Team Contest Reference

Team: Romath

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Contents

n	Runtime $100 \cdot 10^6$ in 3s
$[10, 11]$	$\mathcal{O}(n!)$
< 22	$\mathcal{O}(n2^n)$
≤ 100	$\mathcal{O}(n^4)$
≤ 400	$\mathcal{O}(n^3)$
≤ 2.000	$\mathcal{O}(n^2 \log n)$
≤ 10.000	$\mathcal{O}(n^2)$
$\leq 1.000.000$	$\mathcal{O}(n \log n)$
$\leq 100.000.000$	$\mathcal{O}(n)$

byte (8 Bit, signed): -128 …127
short (16 Bit, signed): -32.768 …23.767
integer (32 Bit, signed): -2.147.483.648 …2.147.483.647
long (64 Bit, signed): $-2^{63} \ldots 2^{63} - 1$

MD5: cat <string>| tr -d [:space:] | md5sum

1 DP

1.1 LongestIncreasingSubsequence

Computes the length of the longest increasing subsequence and is easy to be adapted.
Input: array arr containig a sequence of length N
Output: lenght of the longest increasing subsequence in arr

```java
// This has not been tested yet
// (adapted from tested C++ Murcia Code)
public static int LISeasy(int[] arr, int N) {
  int[] m = new int[N];
  for (int i = N - 1; i >= 0; i--) {
    m[i] = 1; //init table
    for (int j = i + 1; j < N; j++) {
      // if arr[i] increases the length
      // of subsequence from array[j]
      if (arr[j] > arr[i])
        if (m[i] < m[j] + 1)
          // store lenght of new subseq
          m[i] = m[j] + 1;
    }
  }
  // find max in array
  int longest = 0;
  for (int i = 0; i < N; i++) {
    if (m[i] > longest)
      longest = m[i];
  }
  return longest;
}
```

MD5: 7561f576d50b1dc6262568c0fc6c42dd | $\mathcal{O}(n^2)$

1.2 LongestIncreasingSubsequence

Computes the longest increasing subsequence using binary search.
Input: array arr containing a sequence and empty array p of length $arr.length$ for storing indices of the LIS (might be usefull to have)
Output: array s containing the longest increasing subsequence

```java
public static int[] LISfast(int[] arr, int[] p) {
  // p[k] stores index of the predecessor of arr[k]
  // in the LIS ending at arr[k]
  // m[j] stores index k of smallest value arr[k]
  // so there is a LIS of length j ending at arr[k]
```

```java
  int[] m = new int[arr.length+1];
  int l = 0;
  for(int i = 0; i < arr.length; i++) {
    // bin search for the largest positive j <= l
    // with arr[m[j]] < arr[i]
    int lo = 1;
    int hi = l;
    while(lo <= hi) {
      int mid = (int) (((lo + hi) / 2.0) + 0.6);
      if(arr[m[mid]] <= arr[i])
        lo = mid+1;
      else
        hi = mid-1;
    }
    // lo is 1 greater than length of the
    // longest prefix of arr[i]
    int newL = lo;
    p[i] = m[newL-1];
    m[newL] = i;
    // if LIS found is longer than the ones
    // found before, then update l
    if(newL > l)
      l = newL;
  }
  // reconstruct the LIS
  int[] s = new int[l];
  int k = m[l];
  for(int i= l-1; i>= 0; i--) {
    s[i] = arr[k];
    k = p[k];
  }
  return s;
}
```

MD5: `1d75905f78041d832632cb76af985b8e` | $\mathcal{O}(n \log n)$

2 DataStructures

2.1 Fenwick-Tree

Can be used for computing prefix sums.

```java
//note that 0 can not be used
int[] fwktree = new int[m + n + 1];
public static int read(int index, int[] fenwickTree) {
   int sum = 0;
   while (index > 0) {
      sum += fenwickTree[index];
      index -= (index & -index);
   }
   return sum;
}
public static int[] update(int index, int addValue, int[] fenwickTree) {
   while (index <= fenwickTree.length - 1) {
      fenwickTree[index] += addValue;
      index += (index & -index);
   }
   return fenwickTree;
}
```

MD5: `410185d657a3a5140bde465090ff6fb5` | $\mathcal{O}(\log n)$

2.2 Range Maximum Query

$process$ processes an array A of length N in O($N \log N$) such that $query$ can compute the maximum value of A in interval $[i, j]$. Therefore $M[a, b]$ stores the maximum value of interval $[a, a + 2^b - 1]$.

