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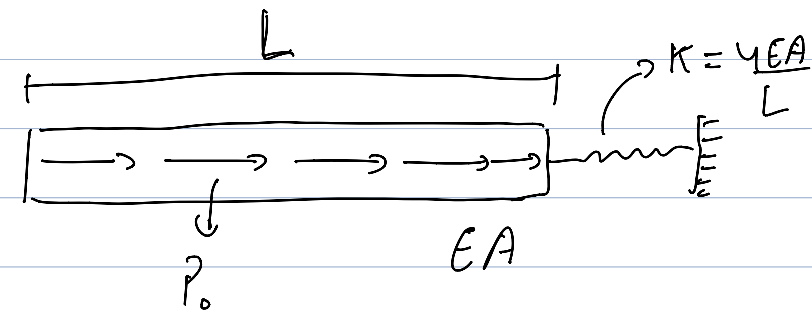
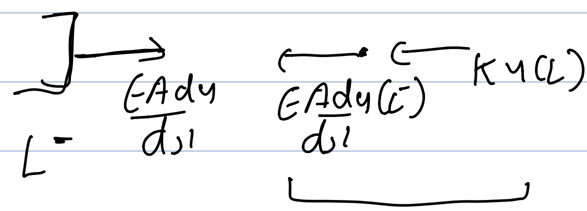
a)




Consider the other side of the bar as free, then we would have the following scenario at $x = L^-$




$$\Rightarrow EA\frac{du}{dx}(L) + Ku(L) = 0$$

$\rightarrow$ Mixed B.C.

The strong form will be :-

$$\left\{ \begin{array}{l} EA\frac{d^2u}{dx^2} + P_0 = 0 \\ EA\frac{du}{dx}(0) = 0 \text{ and } EA\frac{du}{dx}(L) + Ku(L) = 0 \end{array} \right.$$

Solution:-
$$\frac{d^2u}{dx^2} = \frac{-P_0}{EA}$$

$$\frac{du}{dx} = -\frac{P_0}{EA}x + C$$

$$u(x) = \frac{-P_0 x^2}{2EA} + C_1 x + C_2$$

$$\frac{du}{dx}(L) = \frac{-P_0 L}{EA} \Rightarrow EA\frac{du}{dx}(L) = -P_0 L$$

Also, $u(L) = -\frac{P_0 L^2}{2EA} + C_2$

$$\frac{P_0 L}{K} = \frac{-P_0 L^2}{2EA} + C_2$$

$$\Rightarrow C_2 = \frac{P_0 L}{K} + \frac{P_0 L^2}{2EA}$$

$$= \frac{P_0 L^2}{4EA} + \frac{P_0 L^2}{2EA} = \frac{3P_0 L^2}{4EA}$$

$$\Rightarrow u(x) = \frac{-P_0 x^2}{2EA} + \frac{3P_0 L^2}{4EA}$$

$$= \frac{P_0 L^2}{4EA}\left[3 - 2\frac{x^2}{L^2}\right]$$ (Verified)

b). $$R(x) = \left(EA\frac{d^2u}{dx^2} + P_0\right) = 0$$

Take Test function $v(x)$ & multiply with $R(x)$ and take integral over the domain :-

$$\int_0^L v(x) R(x)\, dx = 0$$

$$\Rightarrow \int_0^L v(x)\left[EA\frac{d^2u}{dx^2} + P_0\right] dx = 0$$

$$= \int_0^L \left(EA\, v u'' + P_0 v\right) dx = 0$$

$$= EA\int_0^L v u'' dx = -\int_0^L P_0 v\, dx$$

$$\Rightarrow EA\left[\int_0^L (v u')' dx - \int_0^L v' u'\, dx\right] = -\int_0^L P_0 v\, dx$$

$$\Rightarrow EA\int_0^L v' u' dx = EA\left[v u'\right]_0^L + \int_0^L P_0 v\, dx$$

$$\int_0^L EA v' u' dx = EA\, v(L) u'(L) - EA\, v(0) \cancelto{0}{u'(0)} + \int_0^L P_0 v\, dx$$

$$= \int_0^L EA\, v' u' dx = -K v(L) u(L) + \int_0^L P_0 v\, dx$$

$\Rightarrow$ Weak form :-

$$\int_0^L EA\, v' u'\, dx + k\, v(L)\, u(L) = \int_0^L p_0\, v\, dx$$

and $\int_0^L (v')^2 dx < \infty$, $\int_0^L (u')^2 dx < \infty$

c) Now take $u(x) \approx \hat{u}(x) = \underset{\sim}{\varphi}^T \underset{\sim}{U}$

