# Team Contest Reference
# Team:
System.out.println(42);

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Contents

n	Runtime $100 \cdot 10^6$ in 3s
$[10, 11]$	$\mathcal{O}(n!)$
< 22	$\mathcal{O}(n2^n)$
≤ 100	$\mathcal{O}(n^4)$
≤ 400	$\mathcal{O}(n^3)$
≤ 2.000	$\mathcal{O}(n^2 \log n)$
≤ 10.000	$\mathcal{O}(n^2)$
$\leq 1.000.000$	$\mathcal{O}(n \log n)$
$\leq 100.000.000$	$\mathcal{O}(n)$

`byte` (8 Bit, signed): -128 …127
`short` (16 Bit, signed): -32.768 …23.767
`integer` (32 Bit, signed): -2.147.483.648 …2.147.483.647
`long` (64 Bit, signed): $-2^{63} \ldots 2^{63} - 1$

MD5: `cat <string>| tr -d [:space:] | md5sum`

1 DataStructures

1.1 Fenwick-Tree

Can be used for computing prefix sums.

```java
int[] fwktree = new int[m + n + 1];
public static int read(int index, int[] fenwickTree) {
   int sum = 0;
   while (index > 0) {
      sum += fenwickTree[index];
      index -= (index & -index);
   }
   return sum;
}
public static int[] update(int index, int addValue,
    int[] fenwickTree) {
   while (index <= fenwickTree.length - 1) {
      fenwickTree[index] += addValue;
      index += (index & -index);
   }
   return fenwickTree;
}
```

MD5: `97fd176a403e68cb76a82196191d5f19` | $\mathcal{O}(\log n)$

1.2 Range Maximum Query

process processes an array A of length N in O($N \log N$) such that *query* can compute the maximum value of A in interval $[i, j]$. Therefore $M[a, b]$ stores the maximum value of interval $[a, a + 2^b - 1]$.
Input: dynamic table M, array to search A, length N of A, start index i and end index j

Output: filled dynamic table M or the maximum value of A in interval $[i, j]$

```java
public static void process(int[][] M, int[] A, int N)
    {
  for(int i = 0; i < N; i++)
    M[i][0] = i;
  // filling table M
  // M[i][j] = max(M[i][j-1], M[i+(1<<(j-1))][j-1]),
  // cause interval of length 2^j can be partitioned
  // into two intervals of length 2^(j-1)
  for(int j = 1; 1 << j <= N; j++) {
    for(int i = 0; i + (1 << j) - 1 < N; i++) {
      if(A[M[i][j-1]] >= A[M[i+(1 << (j-1))][j-1]])
        M[i][j] = M[i][j-1];
      else
        M[i][j] = M[i + (1 << (j-1))][j-1];
    }
  }
}

public static int query(int[][] M, int[] A, int N,
                                    int i, int j) {
  // k = |_ log_2(j-i+1) _|
  int k = (int) (Math.log(j - i + 1) / Math.log(2));
  if(A[M[i][k]] >= A[M[j- (1 << k) + 1][k]])
    return M[i][k];
  else
    return M[j - (1 << k) + 1][k];
}
```

MD5: `db0999fa40037985ff27dd1a43c53b80` | $\mathcal{O}(N \log N, 1)$

2 Graph

2.1 Breadth First Search

Iterative BFS. Needs testing. Uses ref Vertex class, no Edge class needed. In this version we look for a shortest path from s to t though we could also find the BFS-tree by leaving out t. *Input:* IDs of start and goal vertex and graph as AdjList *Output:* `true` if there is a connection between s and g, `false` otherwise

```java
public static boolean BFS(Vertex[] G, int s, int t) {

    //make sure that all Vertices vis values are false
         etc

    Queue<Vertex> q = new LinkedList<Vertex>();

    G[s].vis = true;
    G[s].dist = 0;
    G[s].pre = -1;
    q.add(G[s]);

    //expand frontier between undiscovered and
        discovered vertices
    while(!q.isEmpty()) {
  Vertex u = q.poll();
```

```
  //when reaching the goal, return true
  //if we want to construct a BFS-tree delete this
      line
  if(u.id = t) return true;
  //else add adj vertices if not visited
  for(Vertex v : u.adj) {
      if(!v.vis) {
    v.vis = true;
    v.dist = u.dist + 1;
    v.pre = u.id;
    q.add(v);
      }
  }
    }

}
```

MD5: `01c4dadba37bb0e95625e8522e3f6362` | $\mathcal{O}(|V| + |E|)$

2.2 BellmanFord

Finds shortest pathes from a single source. Negative edge weights are allowed. Can be used for finding negative cycles.

```
public static boolean bellmanFord(Vertex[] G) {
   //source is 0
   G[0].dist = 0;
   //calc distances
   //the path has max length |V|-1
   for(int i = 0; i < G.length-1; i++) {
       //each iteration relax all edges
      for(int j = 0; j < G.length; j++) {
         for(Edge e : G[j].adj) {
            if(G[j].dist != Integer.MAX_VALUE
               && e.t.dist > G[j].dist + e.w) {
               e.t.dist = G[j].dist + e.w;
            }
         }
      }
   }
   //check for negative-length cycle
   for(int i = 0; i < G.length; i++) {
      for(Edge e : G[i].adj) {
         if(G[i].dist != Integer.MAX_VALUE && e.t.dist
              > G[i].dist + e.w) {
            return true;
         }
      }
   }
   return false;
}
```