Input: dynamic table M, array to search A, length N of A, start index i and end index j

Output: filled dynamic table M or the maximum value of A in interval $[i, j]$

```java
public static void process(int[][] M, int[] A, int N) {
  for(int i = 0; i < N; i++)
    M[i][0] = i;
  // filling table M
  // M[i][j] = max(M[i][j-1], M[i+(1<<(j-1))][j-1]),
  // cause interval of length 2^j can be partitioned
  // into two intervals of length 2^(j-1)
  for(int j = 1; 1 << j <= N; j++) {
    for(int i = 0; i + (1 << j) - 1 < N; i++) {
      if(A[M[i][j-1]] >= A[M[i+(1 << (j-1))][j-1]])
        M[i][j] = M[i][j-1];
      else
        M[i][j] = M[i + (1 << (j-1))][j-1];
    }
  }
}

public static int query(int[][] M, int[] A, int N,
                                   int i, int j) {
  // k = |_ log_2(j-i+1) _|
  int k = (int) (Math.log(j - i + 1) / Math.log(2));
  if(A[M[i][k]] >= A[M[j- (1 << k) + 1][k]])
    return M[i][k];
  else
    return M[j - (1 << k) + 1][k];
}
```

MD5: `db0999fa40037985ff27dd1a43c53b80` | $\mathcal{O}(N \log N, 1)$

2.3 Union-Find

Union-Find is a data structure that keeps track of a set of elements partitioned into a number of disjoint subsets. $UnionFind$ creates n disjoint sets each containing one element. $union$ joins the sets x and y are contained in. $find$ returns the representative of the set x is contained in.

Input: number of elements n, element x, element y

Output: the representative of element x or a boolean indicating whether sets got merged.

```java
class UnionFind {
  private int[] p = null;
  private int[] r = null;
  private int count = 0;

  public int count() {
    return count;
  } // number of sets

  public UnionFind(int n) {
    count = n; // every node is its own set
```

```java
    r = new int[n]; // every node is its own tree with
         height 0
    p = new int[n];
    for (int i = 0; i < n; i++)
      p[i] = -1; // no parent = -1
  }

  public int find(int x) {
    int root = x;
    while (p[root] >= 0) { // find root
      root = p[root];
    }
    while (p[x] >= 0) { // path compression
      int tmp = p[x];
      p[x] = root;
      x = tmp;
    }
    return root;
  }

  // return true, if sets merged and false, if already
       from same set
  public boolean union(int x, int y) {
    int px = find(x);
    int py = find(y);
    if (px == py)
      return false; // same set -> reject edge
    if (r[px] < r[py]) { // swap so that always h[px
        ]>=h[py]
      int tmp = px;
      px = py;
      py = tmp;
    }
    p[py] = px; // hang flatter tree as child of
        higher tree
    r[px] = Math.max(r[px], r[py] + 1); // update (
        worst-case) height
    count--;
    return true;
  }
}
```

MD5: `5c507168e1ffd9ead25babf7b3769cfd` | $\mathcal{O}(\alpha(n))$

3 Graph

3.1 2SAT

```java
//We assume that ind(not a) = ind(a) + N, with N being
     the number of variables
//could however be changed easily
public static boolean 2SAT(Vertex[] G) {
  //call SCC
  double DFS(G);
  //check for contradiction
  boolean poss = true;
  for(int i = 0; i < S+A; i++) {
    if(G[i].comp == G[i + (S+A)].comp) {
      poss = false;
    }
  }
  return poss;
}
```

