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Lecture Notes for the Course in Water Wave Mechanics

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WATER WAVE MECHANICS

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LECTURE NOTES FOR THE COURSE IN WATER WAVE MECHANICS

by

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Preface

The present notes are written for the course in water wave mechanics given on the 7th semester of the education in civil engineering at Aalborg University.

The prerequisites for the course are the course in fluid dynamics also given on the 7th semester and some basic mathematical and physical knowledge. The course is at the same time an introduction to the course in coastal hydraulics on the 8th semester. The notes cover the first four lectures of the course:

- Definitions. Governing equations and boundary conditions.

- Derivation of velocity potential for linear waves. Dispersion relationship. Particle velocities and accelerations.

- Particle paths, pressure variation, deep and shallow water waves, wave energy and group velocity.

- Shoaling, refraction, diffraction and wave breaking.

The last part of the course is on analysis of irregular waves and was included in the first two editions of the present note but is now covered by the note of Frigaard et al. (2012).
The present notes are based on the following existing notes and books:

- H.F.Burcharth: Bølgehydraulik, AaU (1991)

- H.F.Burcharth og Torben Larsen: Noter i bølgehydraulik, AaU (1988).

- Peter Frigaard and Tue Hald: Noter til kurset i bølgehydraulik, AaU (2004)

- Ib A.Svendsen and Ivar G.Jonsson: Hydrodynamics of Coastal Regions, Den private ingeniørfond, DtU.(1989).

- Leo H. Holthuijsen: Waves in ocean and coastal waters, Cambridge University Press (2007).

Contents

Chapter 1

Phenomena, Definitions and Symbols

1.1 Wave Classification

Various types of waves can be observed at the sea that generally can be divided into different groups depending on their frequency and the generation method.

Phenomenon	Origin	Period
Surges	Atmospheric pressure and wind	1 – 30 days
Tides	Gravity forces from the moon and the sun	app. 12 and 24 h
Barometric wave	Air pressure variations	1 – 20 h
Tsunami	Earthquake, submarine land slide or submerged volcano	5 – 60 min.
Seiches (water level fluctuations in bays and harbour basins)	Resonance of long period wave components	1 – 30 min.
Surf beat, mean water level fluctuations at the coast	Wave groups	0.5 – 5 min.
Swells	Waves generated by a storm some distance away	< 40 sec.
Wind generated waves	Wind shear on the water surface	< 25 sec.

The phenomena in the first group are commonly not considered as waves, but as slowly changes of the mean water level. These phenomena are therefore also characterized as water level variations and are described by the mean water level MWL.

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In the following is only considered short-period waves. Short-period waves are wind generated waves with periods less than approximately 40 seconds. This group of waves includes also for danish waters the most important phenomena.

1.2 Description of Waves

Wind generated waves starts to develop at wind speeds of approximately 1 m/s at the surface, where the wind energy is partly transformed into wave energy by surface shear. With increasing wave height the wind-wave energy transformation becomes even more effective due to the larger roughness.

A wind blown sea surface can be characterized as a very irregular surface, where waves apparently continuously arise and disappear. Smaller ripples are superimposed on larger waves and the waves travel with different speed and partly also different direction. A detailed description seems impossible and it is necessary to make some simplifications, which makes it possible to describe the larger changes in characteristics of the wave pattern.

Waves are classified into one of the following two classes depending on their directional spreading:

Long-crested waves: 2-dimensional (plane) waves (e.g. swells at mild sloping coasts). Waves are long crested and travel in the same direction (e.g. perpendicular to the coast)

Short-crested waves: 3-dimensional waves (e.g. wind generated storm waves). Waves travel in different directions and have a relative short crest.

In the rest of these notes only long-crested (2D) waves are considered, which is a good approximation in many cases. However, it is important to be aware that in reality waves are most often short-crested, and only close to the coast the waves are close to be long crested. Moreover, the waves are in the present note described using the linear wave theory, the so-called Stokes 1. order theory. This theory is only valid for low steepness waves in relative deep water.

1.3 Definitions and Symbols

H	wave height
a	wave amplitude
η	water surface elevations from MWL (posituve upwards)
L	wave length
$s = \dfrac{H}{L}$	wave steepness
$c = \dfrac{L}{T}$	phase velocity of wave
T	wave period, time between two crests passage of same vertical section
u	horizontal particle velocity
w	vertical particle velocity
$k = \dfrac{2\pi}{L}$	wave number
$\omega = \dfrac{2\pi}{T}$	cyclic frequency, angular frequency
h	water depth

Wave fronts

Wave orthogonals

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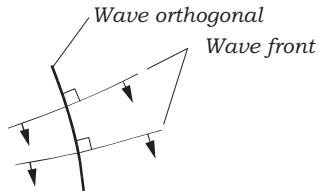
Chapter 2

Governing Equations and Boundary Conditions

In the present chapter the basic equations and boundary conditions for plane and regular surface gravity waves on constant depth are given. An analytical solution of the problem is found to be impossible due to the non-linear boundary conditions at the free surface. The governing equations and the boundary conditions are identical for both linear and higher order Stokes waves, but the present note covers only the linear wave theory, where the boundary conditions are linearized so an analytical solution is possible, cf. chapter 3.

We will start by analysing the influence of the bottom boundary layer on the ambient flow. Afterwards the governing equations and the boundary conditions will be discussed.

2.1   Bottom Boundary Layer

It is well known that viscous effects are important in boundary layers flows. Therefore, it is important to consider the bottom boundary layer for waves the effects on the flow outside the boundary layer. The observed particle motions in waves are given in Fig. 2.1. In a wave motion the velocity close to the bottom is a horizontal oscillation with a period equal to the wave period. The consequence of this oscillatory motion is the boundary layer always will remain very thin as a new boundary layer starts to develop every time the velocity changes direction.

As the boundary layer is very thin dp/dx is almost constant over the boundary layer. As the velocity in the boundary layer is smaller than in the ambient flow the particles have little inertia reacts faster on the pressure gradient. That is the reason for the velocity change direction earlier in the boundary layer than in the ambient flow. A consequence of that is the boundary layer seems to

be moving away from the wall and into the ambient flow (separation of the boundary layer). At the same time a new boundary starts to develop.
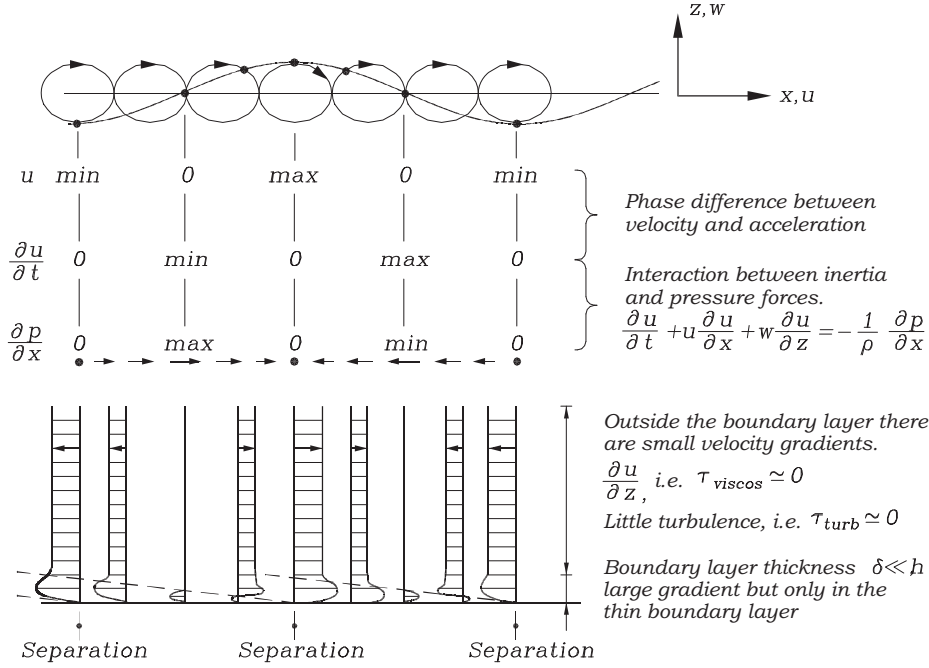
Figure 2.1: Observed particle motions in waves.

In the boundary layer is generated vortices that partly are transported into the ambient flow. However, due to the oscillatory flow a large part of the vortices will be destroyed during the next quarter of the wave cycle. Therefore, only a very small part of the generated vortices are transported into the ambient wave flow and it can be concluded that the boundary layer does almost not affect the ambient flow.

The vorticity which often is denoted $rot\vec{v}$ or $curl\vec{v}$ is in the boundary layer: $rot\vec{v} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \simeq \frac{\partial u}{\partial z}$, as $w \simeq 0$ and hence $\frac{\partial w}{\partial x} \simeq 0$. $\frac{\partial u}{\partial z}$ is large in the boundary layer but changes sign twice for every wave period. Therefore, inside the boundary layer the flow has vorticity and the viscous effects are important. Outside the boundary layer the flow is assumed irrotational as:

> The viscous forces are neglectable and the external forces are essentially conservative as the gravitation force is dominating. Therefore, we neglect surface tension, wind-induced pressure and shear stresses and the Coriolis force. This means that if we consider waves longer than a few centimeters and shorter than a few kilometers we can assume that the external forces are conservative. As

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a consequence of that and the assumption of an inviscid fluid, the vorticity is constant cf. Kelvin's theorem. As $rot\vec{v} = 0$ initially, this will remain the case.

The conclusion is that the ambient flow (the waves) with good accuracy could be described as a *potential flow.*

The velocity potential is a function of x, z and t, $\varphi = \varphi(x, z, t)$. Note that both $\varphi(x, z, t)$ and $\varphi(x, z, t) + f(t)$ will represent the same velocity field (u, w), as $\left(\frac{\partial \varphi}{\partial x} , \frac{\partial \varphi}{\partial z} \right)$ is identical. However, the reference for the pressure is different.

With the introduction of φ the number of variables is reduced from three (u, w, p) to two (φ, p).

2.2   Governing Hydrodynamic Equations

From the theory of fluid dynamics the following basic balance equations are taken:

Continuity equation for plane flow and incompresible fluid with constant density (mass balance equation)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad div \, \vec{v} = 0 \tag{2.1}$$

The assumption of constant density is valid in most situations. However, vertical variations may be important in some special cases with large vertical differences in temperature or salinity. Using the continuity equation in the present form clearly reduces the validity to non-breaking waves as wave breaking introduces a lot of air bubbles in the water and in that case the body is not continuous.

Laplace-equation (plane irrotational flow)
In case of irrotational flow Eq. 2.1 can be expressed in terms of the velocity potential φ and becomes the Laplace equation as $v_i = \frac{\partial \varphi}{\partial x_i}$.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{2.2}$$

Equations of motions (momentum balance)
Newton's 2. law for a *particle* with mass m with external forces $\sum \overline{K}$ acting on the particle is, $m \frac{d\vec{v}}{dt} = \sum \overline{K}$. The general form of this is the Navier-Stoke

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equations which for an ideal fluid (inviscid fluid) can be reduced to the Euler equations as the viscous forces can be neglected.

$$\rho \frac{d\vec{v}}{dt} = -grad\, p + \rho\vec{g} \quad (+ \text{ viscous forces}) \tag{2.3}$$

Bernoulli's generalized equation (plane irrotational flow)
In case of irrotational flow the Euler equations can be rewritten to get the generalized Bernoulli equation which is an integrated form of the equations of motions.

$$g\, z + \frac{p}{\rho} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{\partial\varphi}{\partial t} = C(t)$$

$$g\, z + \frac{p}{\rho} + \frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right) + \frac{\partial\varphi}{\partial t} = C(t) \tag{2.4}$$

Note that the velocity field is independent of $C(t)$ but the reference for the pressure will depend on $C(t)$.

Summary on system of equations:
Eq. 2.2 and 2.4 is two equations with two unknowns (φ, p). Eq. 2.2 can be solved separately if only $\varphi = \varphi(x, z, t)$ and not $p(x, z, t)$ appear explicitly in the boundary conditions. This is usually the case, and we are left with $\varphi(x, z, t)$ as the only unknown in the governing Laplace equation. Hereafter, the pressure $p(x, z, t)$ can be found from Eq. 2.4. Therefore, the pressure p can for potential flows be regarded as a reaction on the already determined velocity field. A reaction which in every point obviously must fulfill the equations of motion (Newton's 2. law).

2.3 Boundary Conditions

Based on the previous sections we assume incompressible fluid and irrotational flow. As the Laplace equation is the governing differential equation for all potential flows, the character of the flow is determined by the boundary conditions. The boundary conditions are of kinematic and dynamic nature. The kinematic boundary conditions relate to the motions of the water particles while the dynamic conditions relate to forces acting on the particles. Free surface flows require one boundary condition at the bottom, two at the free surface and boundary conditions for the lateral boundaries of the domain.

I case of waves the lateral boundary condition is controlled by the assumption that the waves are periodic and long-crested. The boundary conditions at

the free surface specify that a particle at the surface remains at the surface (kinematic) and that the pressure is constant at the surface (dynamic) as wind induced pressure variations are not taken into account. In the following the mathematical formulation of these boundary conditions is discussed. The boundary condition at the bottom is that there is no flow flow through the bottom (vertical velocity component is zero). As the fluid is assumed ideal (no friction) there is not included a boundary condition for the horizontal velocity at the bottom.

2.3.1   Kinematic Boundary Condition at Bottom

Vertical velocity component is zero as there should not be a flow through the bottom:

$$w = 0 \ \text{ or } \ \frac{\partial \varphi}{\partial z} = 0$$
$$\text{for } z = -h \qquad (2.5)$$
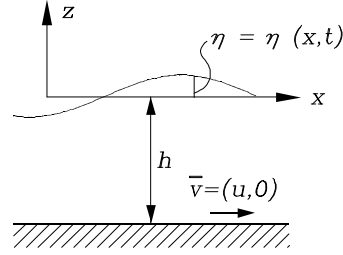
2.3.2   Boundary Conditions at the Free Surface

One of the two surface conditions specify that a particle at the surface remains at the surface (kinematic boundary condition). This *kinematic boundary condition* relates the vertical velocity of a particle at the surface to the vertical velocity of the surface, which can be expressed as:

$$
\begin{aligned}
w &= \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{dx}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} u \ , \quad \text{or} \\
\frac{\partial \varphi}{\partial z} &= \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \varphi}{\partial x} \qquad \text{for } \ z = \eta
\end{aligned}
\qquad (2.6)
$$

The following figure shows a geometrical illustration of this problem.
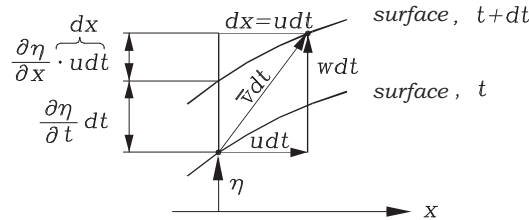
The second surface condition specifies the pressure at free surface (dynamic boundary condition). This *dynamic condition* is that the pressure along the surface must be equal to the atmospheric pressure as we disregard the influence

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of the wind. We assume the atmospheric pressure p_0 is constant which seems valid as the variations in the pressure are of much larger scale than the wave length, i.e. the pressure is only a function of time $p_0 = p_0(t)$. If this is inserted into Eq. 2.4, where the right hand side exactly express a constant pressure divided by mass density, we get:

$$g\,z + \frac{p}{\rho} + \frac{1}{2}\left(u^2 + w^2\right) + \frac{\partial \varphi}{\partial t} = \frac{p_0}{\rho}$$

At the surface $z = \eta$ we have $p = p_0$ and above can be rewritten as the boundary condition:

$$g\eta + \frac{1}{2}\left(\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right) + \frac{\partial \varphi}{\partial t} = 0 \qquad \text{for } z = \eta \qquad (2.7)$$

The same result can be found from Eq. 2.4 by setting p equal to the excess pressure relative to the atmospheric pressure.

