**The Finite Element Method**

CE 381R/CSE 393 - Fall Semester 2020

Assignment 2 – due 10/09/20

Consider the problem of a homogeneous prismatic bar (data: $EA, L$) free at one end and elastically supported at the other end, as shown in the figure below. Let the stiffness of the spring be $k = \frac{4EA}{L}$. The bar is subjected to a uniform axial load $p(x) = p_0$.

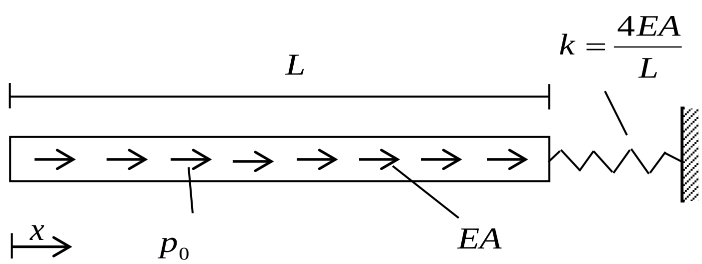


Figure: Elastically supported bar

a. There are at least two ways of casting the strong form of the problem. For example, one could consider the system consisting of both the bar and the spring, with the right end fixed. Alternatively (and preferably), one could consider the bar only, with a free left end, and the following mixed condition on the right end:

$$EA\frac{du}{dx}(L) + ku(L) = 0$$

Complete the **strong form** of the problem, and **verify** that the exact solution is given as:

$$u_{ex}(x) = \frac{p_0 L^2}{4EA}\left[3 - 2\left(\frac{x}{L}\right)^2\right]$$

b. Next, derive the **weak form** of the problem.

c. Solve for the **displacements** using **linear** shape functions corresponding to 2, 4, 8 and 16 elements. In each case, the elements should be of equal size, that is, for the case of 2 elements, the element size should be $h = \frac{L}{2}$, for the case of 4 elements, the element size should be $h = \frac{L}{4}$, and so on and so forth. To this end, compute first the **element stiffness matrix**, and the **element force vector** for a generic element of length $h$.

**Write a code to do the computations.** Your code should take as input the number of elements, and be able to output the displacements at the nodal locations. To develop the code, use any language you want and/or any symbolic or numerical package of your choosing. You should turn in the code as part of the assignment.

d. Plot the displacements (normalized by $\frac{p_0 L^2}{EA}$) against the normalized axial coordinate $\frac{x}{L}$. Plot also the internal axial force (normalized by $p_0 L$) against the normalized axial coordinate, for the cases of **4 elements, 16 elements, and the exact solution**. What do you observe?

e. It can be shown that the approximate strain energy of the system can be expressed as:

$$S_{app} = \frac{1}{2}\mathbf{u}^T \mathbf{F}$$

where $\mathbf{u}$ is the vector of nodal displacements and $\mathbf{F}$ is the force vector.

Compute first the exact strain energy, and then compute the approximate strain energy for 2, 4, 8, and, 16 elements. Plot the normalized approximate strain energy and the exact versus the number of elements; normalize the energies by $\frac{p_0^2 L^3}{EA}$. What do you observe?