SYDE 114 - Final Assignment

Flattening the Curve

Name: Anni Wang

Course: SYDE 114

Professor: Stephen Birkett

Department of Systems Design Engineering

University of Waterloo Waterloo, Ontario Canada

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Question 1

The SIR equations as outlined within the assignment was implemented into Matlab through the following program:

```
function dydt = COVID_19(t,y,b,k)

   ds = -b*y(1)*y(2);
   di = b*y(1)*y(2) - k*y(2);
   dr = k*y(2);

   dydt = [ds; di; dr];
   dydt = dydt(:);

end
```

Question 2

The conditions given by Question 2 was solved using the Matlab program provided in Question 1 along with the following command:

```
%% Differential Equation
N = 10^7; % Susceptible Population
I_0 = 10; %Initial infectious individuals
tspan = [0 70]; %From 0 to 70 days
b = 0.8; %Number of close contacts per day
k = 1/3; %Portion of I that recovers per day
y0 = [1, I_0/N, 0]; %Initial Values for s, i, r

[t,y] = ode45( @(t,y)COVID_19(t,y,b,k), tspan, y0);

%% Plot
plot(t,y)
axis([0 70 0 1.15])
yticks(0:0.1:1.2)
legend('Susceptible','Infected','Recovered')
title('SIR Model for Question 2')
xlabel('t (Days)'), ylabel('s(t), i(t), r(t)')
grid on;
grid minor;
```

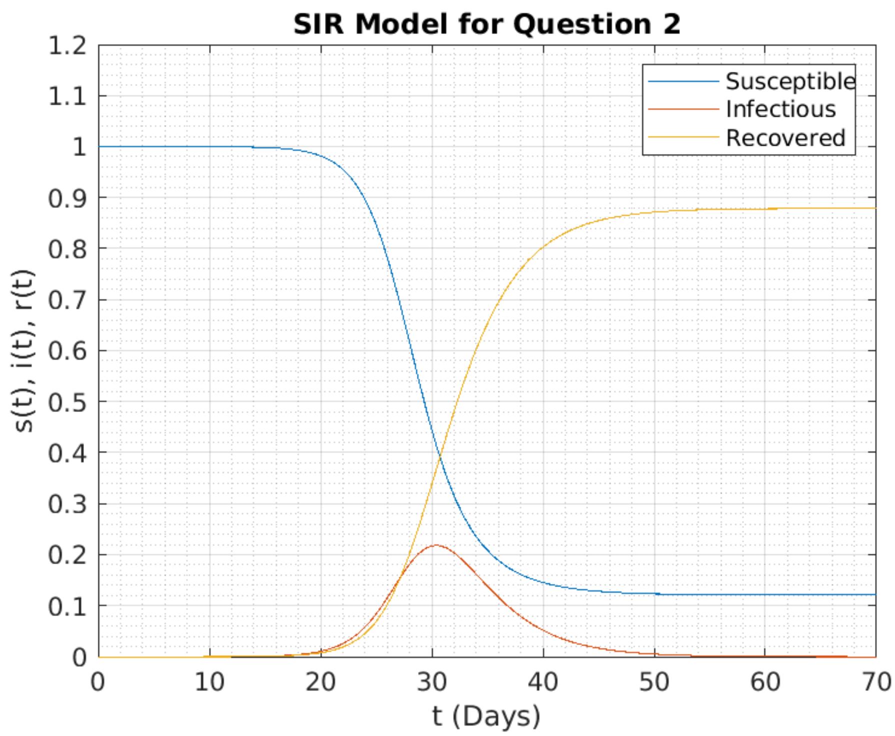

Figure 1. SIR Model for $b = 0.8$, $k = 1/3$, $I_0 = 10$, $N = 10^7$ initial conditions, with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

The findings as predicted by the SIR Model for the conditions given by Question 2 can be summarized as follows:

The population starts off with 10 infectious individuals, and a population of roughly 10^7 individuals or susceptible individuals. As the number of days increase, the number of infectious individuals would begin to increase at a faster rate, the number of recovered individuals would also increase at a faster rate, and the number of susceptible individuals would decrease at a faster rate. Noticeable changes to the population in each group would begin on roughly the 17th day.

At 27.5 days, the number of recovered individuals would surpass the number of infectious individuals at 1.55×10^6 individuals in each group or 15.5% of the population. The number of infectious individuals peaks at roughly 30.5 days after the initial outbreak, with 22% of the population infectious or around 2.2×10^6 infectious individuals.

After the peak, the number of infectious individuals would decrease, the rate of change in the number of recovered individuals and the number of susceptible individuals would approach zero. At around 30.7 days, the number of recovered individuals would surpass the number of susceptible individuals with roughly 39% of the population or 3.9×10^{6} individuals susceptible and recovered.

The population would be rid of the infectious disease after roughly 57 days. By then, there would be 0 infectious individuals. 8.8×10^{6} individuals or roughly 88% of the population would have contracted and recovered/died from the disease. 1.2×10^{6} individuals or roughly 12% of the population would remain susceptible, meaning that they would have never contracted the disease.

Question 3

Variable i_0 can be varied by a change in population or a change in the initial number of infectious individuals (I_0). For this question, I have chosen to vary $I_0 = 10$ (as presented in Question 2) by factors of 10, 100, and 1000, while keeping the total population constant at $N = 10^7$. For $I_0 = 1, 10, 100, 1000, 10000$, the respective i_0 values are $i_0 = 10^{-7},\ 10^{-6},\ 10^{-5},\ 10^{-4}, 10^{-3}$. The curves $s(t), i(t), r(t)$ for varying values of i_0 can be plotted on individual graphs to demonstrate, in an organized manner, the effects of a change in i_0 on the SIR Model. For a sample of a complete SIR Model with all three curves refer to *Figure 1*. The three graphs for curves $s(t), i(t), r(t)$ are presented below:

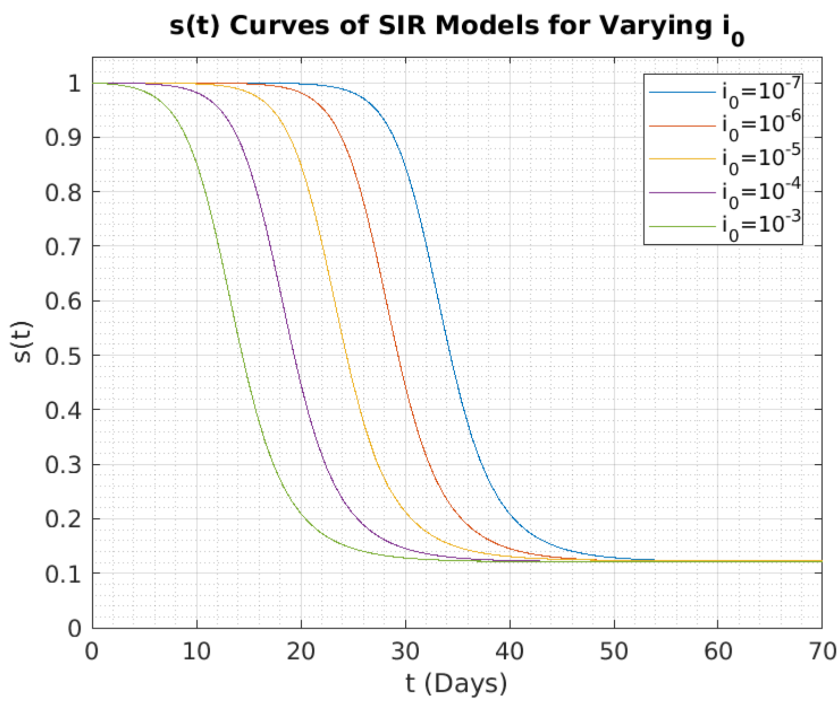

Figure 2. $s(t)$ curves of SIR Models for five values of i_0 that differ by factors of 10, with initial conditions $b = 0.8$, $k = 1/3$, $N = 10^7$. $s(t)$ on the y-axis and t in Days on the x-axis.