2.3.3 Boundary Condition Reflecting Constant Wave Form (Periodicity Condition)

The *periodicity condition* reflects that the wave is a *periodic, progressive wave of constant form*. This means that the wave propagate with constant form in the positive x-direction. The consequence of that is the flow field must be identical in two sections separated by an integral number of wave lengths. This sets restrictions to the variation of η and φ (i.e. surface elevation and velocity field) with t and x (i.e. time and space).

The requirement of constant form can be expressed as:

$$\eta(x, t) = \eta(x + nL, t) = \eta(x, t + nT) \ , \quad \text{where } n = 1, 2, 3, \ldots$$
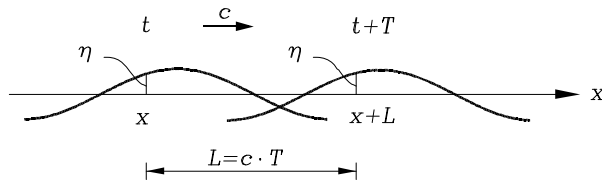
This criteria is fulfilled if (x, t) is combined in the variable $\left(L\frac{t}{T} - x\right)$, as $\eta\left(L\frac{t}{T} - x\right) = \eta\left(L\frac{(t+nT)}{T} - (x+nL)\right) = \eta\left(L\frac{t}{T} - x\right)$. This variable can be expressed in dimensionless form by dividing by the wave length L. $\frac{2\pi}{L}\left(L\frac{t}{T} - x\right) = 2\pi\left(\frac{t}{T} - \frac{x}{L}\right)$, where the factor 2π is added due to the following calculations.

We have thus included the periodicity condition for η and φ by introducing the variable θ.

$$\eta = \eta(\theta) \ \text{ and } \ \varphi = \varphi(\theta, z) \ \text{ where } \theta = 2\pi\left(\frac{t}{T} - \frac{x}{L}\right) \qquad (2.8)$$

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If we introduce the wave number $k = \frac{2\pi}{L}$ and the cyclic frequency $\omega = \frac{2\pi}{T}$ we get:

$$\theta = \omega t - kx \tag{2.9}$$

It is now verified that Eqs. 2.8 and 2.9 corresponds to a wave propagating in the positive x-direction, i.e. for a given value of η should x increase with time t. Eq. 2.9 can be rewritten to:

$$x = \frac{1}{k}(\omega t - \theta)$$

From which it can be concluded that x increases with t for a given value of θ. If we change the sign of the kx term form minus to plus the wave propagation direction changes to be in the negative x-direction.

2.4   Summary of Mathematical Problem

The governing Laplace equation and the boundary conditions (BCs) can be summarized as:

$$\text{Laplace equation} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{2.10}$$

$$\text{Kin. bottom BC} \quad \frac{\partial \varphi}{\partial z} = 0 \ \text{ for } \ z = -h \tag{2.11}$$

$$\text{Kin. surface BC} \quad \frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \varphi}{\partial x} \ \text{ for } \ z = \eta \tag{2.12}$$

$$\text{Dyn. surface BC} \quad g\eta + \frac{1}{2}\left(\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right) + \frac{\partial \varphi}{\partial t} = 0$$

$$\text{for } z = \eta \tag{2.13}$$

$$\text{Periodicity BC} \quad \begin{aligned} &\eta(x,t) \text{ and } \varphi(x,z,t) \Rightarrow \\ &\eta(\theta) \ , \ \varphi(\theta,z) \\ &\text{where } \theta = \omega t - kx \end{aligned}$$

An analytical solution to the problem is impossible. This is due to the two mathematical difficulties:

- Both boundary conditions at the free surface are non-linear.

- The shape and position of the free surface η is one of the unknowns of the problem that we try to solve which is not included in the governing Laplace equation, Eq. 2.10. Therefore, a governing equation with η is missing.

A matematical simplification of the problem is needed.

Chapter 3

Linear Wave Theory

The linear wave theory which is also known as the Airy wave theory (Airy, 1845) or Stokes 1. order theory (Stokes, 1847), is described in the present chapter and the assumptions made are discussed. Based on this theory analytical expressions for the particle velocities, particle paths, particle accelerations and pressure are established.

The linear theory is strictly speaking only valid for non-breaking waves with small amplitude, i.e. when the amplitude is small compared to the wave length and the water depth (H/L and H/h are small). However, the theory is fundamental for understanding higher order theories and for the analysis of irregular waves. Moreover, the linear theory is the simplest possible case and turns out also to be the least complicated theory.

By assuming $H/L << 1$, i.e. small wave steepness, it turns out that the boundary conditions can be linearized and η can be eliminated from the equations. This corresponds to the surface conditions can be taken at $z = 0$ instead of $z = \eta$ and the differential equation can be solved analytically. The linearisation of the boundary conditions is described in the following section.

3.1 Linearisation of Boundary Conditions

The two surface boundary conditions (Eqs. 2.12 and 2.13) are the two non-linear conditions that made an analytical solution to the problem impossible. These are linearised in the following by investigating the importance of the various terms.

3.1.1 Linearisation of Kinematic Surface Condition

The non-linearised kinematic surface condition is, cf. Eq. 2.12:

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \frac{\partial \varphi}{\partial x} \quad for \ z = \eta \tag{3.1}$$

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The magnitude of the different terms is investigated in the following, where σ indicate the order of magnitude. If we consider a deep water wave ($H/h << 1$) observations has shown that the particle paths are circular and as the particles on the surface must remain on the surface the diameter in the circular motion must close to the surface be equal to the wave height H. As the duration of each orbit is equal to the wave period T, the speed of the particles close to the surface can be approximated by $\pi H/T$.

Deep water wave

$$max\frac{\partial\varphi}{\partial x} = u_{max} = \frac{\pi H}{T} = \sigma\left(\frac{H}{T}\right)$$
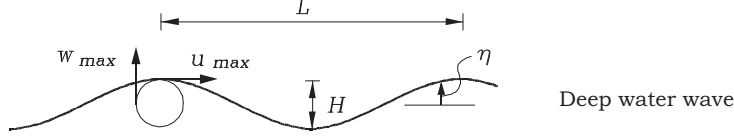
$$max\frac{\partial\varphi}{\partial z} = w_{max} = \frac{\pi H}{T} = \sigma\left(\frac{H}{T}\right)$$

$$\frac{\partial\eta}{\partial x} = \sigma\left(\frac{H}{L}\right) \quad, \text{ as } \eta \text{ varies } H \text{ over the length } L/2$$

$$\frac{\partial\eta}{\partial t} = \sigma\left(\frac{H}{T}\right) \quad, \text{ as } \eta \text{ varies } H \text{ over the time } T/2$$

Therefore, we get from Eq. 3.1:

$$\sigma\left(\frac{H}{T}\right) = \sigma\left(\frac{H}{T}\right) + \sigma\left(\frac{H}{L}\right)\sigma\left(\frac{H}{T}\right)$$

from which it can be seen that the order of magnitude of the last non-linear term is H/L smaller than the order of the linear term. As we assumed $\frac{H}{L} << 1$ we only make a small error by neglecting the non-linear term. However, the argumentation can be risky as we have not said anything about the simoultanousness of the maximum values of each term.

The linaerised kinematic surface boundary condition is thus:

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} \quad, \quad \text{for} \quad z = \eta \tag{3.5}$$

However, we still have the problem that the boundary condition is expressed at $z = \eta$ as the position of the surface is unknown. An additional simplificaion is needed. $\frac{\partial\varphi}{\partial z}$, which is the only term in Eq. 3.5 that depends on z, is expanded in a Taylor series to evaluate the posibilities to discard higher order terms. The general form of the Taylor series is:

$$f(z + \Delta z) = f(z) + \frac{\Delta z}{1!}f'(z) + \frac{(\Delta z)^2}{2!}f''(z) + \ldots + \frac{(\Delta z)^n}{n!}f^{(n)}(z) + R_n(z)$$

where Δz represent a deviation from the variable z. With the Taylor expansion we can get any preassigned accuracy in the approximation of $f(z + \Delta z)$ by

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choosing n large enough. We now make a Taylor series expansion of $\frac{\partial \varphi}{\partial z}$ from $z = 0$ to calculate the values at $z = \eta$, i.e. we set $\Delta z = \eta$ and get:

$$
\begin{aligned}
\frac{\partial \varphi}{\partial z}(x, \eta, t) &= \frac{\partial \varphi}{\partial z}(x, 0, t) + \frac{\eta}{1!} \frac{\partial^2 \varphi(x, 0, t)}{\partial z^2} + \dots \\
&= \frac{\partial \varphi}{\partial z}(x, 0, t) + \frac{\eta}{1!} \left(\frac{-\partial^2 \varphi(x, 0, t)}{\partial x^2} \right) + \dots
\end{aligned}
\tag{3.6}
$$

where $\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial z^2} = 0$ has been used.

As $\eta = \sigma(H)$ and $\dfrac{\partial^2 \varphi}{\partial x^2} = \sigma \left(\dfrac{1}{L} \dfrac{\partial \varphi}{\partial x} \right) = \sigma \left(\dfrac{1}{L} \dfrac{\partial \varphi}{\partial z} \right)$, as $u = \sigma(w)$, we get from Eq. 3.6:

$$
\frac{\partial \varphi}{\partial z}(x, \eta, t) = \frac{\partial \varphi}{\partial z}(x, 0, t) + \overbrace{\sigma(H) \sigma \left(\frac{-1}{L} \frac{\partial \varphi}{\partial z} \right)}^{\sigma\left(\frac{H}{L} \frac{\partial \varphi}{\partial z} \right)} \simeq \frac{\partial \varphi}{\partial z}(x, 0, t) \;, \text{ as } \frac{H}{L} << 1.
$$

The use of $z = 0$ instead of $z = \eta$ in Eqs. 3.1 and 3.5 corresponds thus to neglecting the small second order term with the same magnitude as the non-linear term in the boundary condition removed above. The *linearised kinematic surface boundary condition* is therefore simplified to:

$$
\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{for } z = 0
\tag{3.7}
$$

The error committed by evaluating φ at MWL ($z = 0$) instead of at the surface ($z = \eta$) is thus small and of second order.

3.1.2 Linearisation of Dynamic Surface Condition

The non-linaerised dynamic surface boundary condition reads, cf. Eq. 2.13:

$$
g\eta + \frac{1}{2} \left(\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right) + \frac{\partial \varphi}{\partial t} = 0 \qquad \text{for } z = \eta
\tag{3.8}
$$

The linearisation of the dynamic surface boundary condition follows the same approach as for the kinematic condition. We start by examining the magnitude of the different terms. For the assessment of the magnitude of the term $\frac{\partial \varphi}{\partial t}$, is used $\frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial x} \right) = \frac{\partial u}{\partial t}$ i.e. $\frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t} \right) = \sigma \left(\frac{1}{L} \frac{\partial \varphi}{\partial t} \right) = \frac{\partial u}{\partial t} = \sigma \left(\frac{H/T}{T} \right)$, as $u = \sigma \left(\frac{H}{T} \right)$.

Therefore, we get:

$$
\frac{\partial \varphi}{\partial t} = \sigma \left(L \frac{H}{T^2} \right)
\tag{3.9}
$$

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Moreover we have for the quadratic terms:

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 \simeq \left(\frac{\partial\varphi}{\partial z}\right)^2 = \sigma\left(\frac{H}{T}\right)^2 = \sigma\left(L\frac{H}{T^2}\right)\sigma\left(\frac{H}{L}\right) = \sigma\left(\frac{\partial\varphi}{\partial t}\cdot\frac{H}{L}\right)$$

From this we can conclude that the quadratic terms are small and of higher order and as a consequence they are neglected. Therefore, we can in case of small amplitude waves write the boundary condition as:

$$g\eta + \frac{\partial\varphi}{\partial t} = 0 \qquad \text{for} \quad z = \eta \tag{3.10}$$

However, the problem with the unknown position of the free surface (η) still exists. We use a Taylor expansion of $\frac{\partial\varphi}{\partial t}$ around $z = 0$, which is the only term in Eq. 3.10 that depends on z.

$$\frac{\partial\varphi}{\partial t}(x,\eta,t) = \frac{\partial\varphi}{\partial t}(x,0,t) + \frac{\eta}{1!}\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial t}(x,0,t)\right) + \dots \tag{3.11}$$

$$\frac{\partial\varphi}{\partial t} = \sigma\left(L\frac{H}{T^2}\right) \text{ cf. eq. 3.9,}$$

$$\eta\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial t}\right) = \eta\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial z}\right) = \sigma(H)\sigma\left(\frac{1}{T}\frac{\partial\varphi}{\partial z}\right) =$$

$$\sigma(H)\sigma\left(\frac{1}{T}\right)\sigma\left(\frac{H}{T}\right) = \sigma\left(\frac{H^2}{T^2}\right) = \sigma\left(\frac{H}{L}\right)\sigma\left(L\frac{H}{T^2}\right) \text{ which is}$$

$$\sigma\left(\frac{H}{L}\frac{\partial\varphi}{\partial t}\right) \text{ i.e. } << \frac{\partial\varphi}{\partial t}.$$

The second term in Eq. 3.11 is thus small and of higher order and can be neglected when $H/L << 1$. This corresponds to using $z = 0$ instead of $z = \eta$ in Eq. 3.10. As a consequence the *linearised dynamic surface boundary condition* is simplified to:

$$g\eta + \frac{\partial\varphi}{\partial t} = 0 \qquad \text{for} \quad z = 0 \tag{3.12}$$

3.1.3   Combination of Surface Boundary Conditions

The linearised surface boundary conditions Eqs. 3.7 and 3.12 are now combined in a single surface boundary condition. If we differentiate Eq. 3.12 with respect to t we get:

$$g\frac{\partial\eta}{\partial t} + \frac{\partial^2\varphi}{\partial t^2} = 0 \qquad\qquad \text{for} \quad z = 0 \tag{3.13}$$

which can be rewritten as:

$$\frac{\partial \eta}{\partial t} = -\frac{1}{g}\frac{\partial^2 \varphi}{\partial t^2} \qquad\qquad \text{for} \quad z = 0 \qquad\qquad (3.14)$$

This result is now inserted into Eq. 3.7 and we get the combined surface boundary condition:

$$\frac{\partial \varphi}{\partial z} + \frac{1}{g}\frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad\qquad \text{for} \quad z = 0 \qquad\qquad (3.15)$$

Now η has been eliminated from the boundary conditions and the mathematical problem is reduced enormously.

3.1.4 Summary of Linearised Problem

The mathematical problem can now be summarized as:

3.2 Inclusion of Periodicity Condition

The periodicity condition can as mentioned in section 2.3.3 by inclusion of θ given by Eq. 2.9 instead of the two variables (x, t). Therefore, the Laplace equation and the boundary conditions are rewritten to include $\varphi(\theta, z)$ instead of $\varphi(x, z, t)$. The coordinates are thus changed from (x, t) to (θ) by using the chain rule for differentiation and the definition $\theta = \omega t - kx$ (eq. 2.9).

