COMPSCI 689
Lecture 9: Subgradient Learning for SVMs and the Perceptron

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Outline

The Perceptron Learning Algorithm

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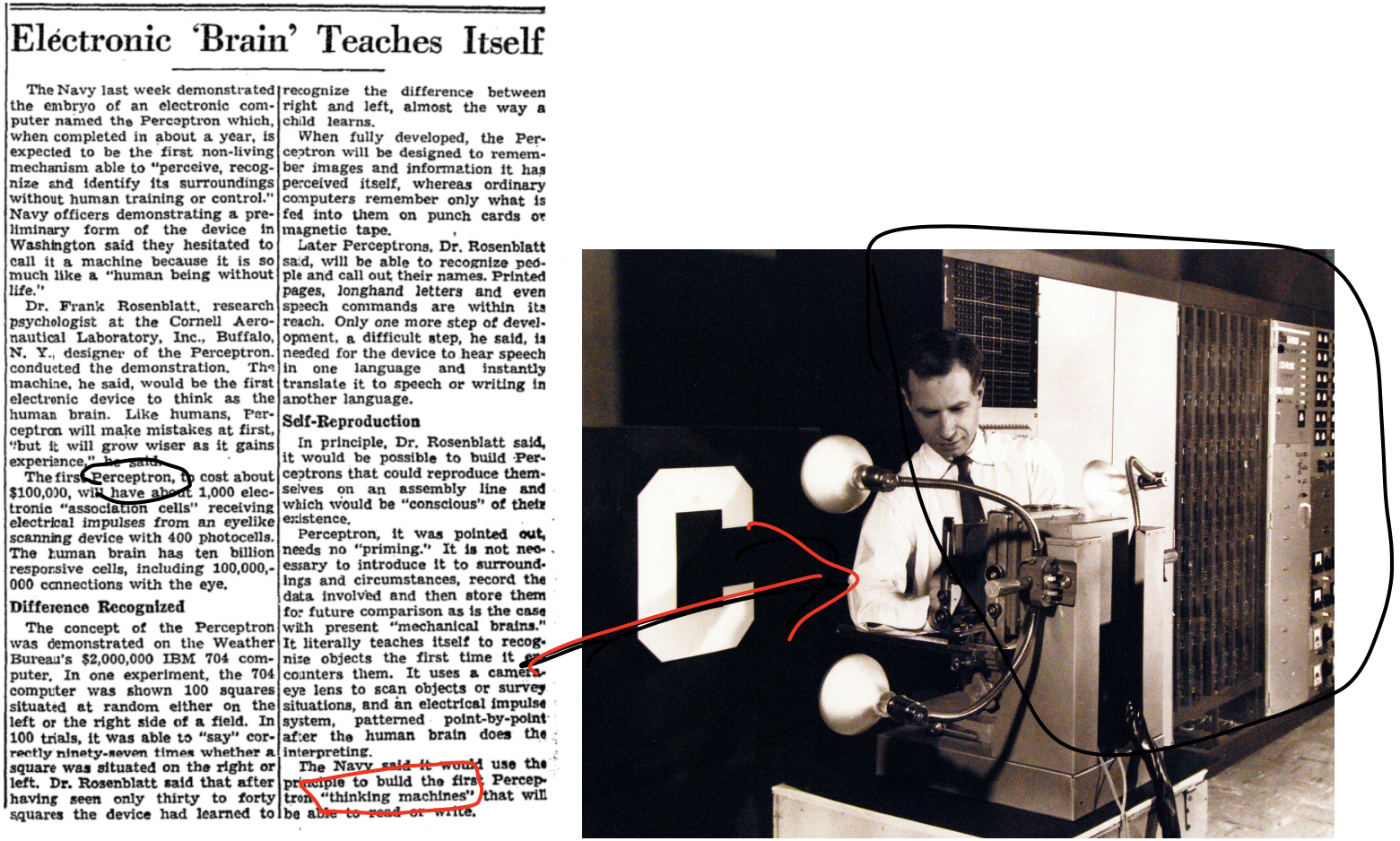


Electronic 'Brain' Teaches Itself

The Navy last week demonstrated the embryo of an electronic computer named the Perceptron which, when completed in about a year, is expected to be the first non-living mechanism able to "perceive, recognize and identify its surroundings without human training or control." Navy officers demonstrating a preliminary form of the device in Washington said they hesitated to call it a machine because it is so much like a "human being without life."

Dr. Frank Rosenblatt, research psychologist at the Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y., designer of the Perceptron, conducted the demonstration. The machine, he said, would be the first electronic device to think as the human brain. Like humans, Perceptron will make mistakes at first, "but it will grow wiser as it gains experience," he said.

The first Perceptron, to cost about $100,000, will have about 1,000 electronic "association cells" receiving electrical impulses from an eyelike scanning device with 400 photocells. The human brain has ten billion responsive cells, including 100,000,000 connections with the eye.

Difference Recognized

The concept of the Perceptron was demonstrated on the Weather Bureau's $2,000,000 IBM 704 computer. In one experiment, the 704 computer was shown 100 squares situated at random either on the left or the right side of a field. In 100 trials, it was able to "say" correctly ninety-seven times whether a square was situated on the right or left. Dr. Rosenblatt said that after having seen only thirty to forty squares the device had learned to recognize the difference between right and left, almost the way a child learns.

When fully developed, the Perceptron will be designed to remember images and information it has perceived itself, whereas ordinary computers remember only what is fed into them on punch cards or magnetic tape.

Later Perceptrons, Dr. Rosenblatt said, will be able to recognize people and call out their names. Printed pages, longhand letters and even speech commands are within its reach. Only one more step of development, a difficult step, he said, is needed for the device to hear speech in one language and instantly translate it to speech or writing in another language.