$v(x) \approx \hat{v}(x) = \underset{\sim}{V}^T \underset{\sim}{\varphi}$

Putting in the weak form, we have :-

$$V^T\left[\left(\int_0^L EA\, \varphi'(x)\, \varphi^{T'}(x)\, dx\right)\underset{\sim}{U} + k\,\varphi(L)\,\varphi^T(L)\,\underset{\sim}{U} - \int_0^L p_0\, \varphi(x)\, dx\right] = 0$$

$$\mathbb{K}\,\underset{\sim}{U} = \underset{\sim}{F}$$

$$\mathbb{K} = \int_0^L \left(EA\, \varphi'(x)\, \varphi^{T'}(x) + k\, \varphi(L)\, \varphi^T(L)\right) dx$$

$$\underset{\sim}{F} = \int_0^L p_0\, \varphi(x)\, dx$$

# Computations are done in MATLAB (solve.m)

u_all

3x1 double

| | 1 |
|---|---|
| 1 | 0.7500 |
| 2 | 0.6250 |
| 3 | 0.2500 |

Displacements for 2 elements

u_all

5x1 double

| | 1 |
|---|---|
| 1 | 0.7500 |
| 2 | 0.7188 |
| 3 | 0.6250 |
| 4 | 0.4688 |
| 5 | 0.2500 |

Displacements for 4 elements

u_all

9x1 double

| | 1 |
|---|---|
| 1 | 0.7500 |
| 2 | 0.7422 |
| 3 | 0.7187 |
| 4 | 0.6797 |
| 5 | 0.6250 |
| 6 | 0.5547 |
| 7 | 0.4687 |
| 8 | 0.3672 |
| 9 | 0.2500 |