MD5: `6c06a2b59fd3a7df3c31b06c58fdaaf5` | $\mathcal{O}(V + E)$

3.2 Breadth First Search

Iterative BFS. Uses ref Vertex class, no Edge class needed. In this version we look for a shortest path from s to t though we could also find the BFS-tree by leaving out t. *Input:* IDs of start and goal vertex and graph as AdjList *Output:* `true` if there is a connection between s and g, `false` otherwise

```java
public static boolean BFS(Vertex[] G, int s, int t) {
  //make sure that Vertices vis values are false etc
  Queue<Vertex> q = new LinkedList<Vertex>();
  G[s].vis = true;
  G[s].dist = 0;
  G[s].pre = -1;
  q.add(G[s]);
  //expand frontier between undiscovered and
      discovered vertices
  while(!q.isEmpty()) {
    Vertex u = q.poll();
    //when reaching the goal, return true
    //if we want to construct a BFS-tree delete this
        line
    if(u.id = t) return true;
    //else add adj vertices if not visited
    for(Vertex v : u.adj) {
      if(!v.vis) {
        v.vis = true;
        v.dist = u.dist + 1;
        v.pre = u.id;
        q.add(v);
      }
    }
  }
  //did not find target
  return false;
}
```

MD5: `71f3fa48b4f1b2abdff3557a27a9a136` | $\mathcal{O}(|V| + |E|)$

3.3 BellmanFord

Finds shortest pathes from a single source. Negative edge weights are allowed. Can be used for finding negative cycles.

```java
public static boolean bellmanFord(Vertex[] G) {
  //source is 0
  G[0].dist = 0;
  //calc distances
  //the path has max length |V|-1
  for(int i = 0; i < G.length-1; i++) {
    //each iteration relax all edges
    for(int j = 0; j < G.length; j++) {
      for(Edge e : G[j].adj) {
        if(G[j].dist != Integer.MAX_VALUE
        && e.t.dist > G[j].dist + e.w) {
          e.t.dist = G[j].dist + e.w;
        }
      }
    }
  }
  //check for negative-length cycle
  for(int i = 0; i < G.length; i++) {
    for(Edge e : G[i].adj) {
      if(G[i].dist != Integer.MAX_VALUE
          && e.t.dist > G[i].dist + e.w) {
        return true;
      }
```

```
    }
  }
  return false;
}
```

MD5: `d101e6b6915f012b3f0c02dc79e1fc6f` | $\mathcal{O}(|V| \cdot |E|)$

3.4 Bipartite Graph Check

Checks a graph represented as adjList for being bipartite. Needs a little adaption, if the graph is not connected.

Input: $graph$ as adjList, amount of nodes N as int

Output: `true` if graph is bipartite, `false` otherwise

```
public static boolean bipartiteGraphCheck(Vertex[] G){
  // use bfs for coloring each node
  G[0].color = 1;
  Queue<Vertex> q = new LinkedList<Vertex>();
  q.add(G[0]);
  while(!q.isEmpty()) {
    Vertex u = q.poll();
    for(Vertex v : u.adj) {
      // if node i not yet visited,
      // give opposite color of parent node u
      if(v.color == -1) {
        v.color = 1-u.color;
        q.add(v);
      // if node i has same color as parent node u
      // the graph is not bipartite
      } else if(u.color == v.color)
        return false;
      // if node i has different color
      // than parent node u keep going
    }
  }
  return true;
}
```

MD5: `e93d242522e5b4085494c86f0d218dd4` | $\mathcal{O}(|V| + |E|)$

3.5 Maximum Bipartite Matching

Finds the maximum bipartite matching in an unweighted graph using DFS.

Input: An unweighted adjacency matrix boolean[M][N] with M nodes being matched to N nodes.

