Lucas Asset Pricing Model

Anindo Sarkar*

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1 Model setup

- There are uncountably many homogenous households whch are infinitely lived.
- Each household owns an asset in form of a tree. Each household can choose how much trees to hold next period. The value k_{it} is the value of asset holdings for household i at **end of period.**
- In each period a tree bears fruit (”dividends”, (d_t). The households consume the fruit (c_{it}). The law of motion of the fruits is given exogenously as:

  $$d_{t+1} = G(d_t)$$

  Here G is assumed to be a Markov process.
- The mass of households is 1. The mass of trees is 1.
- Household has utility from consumption $u(c)$. For this probelm assume power utility $u(c) = c^{-gamma}$.
- If the asset is sold, then it's price is P_t after the houshold has consumed the dividend d_t, that is it is the ex-dividend price.

2 Household problem

- Prices depend on only the current information. The only current information is the endowment process $\{d_t\}$. Moreover, the endowment process is assumed to be Markov. So the solution would be MArkov too and proces only depend on only d_t. So $P_t = P(d_t)$.
- The household maximises lifeltime utility, by taking the solution of prices $P_t = P(d_t)$ as given.

$$\max_{c_{it}, k_{i,t}} \sum_{s=0}^{\infty} E_t \left[\beta^s u(c_{i,t+s}) \right]$$

sub to:

$$c_{it} + P(d_t)k_{i,t} = (P(d_t) + d_t)k_{i,t-1}$$

The Bellman formulation of the problem becomes:

$$V(d_t, k_{i,t-1}) = \max_{c_{i,t}, k_{i,t}} u(c_{it}) + \beta E_t[V(d_{t+1}, k_{i,t})]$$

sub to:

$$c_{it} + P(d_t)k_{it} = (P(d_t) + d_t)k_{i,t-1} \equiv G(d_t)$$

*Alphabetical ordering.

Re-writing the Bellman equation to eliminate a choice variable we get:

$$V(d, k_{-1}) = \max_k \left[u\Big((P(d) + d)k_{-1} - P(d)k \Big) + \beta E[V(d', k)] \right]$$

The FOC wrt k is:

$$u'\Big((P(d) + d)k_{-1} - P(d)k \Big)(-P(d)) + \beta E[V_2(d', k)] = 0$$

The Envelope theorem gives:

$$V_2(d, k_{-1}) = u'\Big((P(d) + d)k_{-1} - P(d)k \Big)(P(d) + d)$$
$$\implies V_2(d', k) = u'\Big((P(d') + d')k - P(d') \Big)(P(d') + d')$$

Combining the 2 equations we arrive at the Euler equation of the Household problem:

$$u'(c_{it})P_t = \beta E_t \Big[u'(c_{i,t+1})(P_{t+1} + d_{t+1}) \Big] \tag{1}$$

3 Market clearing

The goods market clears so that the total fruit endowment is consumed:

$$\int c_{it} di = \int d_t dj$$

where I is the measure of households and J the measure of trees, both being equal to 1. Since the households are homogenous in both preferences and shocks, the decision problem of household i is inedependent of i. That is $c_{it} \equiv c_t$. Thus the market clearing condition becomes:

$$c_t = d_t \tag{2}$$

4 Model Equations

Thus the model equations become:

$$P_t = \beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (P_{t+1} + d_{t+1}) \right] \tag{3}$$

$$c_t = d_t \tag{4}$$

$$d_t = \mu_d + \rho_d d_{t-1} + \epsilon_t \tag{5}$$

where $\epsilon \sim N(0, \sigma^2)$.

5 Parameters

We use the folowing parameters follwing Wouter's code:

$$\mu_d = 0.1$$
$$\rho_d = 0.9$$
$$\sigma = 0.1$$
$$\beta = 0.9$$
$$\gamma = 3$$

6 Model solution

We use projection methods to solve Equation (3). The only state variable is d_t. We asssume the P_t is a function of d_t. We approximate that function with the nth order polynomial:

$$P_t(d_t) = \sum_{n=0}^{N} a_n d_t^n$$

7 Model outputs

We calculate the risk free rate, and the risk premium on the asset paying the dividend stream. The risk free rate is given by:

$$R_{ft} = \frac{1}{\beta E_t(c_{t+1}/c_t)^{-\gamma}}$$

The expected return is given by:

$$r_t = \frac{d_t + P_t}{P_{t-1}}$$
$$x_t = E_t[r_{t+1}]$$

Since all these variables are driven by d_t which is Gaussian, we can caluclate the above expectations using the Gauss Hermite algorithm.