Self-Reproduction

In principle, Dr. Rosenblatt said, it would be possible to build Perceptrons that could reproduce themselves on an assembly line and which would be "conscious" of their existence.

Perceptron, it was pointed out, needs no "priming." It is not necessary to introduce it to surroundings and circumstances, record the data involved and then store them for future comparison as is the case with present "mechanical brains." It literally teaches itself to recognize objects the first time it encounters them. It uses a camera-eye lens to scan objects or survey situations, and an electrical impulse system, patterned point-by-point after the human brain does the interpreting.

The Navy said it would use the principle to build the first Perceptron "thinking machines" that will be able to read or write.

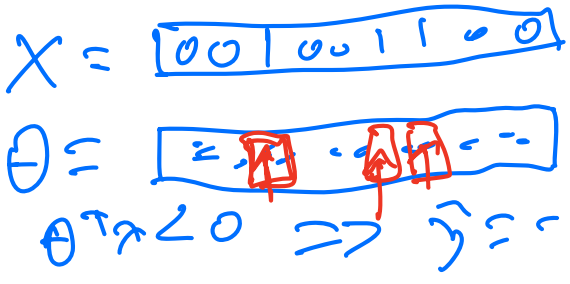
https://timesmachine.nytimes.com/timesmachine/1958/07/13/91396361.html?pageNumber=116

The Perceptron Learning Algorithm

Rosenblatt, 1957

$f(x; \theta) = \text{sign}(\theta^T x)$

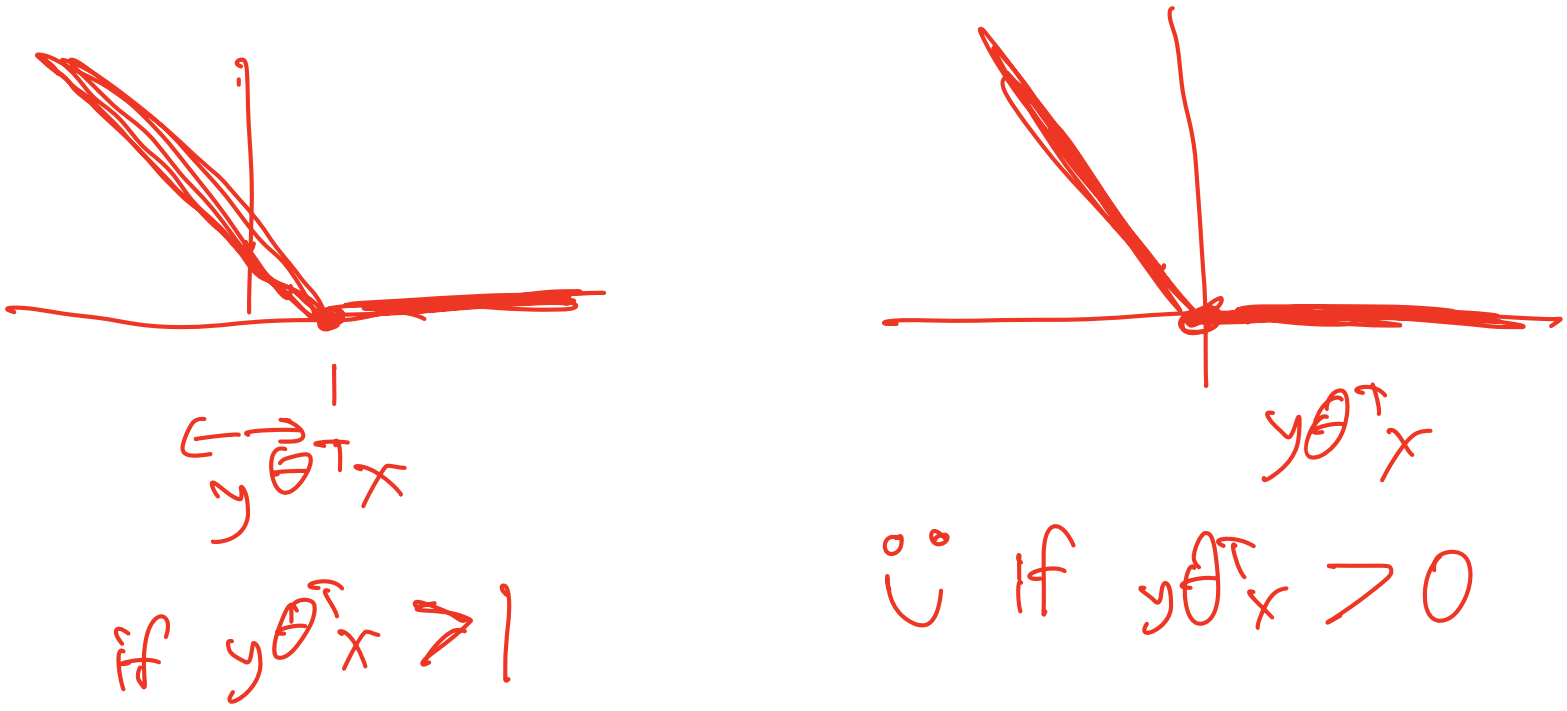
Outer loop (~20 iter):
  for (x_i, y_i) in Dtr:
    $\hat{y} = f(x_i; \theta)$
    if $\hat{y} \neq y_i$:
      $\theta = \theta + y_i x_i$



Convergence: ☹ (in general)

The Perceptron Learning Algorithm

- We view the classic perceptron as an ERM training method for a linear classifier, with the *perceptron loss*, very similar to the SVM hinge loss.



