Team Contest Reference
Team:
System.out.println(42);

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Contents

n	Runtime $100 \cdot 10^6$ in 3s
$[10, 11]$	$\mathcal{O}(n!)$
< 22	$\mathcal{O}(n2^n)$
≤ 100	$\mathcal{O}(n^4)$
≤ 400	$\mathcal{O}(n^3)$
≤ 2.000	$\mathcal{O}(n^2 \log n)$
≤ 10.000	$\mathcal{O}(n^2)$
$\leq 1.000.000$	$\mathcal{O}(n \log n)$
$\leq 100.000.000$	$\mathcal{O}(n)$

`byte` (8 Bit, signed): -128 …127
`short` (16 Bit, signed): -32.768 …23.767
`integer` (32 Bit, signed): -2.147.483.648 …2.147.483.647
`long` (64 Bit, signed): $-2^{63} \ldots 2^{63} - 1$

MD5: `cat <string>| tr -d [:space:] | md5sum`

1 Algorithms

1.1 Binary Search

Binary searchs for an element in a sorted array.

```java
public static boolean BinarySearch(int[] array, int N, int a) {
  int lo = 0;
  int hi = N-1;
  while(lo <= hi) {
    int mid = (int) (((lo + hi) / 2.0) + 0.6);
    if(array[mid] < a) {
      lo = mid+1;
    } else {
      hi = mid-1;
    }
  }
  if(lo < N  && array[lo] == a) {
    return true;
  } else {
    return false;
  }
}
```

MD5: bb87f09a50f05e688706641c26759706 | $\mathcal{O}(log n)$

1.2 Held Karp

Algorithm for TSP

```java
public static int[] tsp(int[][] graph) {
  int n = graph.length;
  if(n == 1) return new int[]{0};
  //C stores the shortest distance to node of the second dimension
  //first dimension is the bitstring of included nodes on the way
  int[][] C = new int[1<<n][n];
  int[][] p = new int[1<<n][n];
  //initialize
  for(int k = 1; k < n; k++) {
     C[1<<k][k] = graph[0][k];
  }
  for(int s = 2; s < n; s++) {
     for(int S = 1; S < (1<<n); S++) {
        if(Integer.bitCount(S)!=s || (S&1) == 1)
           continue;
        for(int k = 1; k < n; k++) {
              if((S & (1 << k)) == 0)
                  continue;

           //Smk is the set of nodes without k
           int Smk = S ^ (1<<k);

           int min = Integer.MAX_VALUE;
           int minprev = 0;
           for(int  m=1; m<n; m++) {
              if((Smk & (1<<m)) == 0)
                 continue;
              //distance to m with the nodes in Smk + connection from m to k
              int tmp = C[Smk][m] +graph[m][k];
              if(tmp < min) {
                 min = tmp;
                 minprev = m;
              }
           }
           C[S][k] = min;
           p[S][k] = minprev;
        }
     }
  }

  //find shortest tour length
  int min = Integer.MAX_VALUE;
  int minprev = -1;
  for(int k = 1; k < n; k++) {
     //Set of all nodes except for the first + cost from 0 to k
     int tmp = C[(1<<n) - 2][k] + graph[0][k];
     if(tmp < min) {
        min = tmp;
        minprev = k;
     }
  }

  //Note that the tour has not been tested yet, only the correctness of the min-tour-value
  //backtrack tour
  int[] tour = new int[n+1];
  tour[n] = 0;
  tour[n-1] = minprev;
  int bits = (1<<n)-2;
  for(int k = n-2; k>0; k--) {
     tour[k] = p[bits][tour[k+1]];
     bits = bits ^ (1<<tour[k+1]);
  }
  tour[0] = 0;
  return tour;
}
```

MD5: 233d98980b1f4dae50ac892d7112dafb | $\mathcal{O}(2^n n^2)$

1.3 LongestIncreasingSubsequence

Computes the longest increasing subsequence and is easy to be adapted.

```java
//This has not been tested yet (adapted from tested C++ Murcia Code)
public static int longestInc(int[] array, int N) {
   int[] m = new int[N];
   for (int i = N - 1; i >= 0; i--) {
      m[i] = 1;
      for (int j = i + 1; j < N; j++) {
         if (array[j] > array[i]) {
            if (m[i] < m[j] + 1) {
               m[i] = m[j] + 1;
            }
         }
      }
   }
   int longest = 0;
   for (int i = 0; i < N; i++) {
      if (m[i] > longest) {
         longest = m[i];
      }
   }
   return longest;
}
```