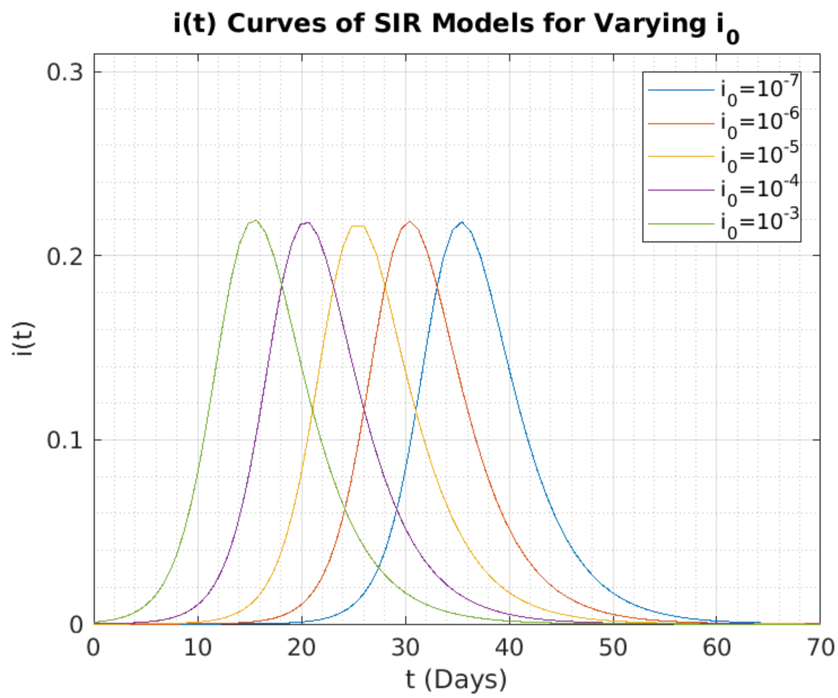

Figure 3. $i(t)$ curves of SIR Models for five values of i_0 that differ by factors of 10, with initial conditions $b = 0.8$, $k = 1/3$, $N = 10^7$. $i(t)$ on the y-axis and t in Days on the x-axis.

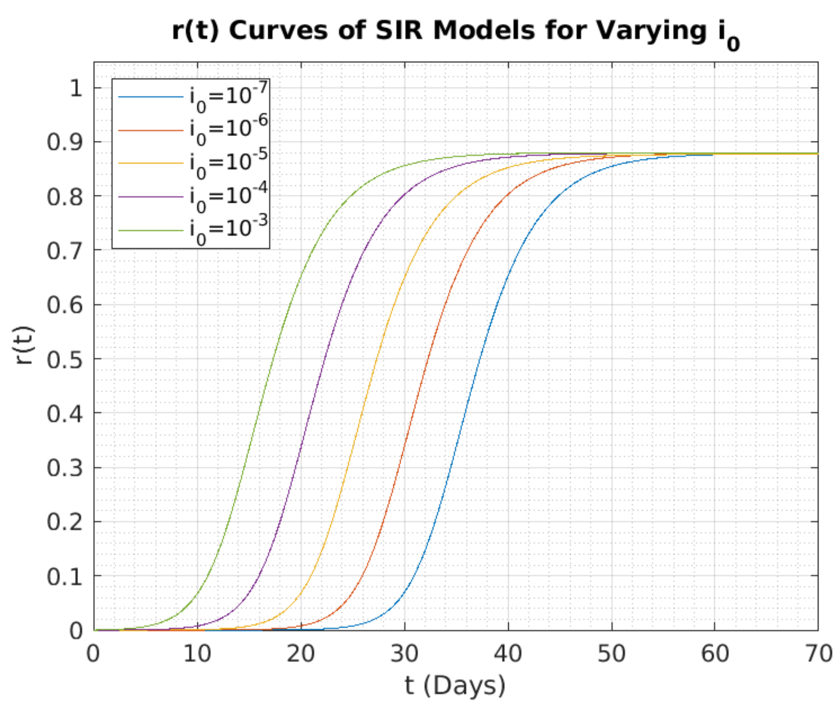

Figure 4. $r(t)$ curves of SIR Models for five values of i_0 that differ by factors of 10, with initial conditions $b = 0.8$, $k = 1/3$, $N = 10^7$. $r(t)$ on the y-axis and t in Days on the x-axis.

As presented in *Figure 2, 3, 4* increasing the i_0 value has the effect of shortening the duration of the initial flat period where $i(t),\ r(t),\ s(t) \approx 0$, for each of the curves or SIR Models. The size and shape of the curves remain the same in all figures. This is akin to saying that an increase in initial cases (I_0) would start the epidemic more quickly, but would not affect the outcome of the outbreak in any other way.

Question 4

To vary b, initial conditions are kept the same to that of Question 2: $k = 1/3$, $I_0 = 10$, $N = 10^7$. The graphs for the SIR Model with $b = 0.5, 1.0, 1.5, 2.0$ are presented below. The SIR Model for $b = 0.8$ is presented in *Figure 1*.

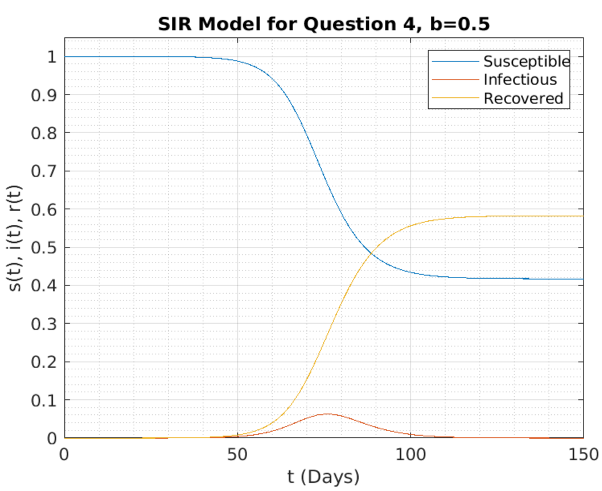

Figure 5. SIR Model for $b = 0.5$, with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

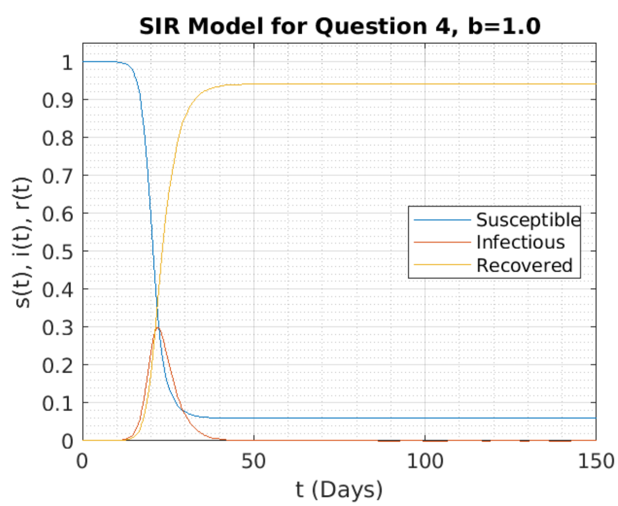

Figure 6. SIR Model for $b = 1.0$, with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

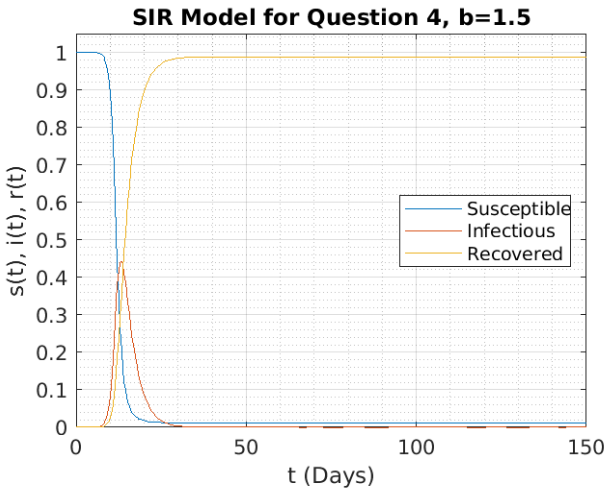

Figure 7. SIR Model for $b = 1.5$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

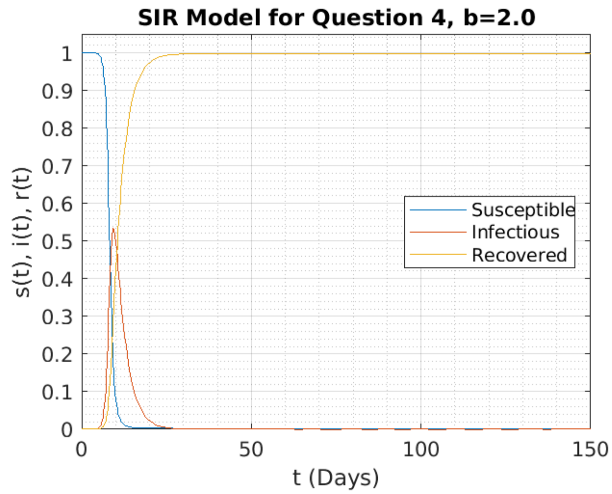

Figure 8. SIR Model for $b = 2.0$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

To better observe the trends that occur with a change in b, *Figure 1, 5, 6, 7, 8* can be plotted together on a single graph:

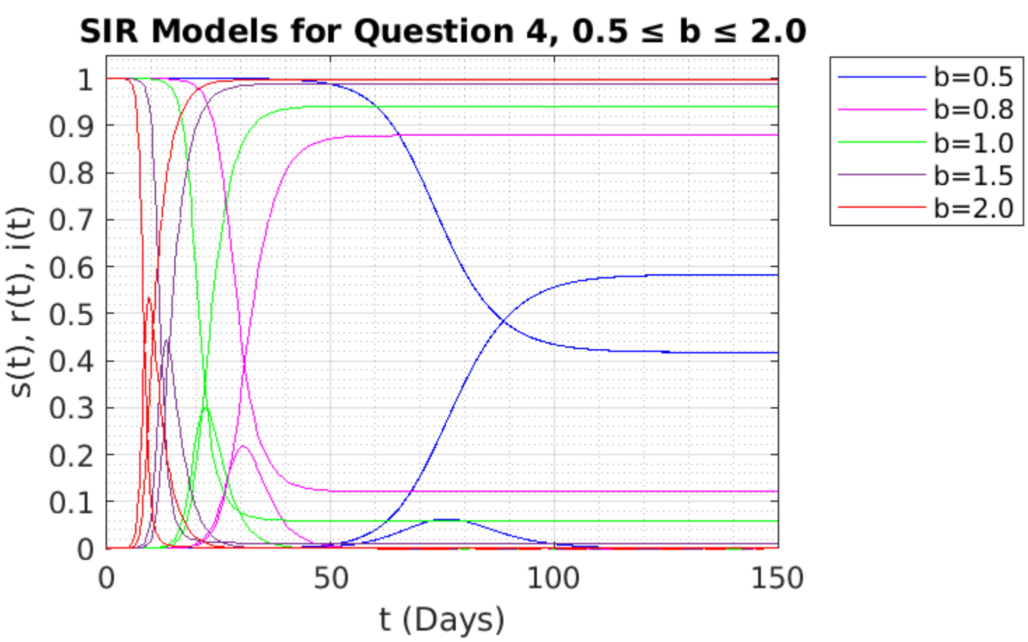

Figure 9. Collection of SIR Models for $b = 0.5, 0.8, 1.0, 1.5, 2.0$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

By examining *Figure 5,6,7,8,9* , it can be observed that with a change in b, the outcome of the epidemic would change in several ways. Firstly, the duration of the initial flat period for the curves would shorten as b increases. Secondly, with an increase in b, the $i(t)$ curves appear to be stretched vertically, reaching a higher peak in a shorter amount of time. As a result, the $r(t)$ curves increase more rapidly and the $s(t)$ curves decrease more rapidly. By the end of the epidemic, the fraction of the population that gets infected and recovers from the disease approaches 1, and the fraction of the population that remains susceptible to the disease by the end of the epidemic approaches 0 with an increase in b. The effects of an increase in b appears to be more drastic when b is increased from a smaller value.

In a real life context, the SIR Models suggest that with an increase in b the epidemic would start more quickly, and the disease would infect a greater fraction of the population at a faster rate. This outcome can be reasoned by the definition of b. Parameter b represents the number of close contacts per day per infected. If infectious individuals come in contact with a greater number of individuals, the probability of them infecting a susceptible individual grows. A greater number of contacts per day per infected would expedite the outbreak cycle along with infecting a greater fraction of the population. Hence, explaining the trends observed in *Figure 9*.

Contact Number Calculation

c represents the total number of close contacts per infected individual. The range of c associated with a change in b value is calculated below. Firstly, the assignment provides that

$$c = b/k \tag{4.1}$$

This property can be used to calculate the c value for endpoints in the range $0.5 \leq b \leq 2.0$, where $b = 0.5$ and $b = 2.0$, with a fixed constant $k = 1/3$.

$$c = 0.5 \div \tfrac{1}{3} = 1.5 \tag{4.2}$$

$$c = 2.0 \div \tfrac{1}{3} = 6.0 \tag{4.3}$$

Thus, it can be concluded that for the range $0.5 \leq b \leq 2.0$, c contains the range $1.5 \leq c \leq 6.0$.

Relation to Public Health Practices

Currently, individuals around the world are practicing social distancing as a means of reducing the number of close contacts for every individual. Social distancing is done by limiting or restricting activities in public areas by remaining at home. This public health practice would help reduce the b and c values of the COVID-19 epidemic by limiting close contacts for the entire population. This action is necessary as there exists an asymptomatic period for this virus where the spread of this disease is still possible. Thus, by practicing social distancing, the fraction of the population affected by COVID-19 can be significantly reduced.

Question 5

To vary k, initial conditions are kept the same to that of Question 2: $b = 0.8$, $I_0 = 10$, $N = 10^7$. The graphs for the SIR Model with $k = 0.1, 0.2, 0.45, 0.6$ are presented below. The SIR Model for $k = 1/3$ is presented in *Figure 1*.

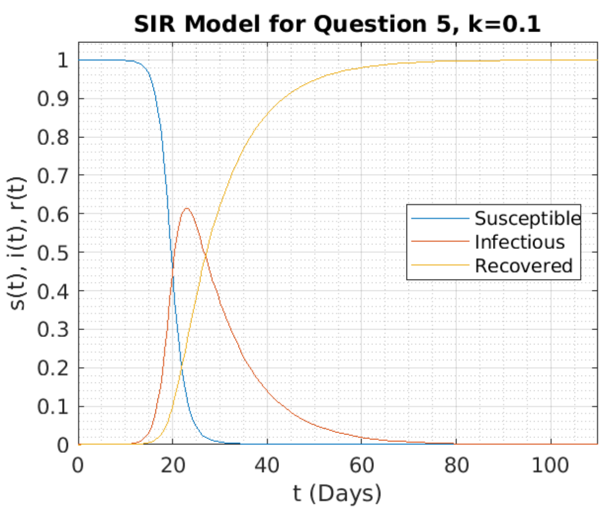

Figure 10. SIR Model for $k = 0.1$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

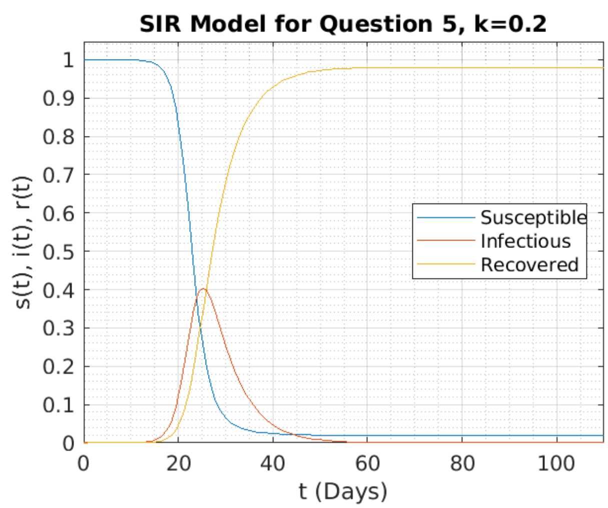

Figure 11. SIR Model for $k = 0.2$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

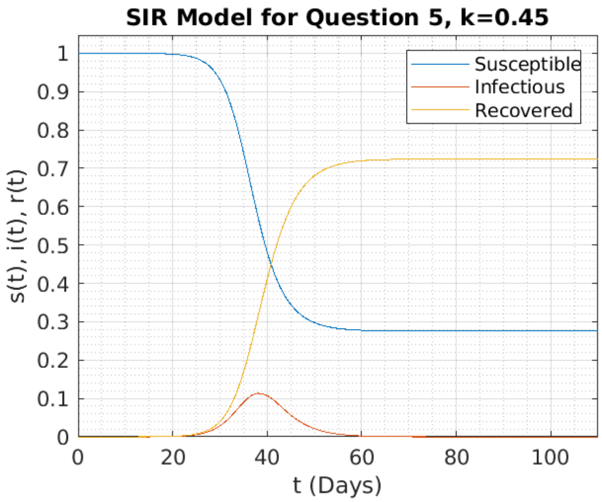

Figure 12. SIR Model for $k = 0.45$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

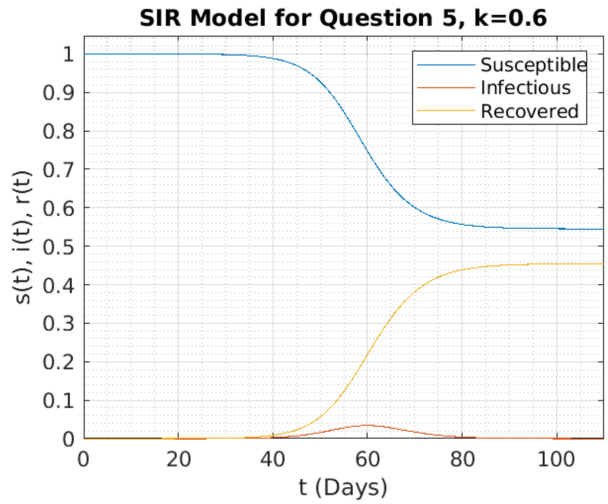

Figure 13. SIR Model for $k = 0.6$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