MD5: `d101e6b6915f012b3f0c02dc79e1fc6f` | $\mathcal{O}(|V| \cdot |E|)$

2.3 Bipartite Graph Check

Checks a graph represented as adjList for being bipartite. Needs a little adaption, if the graph is not connected.
Input: $graph$ as adjList, amount of nodes N as int
Output: `true` if graph is bipartite, `false` otherwise

```
public static boolean bipartiteGraphCheck(Vertex[] G)
    {

    // use bfs for coloring each node
    G[0].color = 1;
    Queue<Vertex> q = new LinkedList<Vertex>();
    q.add(G[0]);
    while(!q.isEmpty()) {
  Vertex u = q.poll();
  for(Vertex v : u.adj) {
      // if node i not yet visited,
      // give opposite color of parent node u
      if(v.color == -1) {
    v.color = 1-u.color;
    q.add(v);
    // if node i has same color as parent node u
    // the graph is not bipartite
      } else if(u.color == v.color)
    return false;
      // if node i has different color
      // than parent node u keep going
  }
    }
    return true;
}
```

MD5: `e93d242522e5b4085494c86f0d218dd4` | $\mathcal{O}(|V| + |E|)$

2.4 Maximum Bipartite Matching

Finds the maximum bipartite matching in an unweighted graph using DFS.
Input: An unweighted adjacency matrix boolean[M][N] with M nodes being matched to N nodes.
Output: The maximum matching. (For getting the actual matching, little changes have to be made.)

```
// A DFS based recursive function that returns true
// if a matching for vertex u is possible
boolean bpm(boolean bpGraph[][], int u,
            boolean seen[], int matchR[]) {
// Try every job one by one
  for (int v = 0; v < N; v++) {
// If applicant u is interested in job v and v
// is not visited
    if (bpGraph[u][v] && !seen[v]) {
      seen[v] = true; // Mark v as visited

// If job v is not assigned to an applicant OR
// previously assigned applicant for job v (which
// is matchR[v]) has an alternate job available.
// Since v is marked as visited in the above line,
// matchR[v] in the following recursive call will
// not get job v again
      if (matchR[v] < 0 ||
          bpm(bpGraph, matchR[v], seen, matchR)) {
        matchR[v] = u;
        return true;
      }
    }
  }
  return false;
}

// Returns maximum number of matching from M to N
int maxBPM(boolean bpGraph[][]) {
// An array to keep track of the applicants assigned
// to jobs. The value of matchR[i] is the applicant
// number assigned to job i, the value -1 indicates
// nobody is assigned.
```

```java
  int matchR[] = new int[N];

// Initially all jobs are available
  for(int i = 0; i < N; ++i)
    matchR[i] = -1;
// Count of jobs assigned to applicants
  int result = 0;
  for (int u = 0; u < M; u++) {
// Mark all jobs as not seen for next applicant.
    boolean seen[] = new boolean[N];
    for(int i = 0; i < N; ++i)
      seen[i] = false;

// Find if the applicant u can get a job
    if (bpm(bpGraph, u, seen, matchR))
      result++;
  }
  return result;
}
```

MD5: e559cef1fc0d34e0ba49b7568cfd480d | $\mathcal{O}(M \cdot N)$

2.5 Single-source shortest paths in dag

```java
public static void dagSSP(Vertex[] G, int s) {
    //calls topological sort method
    LinkedList<Integer> sorting = TS(G);

    G[s].dist = 0;

    //go through vertices in ts order
    for(int u : sorting) {
  for(Edge e : G[u].adj) {
      Vertex v = e.t;
      if(v.dist > u.d + e.w) {
    v.dist = u.d + e.w;
    v.pre = u.id;
      }
  }
    }
}
```

MD5: 3fc829298eb1489b255acd3427d89d1a | $\mathcal{O}(|V| + |E|)$

2.6 Dijkstra

Finds the shortest paths from one vertex to every other vertex in the graph (SSSP).

For negative weights, add |min|+1 to each edge, later subtract from result.

To get a different shortest path when edges are ints, add an $\epsilon = \frac{1}{k+1}$ on each edge of the shortest path of length k, run again.

Input: A source vertex s and an adjacency list G.

Output: Modified adj. list with distances from s and predcessor vertices set.

```java
public static void dijkstra(Vertex[] G, int s) {
    G[s].dist = 0;

    //Tuple class can be found at Prims Alg, maybe we
        should give this class its own space
    Tuple st = new Tuple(s, 0);

    PriorityQueue<Tuple> q = new PriorityQueue<Tuple
        >();
    q.add(G[s]);

    while(!q.isEmpty()) {
  Tuple sm = q.poll();
  Vertex u = G[sm.id];

  if(u.vis) continue;
  if(sm.dist > u.dist) continue;
  u.vis = true;
  for(Edge e : u.adj) {
      Vertex v = e.t;
      if(!v.vis && v.dist > u.dist + e.w) {
    v.pre = u.id;
    v.dist = u.dist + e.w;
    Tuple nt = new Tuple(v.id, v.dist);
    queue.add(nt);
      }
  }
    }
}
```