MD5: 7ee618a580f2736226054b5e106d5635 | $\mathcal{O}(n^2)$

1.4 LongestIncreasingSubsequence

Computes the longest increasing subsequence using binary search.

```
public static int[] LongestIncreasingSubsequencenlogn(
    int[] a, int[] p) {
   int[] m = new int[a.length+1];
   int l = 0;
   for(int i = 0; i < a.length; i++) {
      int lo = 1;
      int hi = l;
      while(lo <= hi) {
         int mid = (int) (((lo + hi) / 2.0) + 0.6);
         if(a[m[mid]] < a[i]) {
            lo = mid+1;
         } else {
            hi = mid-1;
         }
      }
      int newL = lo;
      p[i] = m[newL-1];
      m[newL] = i;
      if(newL > l) {
         l = newL;
      }
   }
   int[] s = new int[l];
   int k = m[l];
   for(int i= l-1; i>= 0; i--) {
      s[i] = a[k];
      k = p[k];
   }
   return s;
}
```

MD5: `e4b7591a2e204809f3e105521a616f70` | $\mathcal{O}(n * logn)$

1.5 NextPermutation

n Returns true if there is another permutation. Can also be used to compute the nextPermutation of an array.

```
public static boolean nextPermutation(char[] a) {
   int i = a.length - 1;
   while(i > 0 && a[i-1] >= a[i]) {
      i--;
   }
   if(i <= 0) {
      return false;
   }
   int j = a.length - 1;
   while (a[j] <=  a[i-1]) {
      j--;
   }
   char tmp = a[i - 1];
   a[i - 1] = a[j];
   a[j] = tmp;

   j = a.length - 1;
   while(i < j) {
      tmp = a[i];
      a[i] = a[j];
      a[j] = tmp;
      i++;
      j--;
   }
   return true;
}
```

MD5: `ca6266722db16f2dc8eae5a6cc5fcacf` | $\mathcal{O}(?)$

2 Graphs

2.1 BellmanFord

DESCRIPTION MISSING

```
public static boolean bellmanFord(Vertex[] vertices) {
   //source is 0
   vertices[0].mindistance = 0;
   //calc distances
   for(int i = 0; i < vertices.length-1; i++) {
      for(int j = 0; j < vertices.length; j++) {
         for(Edge e: vertices[j].adjacencies) {
            if(vertices[j].mindistance != Integer.
                MAX_VALUE
               && e.target.mindistance > vertices[j].
                   mindistance + e.distance) {
               e.target.mindistance = vertices[j].
                   mindistance + e.distance;
            }
         }
      }
   }
   //check for negative-length cycle
   for(int i = 0; i < vertices.length; i++) {
      for(Edge e: vertices[i].adjacencies) {
         if(vertices[i].mindistance != Integer.
             MAX_VALUE && e.target.mindistance >
             vertices[i].mindistance + e.distance) {
            return true;
         }
      }
   }
   return false;
}
```

MD5: `36561a7913a81baf7b7c79b606683819` | $\mathcal{O}(???)$

2.2 BipartiteGraphCheck

DESCRIPTION MISSING

```
public static boolean bipartiteGraphCheck(ArrayList<
    ArrayList<Integer>> graph, int N) {
   int[] color = new int[N];
   for(int i = 0; i < N; i++) color[i] = -1;
   color[0] = 1;
   Queue<Integer> q = new LinkedList<Integer>();
   q.add(0);
   while(!q.isEmpty()) {
      int u = q.poll();
      for(int i : graph.get(u)) {
         if(color[i] == -1) {
            color[i] = 1-color[u];
            q.add(i);
         } else if(color[u] == color[i]) {
            return false;
         }
      }
   }
   return true;
```

```
}
```