The Perceptron Learning Algorithm

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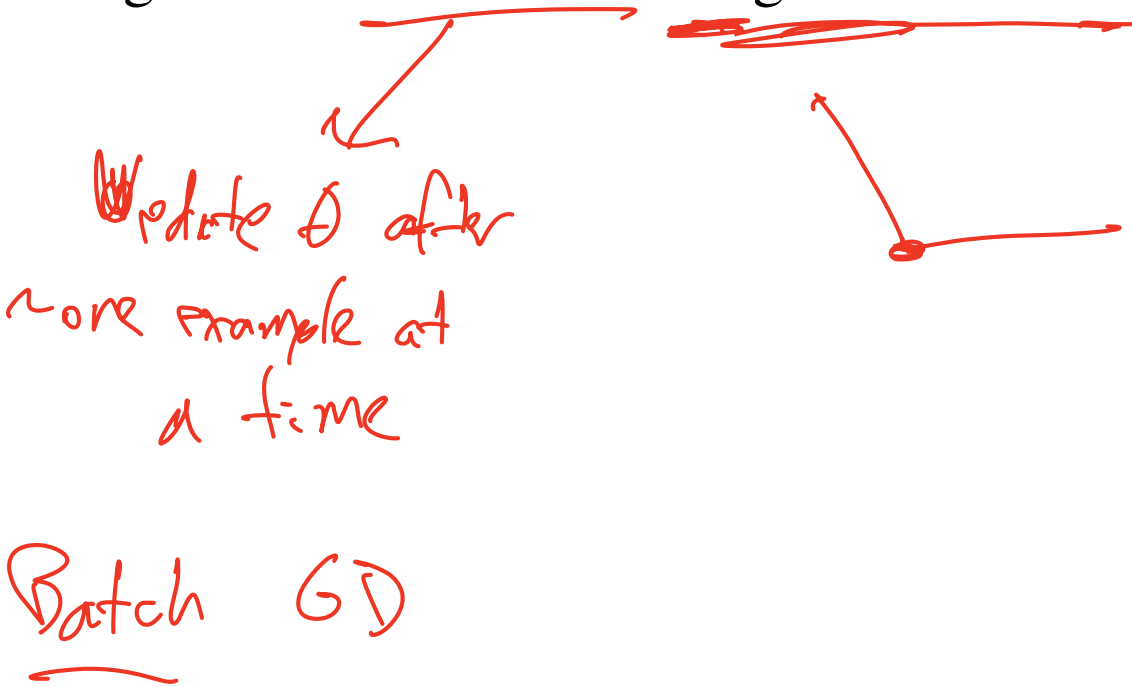
- The algorithm is stochastic subgradient descent for this loss.



Kernelization and Stochastic SubGD

- This isn't the optimal way to learn a kernelized classifier, but it's extremely easy to kernelize/dualize the perceptron.

$i = 1 \ldots N,\ N+1 \ldots 2N, \ldots\ldots NN$

$\theta_i = \theta_{i-1} + y_i x_i 1\{\hat{y}_i \neq y_i\}$

$\theta_i = \sum_{j=1}^{i} y_j x_j 1\{\hat{y}_j \neq y_j\}$

Assume $\theta_0 = \vec{0}$

Warm Start: $\theta_0 \neq \vec{0}$

$f(x^{new}; \theta_i) = \theta_i^T x^{new} = \left(\sum_j y_j x_j 1\{\hat{y}_j = y_j\}\right)^T x^{new}$

$= \sum_j y_j 1\{\hat{y}_j = y_j\}\ x_j^T x^{new}$

$y_j 1\{\hat{y}_j = y_j\} \in \{-1, 0, +1\}$

$x_j^T x^{new} = K(x_j, x^{new})$

Outline

Back to the Hinge Loss Formulation

$$f_\theta(\mathbf{x}) = \text{sign}(g_\theta(\mathbf{x}))$$

$$g_\theta(\mathbf{x}) = \mathbf{x}\theta = \mathbf{x}\mathbf{w} + b$$

$$\hat{\theta} = \arg\min_\theta C \sum_{n=1}^{N} \max(0, 1 - y_n g_\theta(\mathbf{x}_n)) + ||\mathbf{w}||_2^2$$

Global Optimality of Convex Non-Differentiable Functions

- Our existing optimization theory only holds for the case of differentiable functions.

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- However, it turns out that many results generalize to the case of non-differentiable functions that are convex and we can use them to directly minimize the hinge loss.
- To begin, strongly convex non-differentiable functions have a unique global minima, exactly as with convex differentiable functions.
- We will begin with the characterization of the minimizer of a non-differentiable convex function. [1]

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Subgradient

- Let $f : \mathbb{R}^D \to \mathbb{R}$. A vector $\mathbf{g} \in \mathbb{R}^D$ is said to be a sub-gradient of f at a point $\mathbf{x}_o \in \mathbb{R}^D$ if for all $\mathbf{x} \in \mathbb{R}^D$:

$$f(\mathbf{x}) \geq f(\mathbf{x}_o) + \mathbf{g}^T \cdot (\mathbf{x} - \mathbf{x}_o)$$