Displacements for 8 elements

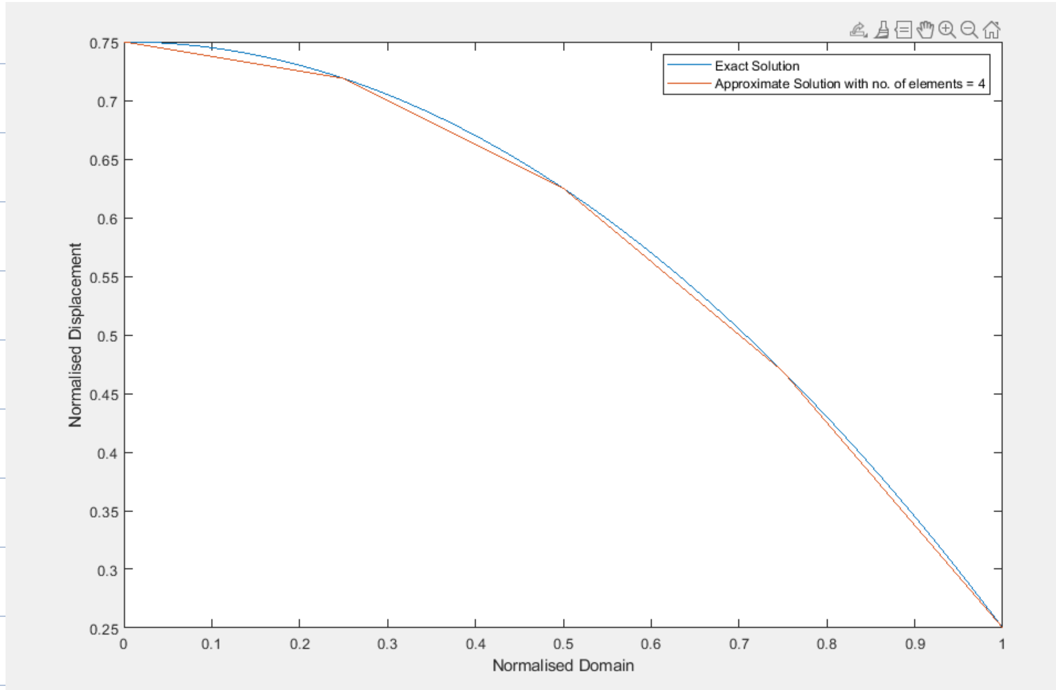
u_all

17x1 double

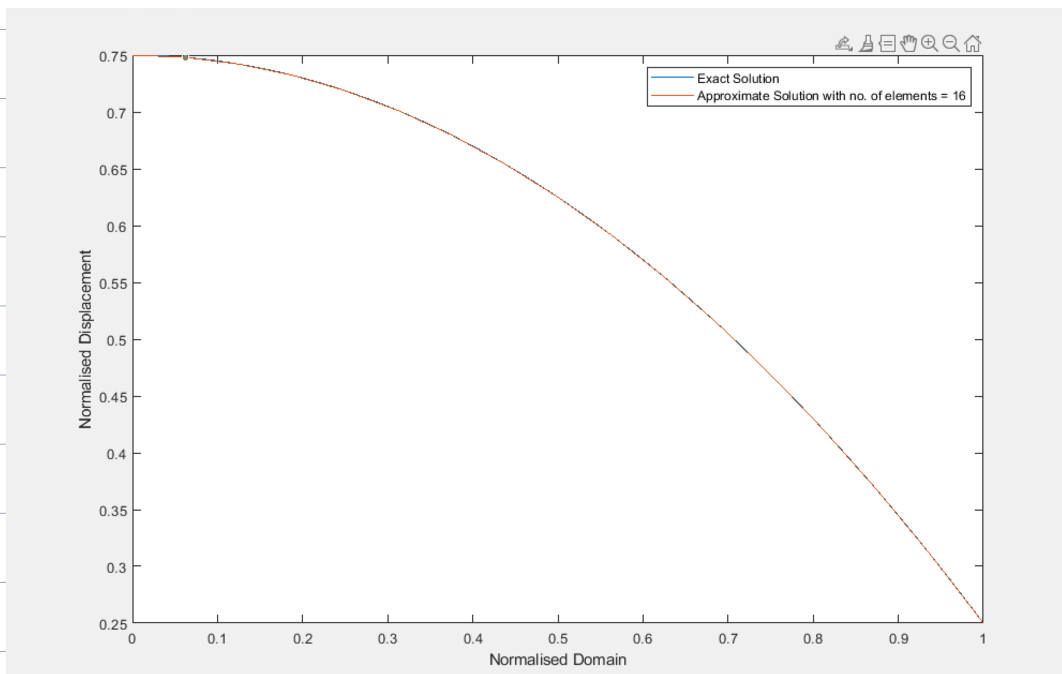
| | 1 |
|---|---|
| 1 | 0.7500 |
| 2 | 0.7480 |
| 3 | 0.7422 |
| 4 | 0.7324 |
| 5 | 0.7188 |
| 6 | 0.7012 |
| 7 | 0.6797 |
| 8 | 0.6543 |
| 9 | 0.6250 |
| 10 | 0.5918 |
| 11 | 0.5547 |
| 12 | 0.5137 |
| 13 | 0.4688 |
| 14 | 0.4199 |
| 15 | 0.3672 |
| 16 | 0.3105 |
| 17 | 0.2500 |
| 18 | |