MD5: 15598cf27ada41bf8cdf83dd5d3301bf | $\mathcal{O}(|E| \log |V|)$

2.7 EdmondsKarp

Finds the greatest flow in a graph. Capacities must be positive.

```java
public static boolean BFS(Vertex[] G, int s, int t) {
   int N = G.length;
   for(int i = 0; i < N; i++) {
      G[i].vis = false;
   }

   Queue<Vertex> q = new LinkedList<Vertex>();
   G[s].vis = true;
   G[s].pre = -1;
   queue.add(G[s]);

   while(!q.isEmpty()) {
      Vertex u = queue.poll();
      if(u.id == t) return true;
      for(int i : u.adj.keySet()) {
    Edge e = u.adj.get(i);
    Vertex v = e.t;
    if(!v.vis) {
        v.vis = true;
        v.pre = u.id;
        q.add(v);
    }
      }
   }
   return (G[t].vis);
}
//We store the edges in the graph in a hashmap
public static int fordFulkerson(Vertex[] G, int s, int
     t) {

   int maxflow = 0;
   while(BFS(rgraph, s, t)) {
      int pathflow = Integer.MAX_VALUE;
      for(int v = t; v!= s; v = v.pre) {
         int u = v.pre;
   pathflow = Math.min(pathflow, G[u].adj.get(v).rw);
      }
```

```java
        for(int v = t; v != s; v = v.pre) {
            int u = v.pre;
    G[u].adj.get(v).rw -= pathflow;
    G[v].adj.get(u).rw += pathflow;
        }

        maxflow += pathflow;
    }
    return maxflow;
}
```

MD5: b5e1ff020addc8138cde5398ec518985 | $\mathcal{O}(|V|^2 \cdot |E|)$

2.8 Reference for Edge classes

Used for example in Dijkstra algorithm, implements edges with weight. Needs testing.

```java
//for Kruskal we need to sort edges, use:
class Edge implements Comparable<Edge> {}

class Edge {

    //for Kruskal it is helpful to store the start as
        well
    //moreover we might not need the vertex class
    int s;
    int t;

    public Edge(int s, int t, int w) {...}
    public int compareTo(Edge other) {
  return Integer.compare(this.w, other.w);
    }

    //for EKarp we also want to store residual weights
    int rw;

    Vertex t;
    int w;

    public Edge(Vertex t, int w) {
  this.t = t;
  this.w = w;
  this.rw = w;
    }

}
```

MD5: fd4ed227f042ee49ef9dac031ad2d5a0 | $\mathcal{O}(?)$

2.9 FloydWarshall

Finds all shortest paths. Paths in array next, distances in ans.

```java
public static void floydWarshall(int[][] graph, int
    [][] next, int[][] ans) {
   for(int i = 0; i < ans.length; i++) {
      for(int j = 0; j < ans.length; j++) {
         ans[i][j] = graph[i][j];
      }
   }
   for (int k = 0; k < ans.length; k++) {
      for (int i = 0; i < ans.length; i++) {
         for (int j = 0; j < ans.length; j++) {
            if (ans[i][k] + ans[k][j] < ans[i][j]
            && ans[i][k] < Integer.MAX_VALUE && ans[k
                ][j] < Integer.MAX_VALUE) {
               ans[i][j] = ans[i][k] + ans[k][j];
               next[i][j] = next[i][k];
            }
         }
      }
   }
}
```

MD5: 4faf8c41a9070f106e68864cc131706d | $\mathcal{O}(|V|^3)$

2.10 terative DFS

Simple iterative DFS, the recursive variant is a bit fancier. Not tested.

```java
//if we want to start the DFS for different connected
    components, there is such a method
//in the recursive variant of DFS

public static boolean ItDFS(Vertex[] G, int s, int t)
    {

    //take care that all the nodes are not visited at
        the beginning

    Stack<Integer> S = new Stack<Integer>();
    s.push(s);
    while(!S.isEmpty()) {
  int u = S.pop();
  if(u.id == t) return true;
  if(!G[u].vis) {
      G[u].vis = true;
      for(Vertex v : G[u].adj) {
    if(!v.vis) S.push(v.id);
      }
  }
    }

    return false;
}
```

MD5: 1f83d8077e6252b6894eb5711298d79c | $\mathcal{O}(|V| + |E|)$

2.11 Johnsons Algorithm

```java
public static int[][] johnson(Vertex[] G) {

    Vertex[] Gd = new Vertex[G.length+1];
    int s = G.length;
    for(int i = 0; i < G.length; i++) {
  Gd[i] = G[i];
    }
    //init new vertex with zero-weight-edges to each
        vertex
    Vertex S = new Vertex(G.length);
    for(int i = 0; i < G.length) {
  S.adj.add(new Edge(Gd[i], 0));
    }

    //bellman-ford to check for neg-weight-cycles and
        to adapt edges to enable running dijkstra
    if(!bellmanFord(G, s)) {
  System.out.println("False");
```

```java
return;
  }
  //change weights
  for(int i = 0; i < G.length; i++) {
for(Edge e : Gd[i].adj) {
    e.w = e.w + Gd[i].dist - e.t.dist;
}
  }
  //store distances to invert this step later
  int[] h = new int[G.length];
  for(int i = 0; i < G.length; i++) {
h[i] = G[i].dist;
  }

  //create shortest path matrix
  int[][] apsp = new int[G.length][G.length];

  //now use original graph G
  //start a dijkstra for each vertex
  for(int i = 0; i < G.length; i++) {
//reset weights, maybe we should put that in the
    dijkstra
for(int j = 0; j < G.length; j++) {
    G[j].vis = false;
    G[j].dist = Integer.MAX_VALUE;
}
dijkstra(G, i);
for(int j = 0; j < G.length; j++) {
    apsp[i][j] = G[j].dist + h[j] - h[i];
}
  }
  return apsp;
}
```