MD5: 5cb4622cf75e4ea5ffae51b0b48abf2b | $\mathcal{O}(???)$

2.3 Dijkstra

Finds the shortest paths from one vertex to every other vertex in the graph (SSSP).
For negative weights, add |min|+1 to each edge, later subtract from result.
To get a different shortest path when edges are ints, add an $\epsilon = \frac{1}{k+1}$ on each edge of the shortest path of length k, run again.
Input: A source vertex s and an adjacency list G.
Output: Modified adj. list with distances from s and predcessor vertices set.

```java
public static void dijkstra(Vertex[] vertices, int src ) {
   vertices[src].mindistance = 0;
   PriorityQueue<Vertex> queue = new PriorityQueue< Vertex>();
   queue.add(vertices[src]);
   while(!queue.isEmpty()) {
      Vertex u = queue.poll();
      if(u.visited)
         continue;
      u.visited = true;
      for(Edge e : u.adjacencies) {
         Vertex v = e.target;
         if(v.mindistance > u.mindistance + e.distance ) {
            v.mindistance = u.mindistance + e.distance ;
            queue.add(v);
         }
      }
   }
}
class Vertex implements Comparable<Vertex> {
   public int id;
   public int mindistance = Integer.MAX_VALUE;
   public LinkedList<Edge> adjacencies = new LinkedList<Edge>();
   public boolean visited = false;

   public int compareTo(Vertex other) {
      return Integer.compare(this.mindistance, other. mindistance);
   }
}
class Edge {
   public Vertex target;
   public int distance;

   public Edge (Vertex target, int distance) {
      this.target = target;
      this.distance = distance;
   }
}
```

MD5: d6882162849418a2541cfc7f6c3ddc58 | $\mathcal{O}(|E| \log |V|)$

2.4 EdmondsKarp

Finds the greatest flow in a graph.

```java
public static boolean BFS(int[][] graph, int s, int t, int[] parent) {
   int N = graph.length;
   boolean[] visited = new boolean[N];
   for(int i = 0; i < N; i++) {
      visited[i] = false;
   }
   Queue<Integer> queue = new LinkedList<Integer>();
   queue.add(s);
   visited[s] = true;
   parent[s] = -1;
   while(!queue.isEmpty()) {
      int u = queue.poll();
      if(u == t) return true;
      for(int v= 0; v < N; v++) {
         if(visited[v] == false && graph[u][v] > 0) {
            queue.add(v);
            parent[v] = u;
            visited[v] = true;
         }
      }
   }
   return (visited[t]);
}
public static int fordFulkerson(int[][] graph, int s, int t) {
   int N = graph.length;
   int[][] rgraph = new int[graph.length][graph.length ];
   for(int u = 0; u < graph.length; u++) {
      for(int v = 0; v < graph.length; v++) {
         rgraph[u][v] = graph[u][v];
      }
   }
   int[] parent = new int[N];
   int maxflow = 0;
   while(BFS(rgraph, s, t, parent)) {
      int pathflow = Integer.MAX_VALUE;
      for(int v = t; v!= s; v = parent[v]) {
         int u = parent[v];
         pathflow = Math.min(pathflow, rgraph[u][v]);
      }

      for(int v = t; v != s; v = parent[v]) {
         int u = parent[v];
         rgraph[u][v] -= pathflow;
         rgraph[v][u] += pathflow;
      }

      maxflow += pathflow;
   }
   return maxflow;
}
```

MD5: 8d85785d45794f20303d9b9f920e80dd | $\mathcal{O}(???)$

2.5 FenwickTree

Can be used for computing prefix sums.

```java
int[] fwktree = new int[m + n + 1];
public static int read(int index, int[] fenwickTree) {
   int sum = 0;
   while (index > 0) {
      sum += fenwickTree[index];
      index -= (index & -index);
   }
```

```java
    return sum;
}
public static int[] update(int index, int addValue,
    int[] fenwickTree) {
    while (index <= fenwickTree.length - 1) {
        fenwickTree[index] += addValue;
        index += (index & -index);
    }
    return fenwickTree;
}
```

MD5: `97fd176a403e68cb76a82196191d5f19` | $\mathcal{O}(\log n)$

2.6 FloydWarshall

Finds all shortest paths. Paths in array next, distances in ans.

```java
public static void floydWarshall(int[][] graph, int
    [][] next, int[][] ans) {
    for(int i = 0; i < ans.length; i++) {
        for(int j = 0; j < ans.length; j++) {
            ans[i][j] = graph[i][j];
        }
    }
    for (int k = 0; k < ans.length; k++) {
        for (int i = 0; i < ans.length; i++) {
            for (int j = 0; j < ans.length; j++) {
                if (ans[i][k] + ans[k][j] < ans[i][j]
                && ans[i][k] < Integer.MAX_VALUE && ans[k
                    ][j] < Integer.MAX_VALUE) {
                    ans[i][j] = ans[i][k] + ans[k][j];
                    next[i][j] = next[i][k];
                }
            }
        }
    }
}
```