Displacements for 16 elements

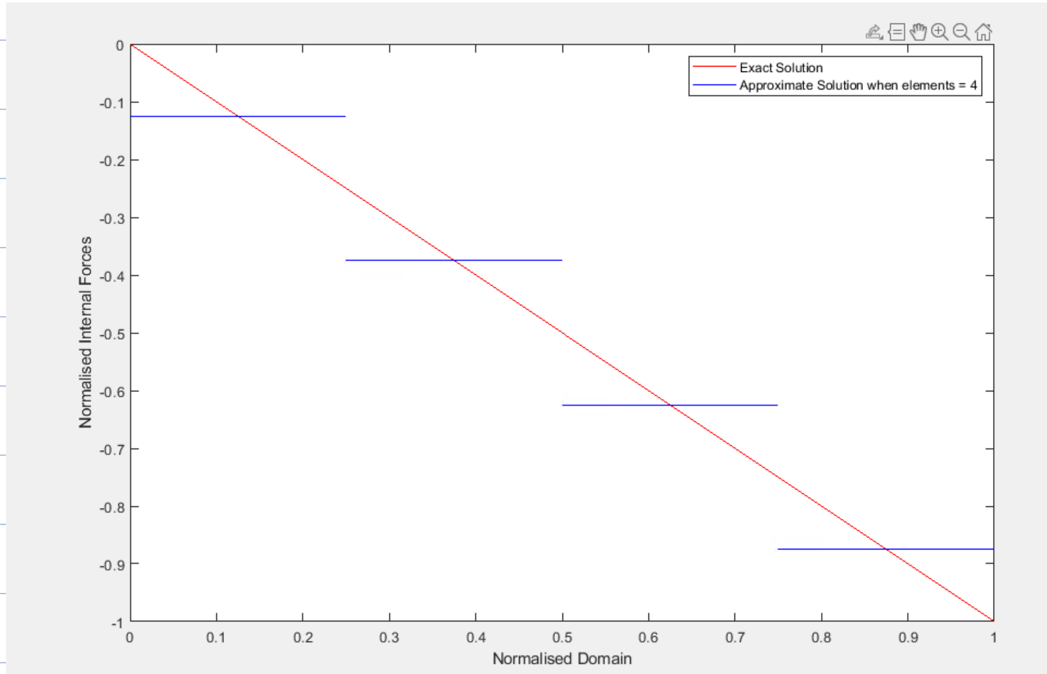
d)

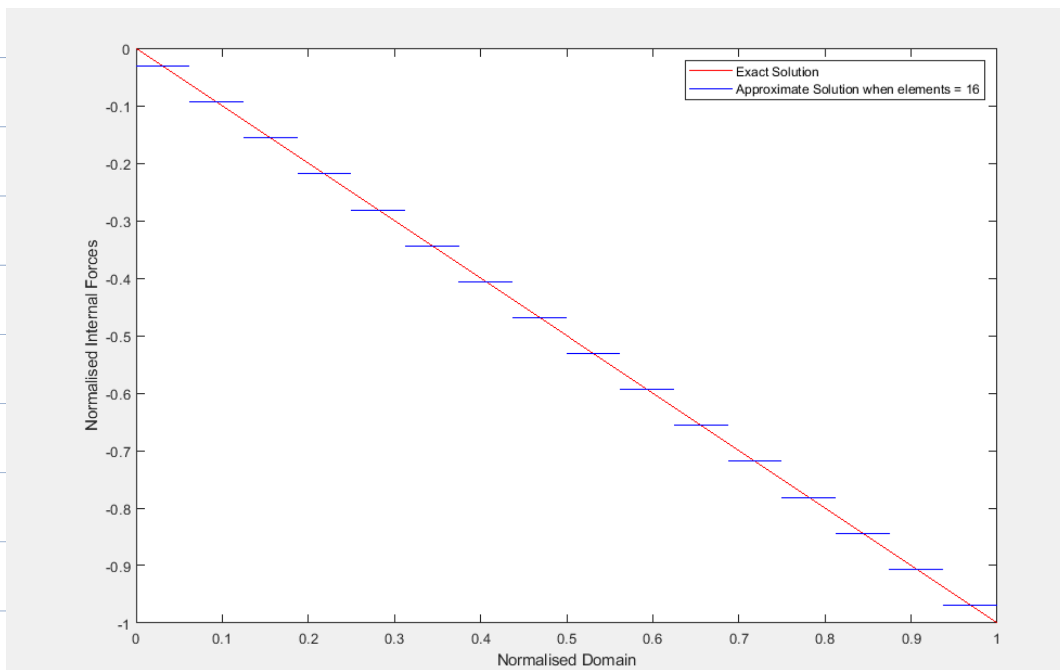


Exact Solution
Approximate Solution with no. of elements = 4
Normalised Displacement
Normalised Domain
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Exact Solution
Approximate Solution with no. of elements = 16
Normalised Displacement
Normalised Domain
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1




Internal axial force for exact soln: $\Rightarrow EA \dfrac{du_{ex}}{dx} = -P_0 L x$