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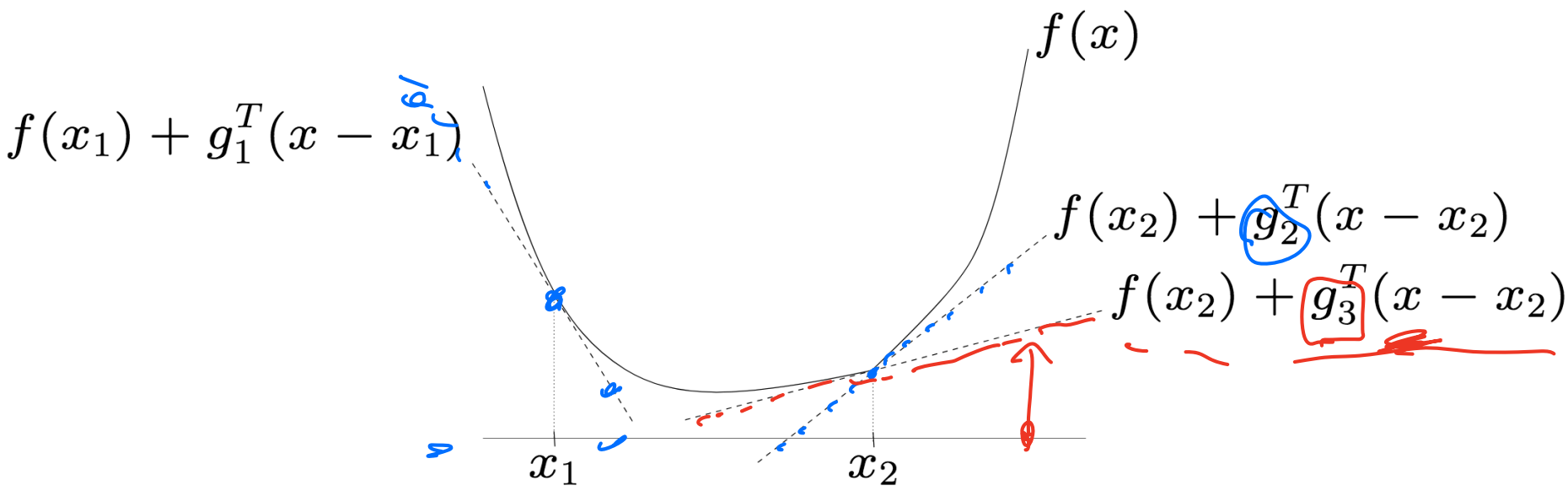
- That is to say, the hyperplane defined by $h(\mathbf{x}) = f(\mathbf{x}_o) + \mathbf{g}^T \cdot (\mathbf{x} - \mathbf{x}_o)$ lies at or below below f everywhere and touches f at $\mathbf{x}_o$.

Example: Subgradients



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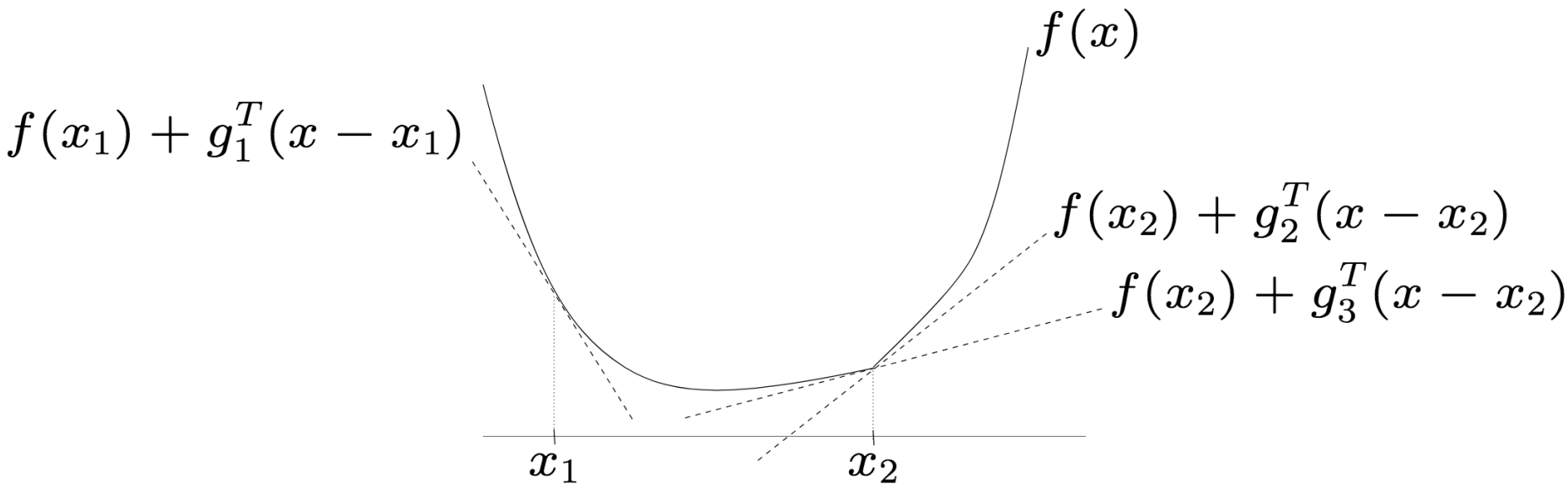

In this example, $\mathbf{g}_1$ is the unique subgradient of f at $\mathbf{x}_1$. Due to f being non-differentiable at $\mathbf{x}_2$, both $\mathbf{g}_2$ and $\mathbf{g}_3$ are subgradients of f at $\mathbf{x}_2$.