8 Output analysis

- Figure (1) shows how price varies with the state variable (dividends). Price increases with dividends as is to be expected from standard asset pricing theory.
- Figure (2) shows how the returns vary with dividends. The expected return varies inversely with price and consequently decreases with dividends.
- The risk free rate and the behaviour of the risk premium is hard to predict from theory.
- Figure (3) shows the simulations of the model.

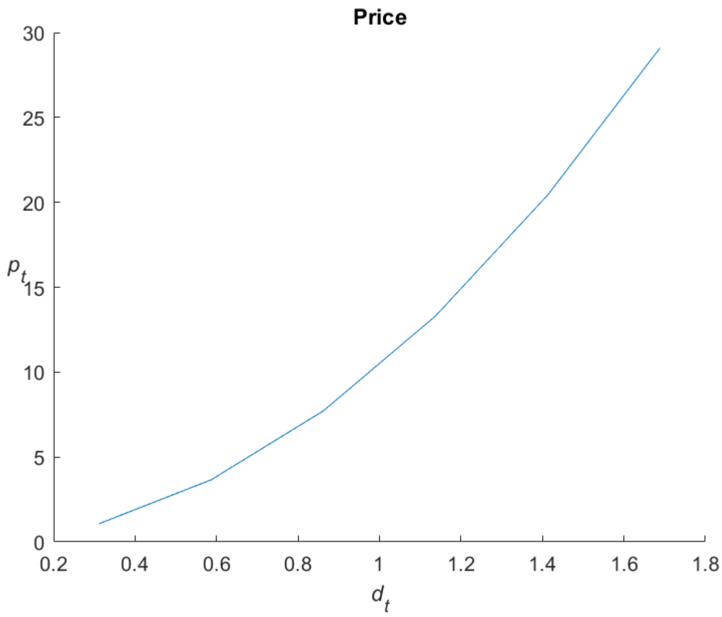

Figure 1: Price with state

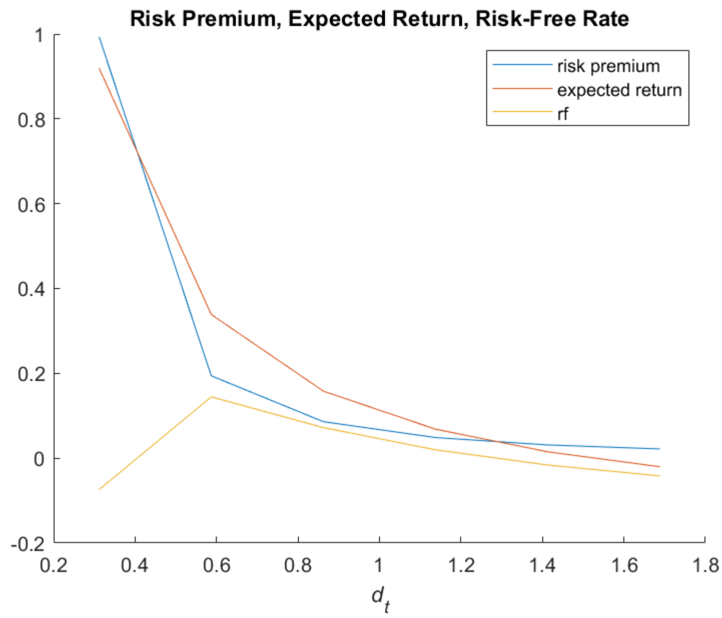

Figure 2: returns with state

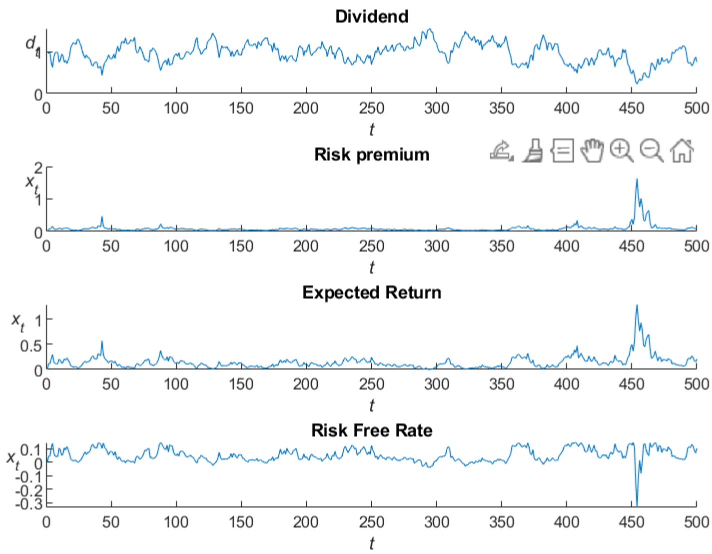

Figure 3: Simulations