Output: The maximum matching. (For getting the actual matching, little changes have to be made.)

```
// A DFS based recursive function that returns true
// if a matching for vertex u is possible
boolean bpm(boolean bpGraph[][], int u,
            boolean seen[], int matchR[]) {
  // Try every job one by one
  for (int v = 0; v < N; v++) {
    // If applicant u is interested in job v and v
    // is not visited
    if (bpGraph[u][v] && !seen[v]) {
      seen[v] = true; // Mark v as visited

      // If job v is not assigned to an applicant OR
      // previously assigned applicant for job v
      // (which is matchR[v]) has an alternate job
      // available. Since v is marked as visited in
      // the above line, matchR[v] in the following
      // recursive call will not get job v again
      if (matchR[v] < 0 ||
      bpm(bpGraph, matchR[v], seen, matchR)) {
        matchR[v] = u;
        return true;
      }
    }
  }
  return false;
}

// Returns maximum number of matching from M to N
int maxBPM(boolean bpGraph[][]) {
  // An array to keep track of the applicants assigned
  // to jobs. The value of matchR[i] is the applicant
  // number assigned to job i, the value -1 indicates
  // nobody is assigned.
  int matchR[] = new int[N];
  // Initially all jobs are available
  for(int i = 0; i < N; ++i)
    matchR[i] = -1;
  // Count of jobs assigned to applicants
  int result = 0;
  for (int u = 0; u < M; u++) {
    // Mark all jobs as not seen for next applicant.
    boolean seen[] = new boolean[N];
    for(int i = 0; i < N; ++i)
      seen[i] = false;
    // Find if the applicant u can get a job
    if (bpm(bpGraph, u, seen, matchR))
      result++;
  }
  return result;
}
```

MD5: `a4cc90bf91c41309ad7aaa0c2514ff06` | $\mathcal{O}(M \cdot N)$

3.6 Bitonic TSP

Input: Distance matrix d with vertices sorted in x-axis direction.

Output: Shortest bitonic tour length

```
public static double bitonic(double[][] d) {
  int N = d.length;
  double[][] B = new double[N][N];
  for (int j = 0; j < N; j++) {
    for (int i = 0; i <= j; i++) {
      if (i < j - 1)
        B[i][j] = B[i][j - 1] + d[j - 1][j];
      else {
        double min = 0;
        for (int k = 0; k < j; k++) {
          double r = B[k][i] + d[k][j];
          if (min > r || k == 0)
            min = r;
        }
        B[i][j] = min;
      }
    }
  }
  return B[N-1][N-1];
}
```

MD5: `49fca508fb184da171e4c8e18b6ca4c7` | $\mathcal{O}(?)$

3.7 Single-source shortest paths in dag

Not tested but should be working fine Similar approach can be used for longest paths. Simply go through ts and add 1 to the largest longest path value of the incoming neighbors

```
public static void dagSSP(Vertex[] G, int s) {
  //calls topological sort method
  LinkedList<Integer> sorting = TS(G);
  G[s].dist = 0;
  //go through vertices in ts order
  for(int u : sorting) {
    for(Edge e : G[u].adj) {
      Vertex v = e.t;
      if(v.dist > u.dist + e.w) {
        v.dist = u.dist + e.w;
        v.pre = u.id;
      }
    }
  }
}
```

MD5: 552172db2968f746c4ac0bd322c665f9 | $\mathcal{O}(|V|+|E|)$

3.8 Dijkstra

Finds the shortest paths from one vertex to every other vertex in the graph (SSSP).
For negative weights, add |min|+1 to each edge, later subtract from result.
To get a different shortest path when edges are ints, add an $\varepsilon = \frac{1}{k+1}$ on each edge of the shortest path of length k, run again.
Input: A source vertex s and an adjacency list G.
Output: Modified adj. list with distances from s and predcessor vertices set.

```
public static void dijkstra(Vertex[] G, int s) {
  G[s].dist = 0;
  Tuple st = new Tuple(s, 0);
  PriorityQueue<Tuple> q = new PriorityQueue<Tuple>();
  q.add(st);

  while(!q.isEmpty()) {
    Tuple sm = q.poll();
    Vertex u = G[sm.id];
    //this checks if the Tuple is still useful, both
        checks should be equivalent
    if(u.vis || sm.dist > u.dist) continue;
    u.vis = true;
    for(Edge e : u.adj) {
      Vertex v = e.t;
      if(!v.vis && v.dist > u.dist + e.w) {
        v.pre = u.id;
        v.dist = u.dist + e.w;
        Tuple nt = new Tuple(v.id, v.dist);
        q.add(nt);
      }
    }
  }
}
```