To better observe the trend that occurs with a change in *k*, *Figure 1, 10, 11, 12, 13* can be plotted together on a single graph:

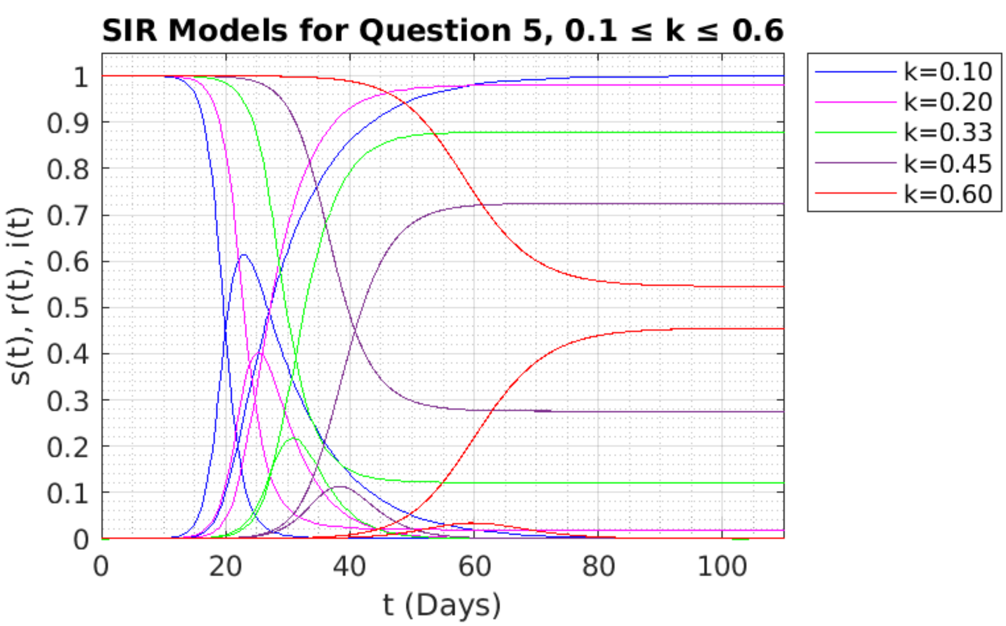

Figure 14. Collection of SIR Models for $b = 0.10, 0.20, 0.\overline{33}, 0.45, 0.6$ with $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

By examining *Figure 10, 11, 12, 13, 14* , it can be observed that with a change in *k*, the outcome of the epidemic would change in several ways. Firstly, the duration of the initial flat period for the curves would shorten as *k* decreases. Secondly, with a decrease in *k*, the $i(t)$ curves would reach a higher peak in a shorter amount of time, however, the duration of the decrease of $i(t)$ from the peak would not shorten. As a result, the $s(t)$ curve would decrease more rapidly, while the rate of change for the $r(t)$ curve may vary. By the end of the epidemic, the fraction of the population that recovers from the disease approaches 1 and the fraction of the population that remains susceptible to the disease approaches 0 with a decrease in *k*.

In a real life context, the SIR Models suggest that with a decrease in *k*, the epidemic would start more quickly, the disease would infect a greater fraction of the population at a faster rate, and the disease may plague the population for a longer period of time. This outcome can be reasoned by the definition of *k*. Parameter *k* represents the numbers of individuals moving to the recovered group per day, and $\frac{1}{k}$ is assumed to be the average number of days an individual remains infected. A decrease in *k* would signify a decrease in the number of individuals that recover each day and an increase in the number of days an individual remains infectious. If the infectious period is prolonged and if the number of close contacts remains constant, then the probability of an infectious individual to infect a susceptible individual increases. A greater number of individuals would be infected at a faster rate, and infectious individuals would remain infectious for a longer time. Hence, this would explain the trends observed in *Figure 14*.

Contact Number Calculation

The range of *c* associated with a change in *k* values can be calculated using Equation 4.1. Given $0.1 \leq k \leq 0.6$, where $k = 0.1$ and $k = 0.6$, and the fixed constant $b = 0.8$,

$$c = 0.8 \div 0.1 = 8.0 \tag{5.2}$$

$$c = 0.8 \div 0.6 = 1.\overline{3} \tag{5.3}$$

Thus, it can be concluded that for the $0.1 \leq k \leq 0.6$, *c* contains the range $1.\overline{3} \leq c \leq 8.0$.

Average Duration of Infection Calculation

For this range of k, the the corresponding average numbers of days an individual is infectious or $\frac{1}{k}$ values would be:

$$\frac{1}{k} = \frac{1}{0.1} = 10 \tag{5.4}$$

$$\frac{1}{k} = \frac{1}{0.6} = \frac{5}{3} \tag{5.5}$$

Thus, for $0.1 \leq k \leq 0.6$, $\frac{5}{3} \leq \frac{1}{k} \leq 10$.

Relation to Public Health Practices

A decrease in k would signify less individuals moving from the I to the R group each day, or a greater quantity of time taken to recover. To manipulate this parameter, it would likely require medical treatment to help reduce the infectious period of the disease. In terms of public health practices, individuals should be encouraged to seek medical care as soon as they observe symptoms, or even when symptoms are speculated. However, this public health practice would only be effective if a potential treatment exists for the disease and if hospitals are not at capacity and care is able to be provided to every patient.

Question 6

Equation 3 as provided by the assignment is,

$$\frac{di}{dt} = bs(t)i(t) - ki(t) \tag{6.1}$$

Given $k = b/c$, and that $i(t)$ can be factored out from both terms, the equation yields

$$\frac{di}{dt} = i(t)(bs(t) - \frac{b}{c}) \tag{6.2}$$

The b term can then be factored out of the Equation 6.2.

$$\frac{di}{dt} = bi(t)(s(t) - \frac{1}{c}) \tag{6.3}$$

Since, $bi(t)$ would always a positive term, because $b \geq 0$ and $i(t) \geq 0$, the sign of $\frac{di}{dt}$ will be dependent on the $(s(t) - \frac{1}{c})$ term. Thus,

$$\frac{di}{dt} < 0 \text{ when } s(t) - \frac{1}{c} < 0 \tag{6.4}$$

Or it can be said that, when

$$\frac{di}{dt} < 0 \text{ when } s < \frac{1}{c} \quad (6.5)$$

The peak value of $i(t)$ would occur at the local maximum of $i(t)$ when,

$$\frac{di}{dt} = 0, \; s = \frac{1}{c} \quad (6.6)$$

The condition for when $\frac{di}{dt} = 0$, $s = \frac{1}{c}$ can be demonstrated using a few cases from the previous questions.

Case 1 for di/dt = 0, s =1/c

Using initial conditions provided in Question 2, $b = 0.8$, $k = ⅓$, $I_0 = 10$, $N = 10^7$.

Given $c = b/k$ and $s = 1/c$, then,

$$s = k/b \quad (6.7)$$

When $b = 0.8$, $k = 1/3$,

$$s = \frac{1}{3} \div 0.8 \quad (6.8)$$

This yields, $s = \frac{5}{12}$. Thus, when $\frac{di}{dt} = 0$, $s = \frac{5}{12} \approx 0.4167$.

The $s(t)$ value for when $s = 1/c$ can be solved by Matlab using the following method.

1. The peak of $i(t)$ can be found using the following command:

```
N = 10^7;
I_0 = 10;
tspan = [0 70];
b = 0.8;
k = 1/3;
y0 = [1, I_0/N, 0];

[t,y] = ode45( @(t,y)COVID_19(t,y,b,k), tspan, y0);
[M, I] = max(y(:,2))
```

   This yields: M = 0.2188, I = 32.

2. Using the I and M values, where $M = i_{max}$ or $i(t)$ at its peak, the corresponding $s(t)$ value for when $s(t) = 1/c$ can be found through inspection of the *y* values using WORKSPACE. $s(t)$ values are displayed in the '1' column, and $i(t)$ are displayed in the '2' column.