MD5: 6bce8e864871064f450e0115a9ab77df | $\mathcal{O}(|V|^2 \log V + VE)$

2.12 Kruskal

Computes a minimum spanning tree for a weighted undirected graph.

```java
public static int kruskal(Edge[] edges, int n) {
    Arrays.sort(edges);
    //n is the number of vertices
    UnionFind uf = new UnionFind(n);
    //we will only compute the sum of the MST, one
        could of course also store the edges
    int sum = 0;
    int cnt = 0;
    for(int i = 0; i < edges.length; i++) {
  if(cnt == n-1) break;
  if(uf.union(edges[j].s, edges[j].t)) {
      sum += edges[j].w;
      cnt++;
  }
    }
    return sum;
}
```

MD5: aa6cc91ea8a00f6b38aa0433130d1be9 | $\mathcal{O}(|E| + \log |V|)$

2.13 Prim

```java
//s is the startpoint of the algorithm, in general not
     too important
//we assume that graph is connected
public static int prim(Vertex[] G, int s) {

    //make sure dists are maxint
    G[s].dist = 0;
    Tuple st = new Tuple(s, 0);

    PriorityQueue<Tuple> q = new PriorityQueue<Tuple
        >();
    q.add(st);

    //we will store the sum and each nodes predecessor
    int sum = 0;

    while(!q.isEmpty()) {
  Tuple sm = q.poll();
  Vertex u = G[sm.id];
  //u has been visited already
  if(u.vis) continue;
  //this is not the latest version of u
  if(sm.dist > u.dist) continue;
  u.vis = true;
  //u is part of the new tree and u.dist the cost of
      adding it
  sum += u.dist;
  for(Edge e : u.adj) {
      Vertex v = e.t;
      if(!v.vis && v.dist > e.w) {
    v.pre = u.id;
    v.dist = e.w;
    Tuple nt = new Tuple(v.id, e.w);
    q.add(nt);
      }
  }

    }
    return sum;

}

class Tuple implements Comparable<Tuple> {

    int id;
    int dist;

    public Tuple(int id, int dist) {
  this.id = id;
  this.dist = dist;
    }

    public int compareTo(Tuple other) {
  return Integer.compare(this.dist, other.dist);
    }
}
```

MD5: 1c35fcc2a3f44ab7c1658d2716805ee1 | $\mathcal{O}()$

2.14 Recursive Depth First Search

Recursive DFS with different options (storing times, connected/unconnected graph). Needs testing. *Input:* A source vertex s, a target vertex t, and adjlist G and the time (0 at the start) *Output:* Indicates if there is connection between s and t.

```java
//if we want to visit the whole graph, even if it is
    not connected we might use this
```

```java
public static void DFS(Vertex[] G) {

   //make sure all vertices vis value is false etc

   int time = 0;
   for(int i = 0; i < G.length; i++) {
 if(!G[i].vis) {
     //note that we leave out t so this does not work
          with the below function
     //adaption will not be too difficult though
     //fix time
     recDFS(i, G, 0);
 }
   }
}

//first call with time = 0
public static boolean recDFS(int s, int t, Vertex[] G,
     int time){

   //it might be necessary to store the time of
       discovery
   time = time + 1;
   G[s].dtime = time;

   G[s].vis = true; //new vertex has been discovered
   //when reaching the target return true
   //not necessary when calculating the DFS-tree
   if(s == t) return true;
   for(Vertex v : G[s].adj) {
 //exploring a new edge
 if(!v.vis) {
     v.pre = u.id;
     if(recDFS(v.id, t, G)) return true;
 }
   }

   //storing finishing time
   time = time + 1;
   G[s].ftime = time;

   return false;
}
```

MD5: e11b8416945db1004b13346a22341c87 | $\mathcal{O}(|V| + |E|)$

2.15 Strongly Connected Components

```java
public static void fDFS(Vertex u, LinkedList<Integer>
    sorting) {
  //compare with TS
  u.vis = true;
  for(Vertex v : u.out) {
    if(!v.vis)
      fDFS(v, sorting);
  }
  sorting.addFirst(u.id);
  return sorting;
}

public static void sDFS(Vertex u, int cnt) {
  //basic DFS, all visited vertices get cnt
  u.vis = true;
  u.comp = cnt;
  for(Vertex v : u.in) {
    if(!v.vis)
      sDFS(v, cnt);
  }
}

public static void doubleDFS(Vertex[] G) {
  //first calc a topological sort by first DFS
  LinkedList<Integer> sorting = new LinkedList<Integer
      >();
  for(int i = 0; i < G.length; i++) {
    if(!G[i].vis)
      fDFS(G[i], sorting);
  }
  for(int i = 0; i < G.length; i++){
    G[i].vis = false;
  }
  //then go through the sort and do another DFS on G^T
  //each tree is a component and gets a unique number
  int cnt = 0;
  for(int i : sorting) {
    if(!G[i].vis)
      sDFS(G[i], cnt++);
  }
}
```

MD5: 67ac4aac19ee3ce07f23dd8ed9877b23 | $\mathcal{O}(|V| + |E|)$

2.16 Topological Sort

```java
public static LinkedList<Integer> TS(Vertex[] G) {
  LinkedList<Integer> sorting = new LinkedList<Integer
      >();
  for(int i = 0; i < G.length; i++) {
    if(!G[i].vis)
      recTS(G[i], sorting);
  }
    //check sorting for a -1 if the graph is not
        necessarily dag
    //maybe checking if there are too many values in
        sorting is easier?!
    return sorting;
}

public static LinkedList<Integer> recTS(Vertex u,
    LinkedList<Integer> sorting) {
  u.vis = true;
    for(Vertex v : u.adj) {
    if(v.vis)
        //the -1 indicates that it will not be
            possible to find an TS
        //there might be a much faster and elegant way
             (flag?!)
        sorting.addFirst(-1);
    else
        recTS(v, sorting);
    }
    sorting.addFirst(u.id);
    return sorting;
}
```