MD5: `4faf8c41a9070f106e68864cc131706d` | $\mathcal{O}(n^3)$

2.7 BFS AdjMtrx Iterativ

Iterative BFS on adjacency matrix. Returns true or false, depending on whether there is a connection between s and g or not.

```java
public static boolean BFSWithoutPathForAdjMatr(int s,
    int g, int[][] graph) {
    //s being the start and g the goal
    boolean[] visited = new boolean[graph.length];
    for(int i = 0; i < visited.length; i++)
        visited[i] = false;
    Queue<Integer> queue = new LinkedList<Integer>();
    queue.add(s);
    visited[s] = true;
    while(!queue.isEmpty()) {
        int node = queue.poll();
        if(node == g)
            return true;
        for(int i = 0; i < graph.length; i++) {
            if(graph[node][i] > 0 && !visited[i]) {
                queue.add(i);
                visited[i] = true;
            }
        }
    }
    return false;
}
```

MD5: `754e7dfa0a691a2511464e16104b8880` | $\mathcal{O}(V+E?)$

2.8 Topologische Sortierung

Sorts a graph (represented as adjMtrx) topologically

```java
// l enthaelt alle Knoten topologisch sortiert (Start:
    0, Ende= n)
int[] l = new int[n];
int idx = 0;
// s enthaelt alle Knoten, die keine eingehende Kante
    haben
ArrayList<Integer> s = new ArrayList<Integer>();
// initialisiere s
for (int i = 0; i < n; i++) {
if (edgesIn[i] == 0) {
s.add(i);
}
}
// Algo Beginn
while (!s.isEmpty()) {
    int node = s.remove(0);
    l[idx++] = node;
    for (int i = 0; i < n; i++) {
        if (adjMtrx[node][i]) {
            adjMtrx[node][i] = false;
            edgesIn[i] -= 1;
            if (edgesIn[i] == 0) {
                s.add(i);
            }
        }
    }
}
```

MD5: `01974f4bab4e48916ecdc48531a79c84` | $\mathcal{O}(O(V+E))$

3 Math

3.1 GrahamScan

GrahamScan finds convex hull. Still has collinear point problematic at the last diagonal.

```java
public static int ccw(Point src, Point q1, Point q2) {
    return (q1.x - src.x) * (q2.y - src.y) - (q2.x -
        src.x) * (q1.y - src.y);
}

public static boolean isColl(Point a, Point b, Point c
    ) {
    if((b.y - a.y) * (c.x - b.x) == (c.y - b.y) * (b.x
        - a.x)) {
        return true;
    } else {
        return false;
    }
}

public static double calcDist(Point src, Point target)
    {
    return Math.sqrt((src.x + target.x) * (src.x +
        target.x) + (src.y + target.y) * (src.y *
        target.y));
```

```java
}

//Expects a array sorted with PolarComp as Comparator
//IMPORTANT! before sorting put lowest, and if two are
    the same leftmost, element at position 0 in array
public static void grahamScan(Point[] points) {
   int m = 1;
   for(int i = 2; i < points.length; i++) {
      while(ccw(points[m-1], points[m], points[i]) <
          0) {
         if(m > 1) m--;
         else if(i == points.length) break;
         else i++;
      }
      m++;
      Point tmp = points[i];
      points[i] = points[m];
      points[m] = tmp;
   }
}

class Point {
   int x;
   int y;
   public Point(int x, int y) {
      this.x = x;
      this.y = y;
   }
}

class PolarComp implements Comparator<Point> {
   Point src;

   public PolarComp(Point source) {
      src = source;
   }

   public double calcDist(Point q1, Point q2) {
      return Math.sqrt((q1.x - q2.x) * (q1.x - q2.x) +
           (q1.y - q2.y) * (q1.y - q2.y));
   }

   public int ccw(Point q1, Point q2) {
      return (q1.x - src.x) * (q2.y - src.y) - (q2.x -
           src.x) * (q1.y - src.y);
   }

   public int compare(Point q1, Point q2) {
      int res = ccw(q1, q2);
      double dist1 = calcDist(src, q1);
      double dist2 = calcDist(src, q2);
      if(res > 0) return -1;
      else if(res < 0) return 1;
      else if(res == 0 && dist1 < dist2) return 1;
      else if(res == 0 && dist1 > dist2) return -1;
      else  return 0;
   }
}
```