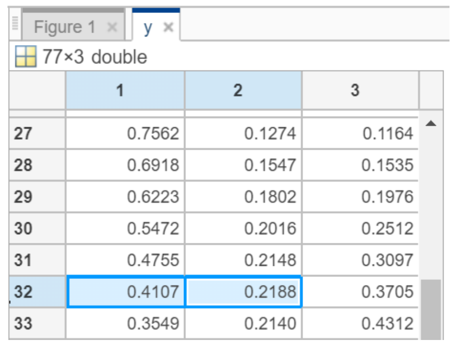
Figure 1 × | y ×
77×3 double

	1	2	3
27	0.7562	0.1274	0.1164
28	0.6918	0.1547	0.1535
29	0.6223	0.1802	0.1976
30	0.5472	0.2016	0.2512
31	0.4755	0.2148	0.3097
32	0.4107	0.2188	0.3705
33	0.3549	0.2140	0.4312

Figure 15. Display of Matlab WORKSPACE for *y* values of the SIR Model for question 2. The $s(t)$ and $i(t)$ values for when $\frac{di}{dt} = 0$ and $s = \frac{1}{c}$ is highlighted in blue.

3. The $s(t)$ and $i(t)$ can be verified graphically to ensure that $i(t)$ is located at $\frac{di}{dt} = 0$, and the points where $i(t)$ and $s(t)$ land contains the same *t* value (vertical line test). This can be done by finding the points for the $s(t)$ and $i(t)$ values on their respective curves on the SIR Model. The two points are plotted in the figure below:

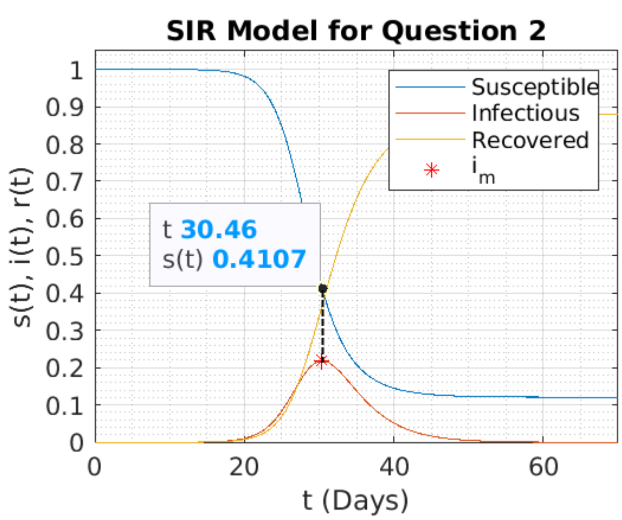

Figure 16. SIR Model for Question 2, with local max of *i(t)* plotted as i_m and the corresponding $s(t)$ point where $s = 1/c$ displayed as a black dot with its coordinates shown.

The conditions mentioned above can be verified to be true.

As can be seen in *Figure 16*, when $\frac{di}{dt} = 0$, $s = 0.4107$ which is roughly equal to 0.4167, the value calculated above for when $s = 1/c$. The discrepancy between the two s values can be explained by floating point arithmetics, and the size of dt being too large at $dt = 1$ day. Therefore, Case 1 verifies that the conditions calculated in equation 6.6 is true.

Case 2 for di/dt = 0, s =1/c

Using initial conditions provided in question 4, $b = 2.0$, $k = ⅓$, $I_0 = 10$, $N = 10^7$.

Given Equation 6.7, when $b = 2.0$, $k = 1/3$,

$$s = \frac{1}{3} \div 2.0 \tag{6.9}$$

This yields, $s = \frac{1}{6}$. Thus, when $\frac{di}{dt} = 0$, $s = \frac{1}{6} \approx 0.167$.

Using the same method as described in Case 1, the following value and graph can be found:

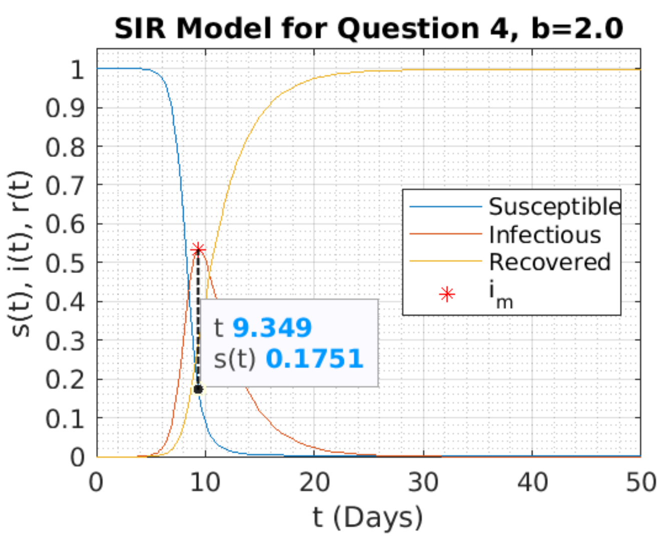

Figure 17. SIR Model for Question 4 where $b = 2.0$, with local max of $i(t)$ plotted as i_m and the corresponding $s(t)$ point where $s = 1/c$ displayed as a black dot with its coordinates shown.

As can be seen in *Figure 17*, when $\frac{di}{dt} = 0$, $s = 0.1751$ which is roughly equal to 0.167, the value calculated above for when $s = 1/c$. Therefore, Case 2 verifies that the conditions calculated in equation 6.6 is true.

Case 3 for di/dt = 0, s =1/c

Using initial conditions provided in question 5, $b = 0.8$, $k = 0.6$, $I_0 = 10$, $N = 10^7$

Given equation 6.7, when $b = 0.8$, $k = 0.6$,

$$s = 0.6 \div 0.8 \tag{6.10}$$

This yields, $s = \frac{3}{4}$. Thus, when $\frac{di}{dt} = 0$, $s = \frac{3}{4} \approx 0.75$.

Using the same method as described in Case 1, the following value and graph can be found:

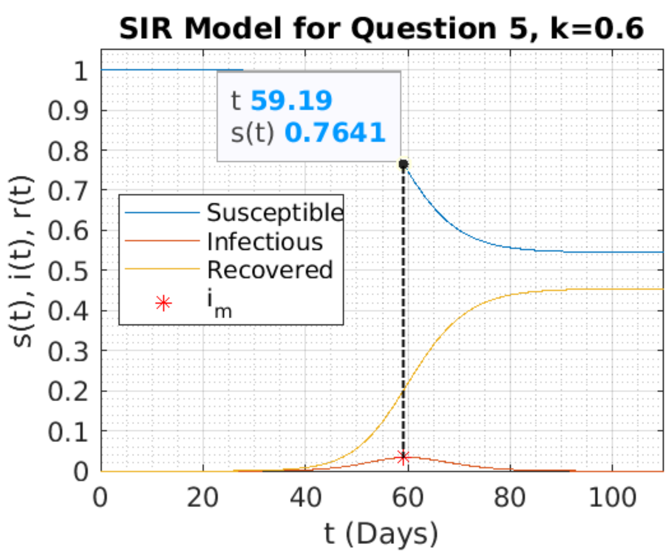

Figure 18. SIR Model for question 5 where $k = 0.6$, with local max of $i(t)$ plotted as i_m and the corresponding *s(t)* point where $s = 1/c$ displayed as a black dot with its coordinates.

As can be seen in *Figure 18*, when $\frac{di}{dt} = 0$, $s = 0.7641$ which is roughly equal to 0.75, the value calculated above for when $s = 1/c$. Therefore Case 3 verifies that the conditions calculated in equation 6.6 is true.

Case for di/dt < 0

To find a case where the initial di/dt is negative, Equation 6.5 can be used: $\frac{di}{dt} < 0$, when $s < \frac{1}{c}$. Since $c = b/k$, the equation can be written as,

$$\frac{di}{dt} < 0 \text{, when } s < \frac{k}{b} \tag{6.11}$$

The initial conditions for s when $t = 0$ is given as $s = 1$. Equation 6.11 can be rewritten as,

$$\frac{di}{dt} < 0 \text{, when } b < k \tag{6.12}$$

Given the conditions provided in Question 2 and 3, $b = 0.8$ and $k = 1/3$, it cannot satisfy equation 6.12, since $0.8 > 1/3$, thus, $b > k$.
Given the conditions provided in Question 4, $k = 1/3$ is a fixed value and $0.5 \leq b \leq 2.0$, it cannot satisfy equation 6.12, since there exists no values in $0.5 \leq b \leq 2.0$, where $b < k$.
Given the conditions provided in Question 5, $b = 0.8$ is a fixed value and $0.1 \leq k \leq 0.6$, it cannot satisfy equation 6.12, since there exists no values in $0.1 \leq k \leq 0.6$, where $b < k$.

There are no cases in the answers above where the rate of change of i is negative initially and no epidemic occurs. This is because if $b < k$, the total number of close contacts per infected individual or c would be less than 1, meaning that the probability of the disease infecting a fraction of the population greater than the initial cases is very low. The disease would eventually disappear, causing no epidemics to occur. All the above cases depict growth in the infectious fraction of the population or i, thus, epidemics would occur.