MD5: b4fb592469cf03dcb788aba03b98263e | $\mathcal{O}(|V| + |E|)$

2.17 Reference for Vertex classes

Used in many graph algorithms, implements a vertex with its edges. Needs testing.

```java
class Vertex {

    int id;
    boolean vis = false;
    int pre = -1;

    //for dijkstra and prim
    int dist = Integer.MAX_VALUE;

    //for SCC store number indicating the dedicated
        component
    int comp = -1;

    //for DFS we could store the start and finishing
        times
    int dtime = -1;
    int ftime = -1;

    //use an ArrayList of Edges if those information
        are needed
    ArrayList<Edge> adj = new ArrayList<Edge>();
    //use an ArrayList of Vertices else
    ArrayList<Vertex> adj = new ArrayList<Vertex>();
    //use two ArrayLists for SCC
    ArrayList<Vertex> in = new ArrayList<Vertex>();
    ArrayList<Vertex> out = new ArrayList<Vertex>();

    //for EdmondsKarp we need a HashMap to store Edges
    HashMap<Integer, Edge> adj = new HashMap<Integer,
        Edge>();

    //for bipartite graph check
    int color = -1;

    //we store as key the target
    public Vertex(int id) {
  this.id = id;
    }

}
```

MD5: f41108043e72983fc088f5851de6b932 | $\mathcal{O}(?)$

3 Math

3.1 Binomial Coefficient

Gives binomial coefficient (n choose k)

```java
public static long bin(int n, int k) {
  if (k == 0) {
    return 1;
  } else if (k > n/2) {
    return bin(n, n-k);
  } else {
    return n*bin(n-1, k-1)/k;
  }
}
```

MD5: ceca2cc881a9da6269c143a41f89cc12 | $\mathcal{O}(k)$

3.2 Binomial Matrix

Gives binomial coefficients for all K <= N.

```java
public static long[][] binomial_matrix(int N, int K) {
  long[][] B = new long[N+1][K+1];
  for (int k = 1; k <= K; k++) {
    B[0][k] = 0;
  }
  for (int m = 0; m <= N; m++) {
    B[m][0] = 1;
  }
  for (int m = 1; m <= N; m++) {
    for (int k = 1; k <= K; k++) {
      B[m][k] = B[m-1][k-1] + B[m-1][k];
    }
  }
  return B;
}
```

MD5: 0754f4e27d08a1d1f5e6c0cf4ef636df | $\mathcal{O}(N \cdot K)$

3.3 Divisability

Calculates (alternating) k-digitSum for integer number given by M.

```java
public static long digit_sum(String M, int k, boolean
    alt) {
  long dig_sum = 0;
  int vz = 1;
  while (M.length() > k) {
    if (alt) vz *= -1;
    dig_sum += vz*Integer.parseInt(M.substring(M.
        length()-k));
    M = M.substring(0, M.length()-k);
  }
  if (alt) vz *= -1;
  dig_sum += vz*Integer.parseInt(M);
  return dig_sum;
}
// example: divisibility of M by 13
public static boolean divisible13(String M) {
  return digit_sum(M, 3, true)%13 == 0;
}
```

MD5: 33b3094ebf431e1e71cd8e8db3c9cdd6 | $\mathcal{O}(?)$

3.4 Iterative EEA

Berechnet den ggT zweier Zahlen a und b und deren modulare Inverse $x = a^{-1} \mod b$ und $y = b^{-1} \mod a$.

```java
// Extended Euclidean Algorithm - iterativ
public static long[] eea(long a, long b) {
  if (b > a) {
    long tmp = a;
    a = b;
    b = tmp;
  }
  long x = 0, y = 1, u = 1, v = 0;
  while (a != 0) {
    long q = b / a, r = b % a;
    long m = x - u * q, n = y - v * q;
    b = a; a = r; x = u; y = v; u = m; v = n;
  }
  long gcd = b;
  // x = a^-1 % b, y = b^-1 % a
  // ax + by = gcd
```

```java
  long[] erg = { gcd, x, y };
  return erg;
}
```