MD5: e46eb1b919179dab6a42800376f04d7a | $\mathcal{O}(|E|\log|V|)$

3.9 EdmondsKarp

Finds the greatest flow in a graph. Capacities must be positive.

```
public static boolean BFS(Vertex[] G, int s, int t) {
  int N = G.length;
  for(int i = 0; i < N; i++) {
    G[i].vis = false;
  }

  Queue<Vertex> q = new LinkedList<Vertex>();
  G[s].vis = true;
  G[s].pre = -1;
  q.add(G[s]);

  while(!q.isEmpty()) {
    Vertex u = q.poll();
    if(u.id == t) return true;
    for(int i : u.adj.keySet()) {
      Edge e = u.adj.get(i);
      Vertex v = e.t;
      if(!v.vis && e.rw > 0) {
        v.vis = true;
        v.pre = u.id;
        q.add(v);
      }
    }
  }
  return (G[t].vis);
}
//We store the edges in the graph in a hashmap
public static int edKarp(Vertex[] G, int s, int t) {
  int maxflow = 0;
  while(BFS(G, s, t)) {
    int pflow = Integer.MAX_VALUE;
    for(int v = t; v!= s; v = G[v].pre) {
      int u = G[v].pre;
      pflow = Math.min(pflow, G[u].adj.get(v).rw);
    }
    for(int v = t; v != s; v = G[v].pre) {
      int u = G[v].pre;
      G[u].adj.get(v).rw -= pflow;
      G[v].adj.get(u).rw += pflow;
    }
    maxflow += pflow;
  }
  return maxflow;
}
```

MD5: 6067fa877ff237d82294e7511c79d4bc | $\mathcal{O}(|V|^2 \cdot |E|)$

3.10 Reference for Edge classes

Used for example in Dijkstra algorithm, implements edges with weight. Needs testing.

```
//for Kruskal we need to sort edges, use: java.lang.
    Comparable
class Edge implements Comparable<Edge> {}

class Edge {
  //for Kruskal it is helpful to store the start as
  //well, moreover we might not need the vertex class
  int s;
  int t;

  //for EdKarp we also want to store residual weights
```

```java
  int rw;

  Vertex t;
  int w;

  public Edge(Vertex t, int w) {
    this.t = t;
    this.w = w;
    this.rw = w;
  }

  public Edge(int s, int t, int w) {...}

  public int compareTo(Edge other) {
    return Integer.compare(this.w, other.w);
  }
}
```

MD5: aae80ac4bfbfcc0b9ac4c65085f6f123 | $\mathcal{O}(1)$

3.11 FloydWarshall

Finds all shortest paths. Paths in array next, distances in ans.

```java
public static void floydWarshall(int[][] graph,
                     int[][] next, int[][] ans) {
  for(int i = 0; i < ans.length; i++)
    for(int j = 0; j < ans.length; j++)
      ans[i][j] = graph[i][j];

  for (int k = 0; k < ans.length; k++)
    for (int i = 0; i < ans.length; i++)
      for (int j = 0; j < ans.length; j++)
        if (ans[i][k] + ans[k][j] < ans[i][j]
                 && ans[i][k] < Integer.MAX_VALUE
                 && ans[k][j] < Integer.MAX_VALUE) {
          ans[i][j] = ans[i][k] + ans[k][j];
          next[i][j] = next[i][k];
        }
}
```