Question 7

Using the chain rule, $\frac{di}{dt}$ can be written as,

$$\frac{di}{dt} = \frac{di}{ds} \cdot \frac{ds}{dt} \tag{7.1}$$

This can rearranged to produce,

$$\frac{di}{ds} = \frac{di}{dt} \div \frac{ds}{dt} \tag{7.2}$$

Given that $\frac{di}{dt} = bi(t)(s(t) - \frac{1}{c})$ and $\frac{ds}{dt} =- bs(t)i(t)$, this yields,

$$\frac{di}{ds} = \frac{-s(t) + \frac{1}{c}}{s(t)} \tag{7.3}$$

Equation 7.3 can be rearranged to produce,

$$\frac{di}{ds} = -1 + 1/cs \tag{7.4}$$

Multiplying both sides by ds, and integrating both sides gives,

$$\int di = \int (-1 + \frac{1}{cs})\, ds \tag{7.5}$$

Integrating Equation 7.5 will yield,

$$i(s) = \frac{ln(|s|)}{c} - s + C \tag{7.6}$$

Rearranging Equation 7.6 and plugging in values for the initial values of i and s at $t = 0$, where $t = 0$, $s = 1$ and $i = I_0/N$ gives,

$$C = \frac{I_0}{N} - \frac{ln(1)}{c} + 1 \tag{7.7}$$

Since $ln(1) = 0$, and $N >> I_0$ then $\frac{I_0}{N} \approx 0$, thus,

$$C = \frac{I_0}{N} + 1 \approx 1 \tag{7.8}$$

Plugging the arbitrary constant C back into Equation 7.6 yields,

$$i(s) = \frac{ln(|s|)}{c} - s + 1 \tag{7.9}$$

Rearranging Equation 7.9 in terms of s gives,

$$c = \frac{ln(|s|)}{i(s)+s-1} \tag{7.10}$$

Using Equation 7.10, as $t \to \infty$, $i \to 0$ and $s \to s_\infty$, thus, this gives,

$$c = \frac{ln(s_\infty)}{s_\infty - 1} \tag{7.11}$$

Therefore, Equation 7.10 yields Equation 4 provided within the assignment.

This property can be illustrated using a few cases from the previous answers as described below.

Case 1 for finding c

Using initial conditions provided in Question 2, $b = 0.8$, $k = ⅓$, $I_0 = 10$, $N = 10^7$.

Given that,

$$c = b/k \tag{7.12}$$

When $b = 0.8$, $k = 1/3$,

$$c = 0.8 \div \frac{1}{3} \tag{7.13}$$

This yields, $c = \frac{12}{5} = 2.4$.

To verify Equation 7.11, the value for s_∞ can be found using Matlab.

1. The span of t can be set to a large number where s has settled to its steady state value.
2. Using Matlab WORKSPACE for y values, the last displayed $s(t)$ value can be used as s_∞. The s_∞ value is shown in the graph below.

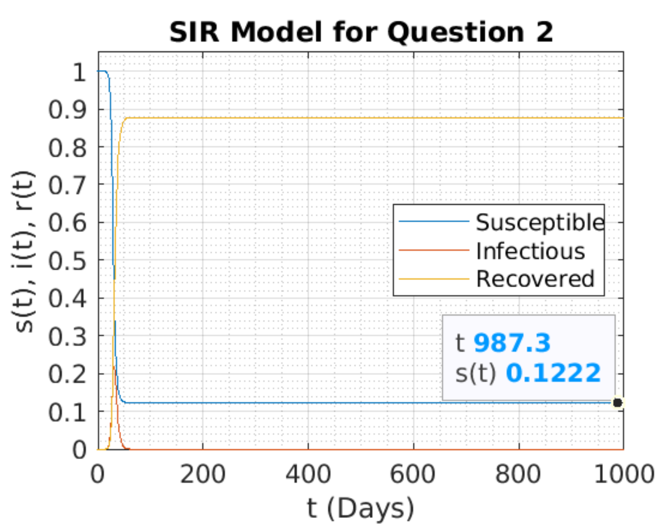

Figure 19. SIR Model for Question 2, with s_∞ displayed as $s_{987.3}$

Given Equation 7.11 and $s_\infty = 0.1222$,

$$c = \frac{ln(0.1222)}{0.1222-1} = 2.39473255 \tag{7.14}$$

This confirms equation 7.11, since given c as calculated in Equation 7.13, $c = 2.4 \approx 2.39473255$. The discrepancies can be attributed to floating point arithmetic error, and/or that s have not completely settled at a steady state, it can be taken at an even larger value.

Case 2 for finding c

Using initial conditions provided in Question 4, $b = 2.0$, $k = ⅓$, $I_0 = 10$, $N = 10^7$.

Given equation 7.12, and $b = 2.0$, $k = 1/3$,

$$c = 2.0 \div \frac{1}{3} \tag{7.15}$$

This yields, $c = 6$.

Using the same process as described in Case 1, the following graph can be produced.

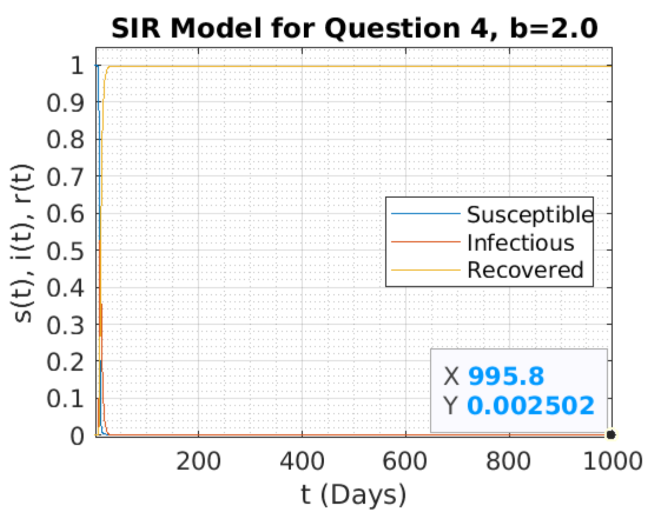

Figure 20. SIR Model for Question 4 $b = 2.0$, with s_∞ value displayed as $s_{995.8}$

Given Equation 7.11 and $s_\infty = 0.002502$,

$$c = \frac{ln(0.002502)}{0.002502-1} = 6.005691106 \tag{7.16}$$

This confirms equation 7.11, since given c as calculated in Equation 7.15, $c = 6 \approx 6.005691106$.

Case 3 for finding c

Using initial conditions provided in Question 5, $b = 0.8$, $k = 0.6$, $I_0 = 10$, $N = 10^7$.

Given equation 7.12, and $b = 0.8$, $k = 0.6$,

$$c = 0.8 \div 0.6 \tag{7.17}$$

This yields, $c = \frac{4}{3} = 1.\overline{3}$.

Using the same process as described in Case 1, the following graph can be produced.

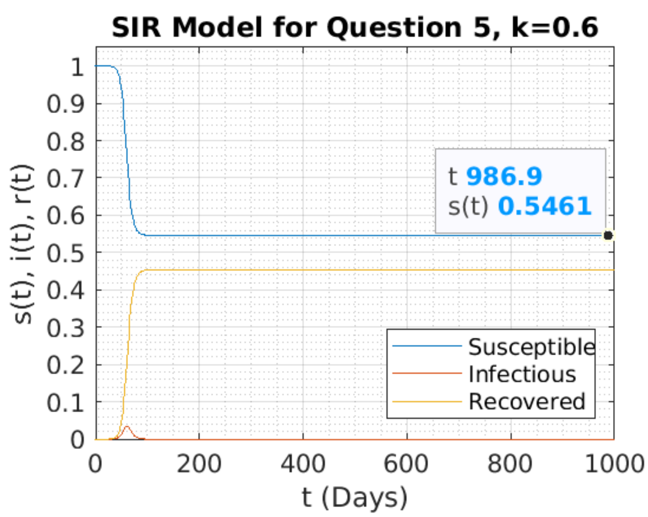

Figure 21. SIR Model for Question 5 $k = 0.6$, with s_∞ value displayed as $s_{986.9}$

Given $c = \frac{ln(s_\infty)}{s_\infty - 1}$ and $s_\infty = 0.5461$,

$$c = \frac{ln(0.5461)}{0.5461 - 1} = 1.332789535 \tag{7.18}$$

This confirms equation 7.11,since given c as calculated in Equation 7.17, $c = 1.\overline{3} \approx 1.33278953$.

Question 8

To vary *s* and *r*, initial conditions are kept the same to that of Question 2: $b = 0.8$, $k = 1/3$, $I_0 = 10$, $N = 10^7$. It should be noted that $s + i + r = 1$, where $i = I_0/N \approx 0$, thus, $s + r = 1$. The SIR Model for $s = 1$, and $r = 0$ is presented in *Figure 1*. Permutations for SIR Models with $s = 0.9,\ 0.8,\ 0.7,\ 0.6,\ 0.5,\ 0.4$ and $r = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6$ respectively are displayed below.