MD5: 81fe8cd4adab21329dcbe1ce0499ee75 | $\mathcal{O}(\log a + \log b)$

3.5 Polynomial Interpolation

```java
public class interpol {

  // divided differences for points given by vectors x
     and y
  public static rat[] divDiff(rat[] x, rat[] y) {
    rat[] temp = y.clone();
    int n = x.length;
    rat[] res = new rat[n];
    res[0] = temp[0];
    for (int i=1; i < n; i++) {
      for (int j = 0; j < n-i; j++) {
        temp[j] = (temp[j+1].sub(temp[j])).div(x[j+i].
            sub(x[j]));
      }
      res[i] = temp[0];
    }
    return res;
  }

  // evaluates interpolating polynomial p at t for
      given
  // x-coordinates and divided differences
  public static rat p(rat t, rat[] x, rat[] dD) {
    int n = x.length;
    rat p = new rat(0);
    for (int i = n-1; i > 0; i--) {
      p = (p.add(dD[i])).mult(t.sub(x[i-1]));
    }
    p = p.add(dD[0]);
    return p;
  }

  public static void main(String[] args) {

    rat[] test = {new rat(4,5), new rat(7,10), new rat
        (3,4)};
    test = rat.commonDenominator(test);
    for (int i = 0; i < test.length; i++) {
      System.out.println(test[i].toString());
    }

    rat[] x = {new rat(0),new rat(1), new rat(2), new
        rat(3), new rat(4), new rat(5)};
    rat[] y = {new rat(-10), new rat(9), new rat(0),
        new rat(1), new rat(1,2), new rat(1,80)};
    rat[] dD = divDiff(x,y);
    System.out.println("p("+7+") = "+p(new rat(7), x,
        dD));
  }

}
// implementation of rational numbers
class rat {

  public long c;
  public long d;

  public rat (long c, long d) {
    this.c = c;
    this.d = d;
    this.shorten();
  }

  public rat (long c) {
    this.c = c;
    this.d = 1;
  }

  public static long ggT(long a, long b) {
    while (b != 0) {
      long h = a%b;
      a = b;
      b = h;
    }
    return a;
  }

  public static long kgV(long a, long b) {
    return a*b/ggT(a,b);
  }

  public static rat[] commonDenominator(rat[] c) {
    long kgV = 1;
    for (int i = 0; i < c.length; i++) {
      kgV = kgV(kgV, c[i].d);
    }
    for (int i = 0; i < c.length; i++) {
      c[i].c *= kgV/c[i].d;
      c[i].d *= kgV/c[i].d;
    }
    return c;
  }

  public void shorten() {
    long ggT = ggT(this.c, this.d);
    this.c = this.c / ggT;
    this.d = this.d / ggT;
    if (d < 0) {
      this.d *= -1;
      this.c *= -1;
    }
  }

  public String toString() {
    if (this.d == 1) return ""+c;
    return ""+c+"/"+d;
  }

  public rat mult(rat b) {
    return new rat(this.c*b.c, this.d*b.d);
  }

  public rat div(rat b) {
    return new rat(this.c*b.d, this.d*b.c);
  }

  public rat add(rat b) {
    long new_d = kgV(this.d, b.d);
    long new_c = this.c*(new_d/this.d) + b.c*(new_d/b.
        d);
    return new rat(new_c, new_d);
  }

  public rat sub(rat b) {
    return this.add(new rat(-b.c, b.d));
  }
```

```
}
```

MD5: `d98bd247b95395d8596ff1d5785ee06b` | $\mathcal{O}(?)$

3.6 Sieve of Eratosthenes

Calculates Sieve of Eratosthenes.
Input: A integer N indicating the size of the sieve.
Output: A boolean array, which is true at an index i iff i is prime.

```
public static boolean[] sieveOfEratosthenes(int N) {
  boolean[] isPrime = new boolean[N+1];
    for (int i=2; i<=N; i++) isPrime[i] = true;
    for (int i = 2; i*i <= N; i++)
        if (isPrime[i])
            for (int j = i*i; j <= N; j+=i)
                isPrime[j] = false;
    return isPrime;
}
```

MD5: `95704ae7c1fe03e91adeb8d695b2f5bb` | $\mathcal{O}(n)$

4 Misc

4.1 Binary Search

Binary searchs for an element in a sorted array.
Input: sorted $array$ to search in, amount N of elements in $array$, element to search for a
Output: returns the index of a in $array$ or -1 if $array$ does not contain a

```
public static int BinarySearch(int[] array,
                                   int N, int a) {
  int lo = 0;
  int hi = N-1;
  // a might be in interval [lo,hi] while lo <= hi
  while(lo <= hi) {
    int mid = (lo + hi) / 2;
    // if a > elem in mid of interval,
    // search the right subinterval
    if(array[mid] < a)
      lo = mid+1;
    // else if a < elem in mid of interval,
    // search the left subinterval
    else if(array[mid] > a)
      hi = mid-1;
    // else a is found
    else
      return mid;
  }
  // array does not contain a
  return -1;
}
```

MD5: `203da61f7a381564ce3515f674fa82a4` | $\mathcal{O}(\log n)$

4.2 Next number with n bits set

From x the smallest number greater than x with the same amount of bits set is computed. Little changes have to be made, if the calculated number has to have length less than 32 bits.

Input: number x with n bits set $(x = (1 << n) - 1)$
Output: the smallest number greater than x with n bits set

```
public static int nextNumber(int x) {
  //break when larger than limit here
  if(x == 0) return 0;
  int smallest = x & -x;
  int ripple = x + smallest;
  int new_smallest = ripple & -ripple;
  int ones  = ((new_smallest/smallest) >> 1) - 1;
  return ripple | ones;
}
```

MD5: `2d8a79cb551648e67fc3f2f611a4f63c` | $\mathcal{O}(1)$

5 Math Roland

5.1 Divisability Explanation

$D \mid M \Leftrightarrow D \mid$ `digit_sum(M, k, alt)`, refer to table for values of D, k, alt.