MD5: a98bbda7e53be8ee0df72dbd8721b306 | $\mathcal{O}(|V|^3)$

3.12 Held Karp

Algorithm for TSP

```java
public static int[] tsp(int[][] graph) {
  int n = graph.length;
  if(n == 1) return new int[]{0};
  //C stores the shortest distance to node of the
      second dimension, first dimension is the
      bitstring of included nodes on the way
  int[][] C = new int[1<<n][n];
  int[][] p = new int[1<<n][n];
  //initialize
  for(int k = 1; k < n; k++) {
    C[1<<k][k] = graph[0][k];
  }
  for(int s = 2; s < n; s++) {
    for(int S = 1; S < (1<<n); S++) {
      if(Integer.bitCount(S)!=s || (S&1) == 1)
          continue;
      for(int k = 1; k < n; k++) {
        if((S & (1 << k)) == 0) continue;

        //Smk is the set of nodes without k
        int Smk = S ^ (1<<k);

        int min = Integer.MAX_VALUE;
        int minprev = 0;
        for(int  m=1; m<n; m++) {
          if((Smk & (1<<m)) == 0) continue;
          //distance to m with the nodes in Smk +
              connection from m to k
          int tmp = C[Smk][m] +graph[m][k];
          if(tmp < min) {
            min = tmp;
            minprev = m;
          }
        }
        C[S][k] = min;
        p[S][k] = minprev;
      }
    }
  }

  //find shortest tour length
  int min = Integer.MAX_VALUE;
  int minprev = -1;
  for(int k = 1; k < n; k++) {
    //Set of all nodes except for the first + cost
        from 0 to k
    int tmp = C[(1<<n) - 2][k] + graph[0][k];
    if(tmp < min) {
      min = tmp;
      minprev = k;
    }
  }

  //Note that the tour has not been tested yet, only
      the correctness of the min-tour-value backtrack
      tour
  int[] tour = new int[n+1];
  tour[n] = 0;
  tour[n-1] = minprev;
  int bits = (1<<n)-2;
  for(int k = n-2; k>0; k--) {
    tour[k] = p[bits][tour[k+1]];
    bits = bits ^ (1<<tour[k+1]);
  }
  tour[0] = 0;
  return tour;
}
```

MD5: f3e9730287dcbf2695bf7372fc4bafe0 | $\mathcal{O}(2^n n^2)$

3.13 Iterative DFS

Simple iterative DFS, the recursive variant is a bit fancier. Not tested.

```java
//if we want to start the DFS for different connected
    components, there is such a method in the
    recursive variant of DFS
public static boolean ItDFS(Vertex[] G, int s, int t){
  //take care that all the nodes are not visited at
      the beginning
  Stack<Integer> S = new Stack<Integer>();
  s.push(s);
  while(!S.isEmpty()) {
    int u = S.pop();
    if(u.id == t) return true;
```

```
    if(!G[u].vis) {
      G[u].vis = true;
      for(Vertex v : G[u].adj) {
        if(!v.vis)
          S.push(v.id);
      }
    }
  }
  return false;
}
```

MD5: 80f28ea9b2a04af19b48277e3c6bce9e | $\mathcal{O}(|V| + |E|)$

3.14 Johnsons Algorithm

```
public static int[][] johnson(Vertex[] G) {
  Vertex[] Gd = new Vertex[G.length+1];
  int s = G.length;
  for(int i = 0; i < G.length; i++)
    Gd[i] = G[i];
  //init new vertex with zero-weight-edges to each
      vertex
  Vertex S = new Vertex(G.length);
  for(int i = 0; i < G.length; i++)
    S.adj.add(new Edge(Gd[i], 0));
  Gd[G.length] = S;

  //bellman-ford to check for neg-weight-cycles and to
      adapt edges to enable running dijkstra
  if(bellmanFord(Gd, s)) {
    System.out.println("False");
    //this should not happen and will cause troubles
    return null;
  }
  //change weights
  for(int i = 0; i < G.length; i++)
    for(Edge e : Gd[i].adj)
      e.w = e.w + Gd[i].dist - e.t.dist;
  //store distances to invert this step later
  int[] h = new int[G.length];
  for(int i = 0; i < G.length; i++)
    h[i] = G[i].dist;

  //create shortest path matrix
  int[][] apsp = new int[G.length][G.length];

  //now use original graph G
  //start a dijkstra for each vertex
  for(int i = 0; i < G.length; i++) {
    //reset weights
    for(int j = 0; j < G.length; j++) {
      G[j].vis = false;
      G[j].dist = Integer.MAX_VALUE;
    }
    dijkstra(G, i);
    for(int j = 0; j < G.length; j++)
      apsp[i][j] = G[j].dist + h[j] - h[i];
  }
  return apsp;
}
```

