optimization

- Approximation: Near-optimal polynomial-time
- Probability: Randomization helps average case
- Optimization: Relaxations, metaheuristics
- AI: Heuristics, search, etc.

Field	**Why NP matters here**	**Techniques used**
Approximation	Exact solutions impossible in P	Guaranteed near-optimal algorithms
Probability	Randomized solutions beat worst-case	Monte Carlo, stochastic search
Optimization	Most optimization ≈ NP-hard	Relaxation, metaheuristics
AI	Search, reasoning, learning tasks	Heuristics, pruning, knowledge base

P and NP

- What is P?
  - Problems solvable efficiently (polynomial time) using deterministic algorithms
  - Example: Sorting numbers

- What is NP?
  - Solutions can be verified efficiently, but finding the solution is in-efficient
  - Example: Verifying TSP route length

- P vs NP:
  - Can every problem whose solution is easy to check also be solved quickly?
  - Unknown whether $P = NP$
  - Believed $P \neq NP$, but unproven

AI's Historical Approach to Complex Problems

- Approximation Algorithms:
  - Guarantee bounds near optimal
  - Algorithms for NP-complete problems provide a solution that is guaranteed to be within a certain factor of the optimal answer for real-world applications.

- Rule-based AI:
  - Rules struggled with combinatorial exponential growth

- Heuristics and Approximations:
  - Provided near-optimal solutions or "calculated guesses" to NP-hard problems
  - Example: Heuristic search algorithms to find a near-optimal route for the TSP instead of exhaustively checking all routes

How Modern AI Tackles NP-Hard Problems

- Machine Learning and Optimization:
  - Modern AI, Rather than finding the exact solution, AI finds optimal solution.
  - Techniques like Neural Networks learn near-optimal solutions to NP-hard problems.

- Reinforcement Learning (RL):
  - Learns strategies for complex domains (e.g., Go)
  - RL agents can be trained to solve complex problems like playing chess or Go by exploring a huge number of potential moves, eventually learning optimal strategies.
  - RL is a practical solution to problems that were previously intractable.

Artificial Neural Networks & Complexity

- Training neural networks = Optimization problem (NP-Hard)
- Inference depends on probability (stochastic models)
- Universal Approximation Theorem → Approximation category

- Artificial Neural Networks has the intersection of
  - Optimization + Probability + Approximation

Future Intersection of AI

- Quantum Computing + AI:
  - Computational power to tackle NP problems that are hard with classical computers.

- AI for Scientific Discovery: To model and predict complex systems in sciences.
  - Example: Progress on protein folding and simulations.
  - This could lead to breakthroughs in solving NP-hard problems.

- Explainable AI: Transparency helps guide complexity reasoning
  - Understanding why a model makes a certain decision "the black box problem".
  - Interactive AI that focuses on making AI more transparent on complex problems.

Summary

- **P** → Efficient & deterministic → "Easy problems"
- **NP** → Solutions easy to verify → Many search/decision problems in AI
- **NP-Complete** → Hardest in NP → Guide complexity barriers
- **NP-Hard** → At least as hard as NP → Optimization & AI problems
- In practice, NP-Hardness **drives** research in:
  - Heuristics → AI
  - Approximation → Algorithms
  - Randomization → Probability
  - Relaxations → Optimization
- Artificial Neural Networks → Optimization + Approximation + Probability

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  - https://qr.ae/pvU11n
  - https://en.wikipedia.org/wiki/Chaos_theory
  - https://qr.ae/pvU11A

תודה רבה
Hebrew

Ευχαριστώ
Greek

Спасибо
Russian

Danke

German

धन्यवादः
Sanskrit

Merci

French

ধন্যবাদ
Bangla

நன்றி
Tamil

شكراً
Arabic

ಧನ್ಯವಾದಗಳು
Kannada

Thank You
English

നന്ദി
Malayalam

多謝

Traditional Chinese

Grazie

Italian

ధన్యవాదాలు
Telugu

આભાર
Gujarati

ਧੰਨਵਾਦ
Punjabi

धन्यवाद
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