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial \varphi}{\partial \theta}(-k) \qquad\qquad (3.16)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial\left(\frac{\partial \varphi}{\partial x}\right)}{\partial x} = \frac{\partial\left(\frac{\partial \varphi}{\partial x}\right)}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial\left(\frac{\partial \varphi}{\partial \theta}(-k)\right)}{\partial \theta}(-k) = k^2\frac{\partial^2 \varphi}{\partial \theta^2} \qquad (3.17)$$

A similar approach for the time derivatives give:

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \frac{\partial \varphi}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial \varphi}{\partial \theta}\omega \\
\frac{\partial^2 \varphi}{\partial t^2} &= \omega^2\frac{\partial^2 \varphi}{\partial \theta^2}
\end{aligned} \qquad\qquad (3.18)$$

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Eq. 3.17 is now inserted into the Laplace equation (Eq. 2.2) and we get:

$$k^2 \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{3.19}$$

Eq. 3.18 is inserted into Eq. 3.15 to get the free surface condition with θ included:

$$\frac{\partial \varphi}{\partial z} + \frac{\omega^2}{g} \frac{\partial^2 \varphi}{\partial \theta^2} = 0 \qquad \text{for} \quad z = 0 \tag{3.20}$$

The boundary condition at the bottom is unchanged ($\frac{\partial \varphi}{\partial z} = 0$).

The periodicity condition $\frac{\partial \varphi}{\partial x}(0, z, t) = \frac{\partial \varphi}{\partial x}(L, z, t)$ is changed by considering the values of θ for $x = 0$ and $x = L$:

For $x = 0$ and $t = t$ we get, $\theta = 2\pi \frac{t}{T}$.

For $x = L$ and $t = t$ — $\theta = 2\pi \frac{t}{T} - 2\pi$.

It can be shown that it is sufficient to to impose the periodicity condition on the horizontal velocity ($u = \frac{\partial \varphi}{\partial x}$), which yields by inclusion of Eq. 3.16:

$$-k \frac{\partial \varphi}{\partial \theta}\left(2\pi \frac{t}{T}, \ z\right) = -k \frac{\partial \varphi}{\partial \theta}\left(2\pi \frac{t}{T} - 2\pi, \ z\right),$$

which should be valid for all values of t and thus also for $t = 0$. As the periodicity condition could just as well been expressed for $x = -L$ instead of $x = L$ it can be concluded that the sign of 2π can be changed and we get:

$$-k \frac{\partial \varphi}{\partial \theta}(0, z) = -k \frac{\partial \varphi}{\partial \theta}(2\pi, z) \tag{3.21}$$

which is the reformulated periodicity condition.

3.3   Summary of Mathematical Problem

The mathematical problem from section 2.4 has now been enormously simplified by linearisation of the boundary conditions and inclusion of θ instead of x, t. The mathematical problem can now be solved analytically and summarized as:

$$\text{Laplace equation:} \qquad k^2 \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{3.22}$$

$$\text{Bottom BC:} \qquad \frac{\partial \varphi}{\partial z} = 0 \ \text{ for } z = -h \tag{3.23}$$

$$\text{Linearised Surface BC:} \qquad \frac{\partial \varphi}{\partial z} + \frac{\omega^2}{g} \frac{\partial^2 \varphi}{\partial \theta^2} = 0 \ \text{ for } z = 0 \tag{3.24}$$

$$\text{Periodicity BC:} \qquad -k \frac{\partial \varphi}{\partial \theta}(0, z) = -k \frac{\partial \varphi}{\partial \theta}(2\pi, z) \tag{3.25}$$

3.4 Solution of Mathematical Problem

The linear wave theory is based on an exact solution to the Laplace equation but with the use of linear approximations of the boundary conditions. The solution to the problem is straight forward and can be found by the method of separation of variables. Hence we introduce:

$$\varphi(\theta, z) = f(\theta) \cdot Z(z) \tag{3.26}$$

which inserted in Eq. 3.22 leads to:

$$k^2 f'' Z + Z'' f = 0$$

We then divide by $\varphi = fZ$ on both sides to get:

$$-k^2 \frac{f''}{f} = \frac{Z''}{Z} \tag{3.27}$$

As the left hand side now only depends on θ and the right hand only depends on z they must be equal to the same constant which we call λ^2 as the constant is assumed positive. Therefore, we get the following two differential equations:

$$f'' + \frac{\lambda^2}{k^2} f = 0 \tag{3.28}$$

$$Z'' - \lambda^2 Z = 0 \tag{3.29}$$

Eq. 3.28 has the solution:

$$f = A_1 cos\left(\frac{\lambda}{k}\theta\right) + A_2 sin\left(\frac{\lambda}{k}\theta\right) = A\,sin\left(\frac{\lambda}{k}\theta + \delta\right) \tag{3.30}$$

where A, λ and δ are constants to be determined from the boundary conditions. However, we can set δ equal to zero corresponding to an appropriate choice of the origin of $\theta = (x, t)$. Therefore, we can write:

$$f = A\,sin\left(\frac{\lambda}{k}\theta\right) \tag{3.31}$$

If we insert the definition in Eq. 3.26 into the periodicity condition (Eq. 3.25) we get the following condition:

$$f'(0) = f'(2\pi)$$

From Eq. 3.31 we get $f' = A\frac{\lambda}{k} cos\left(\frac{\lambda}{k}\theta\right)$ and hence the above condition gives:

$$A\frac{\lambda}{k} cos\left(\frac{\lambda}{k}0\right) = A\frac{\lambda}{k} = A\frac{\lambda}{k} cos\left(\frac{\lambda}{k}2\pi\right) , \quad \text{i.e.}$$

$$\frac{\lambda}{k} = n \ , \quad \text{where} \ \ n = 1, 2, 3 \ldots \qquad (n \neq 0, \ \text{as} \ \lambda \neq 0)$$

This condition is now inserted into Eq. 3.31 and the solution becomes:

$$f = A \, sin(n\theta) = A \, sin\left(n\left(\omega t - \frac{2\pi}{L}x\right)\right)$$

As $x = L$ must correspond to one wave length we get $n = \frac{\lambda}{k} = 1$ as the only solution and $n = 2, 3, 4, \ldots$ must be disregarded. The result can also be written as $\lambda = k$ which is used later for the solution of the second differential equation. The result can also be obtained from $\theta = 2\pi$ by definition corresponds to one wave length. Therefore, we get the following solution to the f-function:

$$f = A \, sin\theta \qquad (3.32)$$

The second differential equation, Eq. 3.29, has the solution:

$$Z = B_1 \, e^{\lambda z} + C_1 \, e^{-\lambda z} \qquad (3.33)$$

As $sinh \, x = \frac{e^x - e^{-x}}{2}$ and $cosh \, x = \frac{e^x + e^{-x}}{2}$ and we choose $B_1 = \frac{B+C}{2}$ and $C_1 = \frac{B-C}{2}$ and at the same time introduce $\lambda = k$ as found above, we get:

$$Z = B \, cosh \, kz + C \, sinh \, kz \qquad (3.34)$$

The three integration constants A, B and C left in Eqs. 3.32 and 3.34 are determined from the bottom and surface boundary conditions. We start by inserting Eq. 3.26 into the bottom condition (Eq. 3.23), $\frac{\partial \varphi}{\partial z} = 0$ for $z = -h$, and get:

$$Z' = 0 \quad \text{for} \ \ z = -h$$

We now differentiate Eq. 3.34 with respect to z and insert the above given condition:

$$B \, k \, sinh(-kh) + C \, k \, cosh(-kh) = 0 \quad \text{or} \quad B = C \, coth \, kh$$

as $sinh(-x) = -sinh(x)$, $cosh(-x) = cosh(x)$ and $coth(x) = \frac{cosh(x)}{sinh(x)}$.

This result is now inserted into Eq. 3.34 to get:

$$\begin{aligned} Z \ &= \ C \, (coth \, kh \, cosh \, kz + sinh \, kz) \\ &= \ \frac{C}{sinh \, kh} \, (cosh \, kh \, cosh \, kz + sinh \, kh \, sinh \, kz) \\ &= \ C \, \frac{cosh \, k(z+h)}{sinh \, kh} \end{aligned} \qquad (3.35)$$

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We now combine the solutions to the two differential equations by inserting Eqs. 3.32 and 3.35 into Eq. 3.26:

$$\varphi = f \cdot Z = AC \frac{\cosh k(z+h)}{\sinh kh} \sin\theta \qquad (3.36)$$

The product of the constants A and C is now determined from the linearised dynamic surface boundary condition (Eq. 3.12), $\eta = -\frac{1}{g} \frac{\partial\varphi}{\partial t}$ for $z = 0$, which express the surface form. We differentiate Eq. 3.36 with respect to t and insert the result into the dynamic surface condition to get:

$$\eta = -\frac{\omega}{g} AC \frac{\cosh kh}{\sinh kh} \cos\theta \quad , \qquad (3.37)$$

where $-\frac{\omega}{g} AC \frac{\cosh kh}{\sinh kh}$ must represent the wave amplitude $a \equiv \frac{H}{2}$. Therefore, the wave form must be given by:

$$\eta = a\cos\theta = \frac{H}{2} \cos(\omega t - kx) \qquad (3.38)$$

The velocity potential is found by inserting the expression for AC and θ into Eq. 3.36:

$$\varphi = -\frac{a\,g}{\omega} \frac{\cosh k\,(z+h)}{\cosh kh} \sin(\omega t - kx) \qquad (3.39)$$

3.5   Dispersion Relationship

If we take a look on the velocity potential, Eq. 3.39, then we observe that the wave motion is specified by the four parameters a, ω, h and k or alternatively we can use the parameters H, T, h and L. However, these four parameters are dependent on each other and it turns out we only need to specify three parameters to uniquely specify the wave. This is because a connection between the wave length and the wave period exists, i.e. the longer the wave period the longer the wave length for a given water depth. This relationship is called the *dipersion relationship* which is derived in the following.

The dispersion relationship is determined by inserting Eq. 3.36 into the linearised free surface boundary condition (Eq. 3.24), $\frac{\partial\varphi}{\partial z} + \frac{\omega^2}{g} \frac{\partial^2\varphi}{\partial\theta^2} = 0$ for $z = 0$.

$$\text{As} \quad \frac{\partial\varphi}{\partial z} = AC k \frac{\sinh \,, k\,(z+h)}{\sinh kh} \sin\theta$$

$$\text{and} \quad \frac{\partial^2\varphi}{\partial\theta^2} = AC \frac{\cosh k\,(z+h)}{\sinh kh} (-\sin\theta)$$

We find by substitution into Eq. 3.24 and division by AC:

$$\omega^2 = g\,k\,\tanh kh \qquad (3.40)$$

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which could be rewritten by inserting $\omega = \frac{2\pi}{T}$, $k = \frac{2\pi}{L}$ and $L = c \cdot T$ to get:

$$c = \sqrt{\frac{g\,L}{2\pi} \, tanh \, \frac{2\pi h}{L}} \tag{3.41}$$

This equation shows that waves with different wave length in general have different propagation velocities, i.e. the waves are dispersive. Therefore, this equation is often refered to as the *dispersion relationship*, no matter if the formulation in Eq. 3.40 or Eq. 3.41 is used. We can conclude that if h and H are given, which is the typical case, it is enough to specify only one of the parameters c, L and T. The simplest case is if h, H and L are specified (geometry specified), as we directly from Eq. 3.41 can calculate c and afterwards $T = \frac{L}{c}$. However, it is much easier to measure the wave period T than the wave length L, so the typical case is that h, H and T are given. However, this makes the problem somewhat more complicated as L cannot explicitly be determined for a given set of h, H and T. This can be see by rewriting the dispersion relation (Eq. 3.41) to the alternative formulation:

$$L = \frac{g\,T^2}{2\pi} \, tanh \, \frac{2\pi h}{L} \tag{3.42}$$

From this we see that L has to be found by iteration. In the literature it is possible to find many approximative formulae for the wave length, e.g. the formula by Hunt, 1979 or Guo, 2002. However, the iteration procedure is simple and straight forward but the approximations can be implemented as the first guess in the numerical iteration. The Guo, 2002 formula is based on logarithmic matching and reads:

$$L = \frac{2\pi h}{x^2(1 - exp(-x^\beta))^{-1/\beta}} \tag{3.43}$$

where $x = h\omega/\sqrt{gh}$ and $\beta = 2.4908$.

The velocity potential can be rewritten in several waves by including the dispersion relation. One version is found by including Eq. 3.40 in Eq. 3.39 to get:

$$\varphi = -a\,c \, \frac{cosh\,k(z+h)}{sinh\,kh} \, sin(\omega t - kx) \tag{3.44}$$

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3.6 Particle Velocities and Accelerations

The *velocity field* can be found directly by differentiation of the velocity potential given in Eq. 3.39 or an alternatively form where the dispersion relation has been included (e.g. Eq. 3.44).

$$
\begin{aligned}
u = \frac{\partial \varphi}{\partial x} &= \frac{a\,g\,k}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(\omega t - kx) \\[2mm]
&= a\,c\,k\,\frac{\cosh k(z+h)}{\sinh kh}\cos(\omega t - kx) \\[2mm]
&= a\,\omega\,\frac{\cosh k(z+h)}{\sinh kh}\cos(\omega t - kx) \qquad (3.45)\\[2mm]
&= \frac{\pi\,H}{T}\frac{\cosh k(z+h)}{\sinh kh}\cos(\omega t - kx)
\end{aligned}
$$

$$
\begin{aligned}
w = \frac{\partial \varphi}{\partial z} &= -\frac{a\,g\,k}{\omega}\frac{\sinh k(z+h)}{\cosh kh}\sin(\omega t - kx) \\[2mm]
&= -a\,c\,k\,\frac{\sinh k(z+h)}{\sinh kh}\sin(\omega t - kx) \\[2mm]
&= -a\,\omega\,\frac{\sinh k(z+h)}{\sinh kh}\sin(\omega t - kx) \qquad (3.46)\\[2mm]
&= -\frac{\pi\,H}{T}\frac{\sinh k(z+h)}{\sinh kh}\sin(\omega t - kx)
\end{aligned}
$$

The *acceleratation field* for the particles is found by differentiation of Eqs. 3.45 and 3.46 with respect to time. It turns out that for the linear theory the total accelerations can be approximated by the local acceleration as the convective part are of higher order.

$$
\frac{du}{dt} \approx \frac{\partial u}{\partial t} = -a\,g\,k\,\frac{\cosh k(z+h)}{\cosh kh}\sin(\omega t - kx) \qquad (3.47)
$$

$$
\frac{dw}{dt} \approx \frac{\partial w}{\partial t} = -a\,g\,k\,\frac{\sinh k(z+h)}{\cosh kh}\cos(\omega t - kx) \qquad (3.48)
$$
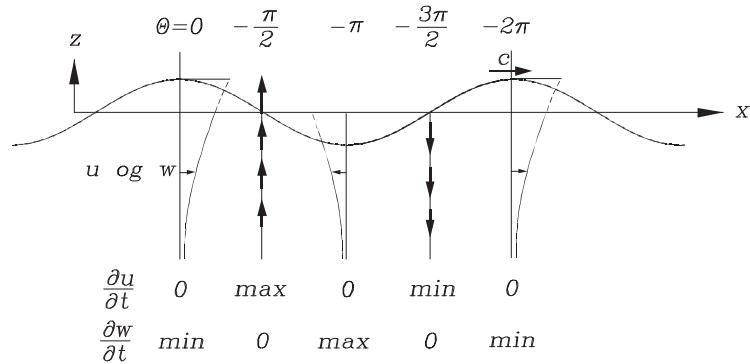
Theoretically the expressions in Eqs. 3.46 to 3.48 is only valid for $\frac{H}{L} << 1$, i.e. in the interval $-h < z \simeq 0$. However, it is quite common practise to use the expressions for finite positive and negative values of η, i.e. also for $z = \eta$. However, this can only give a very crude approximation as the theory breaks down near the surface. Alternatively the so-called Wheeler stretching of the velocity and acceleration profiles can be applied, where the profiles are stretched and compressed so that the evaluation coordinate (z_c) is never positive. The evaluation coordinate is given by $z_c = \frac{h(z-\eta)}{h+\eta}$ where η is the instantaneous water surface elevation. This type of stretching is commonly used for irregular linear waves where the velocity of each component is stretched to the real surface, i.e. the sum of all η components. Alternatively is also commonly used extrapolation of the velocity profile from SWL.