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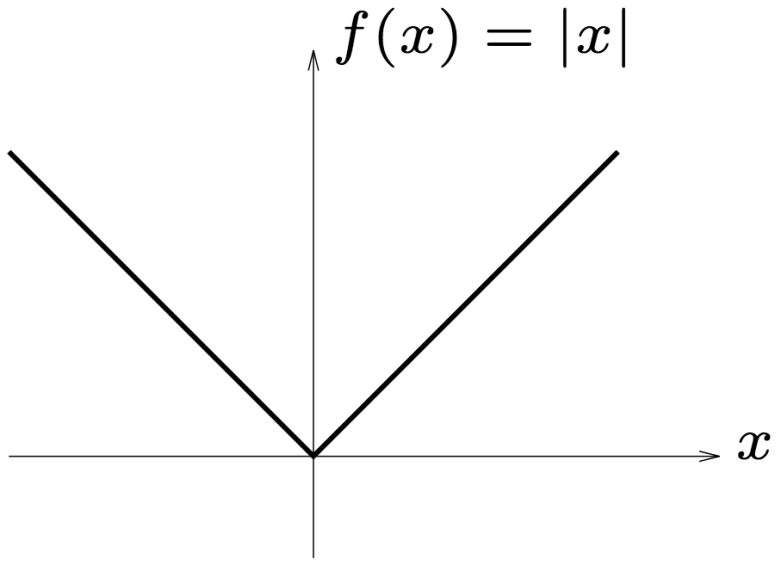
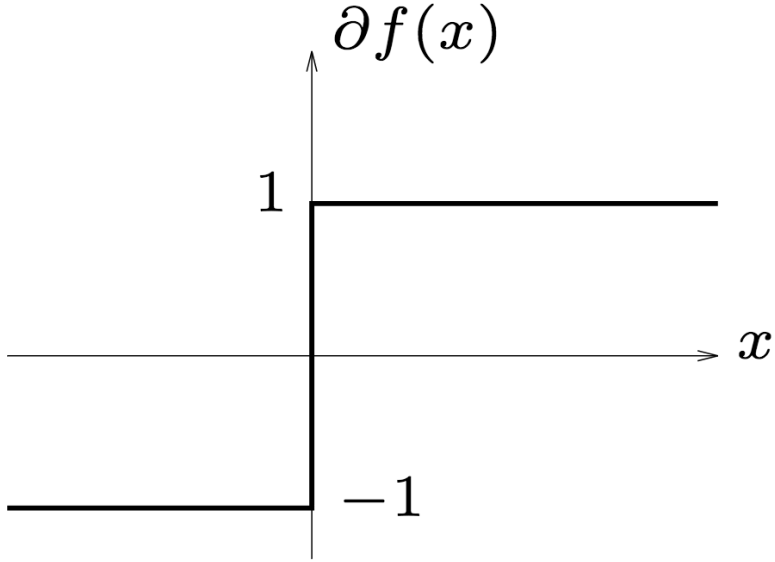
- $\partial f(\mathbf{x}_o)$ is a closed, convex set in $\mathbb{R}^D$. If f is convex, $\partial f(\mathbf{x}_o)$ is always non-empty.

Example: Subdifferentials





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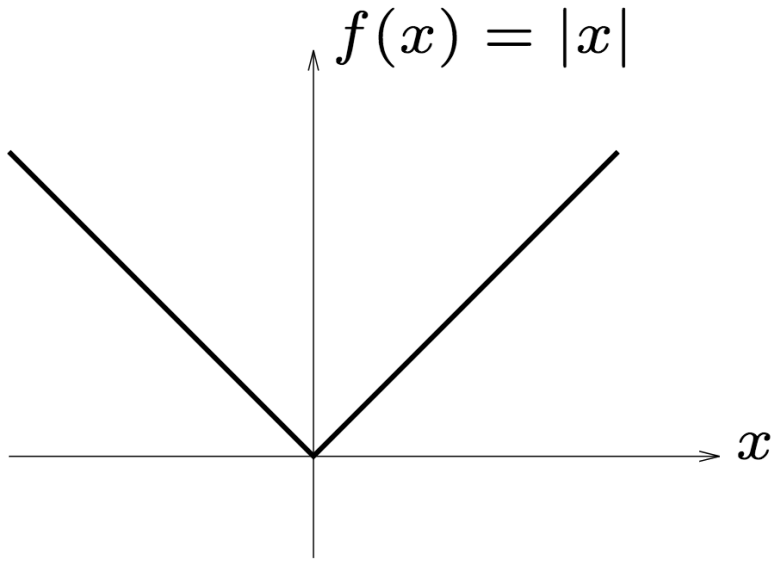

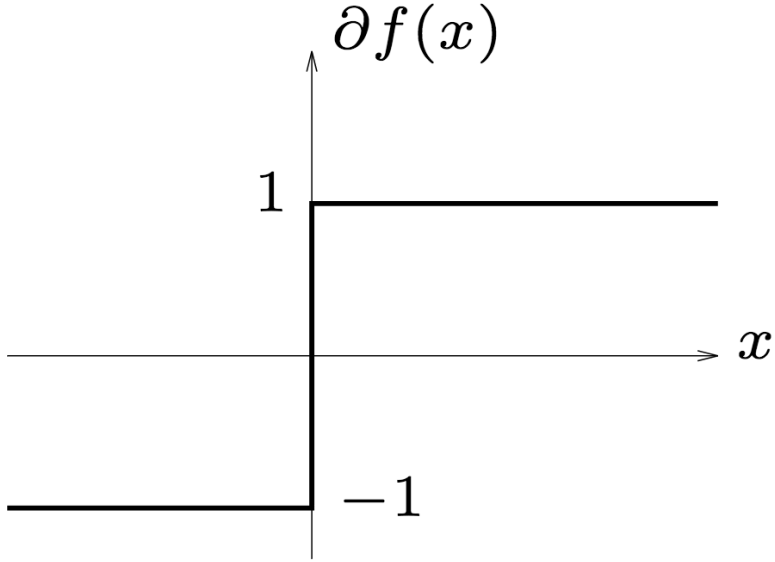

The righthand plot shows $\partial f(x)$ for $f(x) = |x|$.