MD5: 97ad3ab5efa1cbfa7374a86aa2db7f62 | $\mathcal{O}(???)$

4 Java Knowhow

4.1 System.out.printf() und String.format()

Syntax: `%[flags][width][.precision][conv]`

flags:

`-`	left-justify (default: right)
`+`	always output number sign
`0`	zero-pad numbers
(space)	space instead of minus for pos. numbers
,	group triplets of digits with ,

width specifies output width

precision is for floating point precision

conv:

d	byte, short, int, long
f	float, double
c	char (use C for uppercase)
s	String (use S for all uppercase)

4.2 Speed up IO

Use `BufferedReader br = new BufferedReader(new InputStreamReader(System.in));`

Use `Double.parseDouble(Scanner.next())`

Theoretical Computer Science Cheat Sheet

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

Identities

1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\binom{n}{k} = \binom{n}{n-k}$,

4. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\left[{n \atop 1}\right] = (n-1)!$, **15.** $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\left[{n \atop n}\right] = 1$, **17.** $\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0$ for $n \ne 0$,

34. $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix}\binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}} = n!\sum_{k=0}^{n} \frac{1}{k!}\begin{bmatrix} k \\ m \end{bmatrix},$$

39. $$\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$$

40. $$\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k}\left\{ {k+1 \atop m+1} \right\}(-1)^{n-k},$$

41. $$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\binom{k}{m}(-1)^{m-k},$$

42. $$\left\{ {m+n+1 \atop m} \right\} = \sum_{k=0}^{m} k\left\{ {n+k \atop k} \right\},$$

43. $$\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} k(n+k)\begin{bmatrix} n+k \\ k \end{bmatrix},$$

44. $$\binom{n}{m} = \sum_k \left\{ {n+1 \atop k+1} \right\}\begin{bmatrix} k \\ m \end{bmatrix}(-1)^{m-k},$$

45. $$(n-m)!\binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\left\{ {k \atop m} \right\}(-1)^{m-k}, \quad \text{for } n \ge m,$$

46. $$\left\{ {n \atop n-m} \right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\begin{bmatrix} m+k \\ k \end{bmatrix},$$

47. $$\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{ {m+k \atop k} \right\},$$

48. $$\left\{ {n \atop \ell+m} \right\}\binom{\ell+m}{\ell} = \sum_k \left\{ {k \atop \ell} \right\}\left\{ {n-k \atop m} \right\}\binom{n}{k},$$

49. $$\begin{bmatrix} n \\ \ell+m \end{bmatrix}\binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix}\begin{bmatrix} n-k \\ m \end{bmatrix}\binom{n}{k}.$$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:
$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$
and equality holds only if every internal node has 2 sons.

Recurrences

Master method:
$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then
$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then
$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then
$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence
$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have
$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get
$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find
$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence
$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side
$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get
$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have
$$\begin{aligned} n \sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider
$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that
$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find
$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:
$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:
$$\sum_{i \ge 0} g_{i+1} x^i = \sum_{i \ge 0} 2g_i x^i + \sum_{i \ge 0} x^i.$$

We choose $G(x) = \sum_{i \ge 0} x^i g_i$. Rewrite in terms of $G(x)$:
$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \ge 0} x^i.$$

Simplify:
$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:
$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:
$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i \ge 0} 2^i x^i - \sum_{i \ge 0} x^i\right) \\ &= \sum_{i \ge 0} (2^{i+1} - 1)x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$
$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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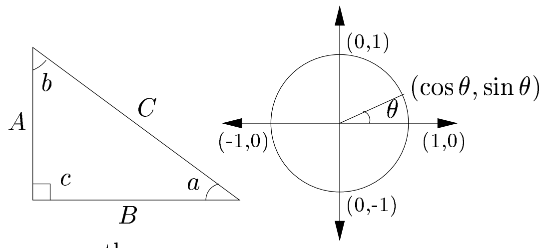
Trigonometry



Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \qquad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

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Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

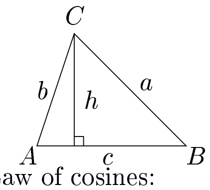
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$\begin{aligned} C &\equiv r_1 \bmod m_1 \\ \vdots\ &\ \vdots\ \ \vdots \\ C &\equiv r_n \bmod m_n \end{aligned}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n\frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embedded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k,\ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

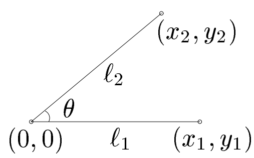
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p},$$

$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

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π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:
$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1-u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$,

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\, dx = \tan x,$

20. $\int \csc^2 x\, dx = -\cot x,$

21. $\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx,$

22. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx,$

23. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \quad n \neq 1,$

24. $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \quad n \neq 1,$

25. $\int \sec^n x\, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx, \quad n \neq 1,$

26. $\int \csc^n x\, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x\, dx, \quad n \neq 1,$

27. $\int \sinh x\, dx = \cosh x,$

28. $\int \cosh x\, dx = \sinh x,$

29. $\int \tanh x\, dx = \ln|\cosh x|,$

30. $\int \coth x\, dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x\, dx = \ln\left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x\, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln|a^2 - x^2|,$

38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$

39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

41. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right|,$

45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln\left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln\left|\frac{x}{a + bx}\right|,$

49. $\int x\sqrt{a + bx}\, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

51. $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

53. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

57. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a,} \quad a > 0,$

58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln\left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

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Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac,\\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac,\end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0,\\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0,\end{cases}$$

68. $$\int\sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int\frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0,\\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0,\end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m-\frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$\begin{array}{rclcl}
x^1 &=& x^{\underline{1}} &=& x^{\overline{1}}\\
x^2 &=& x^{\underline{2}}+x^{\underline{1}} &=& x^{\overline{2}}-x^{\overline{1}}\\
x^3 &=& x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} &=& x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}\\
x^4 &=& x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} &=& x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}\\
x^5 &=& x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} &=& x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{rclrcl}
x^{\overline{1}} &=& x^1 & x^{\underline{1}} &=& x^1\\
x^{\overline{2}} &=& x^2+x^1 & x^{\underline{2}} &=& x^2-x^1\\
x^{\overline{3}} &=& x^3+3x^2+2x^1 & x^{\underline{3}} &=& x^3-3x^2+2x^1\\
x^{\overline{4}} &=& x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} &=& x^4-6x^3+11x^2-6x^1\\
x^{\overline{5}} &=& x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} &=& x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x)=\Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu)=c\Delta u, \qquad \Delta(u+v)=\Delta u+\Delta v,$

$\Delta(uv)=u\Delta v+\mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}})=nx^{\underline{n-1}},$

$\Delta(H_x)=x^{\underline{-1}}, \qquad \Delta(2^x)=2^x,$

$\Delta(c^x)=(c-1)c^x, \qquad \Delta\binom{x}{m}=\binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x+\sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv-\sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}}=1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}}=1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$

$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$

$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^{n}\left\{{n \atop k}\right\}x^{\underline{k}} = \sum_{k=1}^{n}\left\{{n \atop k}\right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^{n}\left[{n \atop k}\right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^{n}\left[{n \atop k}\right]x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1-(F_{n-1}+F_{n+1})x-(-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i, \qquad \left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ i \atop n \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[n \atop i \right]x^i, \qquad (e^x-1)^n = \sum_{i=0}^{\infty}\left\{ i \atop n \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[i \atop n \right]\frac{n!x^i}{i!}, \qquad x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!}, \qquad \zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x}, \qquad \frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

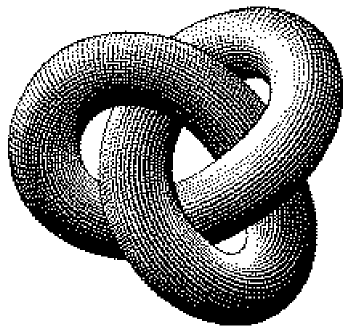
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Fibonacci Numbers

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)