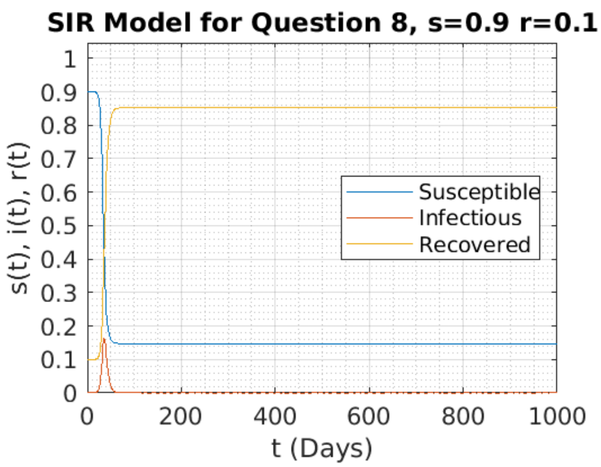

Figure 22. SIR Model for Question 2, with $s = 0.9$ and $r = 0.1$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and *t* in Days on the x-axis.

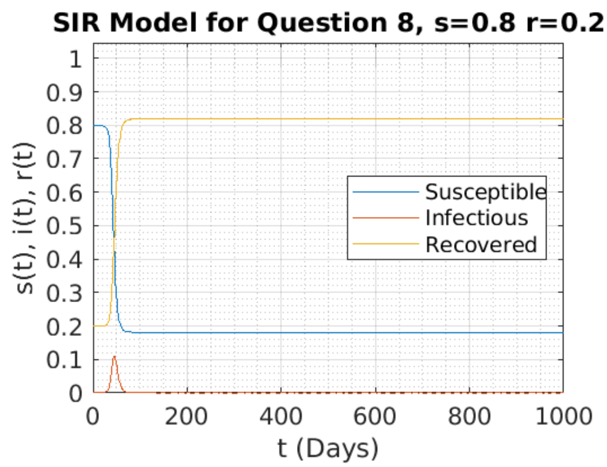

Figure 23. SIR Model for Question 2, with $s = 0.8$ and $r = 0.2$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and *t* in Days on the x-axis.

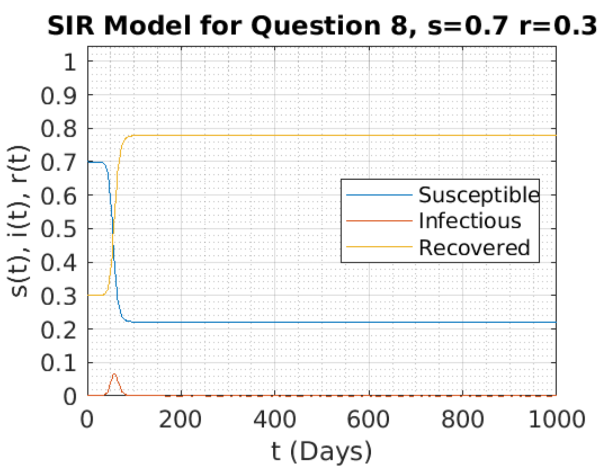

Figure 24. SIR Model for Question 2, with $s = 0.7$ and $r = 0.3$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and *t* in Days on the x-axis.

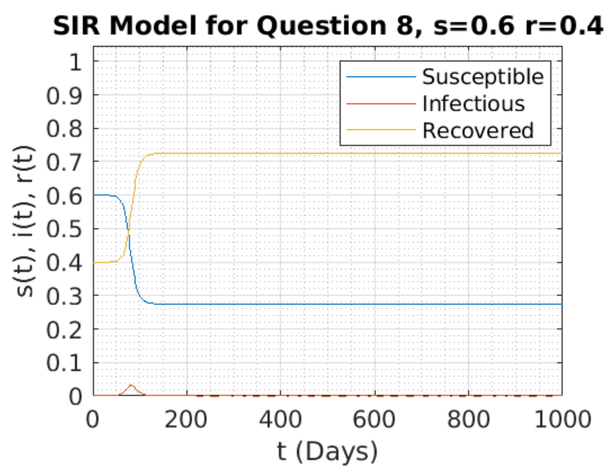

Figure 25. SIR Model for Question 2, with $s = 0.6$ and $r = 0.4$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and *t* in Days on the x-axis.

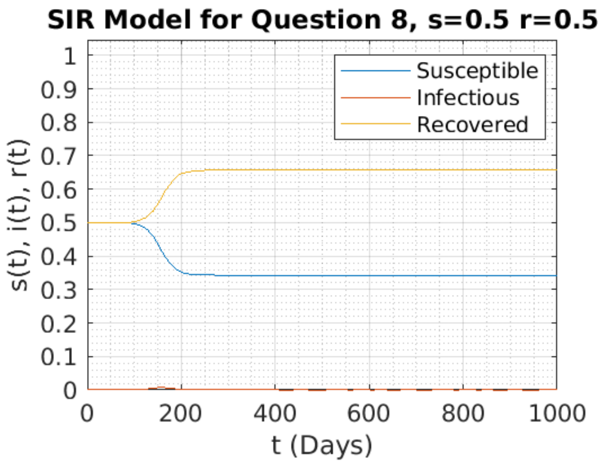

Figure 26. SIR Model for Question 2, with $s = 0.5$ and $r = 0.5$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

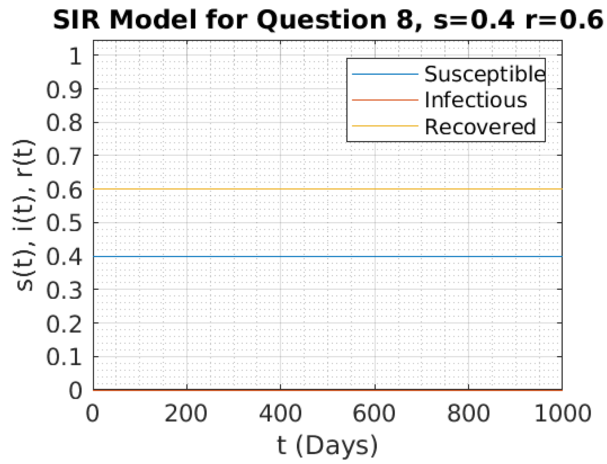

Figure 27. SIR Model for Question 2, with $s = 0.4$ and $r = 0.6$ and $s(t)$, $i(t)$, $r(t)$ on the y-axis and t in Days on the x-axis.

From *Figure 1, 22, 23, 24, 25, 26, 27*, it can be observed that with a decrease in s and an increase in r, the peak of the $i(t)$ curves would begin to flatten. As a result, the $s(t)$ curves and the $r(t)$ curves appear to be compressed vertically. Less individuals would become infected by the disease, and the $s(t)$ and $r(t)$ curves would begin to flatline. All curves display no change (ie. horizontal linear lines), when $s = 0.4$ and $r = 0.6$.

In a real life scenario, an increase in r_0, before the outbreak, alludes to an increase in the fraction of the population that is inoculated for the disease. A greater r_0 decreases the fraction of the population that is still susceptible or s_0. As r_0 grows, the fraction of the population that is inoculated helps prevent the susceptible population from contracting the disease, as inoculated individuals cannot be infected and pass on the disease even if they come in contact with an infectious individual. This phenomenon is called herd immunity.

As displayed in *Figure 27*, the disease described by the initial condition $b = 0.8$, $k = 1/3$, $I_0 = 10$, $N = 10^7$ can be completely avoided if roughly 0.6 or 60% of the population becomes inoculated. This can be observed in *Figure 27*, as after 1000 days, the initial fraction of populations in each group remains the same, thus, no epidemic occurs.

Fraction of Inoculated Population to Prevent Epidemic Calculation

The specific fraction of the population that needs to be inoculated can be calculated using the principle described in Question 6, where no epidemic would occur if $\frac{di}{dt} < 0$.

Using Equation 6.11 and given that $b = 0.8$, $k = 1/3$,

$$s < \frac{1/3}{0.8} \tag{8.1}$$

This would yield $s < \frac{5}{12} \approx 0.4167$. Thus, as long as $\frac{7}{12}$ or roughly 58.3% of the population is inoculated, no epidemic would occur for the given condition.

Question 9

From the information given within the text, several information can be extracted. Firstly, the average number of days an individual remains infected or $1/k$ is thought to be 14 days or between 7-8 days. Thus, $1/k = 14$ or $7 \leq \frac{1}{k} \leq 8$. Next, it is provided that the contact number or c, can be between 2 to 6. Thus, $2 \leq c \leq 6$.