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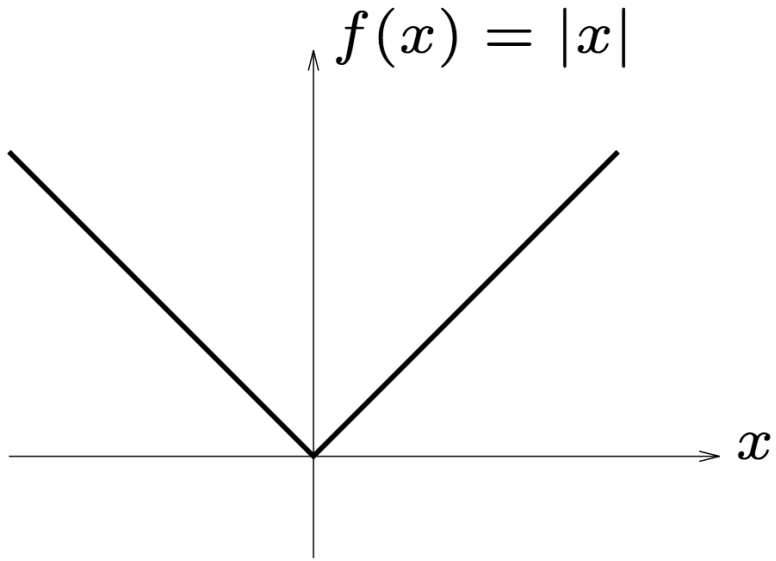

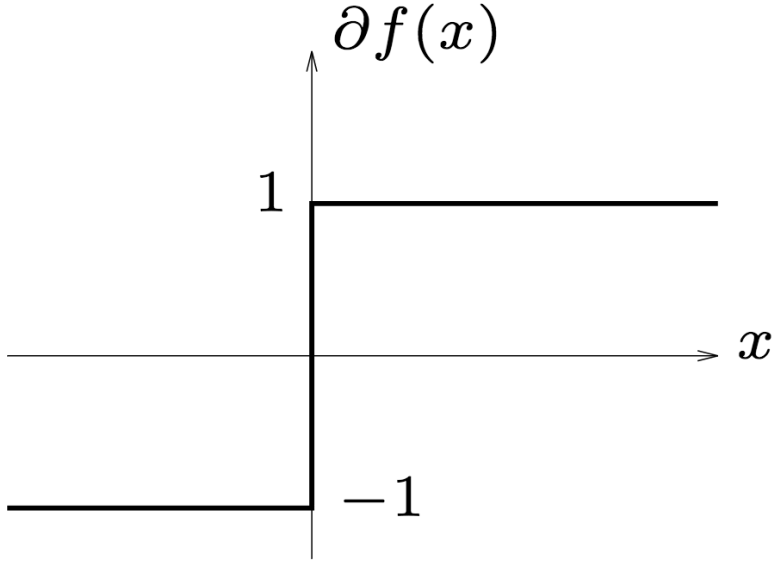

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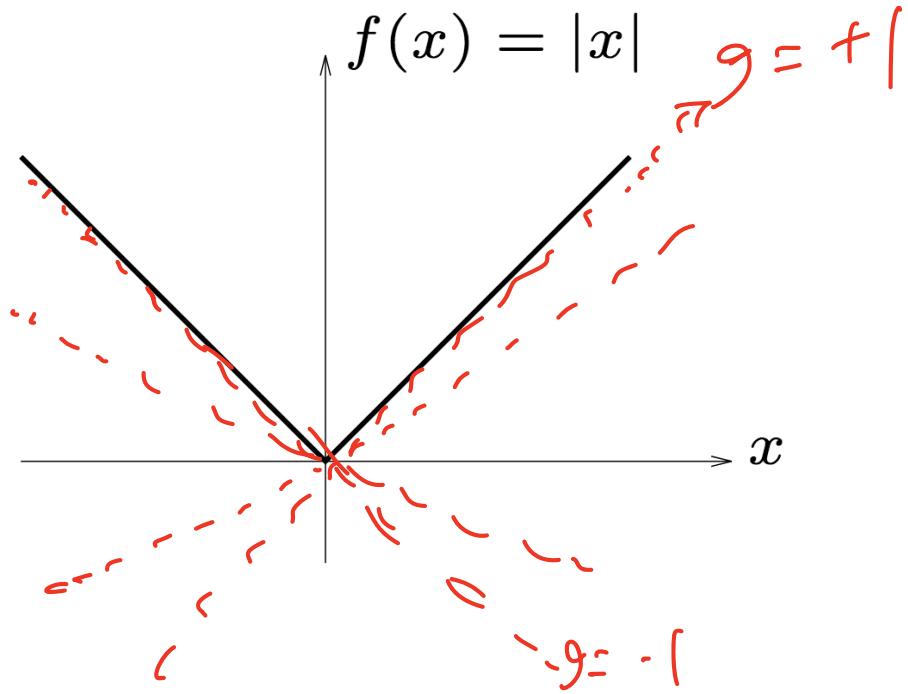

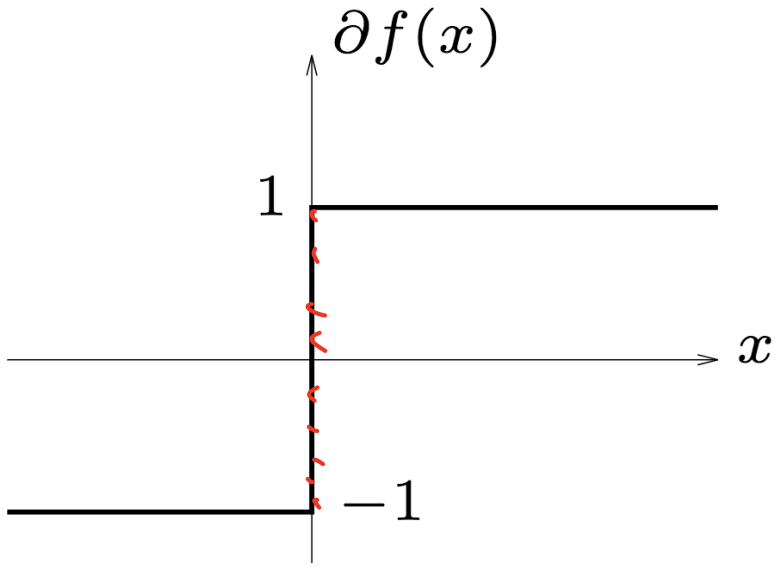

The righthand plot shows $\partial f(x)$ for $f(x) = |x|$.
We have $\partial f(\mathbf{x}) = \{\text{sign}(x)\}$ for $x \neq 0$.
When $x = 0$, the line $|0| + g \cdot (x - 0) = g \cdot x$ will lie below f everywhere only if $g \in [-1, 1]$.