3.7 Pressure Field

The pressure variations are calculated from the Bernoulli equation, Eq. 2.4:

$$gz + \frac{p}{\rho} + \frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right) + \frac{\partial\varphi}{\partial t} = 0 \qquad (3.49)$$

The reference pressure for $z = 0$, i.e. the atmospheric pressure is here set equal to zero. As a consequence the pressure p is the excess pressure relative to the atmospheric pressure. The quadratic terms are small when $H/L << 1$ as shown earlier in the linearisation of the dynamic surface boundary condition. The linearised Bernoulli equation reads:

$$gz + \frac{p}{\rho} + \frac{\partial\varphi}{\partial t} = 0 \qquad (3.50)$$

We now define the dynamic pressure p_d which is the wave induced pressure, i.e. the excess pressure relative to the hydrostatic pressure (and the atmospheric pressure), i.e.:

$$p_d \equiv p - \rho g(-z) = p + \rho g z \qquad (3.51)$$

which when inserted into Eq. 3.50 leads to:

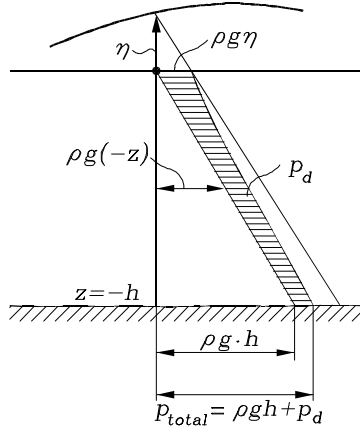$$p_d = -\rho \frac{\partial\varphi}{\partial t} \qquad (3.52)$$

From Eq. 3.51 we get:

$$p_d = \rho g \frac{H}{2} \frac{\cosh k(z+h)}{\cosh kh} \cos(\omega t - kx) \qquad (3.53)$$

As $\eta = \frac{H}{2}\cos(\omega t - kx)$, we can also write Eq. 3.53 as:

$$p_d = \rho g \eta \frac{\cosh k(z+h)}{\cosh kh} \ , \ \text{which at } z = 0 \text{ gives } \ p_d = \rho g \eta \qquad (3.54)$$

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This means the pressure is in phase with the surface elevation and with decreasing amplitude towards the bottom. The figure below shows the pressure variation under the wave crest.

For $z > 0$, where the previous derivations are not valid, we can make a crude approximation and use hydrostatic pressure distribution from the surface, i.e. $p_{total} = \rho g(\eta - z)$ giving $p_d = \rho g \eta$.

Wave height estimations from pressure measurements

Waves in the laboratory and in the prototype can be measured in several ways. The most common in the laboratory is to measure the surface elevation directly by using resistance or capacitance type electrical wave gauges. However, in the prototype this is for practical reasons seldom used unless there is already an existing structure where you can mount the gauge. In the prototype it is more common to use buoys or pressure transducers, which both give rise to some uncertainties. For the pressure transducer you assume that the waves are linear so you can use the linear transfer function from pressure to surface elevations. For a regular wave this is easy as you can use:
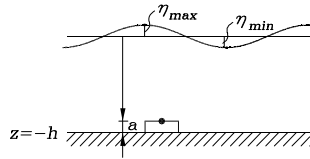
Highest measured pressure (p_{max}):

$$\rho\, g(h-a) + \rho\, g\, \eta_{max}\, \frac{cosh\, k(-(h-a)+h)}{cosh\, kh}$$

Lowest measured pressure (p_{min}):

$$\rho\, g(h-a) + \rho\, g\, \eta_{min}\, \frac{cosh\, k(-(h-a)+h)}{cosh\, kh}$$

$$p_{max} - p_{min} = \rho g \frac{cosh\, ka}{cosh\, kh} \cdot (\eta_{max} - \eta_{min})$$

In case of irregular waves you cannot use the above give procedure as you have a mix of frequencies. In that case you have to split the signal into the different frequencies. The position of the pressure transducer is important as you need to locate it some distance below the lowest surface elevation you expect. Moreover, you need a significant variation in the pressure compared to the noise level for the frequencies considered important. This means that if you have deep water waves you cannot put the pressure gauge close to the bottom as the wave induced pressures will be extremely small.

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3.8 Linear Deep and Shallow Water Waves

In the literature the terms deep and shallow water waves can be found. These terms corresponds to the water depth is respectively large and small compared to the wave length. It turns out the linear equations can be simplified in these cases. The two cases will be discussed in the following sections and the equations will be given.

3.8.1 Deep Water Waves

When the water depth becomes large compared to the wave length $kh = \frac{2\pi h}{L} \to \infty$, the wave is no longer influence by the presence of a bottom and hence the water depth h must vanish from the equations. Therefore, the expressions describing the wave motion can be simplified compared to the general case. The equations are strictly speaking only valid when kh is infinite, but it turns out that these simplified equations are excellent approximations when $kh > \pi$ corresponding to $\frac{h}{L} > \frac{1}{2}$.

Commonly indice 0 is used for deep water waves, i.e. L_0 is the deep water wave length. From Eq. 3.42 we find:

$$L_0 = \frac{g\,T^2}{2\pi} \quad \text{or} \quad T = \sqrt{\frac{2\pi}{g}\,L_0} \quad \text{or} \quad c_0 = \sqrt{\frac{g}{k_0}} \tag{3.55}$$

as $tanh(kh) \to 1$ for $kh \to \infty$. Therefore, we can conclude that in case of deep water waves the wave length only depends on the wave period as the waves doesn't feel the bottom. Note that there is no index on T, as this does not vary with the water depth.

From Appendix A we find $cosh\,\alpha$ and $sinh\,\alpha \to \frac{1}{2}e^\alpha$ for $\alpha \to \infty$ and $tanh\,\alpha$ and $coth\,\alpha \to 1$ for $\alpha \to \infty$. Therefore, we find the following deep water expressions from Eqs. 3.44, 3.45, 3.46 and 3.53:

$$
\begin{aligned}
\varphi &= -\frac{H_0\,L_0}{2\,T}\,e^{k_0 z} sin(\omega t - k_0 x) \\[2mm]
u &= \frac{\pi\,H_0}{T}\,e^{k_0 z}\,cos(\omega t - k_0 x) \\[2mm]
w &= -\frac{\pi\,H_0}{T}\,e^{k_0 z}\,sin(\omega t - k_0 x) \\[2mm]
p_d &= \rho g\,\frac{H_0}{2}\,e^{k_0 z}\,cos(\omega t - k_0\,x)
\end{aligned}
\tag{3.56}
$$

Even though these expressions are derived for $kh \to \infty$ they are very good approximations for $h/L > \frac{1}{2}$.

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3.8.2 Shallow Water Waves

For shallow water waves, i.e. $kh \to 0$, we can also find simplified expressions. As $tanh\,\alpha \to \alpha$ for $\alpha \to 0$ we find from Eq. 3.41 or 3.42:

$$L = \frac{g\,T^2\,h}{L} \quad , \quad T = \sqrt{\frac{L^2}{g\,h}} \quad , \quad L = T\sqrt{g\,h} \;\; , \;\; c = \sqrt{g\,h} \qquad (3.57)$$

From which we can conclude that the phase velocity c depends only of the water depth, and in contrast to the deep water case is thus independent of the wave period. Shallow water waves are thus non-dispersive, as all components propagate with the same velocity.

As $cosh\,\alpha \to 1$, $sinh\,\alpha \to \alpha$ and $tanh\alpha \to \alpha$ for $\alpha \to 0$ we find:

$$\varphi = -\frac{H\,L}{2\,T}\frac{1}{k\,h}sin(\omega t - kx)$$

$$u = \frac{H}{2}\frac{L}{Th}cos(\omega t - kx)$$

$$w = -\frac{\pi\,H}{T}\frac{z+h}{h}sin(\omega t - kx)$$

$$p_d = \rho\,g\,\frac{H}{2}\frac{z+h}{h}cos(\omega t - kx)$$

These equations are good approximations for $h/L < \frac{1}{20}$.

3.9 Particle Paths

The previously derived formulae for the particle velocities (Eqs. 3.45 and 3.46) describe the velocity field with respect to a fixed coordinate, i.e. an Eulerian description. In this section we will describe the particle paths $(x(t), z(t))$, i.e. a Lagrange description. In general the particle paths can be determined by integrating the velocity of the particle in time, which means solving the following two equations:

$$\frac{dx}{dt} = u(x, z, t) \qquad \frac{dz}{dt} = w(x, z, t) \qquad (3.58)$$

where the particle velocity components u and w are given by Eqs. 3.45 and 3.46. These equations (3.58) cannot be solved analytically because of the way u and w depend on x and z.

We utilize the small amplitude assumption, $H/L << 1$, to linearize Eq. 3.58. Based on the expressions of u, w and visual observations we assume the particle paths are closed orbits and we can introduce a mean particle position $(x, z) = (\xi, \zeta)$. Moreover, based on the 1. order theory we assume that, the particle

oscillations $\Delta x, \Delta z$ from respectively ξ and ζ are small compared to the wave length, L, and water depth, h. We can write the instantaneous particle position (x, z) as:

$$x = \xi + \Delta x \qquad \text{and} \qquad z = \zeta + \Delta z \qquad (3.59)$$

We now insert Eq. 3.59 into Eqs. 3.45 and 3.46, and make a Taylor expansion of the sin, cos, sinh and cosh functions from the mean position (ξ, ζ). Terms of higher order are discarded and here after we can solve Eq. 3.58 with respect to x and z.

By using the taylor series expansion:

$$f(a + \Delta a) = f(a) + \frac{f'(a)}{1!} \Delta a + \frac{f''(a)}{2!} \Delta a^2 + \dots$$

we get by introducing Eq. 3.59 the following series expansions of sinh, cosh, sin and cos.

$$
\begin{aligned}
sinh\, k(z + h) &= sinh\, k(\zeta + h) + k\, cosh\, k(\zeta + h) \cdot \Delta z + \dots \\
cosh\, k(z + h) &= cosh\, k(\zeta + h) + k\, sinh\, k(\zeta + h) \cdot \Delta z + \dots
\end{aligned}
$$

$$(3.60)$$

$$
\begin{aligned}
sin(\omega t - kx) &= sin(\omega t - k\xi) + (-k)cos(\omega t - k\xi) \cdot \Delta x + \dots \\
cos(\omega t - kx) &= cos(\omega t - k\xi) - (-k)sin(\omega t - k\xi) \cdot \Delta x + \dots
\end{aligned}
$$

If we insert Eqs. 3.45 and 3.58 we get for the x-coordinate:

$$
\begin{aligned}
\frac{dx}{dt} &\simeq \frac{\pi H}{T} \frac{cosh\, k(\zeta + h) + k\, \Delta z\, sinh\, k(\zeta + h)}{sinh\, kh} \left(cos(\omega t - k\xi)\right. \\
&\quad \left. + k\, \Delta x\, sin(\omega t - k\xi)\right)
\end{aligned}
$$

As $k\Delta z$ and $k\Delta x = \sigma\left(\frac{H}{L}\right) << 1$ we find the linearised expression:

$$\frac{dx}{dt} \simeq \frac{\pi H}{T} \frac{cosh\, k(\zeta + h)}{sinh\, kh} cos(\omega t - k\xi) \qquad (3.61)$$

and equivalent for dz/dt:

$$\frac{dz}{dt} \simeq -\frac{\pi H}{T} \frac{sinh\, k(\zeta + h)}{sinh\, kh} sin(\omega t - k\xi) \qquad (3.62)$$

By integration we find:

$$x = \frac{\pi H}{T\omega} \frac{cosh\, k(\zeta + h)}{sinh\, kh} sin(\omega t - k\xi) + C \qquad (3.63)$$

This equation could also be written as:

$$x = \mathcal{K}\, sin(\omega t - k\xi) + C \quad \text{or} \quad \mathcal{K}\, sin\theta + C\,, \quad \text{where } \theta \text{ has the cycle } 2\pi.$$

The mean position $\xi \equiv \bar{x} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathcal{K}\, sin\theta\, d\theta + C = 0 + C,$ hence $C = \xi.$

$$x = \xi + \frac{H}{2}\frac{cosh\, k(\zeta + h)}{sinh\, kh}\, sin(\omega t - k\xi)$$

and by equivalent calculations we get: (3.64)

$$z = \zeta + \frac{H}{2}\frac{sinh\, k(\zeta + h)}{sinh\, kh}\, cos(\omega t - k\xi)$$

Eq. 3.64 could be written as:

$$x - \xi = A(\zeta)sin\theta$$

$$z - \zeta = B(\zeta)cos\theta$$

By squaring and summation we get, as $\sqrt{cos^2\theta + sin^2\theta} = 1$:

$$\left(\frac{x - \xi}{A(\zeta)}\right)^2 + \left(\frac{z - \zeta}{B(\zeta)}\right)^2 = 1 \quad,$$

Leading to the conclusion that the particle paths are for linear waves elliptical with center (ξ, ζ) and $A(\zeta)$ and $B(\zeta)$ are horizontal and vertical amplitude respectively. Generally speaking the amplitudes are a function of ζ, i.e. the depth. At the surface the vertical amplitude is equal to $H/2$ and the horizontal one is equal to $H/2\, coth\, kh$. Below is a figure showing the particle paths, the foci points and the amplitudes.
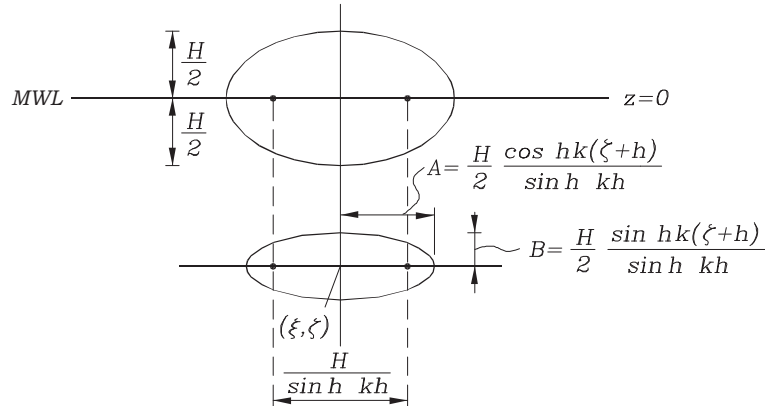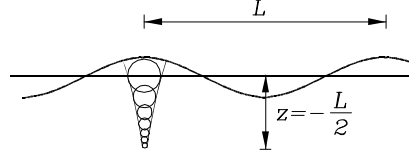

3.9.1 Deep Water Waves

We will now consider the deep water case $\frac{h}{L} > \frac{1}{2}$, corresponding to $kh = \frac{2\pi h}{L} > \pi$ we have $cosh\, kh \simeq \frac{1}{2}e^{kh}$ and $sinh\, kh \simeq \frac{1}{2}e^{kh}$.