Keeping the initial conditions, $I_0 = 10$, $N = 10^7$, variables k and b can be adjusted to potentially simulate COVID-19. Using the above information and Equation 7.12 or $c = b/k$, various combinations of k and b values can be determined. These permutations can be narrowed down by graphing the SIR Models using Matlab, and then finding the rise time of $i(t)$ from when it becomes noticeably more than zero to the peak (this would be estimated by reading the graph). As mentioned by the assignment, the rise time should be about 30-45 days. Combinations of k and b not within that range would be eliminated. Through this process, four different cases of SIR Models that could potentially simulate COVID-19 were selected. The graphs are displayed below.

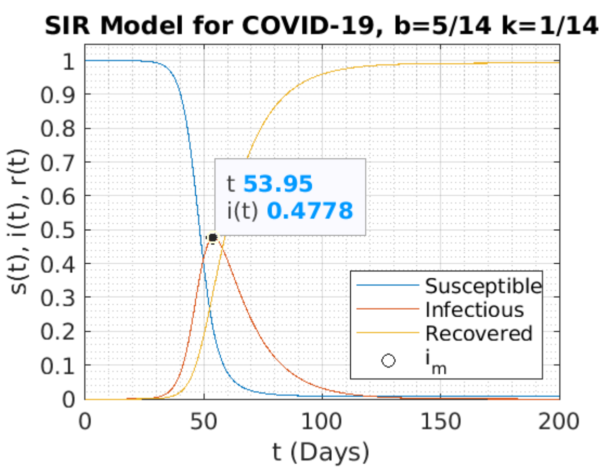

Figure 28. SIR Model to simulate COVID-19, with initial conditions of $c = 5$, and $1/k = 14$. The rise period of $i(t)$ is roughly 31 days. The peak is at roughly $i(t) = 0.4778$.

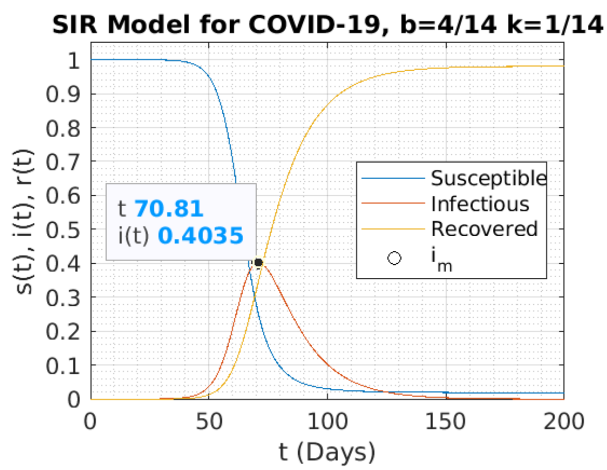

Figure 29. SIR Model to simulate COVID-19, with initial conditions of $c = 4$, and $1/k = 14$. The rise period of $i(t)$ is roughly 37 days. The peak is at roughly $i(t) = 0.4778$.

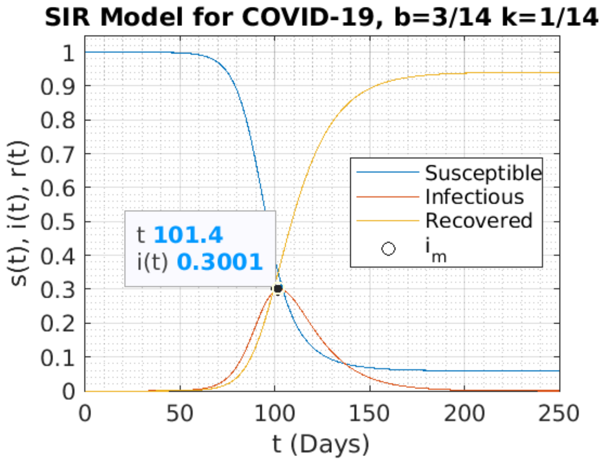

Figure 30. SIR Model to simulate COVID-19, with initial conditions of $c = 3$, and $1/k = 14$. The rise period of $i(t)$ is roughly 45 days. The peak is at roughly $i(t) = 0.3001$.

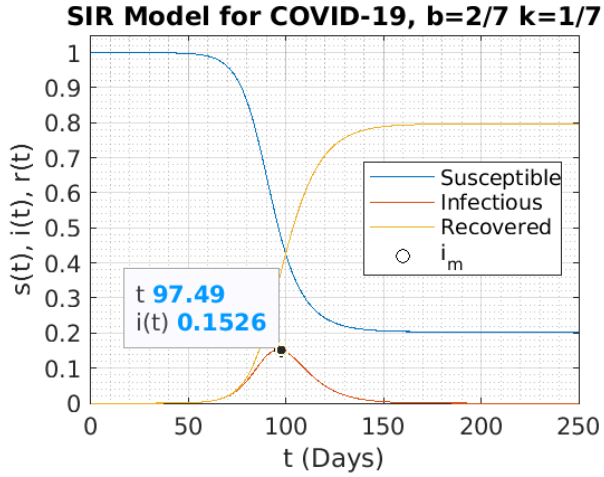

Figure 31. SIR Model to simulate COVID-19, with initial conditions of $c = 2$, and $1/k = 7$. The rise period of $i(t)$ is roughly 45 days. The peak is at roughly $i(t) = 0.1526$.

Comparison to Reality

These prediction models can be compared to reality through statistics for COVID-19 outcomes of countries that are nearing the end of their epidemic cycle. One densely populated country that is nearing the end of their COVID-19 epidemic is China. At the height of the outbreak, China had a total of 58,016 active cases on February 17th [1]. Given that China has a population of roughly $N = 1.4 \times 10^9$ [2], and assuming that the actual population had 100 times more than the reported cases, this means that at the peak of China's $i(t)$ curve, they had $i(t) \approx 0.004144$, or roughly 0.4% of their population infectious. Since the number of active cases began noticeably rising around January 1st [3], the rising period for China's $i(t)$ curve is roughly 47 days. Based only on the peak of $i(t)$, the closest model from *Figure 28, 29, 30, 31* to China's outbreak statistics is probably *Figure 31*. Even then, the fraction of the population that would be impacted by the virus is over predicted by roughly 40 times. This might be explained by China's strict enforcement of their social distancing policies, along with many other factors and variables unaccounted for within the SIR Model.

These models can also be compared to another densely populated country that is nearing the end of their epidemic cycle: South Korea. S. Korea's $i(t)$ peak occurred on March 11th, with 7362 cases [4]. Given that the actual cases are 100 times the statistic, and that S.Korea's population is roughly $N = 52 \times 10^6$ [5], their $i(t)$ at its peak is $i(t) \approx 0.01416$. S. Korea's noticeable rise in active cases occurred on February 1st [6], thus, the duration of the $i(t)$ curve rise would be roughly 39 days. The closest model that mimics this outbreak would again be *Figure 31*. With this model, the fraction of the population that becomes infectious during the peak is overpredicted by 11 times. This discrepancy, again, can be attributed to the many unaccounted variables for within the model.

The SIR Model is one of the basic tools used in epidemiology to help us predict and estimate trends for disease outbreaks around the world. Although not completely accurate, it would allow us to predict and prepare for epidemics like COVID-19, through the understanding of factors that affect epidemics and epidemic dynamics.

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[1] “ China,” *Worldometer*. [Online]. Available: https://www.worldometers.info/coronavirus/country/china/. [Accessed: 14-Apr-2020].

[2] “Population,” *Google*. [Online]. Available: https://www.google.com/publicdata/explore?ds=d5bncppjof8f9_&met_y=sp_pop_totl&idim=country:CHN:IND&hl=en&dl=en. [Accessed: 14-Apr-2020].

[3] D. B. Taylor, “A Timeline of the Coronavirus Pandemic,” *The New York Times*, 13-Feb-2020. [Online]. Available: https://www.nytimes.com/article/coronavirus-timeline.html. [Accessed: 14-Apr-2020].

[4] “ South Korea,” *Worldometer*. [Online]. Available: https://www.worldometers.info/coronavirus/country/south-korea/. [Accessed: 14-Apr-2020].

[5] “Population,” *Google*. [Online]. Available: https://www.google.com/publicdata/explore?ds=d5bncppjof8f9_&met_y=sp_pop_totl&idim=country:KOR:PRK:IRN&hl=en&dl=en. [Accessed: 14-Apr-2020].

[6] “South Korea learned its successful Covid-19 strategy from a previous coronavirus outbreak: MERS,” *Bulletin of the Atomic Scientists*, 20-Mar-2020. [Online]. Available: https://thebulletin.org/2020/03/south-korea-learned-its-successful-covid-19-strategy-from-a-previous-coronavirus-outbreak-mers/. [Accessed: 14-Apr-2020].