Observations:- (1) As the no. of elements are increasing the $\tilde{u}(x)$ is approaching $u_{ex}(x)$

i.e. the error is decreasing

(2) The approximate internal force is piecewise continuous

e) $u_{ex} = \dfrac{P_0 L^2}{4EA}\left[3 - 2\left(\dfrac{x}{L}\right)^2\right]$

$$S[u_{ex}] = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}k\,u(L)^2$$

$$\frac{du_{ex}}{dx} = \frac{P_0 L^2}{4EA}\left[0 - \frac{4x}{L^2}\right]$$

$$= -\frac{P_0 x}{EA}$$

$$\left(\frac{du_{ex}}{dx}\right)^2 = \frac{P_0^2 x^2}{E^2A^2}$$

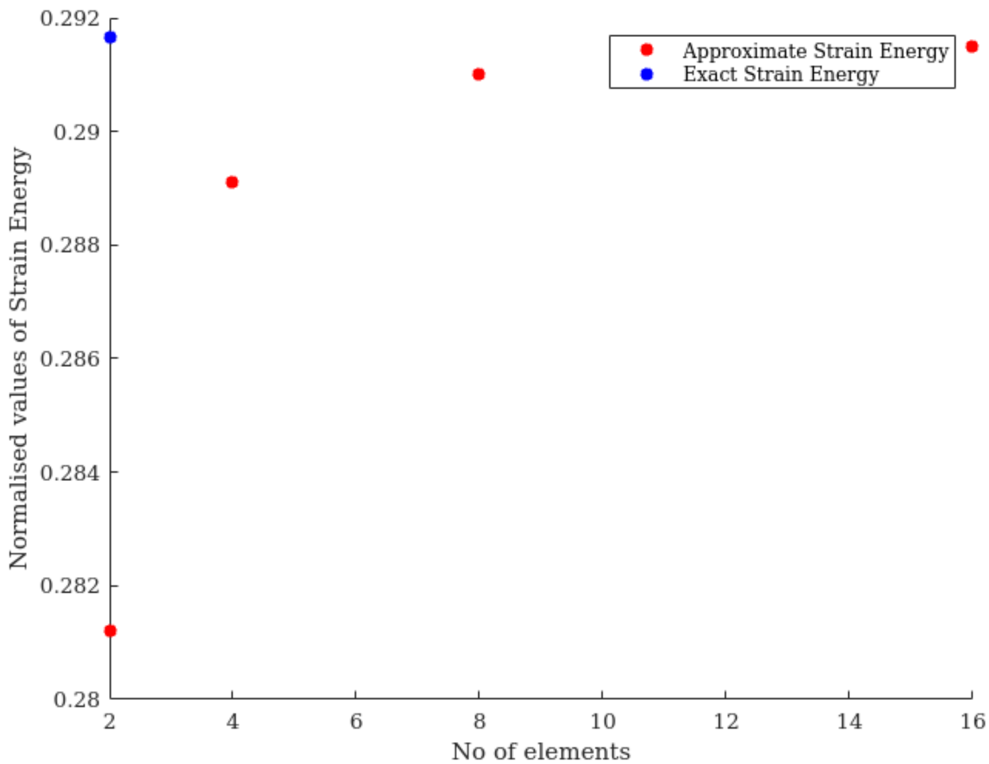
$$S(u_{ex}) = \frac{EA}{2E^2A^2}P_0^2\int_0^L x^2\,dx + \frac{1}{2}\times\frac{4EA}{L}\times\left(\frac{P_0L^2}{4EA}\right)^2$$

$$= \frac{P_0^2 L^3}{6EA} + \frac{1}{8}\frac{P_0^2 L^3}{EA} = \frac{7}{24}\frac{P_0^2 L^3}{EA}$$

$S[\tilde{u}(x)]$ computed using MATLAB code:-

| No. of elements | $S[\tilde{u}(x)]$ |
|---|---|
| 2 | 0.2812 |
| 4 | 0.2891 |
| 8 | 0.2910 |
| 16 | 0.2915 |

& $S[u_{exact}(x)] = \frac{7}{24}$




Observation:- The exact strain energy > Approximate strain energy.