By using $\frac{cosh\,k(\zeta+h)}{sinh\,kh} = \frac{coshk\zeta\,cosh\,kh+sinh\,k\zeta\cdot sinh\,kh}{sinh\,kh}$ we find:

$$A(\zeta) \simeq \frac{H}{2}\,e^{k\zeta}$$

$$B(\zeta) \simeq \frac{H}{2}\,e^{k\zeta}$$

Leading to the conclusion that when we have deep water waves, the particle paths are circular with radius $A = B$. At the surface the diameter is naturally equal to the wave height H. At the depth $z = -\frac{L}{2}$ the diameter is only approx. 4% of H. We can thus conclude that the wave do not penetrate deep into the ocean.

3.9.2 Shallow Water Waves

For the shallow water case $\frac{h}{L} < \frac{1}{20}$, corresponding to $kh = \frac{2\pi h}{L} < \frac{\pi}{10}$ we find, as $cosh\,kh \simeq 1$ and $sinh\,kh \simeq kh$:

$$A(\zeta) \quad \simeq \quad \frac{H}{2}\frac{1}{kh} \qquad\qquad \text{, i.e. constant over depth}$$

$$B(\zeta) \quad \simeq \quad \frac{H}{2}\left(1+\frac{\zeta}{h}\right) \quad \text{, i.e. linearly decreasing with depth.}$$

3.9.3 Summary and Discussions

Below are the particle paths illustrated for three different water depths.
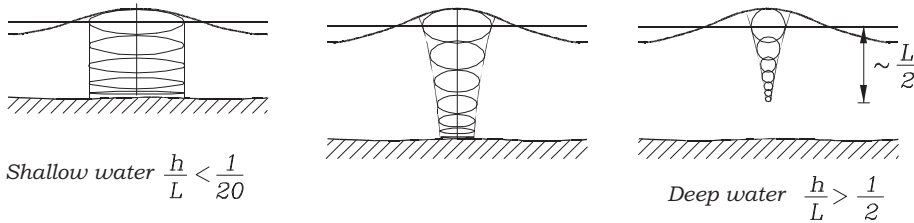
The shown particle paths are for small amplitude waves. In case of finite amplitude waves the particle paths are no longer closed orbits and a net transport of water can be observed. This is because the particle velocity in the upper part of the orbit is larger than in the lower part of the orbit.

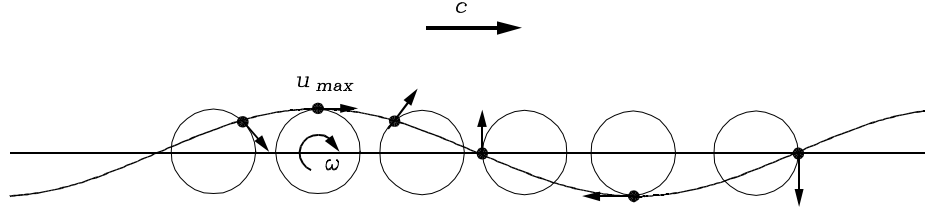When small wave steepness the paths are closed orbits (general ellipses):

When large wave steepness the paths are open orbits, i.e. net mass transport:

However, the transport velocity is even for steep waves smaller than 4% of the phase speed c. Below is the velocity vectors and particle paths drawn for one

wave period.

$u_{max} < c$ for deep water waves. For $H/L = 1/7$ we find $u_{max} \simeq 0,45c$

3.10 Wave Energy and Energy Transportation

When we talk of wave energy we normally think of the mechanical energy content, i.e. kinetic and potential energy. The kinetic energy originates from the movement of the particles and the potential energy originates from the displacement of the water surface from a horizontal plane surface.

The amount of heat energy contained in the fluid is of no interest as the heat energy never can be converted to mechanical wave energy again. However, the transformation of mechanical energy to heat energy is interesting, as it describes the 'loss' of mechanical energy. Wave breaking is in most cases the main contributor to the loss in mechanical energy. In the description of certain phenomena, such as for example wave breaking, it is important to know the amount of energy that is transformed.

The energy in the wave can be shown to propagate in the wave propagation direction. In fact the wave propagation direction is defined as the direction the energy propagate.

3.10.1 Kinetic Energy

As we consider an ideal fluid there is no turbulent kinetic energy present. Therefore, we only consider the particle velocities caused by the wave itself. The instantaneous kinetic energy per unit volume $e_k(\theta)$ is:

$$
\begin{aligned}
e_k(\theta) &= \frac{1}{2}\rho(u^2 + w^2) \\
e_k(\theta) &= \frac{1}{2}\rho(\frac{H\omega}{2sinhkh})^2[cosh^2k(z+h)cos^2\theta + sinh^2k(z+h)sin^2\theta)] \\
e_k(\theta) &= \frac{1}{4}\rho\frac{gkH^2}{sinh2kh}[cos^2\theta + sinh^2k(z+h)] \tag{3.65}
\end{aligned}
$$

The instantaneous kinetic energy per unit area in the horizontal plane $E_k(\theta)$ is found by integrating $e_k(\theta)$ from the bottom $(z = -h)$ to the surface $(z = \eta)$. However, as it mathematically is very complicated to integrate to the surface, is instead chosen to do the integration to the mean water level $(z = 0)$. It can easily be shown that the error related to this is small when $H/L << 1$.

$$E_k(\theta) = \frac{1}{4}\rho\frac{gkH^2}{sinh2kh}\left(h\,cos^2\theta + \frac{1}{2}\int_{-h}^{0}[cosh2k(z+h) - 1]\,dz\right)$$

where $sinh^2(x) = \frac{e^{2x}+e^{-2x}-2}{4} = \frac{1}{2}[cosh(2x)-1]$ has been used. After performing the integration and rearranging we get:

$$E_k(\theta) = \frac{1}{16}\rho gH^2 + \frac{1}{8}\rho gH^2\frac{2kh}{sinh2kh}\left[cos^2\theta - \frac{1}{2}\right] \tag{3.66}$$

If we average over one wave period T or one wave length L (which gives identical results for waves with constant form), we get the mean value of the kinetic energy E_k to:

$$E_k = \frac{1}{16}\rho gH^2 \tag{3.67}$$

as the mean value of $cos^2(\theta)$ over one period is $1/2$.

3.10.2 Potential Energy

As the fluid is assumed incompressible and surface tension is neglected all the potential energy originates from the gravitational forces. Further, we deal only with the energy caused by displacement of the water surface from the mean water level. With these assumptions we can write the instantaneous value of the potential energy $E_p(\theta)$ per unit area in the horizontal plane as:

$$E_p(\theta) = \int_{-h}^{\eta}\rho gz\,dz - \int_{-h}^{0}\rho gz\,dz$$

$$E_p(\theta) = \int_{0}^{\eta}\rho gz\,dz$$

$$E_p(\theta) = \frac{1}{2}\rho g\eta^2 \tag{3.68}$$

Averaging over one wave period T or one wave length L gives the mean value of the potential energy E_p:

$$E_p = \frac{1}{2}\rho g\overline{\eta^2}$$

$$E_p = \frac{1}{2}\rho g\frac{H^2}{4}\overline{cos^2\theta} \quad \text{(for linear waves)}$$

$$E_p = \frac{1}{16}\rho gH^2 \tag{3.69}$$

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3.10.3 Total Energy Density

The total wave energy density per unit area in the horizontal plane E is the sum of the kinetic energy density E_k and the potential energy density E_p.

$$E = E_k + E_p$$

$$E = \frac{1}{8}\rho g H^2 \tag{3.70}$$

3.10.4 Energy Flux

As the waves travel across the ocean they carry their potential an kinetic energy with them. However, the energy density in the waves can not directly be related to an energy equation for the wave motion. In that case we need to consider the average energy (over one period) that is transported through a fixed vertical section and integrated over the depth. If this section is parallel to the wave fronts and has a width of 1 m, it is called the mean transported energy flux or simply the energy flux E_f.
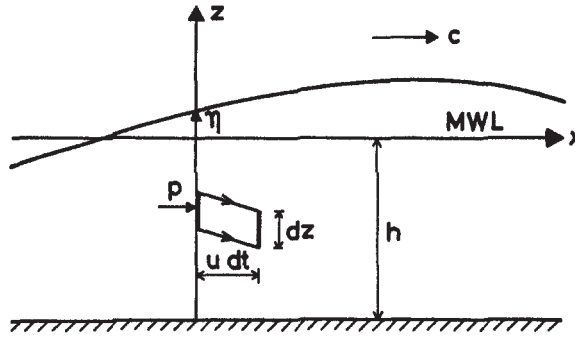
Figure 3.1: Definitions for calculating energy flux.

We now consider the element shown in Fig. 3.1. The energy flux through the shown vertical section consist partly of the transported mechanical energy contained in the control volume, and partly of the increase in kinetic energy, i.e. the work done by the external forces.

Work produced by external forces:
On a vertical element dz acts the horizontal pressure force pdz. During the time interval dt the element moves the distance udt to the right. The work produced per unit width A (force x distance) is thus:

$$A = \Delta E_k = p \, u \, dz \, dt$$

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Mechanical energy:
The transported mechanical energy through the vertical element dz per unit width is calculated as:

$$E_{f,mec} = [\rho g z + \frac{1}{2}\rho(u^2 + w^2)]u \; dz \; dt$$

Energy flux:
The instantaneous energy flux $E_f(t)$ per unit width is:

$$E_f(t) = \int_{-h}^{\eta}[p + \rho g z + \frac{1}{2}\rho(u^2 + w^2)]u dz$$

After neglecting the last term which is of higher order, change of upper integration limit to $z = 0$, and introduction of the dynamic pressure $p_d = p + \rho g z$ we get:

$$E_f(t) = \int_{-h}^{0} p_d \, u \, dz \tag{3.71}$$

Note that the symbol p^+ (excess pressure) can be found in some literature instead of p_d.

The mean energy flux E_f (often just called the energy flux) is calculated by integrating the expression 3.71 over one wave period T, and insertion of the expressions for p_d and u.

$$\begin{aligned}
E_f &= \overline{E}_f(t) \\
E_f &= \frac{1}{16}\rho g H^2 c[1 + \frac{2kh}{sinh2kh}] \tag{3.72} \\
E_f &= Ec_g \tag{3.73}
\end{aligned}$$

where we have introduced the energy propagation velocity $c_g = c(\frac{1}{2} + \frac{kh}{sinh2kh})$. The energy propagation velocity is often called the group velocity as it is related to the velocity of the wave groups, cf. section 3.10.5

If we take a look at the distribution of the transported energy over the depth we will observe that for deep water waves (high kh) most of the energy is close to the free surface. For decreasing water depths the energy becomes more and more evenly distributed over the depth. This is illustrated in Fig. 3.2.
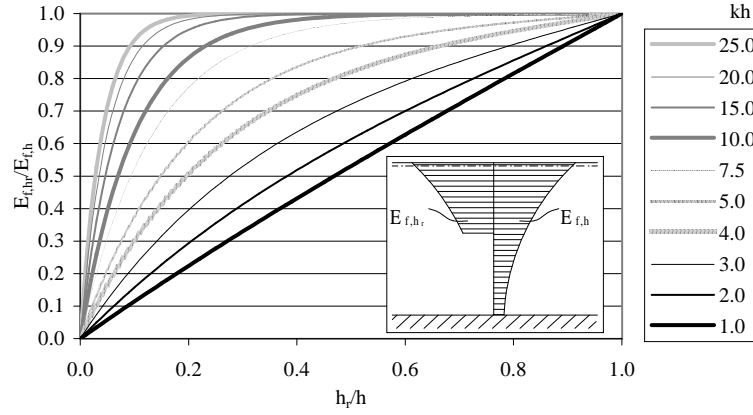
Figure 3.2: Distribution of the transported energy over the water depth.

3.10.5 Energy Propagation and Group Velocity

The energy in the waves travels as mentioned above with the velocity c_g. However, c_g also describes the velocity of the wave groups (wave packets), which is a series of waves with varying amplitude. As a consequence c_g is often called the group velocity. In other words the group velocity is the speed of the envelope of the surface elevations.


c_g group velocity, energy propagation velocity

$$c_g = \frac{1}{2} \cdot c \quad \text{for deep water waves}$$

$$c_g = c \qquad \text{for shallow water waves}$$

This phenomena can easy be illustrated by summing two linear regular waves with slightly different frequencies, but identical amplitudes and direction. These two components travel with different speeds, cf. the dispersion relationship. Therefore, they will reinforce each other at one moment but cancel out in another moment. This will repeat itself over and over again, and we get an infinite number of wave groups formed.

Another way to observe wave groups is to observe a stone dropped into water to generate some few deep water waves.
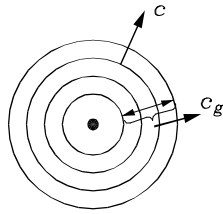
Stone drop in water generates ripples of circular waves, where the individual wave overtake the group and disappear at the front of the group while new waves develop at the tail of the group.

One important effect of deep water waves being dispersive (c and c_g depends on the frequency) is that a field of wind generated waves that normally consist of a spectrum of frequencies, will slowly separate into a sequence of wave fields, as longer waves travel faster than the shorter waves. Thus when the waves after traveling a very long distance hit the coast the longer waves arrive first and then the frequency slowly increases with time. The waves generated in such a way are called swell waves and are very regular and very two-dimensional (long-crested).
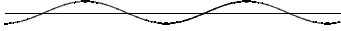
In very shallow water the group velocity is identical to the phase velocity, so the individual waves travel as fast as the group. Therefore, shallow water waves maintain there position in the wave group.

3.11 Evaluation of Linear Wave Theory

In the previous pages is the simplest mathematical model of waves derived and described. It is obvious for everyone, who has been at the coast, that real waves are not regular monochromatic waves (sine-shaped). Thus the question that probably arise is: When and with what accuracy can we use the linear theory for regular waves to describe real waves and their impact on ships, coasts, structures etc.?

The developed theory is based on regular and linear waves. In engineering practise the linear theory is used in many cases. However, then it is in most cases irregular linear waves that are used. In case regular waves are used for design purposes it is most often a non-linear theory that is used as the Stokes 5. order theory or the stream function theory. Waves with finite height (non-linear waves) is outside the scope of this short note, but will be introduced in the next semester.

To distinguish between linear and non-linear waves we classify the waves after their steepness:

$H/L \to 0$, waves with small amplitude

1. order Stokes waves, linear waves, Airy waves, monochromatic waves.

$H/L > 0.01$, waves with finite height

higher order waves, e.g. 5. order Stokes waves.

Even though the described linear theory has some shortcomings, it is important to realise that we already (after two lectures) are able to describe waves in a sensible way. It is actually impressive the amount of problems that can be solved by the linear theory. However, it is also important to be aware of the limitations of the linear theory.

From a physically point of view the difference between the linear theory and higher order theories is, that the higher order theories take into account the influence of the wave itself on its characteristics. Therefore, the shape of the surface, the wave length and the phase velocity all becomes dependent on the wave height.

Linear wave theory predicts that the wave crests and troughs are of the same size. Theories for waves with finite height predicts the crests to significant greater than the troughs. For high steepness waves the trough is only around 30 percent of the wave height. This is very important to consider for design of e.g. top-sites for offshore structure (selection of necessary level). The use of the linear theory will in such cases lead to very unsafe designs. This shows that it is important to understand the differences between the theories and their validity.

Linear wave theory predicts the particle paths to be closed orbits. Theories for waves with finite height predicts open orbits and a net mass flow in the direction of the wave.