5.2 Combinatorics

- Variations (ordered): k out of n objects (permutations for $k = n$)
  - without repetition:
    $M = \{(x_1, \ldots, x_k) : 1 \le x_i \le n,\ x_i \ne x_j \text{ if } i \ne j\}$, $|M| = \frac{n!}{(n-k)!}$
  - with repetition:
    $M = \{(x_1, \ldots, x_k) : 1 \le x_i \le n\}, |M| = n^k$
- Combinations (unordered): k out of n objects
  - without repetition: $M = \{(x_1, \ldots, x_n) : x_i \in \{0, 1\},\ x_1 + \ldots + x_n = k\}, |M| = \binom{n}{k}$
  - with repetition: $M = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, k\},\ x_1 + \ldots + x_n = k\}, |M| = \binom{n+k-1}{k}$
- Ordered partition of numbers: $x_1 + \ldots + x_k = n$ (i.e. 1+3 = 3+1 = 4 are counted as 2 solutions)
  - #Solutions for $x_i \in \mathbb{N}_0$: $\binom{n+k-1}{k-1}$
  - #Solutions for $x_i \in \mathbb{N}$: $\binom{n-1}{k-1}$
- Unordered partition of numbers: $x_1 + \ldots + x_k = n$ (i.e. 1+3 = 3+1 = 4 are counted as 1 solution)
  - #Solutions for $x_i \in \mathbb{N}$: $P_{n,k} = P_{n-k,k} + P_{n-1,k-1}$ where $P_{n,1} = P_{n,n} = 1$
- Derangements (permutations without fixed points): $!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$

5.3 Polynomial Interpolation

5.3.1 Theory

Problem: for $\{(x_0, y_0), \ldots, (x_n, y_n)\}$ find $p \in \Pi_n$ with $p(x_i) = y_i$ for all $i = 0, \ldots, n$.

Solution: $p(x) = \sum_{i=0}^{n} \gamma_{0,i} \prod_{j=0}^{i-1} (x - x_i)$ where $\gamma_{j,k} = y_j$ for $k = 0$ and $\gamma_{j,k} = \frac{\gamma_{j+1,k-1} - \gamma_{j,k-1}}{x_{j+k} - x_j}$ otherwise.
Efficient evaluation of $p(x)$: $b_n = \gamma_{0,n}$, $b_i = b_{i+1}(x - x_i) + \gamma_{0,i}$ for $i = n-1, \dots, 0$ with $b_0 = p(x)$.

5.4 Fibonacci Sequence

5.4.1 Binet's formula

$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow f_n = \frac{1}{\sqrt{5}}(\phi^n - \tilde{\phi}^n)$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\tilde{\phi} = \frac{1-\sqrt{5}}{2}$.

5.4.2 Generalization

$g_n = \frac{1}{\sqrt{5}}(g_0(\phi^{n-1} - \tilde{\phi}^{n-1}) + g_1(\phi^n - \tilde{\phi}^n)) = g_0 f_{n-1} + g_1 f_n$ for all $g_0, g_1 \in \mathbb{N}_0$

5.4.3 Pisano Period

Both $(f_n \bmod k)_{n \in \mathbb{N}_0}$ and $(g_n \bmod k)_{n \in \mathbb{N}_0}$ are periodic.

6 Java Knowhow

6.1 System.out.printf() und String.format()

Syntax: `%[flags][width][.precision][conv]`

flags:

`-`	left-justify (default: right)
`+`	always output number sign
`0`	zero-pad numbers
(space)	space instead of minus for pos. numbers
,	group triplets of digits with ,

width specifies output width
precision is for floating point precision
conv:

d	byte, short, int, long
f	float, double
c	char (use C for uppercase)
s	String (use S for all uppercase)

6.2 Modulo: Avoiding negative Integers

```java
int mod = (((nums[j] % D) + D) % D);
```

6.3 Speed up IO

Use

```java
BufferedReader br = new BufferedReader(new
InputStreamReader(System.in));
```

Use

```java
Double.parseDouble(Scanner.next());
```

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Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[n \atop k \right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{ n \atop k \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle n \atop k \right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1,2,\ldots,n\}$ with k ascents.
$\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{ n \atop 1 \right\} = \left\{ n \atop n \right\} = 1$,

12. $\displaystyle\left\{ n \atop 2 \right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{ n \atop k \right\} = k\left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$,

14. $\displaystyle\left[n \atop 1 \right] = (n-1)!$, **15.** $\displaystyle\left[n \atop 2 \right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[n \atop n \right] = 1$, **17.** $\displaystyle\left[n \atop k \right] \ge \left\{ n \atop k \right\}$,

18. $\displaystyle\left[n \atop k \right] = (n-1)\left[n-1 \atop k \right] + \left[n-1 \atop k-1 \right]$, **19.** $\displaystyle\left\{ n \atop n-1 \right\} = \left[n \atop n-1 \right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[n \atop k \right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle n \atop 0 \right\rangle = \left\langle n \atop n-1 \right\rangle = 1$, **23.** $\displaystyle\left\langle n \atop k \right\rangle = \left\langle n \atop n-1-k \right\rangle$, **24.** $\displaystyle\left\langle n \atop k \right\rangle = (k+1)\left\langle n-1 \atop k \right\rangle + (n-k)\left\langle n-1 \atop k-1 \right\rangle$,

25. $\displaystyle\left\langle 0 \atop k \right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle\left\langle n \atop 1 \right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle n \atop 2 \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{ n \atop m \right\} = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{n}\left\{ n \atop k \right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\!\left\langle n \atop 0 \right\rangle\!\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\!\left\langle n \atop n \right\rangle\!\!\right\rangle = 0$ for $n \ne 0$,

34. $\displaystyle\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = (k+1)\left\langle\!\!\left\langle n-1 \atop k \right\rangle\!\!\right\rangle + (2n-1-k)\left\langle\!\!\left\langle n-1 \atop k-1 \right\rangle\!\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{ x \atop x-n \right\} = \sum_{k=0}^{n}\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{ n+1 \atop m+1 \right\} = \sum_{k}\binom{n}{k}\left\{ k \atop m \right\} = \sum_{k=0}^{n}\left\{ k \atop m \right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[{k \atop m}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\left[{k \atop m}\right],$

39. $\left[{x \atop x-n}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k},$