Characterizing the Global Minimum

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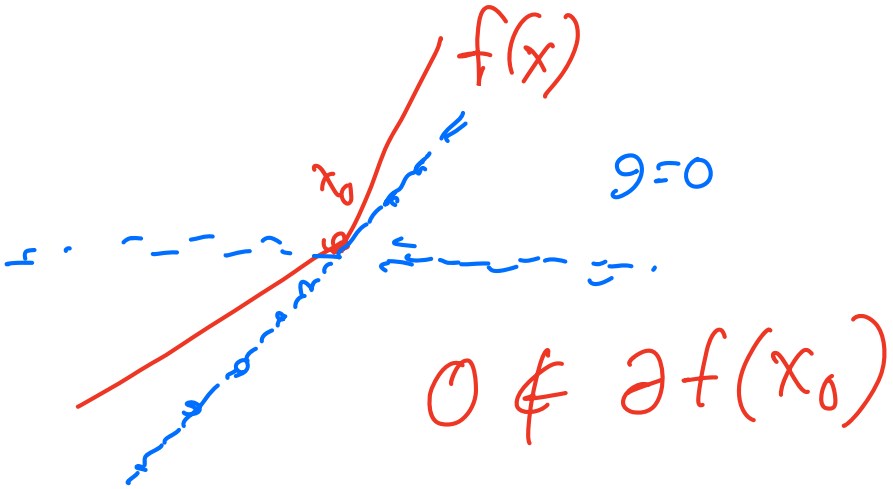
- $\mathbf{x}_*$ is the global minimizer of f if and only if $\mathbf{0} \in \partial f(\mathbf{x}_*)$.
- This is a generalization of the idea of a stationary point to include the case of non-differentiable functions.



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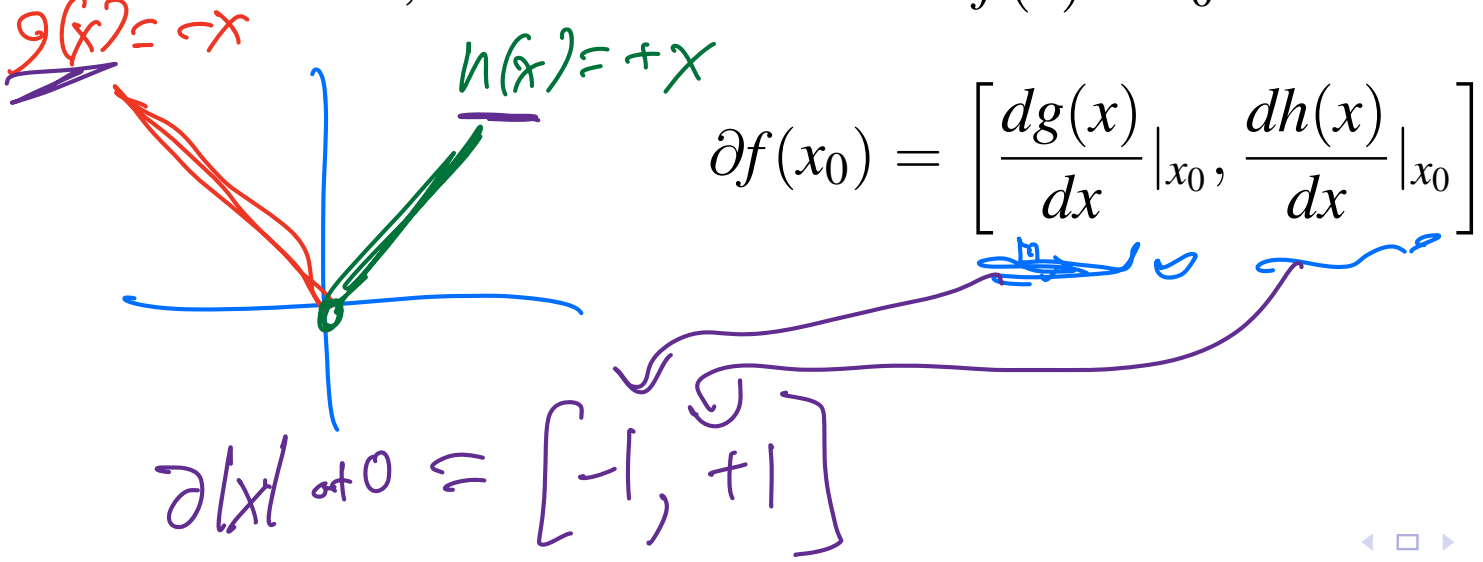
- Then, the subdifferential of $f(x)$ at x_0 is:

$$\partial f(x_0) = \left[\frac{dg(x)}{dx}\Big|_{x_0}, \frac{dh(x)}{dx}\Big|_{x_0} \right]$$



Partial Linearity and Chain Rule

The subdifferential operator satisfies partial linearity and chain rule properties:

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These properties can be used to determine the subdifferentials of more some complex functions if they reduce to certain combinations or compositions of simpler functions.

Outline

Sub-gradient Descent

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- If f is not differentiable at $\mathbf{x}$, then there may exist some $\mathbf{g} \in \partial f(\mathbf{x})$ where $-\mathbf{g}$ is a **not** descent direction.