Chapter 4

Changes in Wave Form in Coastal Waters

Most people have noticed that the waves changes when they approach the coast. The change affect both the height, length and direction of the waves. In calm weather with only small swells these changes are best observed. In such a situation the wave motion far away from the coast will be very limited. If the surface elevation is measured we would find that they were very close to small amplitude linear waves, i.e. sine shaped. Closer to the coast the waves becomes affected by the limited water depth and the waves raises and both the wave height and especially the wave steepness increases. This phenomena is called shoaling. Closer to the coast when the wave steepness or wave height has become too large the wave breaks.

The raise of the waves is in principle caused by three things. First of all the decreasing water depth will decrease the wave propagation velocity, which will lead to a decrease in the wave length and thus the wave steepness increase. Second of all the wave height increases when the propagation velocity decreases, as the energy transport should be the same and as the group velocity decreases the wave height must increase. Finally, does the increased steepness result in a more non-linear wave form and thus makes the impression of the raised wave even more pronounced.

The change in the wave form is solely a result of the boundary condition that the bottom is a streamline. Theoretical calculations using potential theory gives wave breaking positions that can be reproduced in the laboratory. Therefore, the explanation that wave breaking is due to friction at the bottom must be wrong.

Another obvious observation is that the waves always propagate towards the coast. However, we probably all have the feeling that the waves typically propagate in the direction of the wind. Therefore, the presence of the coast must

affect the direction of the waves. This phenomenon is called wave refraction and is due to the wave propagation velocity depends on the water depth.

These depth induced variations in the wave characteristics (height and direction) are usually sufficiently slow so we locally can apply the linear theory for waves on a horizontal bottom. When the non-linear effects are too strong we have to use a more advanced model for example a Boussinesq model.

In the following these shallow water phenomena are discussed. An excellent location to study these phenomena is Skagens Gren (the northern point of Jutland).

4.1   Shoaling

We investigate a 2-dimensional problem with parallel depth contours and where the waves propagate perpendicular to the coast (no refraction). Moreover, we assume:

- Water depth vary so slowly that the bottom slope is everywhere so small that there is no reflection of energy and so we locally can apply the linear theory for progressive waves with the horizontal bottom boundary condition. The relative change in water depth over one wave length should thus be small.

- No energy is propagating across wave orthogonals, i.e. the energy is propagating perpendicular to the coast (in fact it is enough to assume the energy exchange to be constant). This means there must be no current and the waves must be long-crested.

- No wave breaking.

- The wave period T is unchanged and hence f and ω are also unchanged. This seems valid when there is no current and the bottom has a gentle slope.

The energy content in a wave per unit area in the horizontal plane is:

$$E = \frac{1}{8}\rho g H^2 \tag{4.1}$$

The energy flux through a vertical section is E multiplied by the energy propagation velocity c_g:

$$P = E c_g \tag{4.2}$$

Inserting the expressions from Eq. 3.73 gives:

$$P = \frac{1}{8}\rho g H^2 \cdot c \left(\frac{1}{2} + \frac{kh}{sinh(2kh)}\right) \tag{4.3}$$
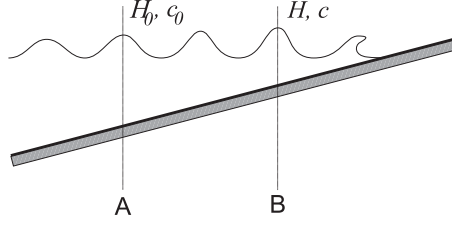
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Figure 4.1: Definitions for calculating 2-dimensional shoaling (section A is assumed to be on deep water).

Due to the assumptions made the energy is conserved in the control volume. Thus the energy amount that enters the domain must be identical to the energy amount leaving the domain. Moreover, as we have no energy exchange perpendicular to the wave orthogonals we can write:

$$E^A \cdot c_g{}^A = E^B \cdot c_g{}^B \tag{4.4}$$

$$H^B = H^A \sqrt{\frac{c_g{}^A}{c_g{}^B}} \tag{4.5}$$

The above equation can be used between two arbitary vertical sections, but remember the assumption of energy conservation (no wave breaking) and small bottom slopes. In many cases it is assumed that section A is on deep water and we get the following equation:

$$\frac{H}{H_0} = K_s = \sqrt{\frac{c_{0,g}}{c_g}} \tag{4.6}$$

The coefficient K_s is called the shoaling coefficient. As shown in Figure 4.1 the shoaling coefficient first drops slightly below one, when the wave approach shallower waters. However, hereafter the coefficient increase dramatically.

All in all it can thus be concluded that the wave height increases as the wave approach the coast. This increase is due to a reduction in the group velocity when the wave approach shallow waters. In fact using the linear theory we can calculate that the group velocity approaches zero at the water line, but then we have really been pushing the theory outside its range of validity.

As the wave length at the same time decreases the wave steepness grows and grows until the wave becomes unstable and breaks.

4.2 Refraction

A consequence of the phase velocity of the waves is decreasing with decreasing water depth (wave length decreases), is that waves propagating at an angle
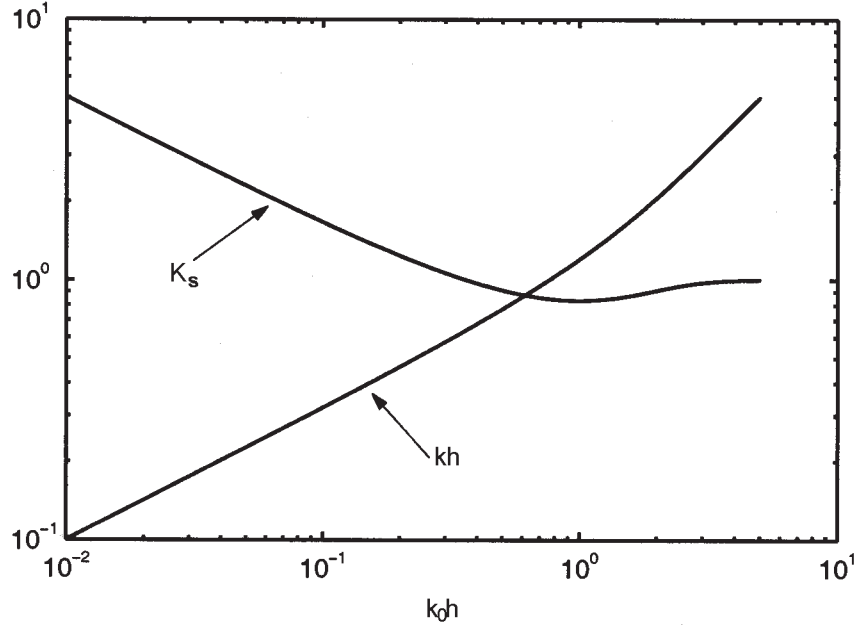
Figure 4.2: Variation of the shoaling coefficient K_s and the dimensionless depth parameter kh, as function of $k_0 h$, where $k_0 = 2\pi/L_0$ is the deep water wave number.

(oblique incidence) toward a coast slowly change direction so the waves at last propagate almost perpendicular to the coast.

Generally the phase velocity of a wave will vary along the wave crest due to variations in the water depths. The crest will move faster in deep water than in more shallow water. A result of this is that the wave will turn towards the region with more shallow water and the wave crests will become more and more parallel to the bottom contours.

Therefore, the wave orthogonals will not be straight lines but curved. The result is that the wave orthogonals could either diverge or converge towards each other depending on the local bottom contours. In case of parallel bottom contours the distance between the wave orthogonals will increase towards the coast meaning that the energy is spread over a longer crest.
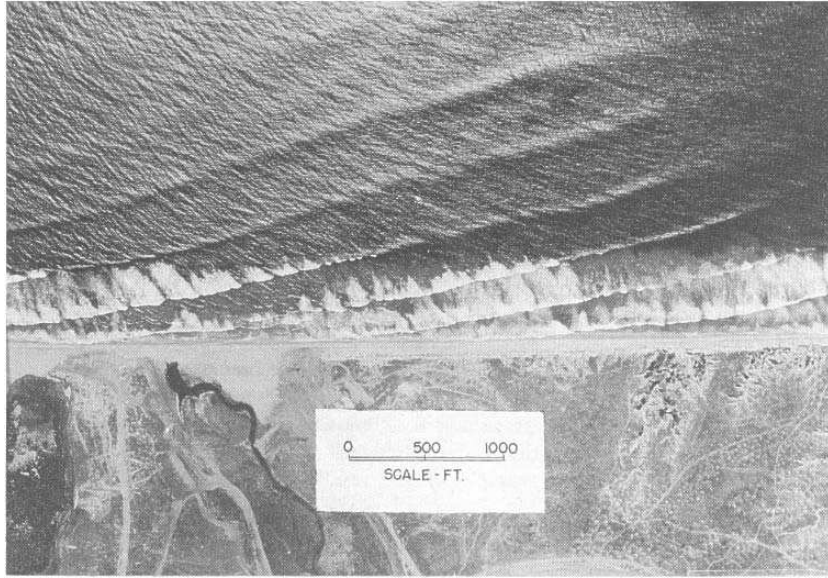
Figure 4.3: Photo showing wave refraction. The waves change direction when they approach the coast.

We will now study a case where oblique waves approach a coast. Moreover, we will just as for shoaling assume:

- Water depth vary so slowly that the bottom slope is everywhere so small that there is no reflection of energy and so we locally can apply the linear theory for progressive waves with the horizontal bottom boundary condition. The relative change in water depth over one wave length should thus be small.

- No energy is propagating across wave orthogonals, i.e. the energy is propagating perpendicular to the coast (in fact it is enough to assume the energy exchange to be constant). This means there must be no current and the waves must be long-crested.

- No wave breaking.

- The wave period T is unchanged and hence f and ω are also unchanged. This seems valid when there is no current and the bottom has a gentle slope.

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Figure 4.4: Refraction of regular waves in case of parallel bottom contours.

The energy flux P_{b_0}, passing section b_0 will due to energy conservation be identical to the energy flux P_b passing section b, cf. Fig. 4.4. The change in wave height due to changing water depth and length of the crest, can be calculated by require energy conservation for the control volume shown in Fig. 4.4:

$$E^{b_0} \cdot c_g{}^{b_0} \cdot b_0 = E^b \cdot c_g{}^b \cdot b \Rightarrow \tag{4.7}$$

$$H^b = H^{b_0} \sqrt{\frac{c_g{}^{b_0}}{c_g{}^b}} \cdot \sqrt{\frac{b_0}{b}} \Rightarrow \tag{4.8}$$

$$H^b = H^{b_0} \cdot K_s \cdot K_r \tag{4.9}$$

$$where, \quad c_g = c \cdot \left(\frac{1}{2} + \frac{kh}{sinh(2kh)}\right)$$

K_r is called the refraction coefficient. In case of parallel depth contours as shown in Fig. 4.4 the refraction coefficient is smaller than unity as the length of the crests increases as the wave turns.

In the following we will shortly go through a method to calculate the refraction coefficient. The method starts by considering a wave front on deep water and then step towards the coast for a given bottom topography. The calcultion is performed by following the wave crest by stepping in time intervals Δt, e.g. 50 seconds. In each time interval is calculated the phase velocity "in each end" of the selected wave front. As the water depths in each of the ends are different the phase velocities are also different. It is now calculated the distance that each end of the wave front has tralled during the time interval Δt. Hereafter we can draw the wave front Δt seconds later. This procedure is continued until the wave front is at the coast line. It is obvious that the above given

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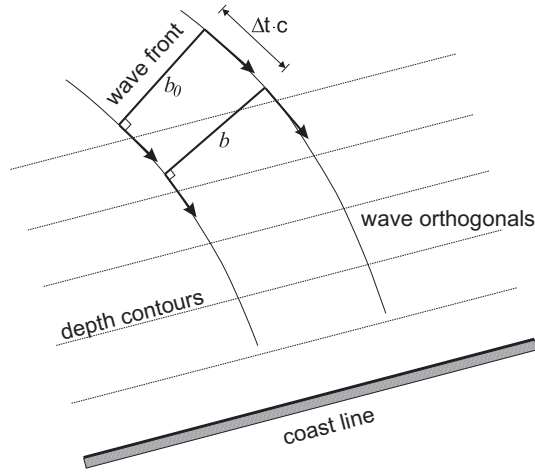procedure requires some calculation and should be solved numerically.

Figure 4.5: Refraction calculation.

As the wave fronts turns it must be evident that the length of the fronts will change. We can thus conclude that this implies that the refraction coefficient is larger than unity where the length of wave front is decreased and visa versa.
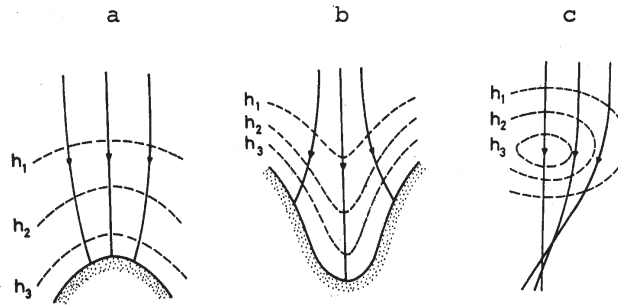

Figure 4.6: Influence of refraction on wave height for three cases. The curves drawn are wave orthogonals and depth contours. a) Increased wave height at a headland due to focusing of energy (converging wave orthogonals). b) Decrease in wave height at bay or fjord (diverging wave orthogaonals). c) Increased wave height behind submerged ridge (converging wave orthogonals).

Figure 4.6 shows that it is a good idea to consider refraction effects when looking for a location for a structure built into the sea. This is the case both if you want small waves (small forces on a structure) or large waves (wave power plant). In fact you will find that many harbours are positioned where you have small waves due to refraction and/or sheltering.

Practically the refraction/shoaling problem is always solved by a large numerical wave propagation model. Examples of such models are D.H.I.'s System21, AaU's MildSim and Delfts freely available SWAN model, just to mention a few of the many models available.

If there is a strong current in an area with waves it can be observed that the current will change the waves as illustrated in Fig. 4.7. The interaction affects both the direction of wave propagation and characteristics of the waves such as height and length. Swell in the open ocean can undergo significant refraction as it passes through major current systems like the Gulf Stream. If the current is in the same direction as the waves the waves become flatter as the wave length will increase. In opposing current conditions the wave length decreases and the waves become steeper. If the wave and current are not co-directional the waves will turn due to the change in phase velocity. The phase velocity is now both a function of the depth and the current velocity and direction. This phenomena is called current refraction. It should be noted that the energy conservation is not valid when the wave propagate through a current field.
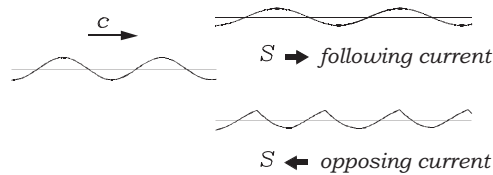
Figure 4.7: Change of wave form due to current.

4.3   Diffraction

If you observe the wave disturbance in a harbour, you will observe wave disturbance also in areas that actually are in shelter of the breakwaters. This wave disturbance is due to the waves will travel also into the shadow of the breakwater in an almost circular pattern of crests with the breakwater head being the center point. The amplitude of the waves will rapidly decrease behind the breakwater. Thus the waves will turn around the head of the breakwater even when we neglect refraction effects. We say the wave diffracts around the breakwater.

If diffraction effects were ignored the wave would propagate along straight orthogonals with no energy crossing the shadow line and no waves would enter into the shadow area behind the breakwater. This is of cause physical impossible as it would lead to a jump in the energy level. Therefore, the energy will spread and the waves diffract.

Figure 4.8: Diffraction around breakwater head.

The wave disturbance in a harbour is determining the motions of moored ships and thus related to both the down-time and the forces in the mooring systems (hawsers and fenders). Also navigation of ships and sediment transport is affected by the diffracted waves. Moreover, diffraction plays a role for forces on offshore large structures and wind mill foundations. It is therefore important to be able to estimate wave diffraction and diffracted wave heights.