41. $\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k},$

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k\left\{{n+k \atop k}\right\},$

43. $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} k(n+k)\left[{n+k \atop k}\right],$

44. $\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k},$

45. $(n-m)!\binom{n}{m} = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}, \quad \text{for } n \ge m,$

46. $\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right],$

47. $\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\},$

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k},$

49. $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_k \left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$\begin{aligned} 1\big(T(n) - 3T(n/2) &= n\big) \\ 3\big(T(n/2) - 3T(n/4) &= n/2\big) \\ \vdots \quad \vdots \quad &\vdots \\ 3^{\log_2 n - 1}\big(T(2) - 3T(1) &= 2\big) \end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \ge 0} g_{i+1} x^i = \sum_{i \ge 0} 2g_i x^i + \sum_{i \ge 0} x^i.$$

We choose $G(x) = \sum_{i \ge 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \ge 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i \ge 0} 2^i x^i - \sum_{i \ge 0} x^i\right) \\ &= \sum_{i \ge 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

Theoretical Computer Science Cheat Sheet

$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i,j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n} \Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n} (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

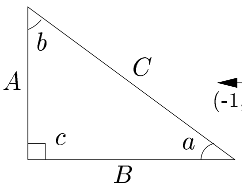
Geometric distribution:

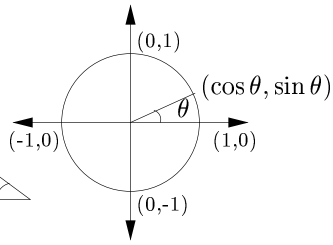
$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

Theoretical Computer Science Cheat Sheet

Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \qquad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

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Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

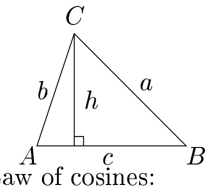
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S\rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embedded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k,\ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

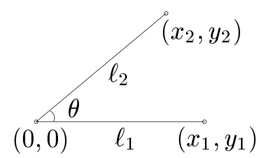
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\tfrac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:
$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1-u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$,

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\, dx = \tan x,$

20. $\int \csc^2 x\, dx = -\cot x,$

21. $\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx,$

22. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx,$

23. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \quad n \neq 1,$

24. $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \quad n \neq 1,$

25. $\int \sec^n x\, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx, \quad n \neq 1,$

26. $\int \csc^n x\, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x\, dx, \quad n \neq 1,$

27. $\int \sinh x\, dx = \cosh x,$

28. $\int \cosh x\, dx = \sinh x,$

29. $\int \tanh x\, dx = \ln |\cosh x|,$

30. $\int \coth x\, dx = \ln |\sinh x|,$

31. $\int \operatorname{sech} x\, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x\, dx = \ln \left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x\, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$

38. $$\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$$

39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

41. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right|,$

45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left|\frac{x}{a + bx}\right|,$

49. $\int x\sqrt{a + bx}\, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

51. $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln \left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

53. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

57. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a,} \quad a > 0,$

58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

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Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \tfrac{1}{a}\arccos\tfrac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac,\\ \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac,\end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0,\\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0,\end{cases}$$

68. $$\int\sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int\frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int\frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0,\\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0,\end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^n e^{ax}\,dx = \frac{x^ne^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$\begin{aligned}
x^1 &= x^{\underline{1}} &&= x^{\overline{1}}\\
x^2 &= x^{\underline{2}}+x^{\underline{1}} &&= x^{\overline{2}}-x^{\overline{1}}\\
x^3 &= x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} &&= x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}\\
x^4 &= x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} &&= x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}\\
x^5 &= x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} &&= x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{aligned}$$

$$\begin{aligned}
x^{\overline{1}} &= x^1 & x^{\underline{1}} &= x^1\\
x^{\overline{2}} &= x^2+x^1 & x^{\underline{2}} &= x^2-x^1\\
x^{\overline{3}} &= x^3+3x^2+2x^1 & x^{\underline{3}} &= x^3-3x^2+2x^1\\
x^{\overline{4}} &= x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} &= x^4-6x^3+11x^2-6x^1\\
x^{\overline{5}} &= x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} &= x^5-10x^4+35x^3-50x^2+24x^1
\end{aligned}$$

Finite Calculus

Difference, shift operators:
$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\, f(x) = f(x+1).$$

Fundamental Theorem:
$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u+\Delta v,$

$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$

$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$

$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right]x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i, \qquad \left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i, \qquad (e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!}, \qquad x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!}, \qquad \zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x}, \qquad \frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

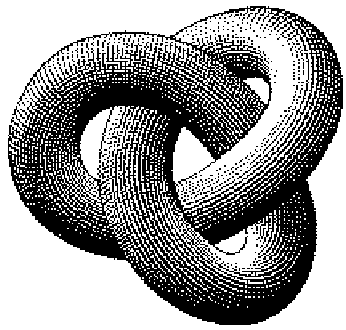
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then
$$\int_a^b G(x)\,dF(x)$$
exists. If $a\le b\le c$ then
$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$
If the integrals involved exist
$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$
$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$
$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$
$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$
If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then
$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:
$$\begin{array}{c} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{array}$$
Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\neq 0$. Let A_i be A with column i replaced by B. Then
$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation
$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$
where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Fibonacci Numbers

$1,1,2,3,5,8,13,21,34,55,89,\ldots$

Definitions:
$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$
$$F_{-i} = (-1)^{i-1}F_i,$$
$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$
Cassini's identity: for $i>0$:
$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$
Additive rule:
$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$
$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$
Calculation by matrices:
$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)