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  - Set $\mathbf{x}_k = \mathbf{x}_{k-1} - \alpha_k \mathbf{g}_k$
  - If $f(\mathbf{x}_k) < f_{min}$ then set $f_{min} = f(\mathbf{x}_k)$ and $\mathbf{x}_* = \mathbf{x}_k$

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- For k from 1 to K:
  - Let $\mathbf{g}_k \in \partial f(\mathbf{x}_{k-1})$
  - Set $\mathbf{x}_k = \mathbf{x}_{k-1} - \alpha_k \mathbf{g}_k$
  - If $f(\mathbf{x}_k) < f_{min}$ then set $f_{min} = f(\mathbf{x}_k)$ and $\mathbf{x}_* = \mathbf{x}_k$
- Return $\mathbf{x}_*$

Sub-gradient Descent

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best solution so far

best value so far

Questions: How to choose α_k? How to choose K?

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- Line search is typically not used for sub-gradient descent procedures. It is more common to use a fixed sequence of step sizes.
- Common step size rules include $\alpha_k = \alpha/(\beta + k)$ or $\alpha_k = \alpha/\sqrt{k}$.
- Under either of these step size rules, we have that the sequence of subgradient descent iterates $\mathbf{x}_k$ satisfies:

$$\lim_{k \to \infty} f(\mathbf{x}_k) = \min_{\mathbf{x}} f(\mathbf{x})$$

Subgradient Descent with Momentum

- Initialize $\mathbf{x}_0 \in \mathbb{R}^D, f_{min} = \infty, \mathbf{x}_* = 0$
- For k from 1 to K:
  - Let $\mathbf{g}_k \in \partial f(\mathbf{x}_{k-1})$
  - Let $\mathbf{d}_k = \gamma \mathbf{d}_{k-1} + \alpha_k \mathbf{g}_k$
  - Set $\mathbf{x}_k = \mathbf{x}_{k-1} - \mathbf{d}_k$
  - If $f(\mathbf{x}_k) < f_{min}$ then set $f_{min} = f(\mathbf{x}_k)$ and $\mathbf{x}_* = \mathbf{x}_k$
- Return $\mathbf{x}_*$

Typically use with $0 < \gamma < 1$. $\gamma = 0.9$ is a common choice.

Nesterov Accelerated Subgradient Descent

- Initialize $\mathbf{x}_0 \in \mathbb{R}^D, f_{min} = \infty, \mathbf{x}_* = 0$
- For k from 1 to K:
  - Let $\mathbf{g}_k \in \partial f(\mathbf{x}_{k-1} - \gamma \mathbf{d}_{k-1})$
  - Let $\mathbf{d}_k = \gamma \mathbf{d}_{k-1} + \alpha_k \mathbf{g}_k$
  - Set $\mathbf{x}_k = \mathbf{x}_{k-1} - \mathbf{d}_k$
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Outline

Example: Finding a Subgradient for SVC

- We begin the risk function:

$$
\begin{aligned}
R(\theta, \mathcal{D}) &= C \sum_{n=1}^{N} \max(0, 1 - y_n g_\theta(\mathbf{x}_n)) + ||\mathbf{w}||_2^2 \\
&= C \sum_{n=1}^{N} L(y_n g_\theta(\mathbf{x}_n)) + ||\mathbf{w}||_2^2
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$$= C \sum_{n=1}^{N} L(y_n g_\theta(\mathbf{x}_n)) + ||\mathbf{w}||_2^2$$

- We are looking for a vector $\mathbf{h}$ such that:

$$\mathbf{h} \in \partial R(\theta, \mathcal{D})$$

Example: Finding a Subgradient for SVC

- By the additivity and scaling properties, we have that if $\mathbf{h}_n \in \partial L(y_n g_\theta(\mathbf{x}_n))$ and $\mathbf{h}_w \in \partial ||\mathbf{w}||_2^2$, then:

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- We will proceed by finding suitable vectors $\mathbf{h}_n$ and $\mathbf{h}_w$ using properties of sub-gradients.

Example: Finding a Subgradient for SVC

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Example: Finding a Subgradient for SVC

- Consider $h_w \in \partial ||\mathbf{w}||_2^2$.
- Since $||\mathbf{w}||_2^2$ is a differentiable function with gradient $2\mathbf{w}$, we have that $h_w = 2[\mathbf{w}; 0]$.

Example: Finding a Subgradient for SVC

- Consider $\mathbf{h}_n \in \partial L(y_n g_\theta(\mathbf{x}_n))$. Recall $L(z) = \max(0, 1 - z)$ and $g_\theta(\mathbf{x}_n) = \mathbf{x}_n\theta$.

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- By the chain rule, we have that $\mathbf{h}_n \in \partial L(y_n \mathbf{x}_n \theta)$ if and only if:

$$\mathbf{h}_n \in \{k y_n \mathbf{x}_n^T \mid k \in \partial L(z), z = y_n \mathbf{x}_n \theta\}$$

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- We next need to figure out what $\partial L(z)$ is.

Example: Finding a Subgradient for SVC

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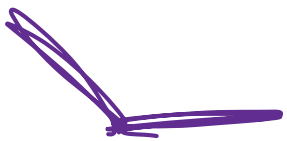
- We choose the following sub-gradient function for the hinge loss:

$$L'(z) = \begin{cases} 0 & \dots z \geq 1 \\ -1 & \dots z < 1 \end{cases}$$



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$$\mathbf{h}_n = L'(y_n \mathbf{x}_n \theta) y_n \mathbf{x}_n^T \in \partial L(y_n \mathbf{x}_n \theta)$$

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- This gives us a final answer for a subgradient of the risk:

$$\mathbf{h} = C \sum_{n=1^N} L'(y_n \mathbf{x}_n \theta) y_n \mathbf{x}_n^T + 2[\mathbf{w}; 0]$$