From the theory of light we know the diffraction phenomena. As the governing equation for most wave phenomena formally are identical, we can profit from the analytical solution developed for diffraction of electromagnetic waves around a half-infinite screen (Sommerfeld 1896).

Fig. 4.9 shows the change in wave height behind a fully absorbing breakwater. The shown numbers are the so-called diffraction coefficient K_d, which is defined as the diffracted wave height divided by the incident wave height. In reallity a breakwater is not fully absorbing as the energy can either be absorbed, transmitted or reflected. For a rubble mound breakwater the main part of the energy is absorbed and most often only a small part of the energy is reflected and transmitted. In case of a vertical breakwater the main part of the energy is reflected. Therefore, these cases are not generally covered by the Sommerfeld
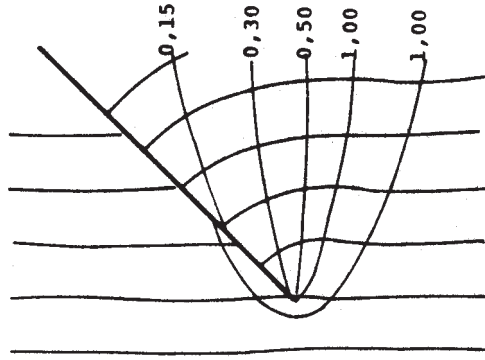
Figure 4.9: Diffraction around absorbing breakwater.

solution, but anyway the solution gives an idea of the diffracted wave height.

The preceding description (the Sommerfeld solution) is based on the assumption of constant phase velocity of the wave. As previously derived the phase velocity depends not only on the wave period but also on the water depth. Therefore, we have implicit assumed constant water depth when we apply the Sommerfeld solution.

In the conceptual design of a harbour or another structure is the diffraction diagram is an essential tool. However, a detailed design should be based on either physical model tests or advanced numerical modelling.

A larger mathematical derivation leads to the so-called Mild-Slope equations and Boussinesq equations. It is outside the scope of these notes to present this derivation, but it should just be mentioned that commercial wave disturbance models are based on these equations.

Generally all the shallow water phenomena (i.e. shoaling, refraction and diffraction) are included in such a numerical model. Examples of such models are as previously mentioned D.H.I.'s Mike21, AaU's MildSim and Delfts SWAN model.
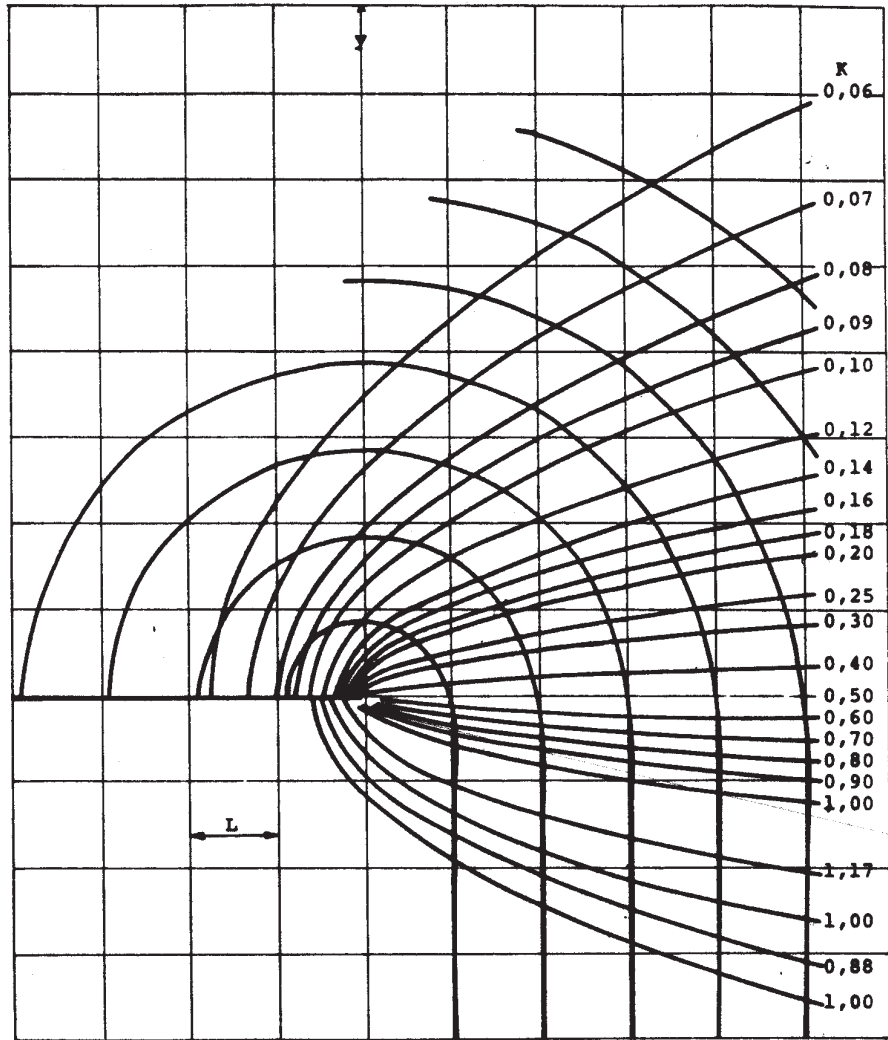
Figure 4.10: Diffraction diagram for fully absorbing breakwater.
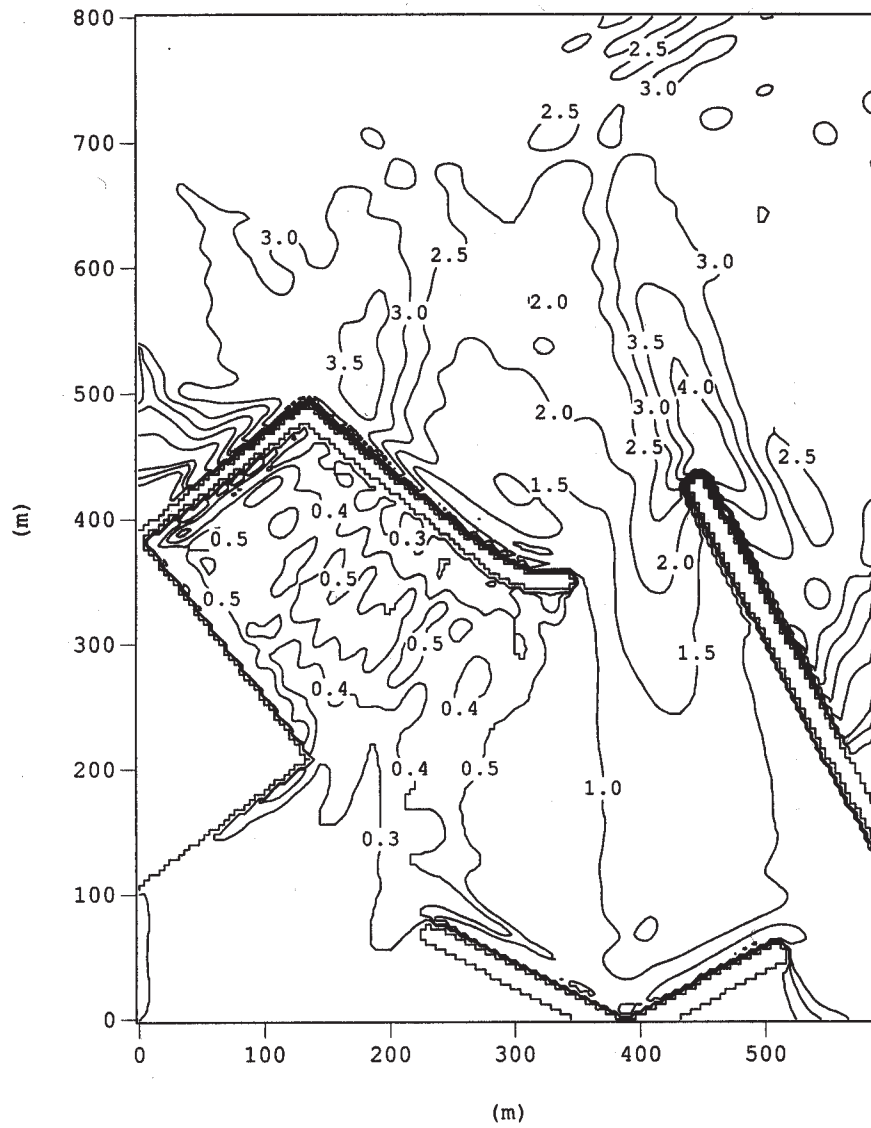
Figure 4.11: Example of wave heights in the outer part of Grenaa harbour calculated by the MildSim model.

4.4 Wave Breaking

Wave measurements during storm periods shows that the wave heights almost never gets higher than approximately 1/10 of the wave length. If we in the laboratory try to generate steep waves we will observe that it is only possible to generate waves with a wave steepness up to between 1/10 and 1/8. If we try to go steeper we will observe that the wave breaks.

Miche (1944) has shown theoretically that the maximum wave is limited by the fact that the particle velocity u cannot be larger then the phase velocity c.

$$u_{max} = c \qquad (4.10)$$

The wave steepness is high when the wave breaks and thus the assumptions in the linear theory are violated too strong to give usable results. Miche (1944) found the maximum steepness from Eq. 4.10 to:

$$\frac{H}{L} = 0.142 \cdot tanh(kh) \qquad (4.11)$$

If we instead apply the linear theory we get when using the velocity at $z = 0$ a coefficient $1/\pi$ instead of 0.142, which shows that the linear theory is pushed way out of its range of validity.

Eq. 4.11 gives for deep water waves $(\frac{h}{L} \geq \frac{1}{2})$ that the maximum steepness is 0.14. In shallow water $(\frac{h}{L} \leq \frac{1}{20})$ Eq. 4.11 is reduced to:

$$H \leq 0.88 \cdot h \qquad (4.12)$$

However, observation shows that this formula is the upper limit and typically the wave breaks around $H/h = 0.6$ to 0.8. In case of irregular waves observations shows that the maximum significant wave height around $H_s/h \approx 0.5$.

In reality the breaking wave height depends in shallow water not only on the depth as suggested by Eq. 4.12 but also on the bottom slope. We can observe at least three different wave breaking forms. Waves on deep water breaks by spilling, when the wind has produced relative steep waves.
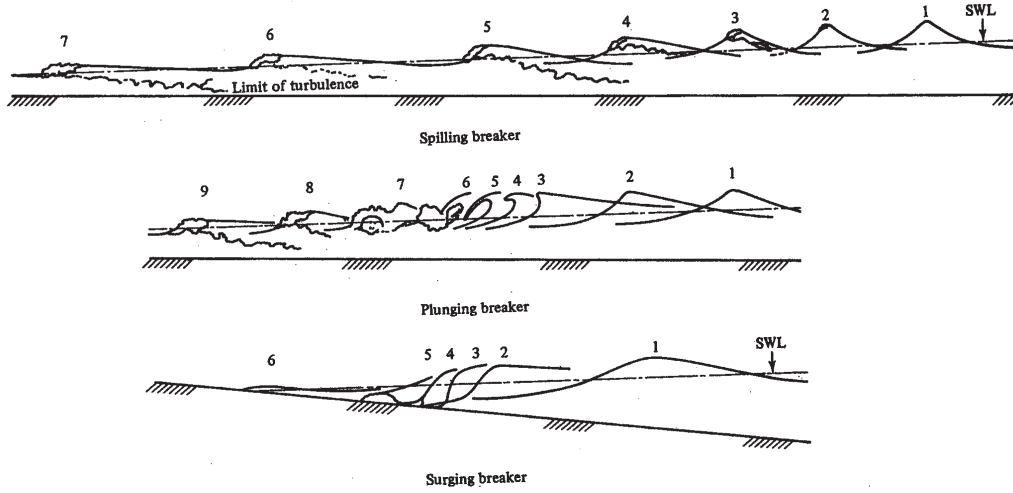
Figure 4.12: Different breaker types.

The type of wave breaking depends on shallow water on both the wave steepness and the bottom slope typically combined in the Iribarren number defined as:

$$\xi = \frac{tan(\alpha)}{\sqrt{H_b/L_0}} = \frac{tan(\alpha)}{\sqrt{s_0}} \tag{4.13}$$

where s_0 is the wave steepness at the breaker point but using the deep water wave length. The Iribarren number is also known as the surf similarity parameter and the breaker parameter. Typical values used for the different breaker types are:

$$\begin{aligned} spilling : \quad & \xi < 0.4 \\ plunging : \quad & 0.4 < \xi < 2.0 \\ surging : \quad & \xi > 2.0 \end{aligned}$$

Fig. 4.13 indicate the breaker type as function of the bottom slope and the wave steepness ($s_0 = H/L_0$) using the above given limits for the breaker parameter. In many cases the reflection from a sloping structure is calculated using the Iribarren number as this determines the breaker process and thus the energy dissipation. Also stability of rubble mound structures depends on the Iribarren number.
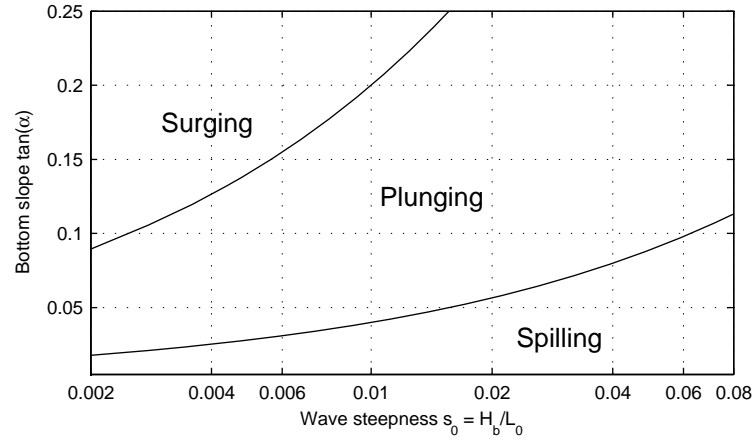
Figure 4.13: Type of wave breaking as function of wave steepness and bottom slope.

Towards the surf zone there are changes in the mean water level. Before the wave breaking point there is a small set-down of the mean water level. From the breaker line and towards the coast line there is a set-up of the water level. These changes in the water level is due to variations in the wave height (wave radiation stress), i.e. before the breaker zone the wave height is increased due to shoaling and causes the set-down. In the breaker zone the wave height is reduced very significantly and leads to set-up, which can be as much as 20% of the water depth at the breaker point.

These water level variation gives also rise to a return flow from the coast towards the breaker zone (cross-shore current). In case of oblique incident waves a long-shore current is also generated. These currents can if they are strong enough be extremely dangerous for swimmers as they occasional can outbreak to the sea and generate what is often refered to as rip currents. Moreover, the wave generated currents are important for the sediment transport at the coast.

Chapter 5

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Appendix A

Hyperbolic Functions

y	y'
$sinh\,\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2} = -sinh\,(-\alpha)$	$cosh\,\alpha$
$cosh\,\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2} = cosh\,(-\alpha)$	$sinh\,\alpha$

Limit for shallow water waves

$\frac{h}{L} = \frac{1}{20}$

correponding to

$kh=\frac{\pi}{10}$

Limit for deep water waves

$\frac{h}{L} = \frac{1}{2}$

correponding to

$kh=\pi$
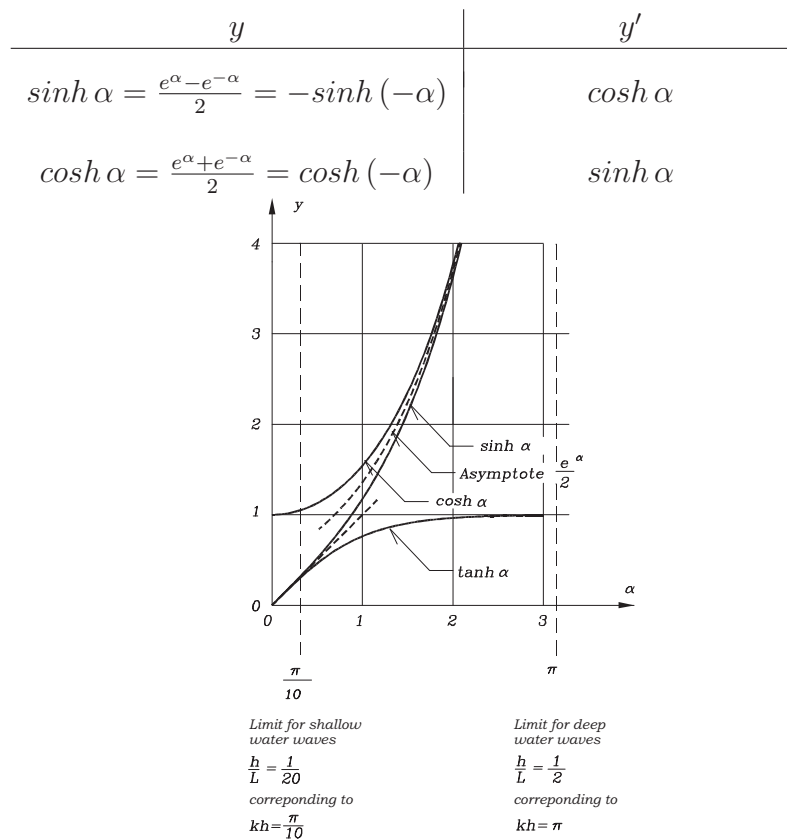
Appendix B

Phenomena, Definitions and Symbols

B.1 Definitions and Symbols

H	wave height
a	wave amplitude
η	water surface elevations from MWL (posituve upwards)
L	wave length
$s = \dfrac{H}{L}$	wave steepness
$c = \dfrac{L}{T}$	phase velocity of wave
T	wave period, time between two crests passage of same vertical section
u	horizontal particle velocity
w	vertical particle velocity
$k = \dfrac{2\pi}{L}$	wave number
$\omega = \dfrac{2\pi}{T}$	cyclic frequency, angular frequency
h	water depth

Wave fronts

Wave front

Wave orthogonals

Wave orthogonal

Wave front

B.2   Particle Paths

Shallow water $\frac{h}{L} < \frac{1}{20}$
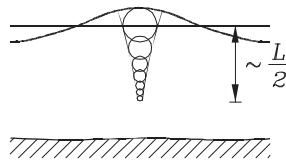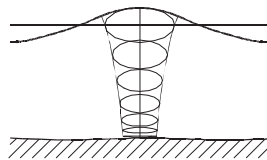
Deep water $\frac{h}{L} > \frac{1}{2}$

$\sim \frac{L}{2}$

When small wave steepness the paths are closed orbits (general ellipses).

When large wave steepness the paths are open orbits, i.e. net mass transport.

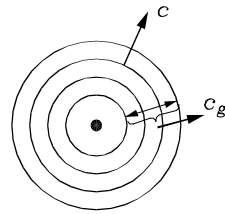However, the transport velocity is even for steep waves smaller than 4% of the phase speed c.

c

u_{max}

ω

$u_{max} < c$ for deep water waves. For $H/L = 1/7$ we find $u_{max} \simeq 0,45c$

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B.3 Wave Groups

c_g group velocity, energy propagation velocity
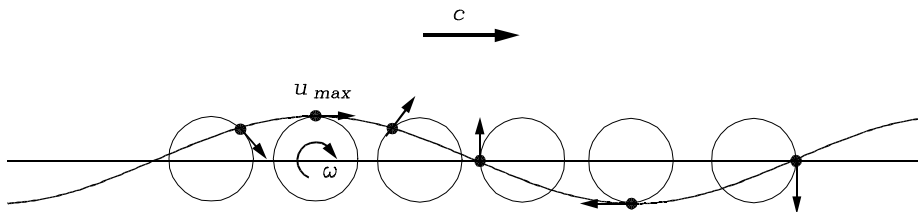
$$c_g = \frac{1}{2} \cdot c \quad \text{for deep water waves}$$

$$c_g = c \qquad \text{for shallow water waves}$$
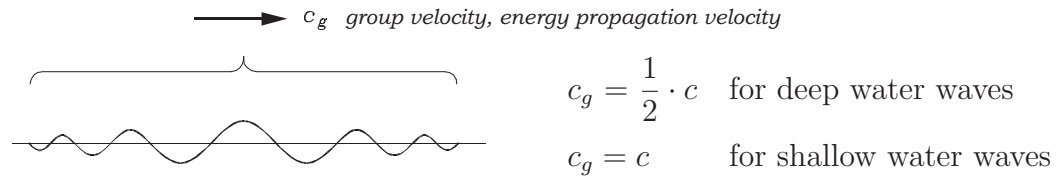
Example:
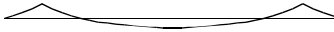
Stone drop in water generates ripples of circular waves, where the individual wave overtake the group and disappear at the front of the group while new waves develop at the tail of the group.

B.4 Wave Classification after Origin

Phenomenon	Origin	Period
Surges	Atmospheric pressure and wind	1 – 30 days
Tides	Gravity forces from the moon and the sun	app. 12 and 24 h
Barometric wave	Air pressure variations	1 – 20 h
Tsunami	Earthquake, submarine land slide or submerged volcano	5 – 60 min.
Seiches (water level fluctuations in bays and harbour basins)	Resonance of long period wave components	1 – 30 min.
Surf beat, mean water level fluctuations at the coast	Wave groups	0.5 – 5 min.
Swells	Waves generated by a storm some distance away	< 40 sec.
Wind generated waves	Wind shear on the water surface	< 25 sec.

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B.5 Wave Classification after Steepness

$H/L \to 0,$ waves with small amplitude

1. order Stokes waves, linear waves, Airy waves, monochromatic waves.

$H/L > 0,01,$ waves with finite height

higher order waves, e.g. 5. order Stokes waves.

B.6 Wave Classification after Water Depth

$h/L < \frac{1}{20}$, shallow water waves

$h/L > \frac{1}{2}$, deep water waves

B.7 Wave Classification after Energy Propagation Directions

Long-crested waves: 2-dimensional (plane) waves (e.g. swells at mild sloping coasts). Waves are long crested and travel in the same direction (e.g. perpendicular to the coast)

Short-crested waves: 3-dimensional waves (e.g. wind generated storm waves). Waves travel in different directions and have a relative short crest.

B.8 Wave Phenomena

In the following is described wave phenonema related to short period waves. Short period waves are here defined as typical wind generated waves with periods less than approximately 30 seconds.
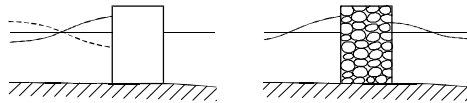
Wind speed > 1 m/s
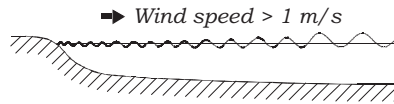
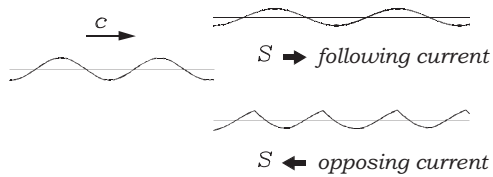H, L and T increases with increasing wind velocity and the distance the wind has acted over (the fetch).
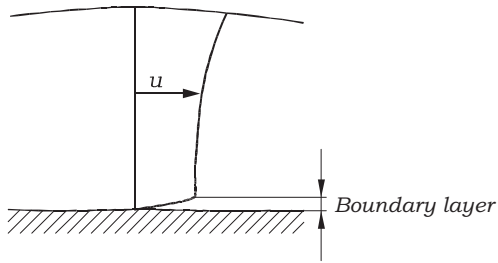
Wind shear current in the surface layer.

Decreasing H
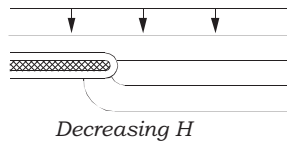
Diffraction

Reflection and transmission

c

S *following current*
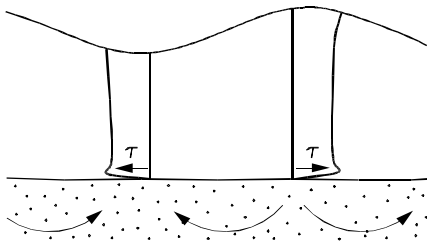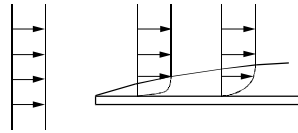
S *opposing current*

Change of wave form due to current. The corresponding change in phase velocity cause current refraction if S and c are not parallel.

Phenomena related to the presence of the bottom:
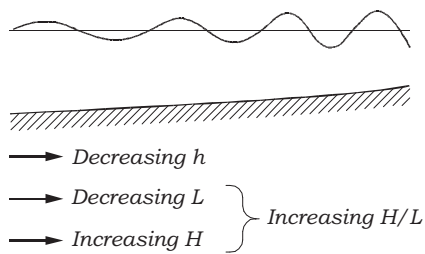

u

Boundary layer

Thin boundary layer due to the oscillating motion. Compare to boundary development at a plate in stationary flow
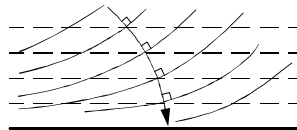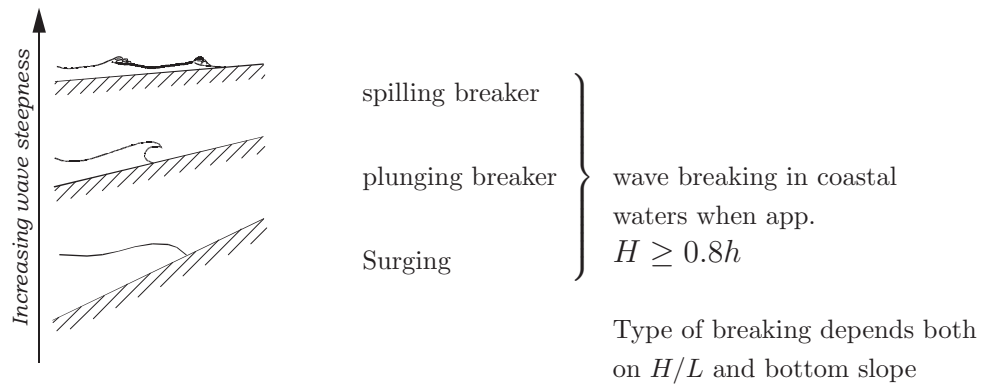

τ τ

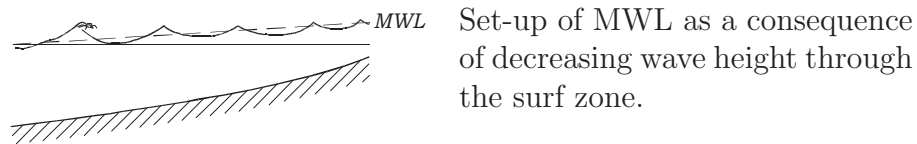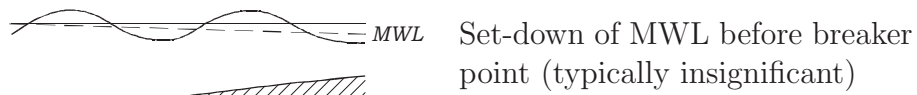Bed shear stress (can generate sediment transport)

Percolation

→ *Decreasing h*

→ *Decreasing L*

→ *Increasing H*

Increasing H/L

Waves shoal and refract in coastal waters. Refraction when wave crests and depth contours are not parallel

Set-down of MWL before breaker point (typically insignificant)

Set-up of MWL as a consequence of decreasing wave height through the surf zone.

Increasing wave steepness

spilling breaker

plunging breaker

Surging

wave breaking in coastal waters when app. $H \geq 0.8h$

Type of breaking depends both on H/L and bottom slope

It should be noted that waves on deep water breaks by spilling when the wind has produced relative steep waves.

Breaking waves generates long-shore currents where the wave orthogonals are not perpendicular to the depth contours.

Appendix C

Equations for Regular Linear Waves

C.1   Linear Wave Theory

$$\varphi = -\frac{a\,g}{\omega}\frac{\cosh k\,(z+h)}{\cosh kh}\,sin(\omega t - kx) \tag{C.1}$$

$$\eta = a\,cos\theta = \frac{H}{2}\,cos(\omega t - kx) \tag{C.2}$$

$$c = \sqrt{\frac{g\,L}{2\pi}\,tanh\,\frac{2\pi h}{L}} \tag{C.3}$$

$$
\begin{aligned}
u = \frac{\partial\varphi}{\partial x} \quad &= \quad -\frac{Hc}{2}(-k)\,\frac{\cosh k(z+h)}{sinh\,kh}\,cos(\omega t - kx) \\
&= \quad \frac{\pi\,H}{T}\,\frac{\cosh k(z+h)}{sinh\,kh}\,cos(\omega t - kx) \\
&= \quad \frac{a\,g\,k}{\omega}\,\frac{\cosh k(z+h)}{\cosh kh}\,cos(\omega t - kx)
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
w = \frac{\partial\varphi}{\partial z} \quad &= \quad -\frac{Hc}{2}\,k\,\frac{sinh\,k(z+h)}{sinh\,kh}\,sin(\omega t - kx) \\
&= \quad -\frac{\pi\,H}{T}\,\frac{sinh\,k(z+h)}{sinh\,kh}\,sin(\omega t - kx) \\
&= \quad -\frac{a\,g\,k}{\omega}\,\frac{sinh\,k(z+h)}{\cosh kh}\,sin(\omega t - kx)
\end{aligned} \tag{C.5}
$$

$$\frac{\partial u}{\partial t} = -a\,g\,k\,\frac{cosh\,k(z+h)}{cosh\,kh}\,sin(\omega t - kx) \tag{C.6}$$

$$\frac{\partial w}{\partial t} = -a\,g\,k\,\frac{sinh\,k(z+h)}{cosh\,kh}\,cos(\omega t - kx) \tag{C.7}$$

$$p_d = \rho\,g\,\eta\,\frac{cosh\,k(z+h)}{cosh\,kh}\;,\;\;\text{which at }z=0\text{ gives }\;\;p_d = \rho\,g\,\eta \tag{C.8}$$

$$L = \frac{g\,T^2}{2\pi}\,tanh\,\frac{2\pi h}{L} \tag{C.9}$$

$$E = \frac{1}{8}\rho_v g H^2 \tag{C.10}$$

$$P = E c_g \tag{C.11}$$

$$P = \frac{1}{8}\rho_v g H^2 \cdot \left(\frac{1}{2} + \frac{kh}{sinh(2kh)}\right) \tag{C.12}$$

C.2 Wave Propagation in Shallow Waters

$$\frac{H}{H_0} = K_s = \sqrt{\frac{c_0}{c}} \tag{C.13}$$

$$\frac{H^b}{H^{b_0}} = \sqrt{\frac{c_g^{b_0}}{c_g^{b}}} \cdot \sqrt{\frac{b_0}{b}} \tag{C.14}$$