MD5: 0a5c741be64b65c5211fe6056ffc1e02 | $\mathcal{O}(|V|^2 \log V + VE)$

3.15 Kruskal

Computes a minimum spanning tree for a weighted undirected graph.

```
public static int kruskal(Edge[] edges, int n) {
  Arrays.sort(edges);
  //n is the number of vertices
  UnionFind uf = new UnionFind(n);
  //we will only compute the sum of the MST, one could
      of course also store the edges
  int sum = 0;
  int cnt = 0;
  for(int i = 0; i < edges.length; i++) {
    if(cnt == n-1) break;
    if(uf.union(edges[i].s, edges[i].t)) {
      sum += edges[i].w;
      cnt++;
    }
  }
  return sum;
}
```

MD5: 91a1657706750a76d384d3130d98e5fb | $\mathcal{O}(|E| + \log |V|)$

3.16 Min Cut

Calculates the min cut using Edmonds Karp algorithm.

MD5: d41d8cd98f00b204e9800998ecf8427e | $\mathcal{O}(?)$

3.17 Prim

```
//s is the startpoint of the algorithm, in general not
    too important; we assume that graph is connected
public static int prim(Vertex[] G, int s) {
  //make sure dists are maxint
  G[s].dist = 0;
  Tuple st = new Tuple(s, 0);

  PriorityQueue<Tuple> q = new PriorityQueue<Tuple>();
  q.add(st);
  //we will store the sum and each nodes predecessor
  int sum = 0;

  while(!q.isEmpty()) {
    Tuple sm = q.poll();
    Vertex u = G[sm.id];
    //u has been visited already
    if(u.vis) continue;
    //this is not the latest version of u
    if(sm.dist > u.dist) continue;
    u.vis = true;
    //u is part of the new tree and u.dist the cost of
        adding it
    sum += u.dist;
    for(Edge e : u.adj) {
      Vertex v = e.t;
      if(!v.vis && v.dist > e.w) {
        v.pre = u.id;
        v.dist = e.w;
        Tuple nt = new Tuple(v.id, e.w);
        q.add(nt);
      }
    }
```

```java
  }
  return sum;
}
```

MD5: `c82f0bcc19cb735b4ef35dfc7ccfe197` | $\mathcal{O}(?)$

3.18 Recursive Depth First Search

Recursive DFS with different options (storing times, connected/unconnected graph). Needs testing.

Input: A source vertex s, a target vertex t, and adjlist G and the time (0 at the start)

Output: Indicates if there is connection between s and t.

```java
//if we want to visit the whole graph, even if it is not connected we might use this
public static void DFS(Vertex[] G) {
  //make sure all vertices vis value is false etc
  int time = 0;
  for(int i = 0; i < G.length; i++) {
    if(!G[i].vis) {
      //note that we leave out t so this does not work with the below function
      //adaption will not be too difficult though
      //time should not always start at zero, change if needed
      recDFS(i, G, 0);
    }
  }
}

//first call with time = 0
public static boolean recDFS(int s, int t, Vertex[] G, int time){
  //it might be necessary to store the time of discovery
  time = time + 1;
  G[s].dtime = time;

  G[s].vis = true; //new vertex has been discovered
  //when reaching the target return true
  //not necessary when calculating the DFS-tree
  if(s == t) return true;
  for(Vertex v : G[s].adj) {
    //exploring a new edge
    if(!v.vis) {
      v.pre = u.id;
      if(recDFS(v.id, t, G)) return true;
    }
  }
  //storing finishing time
  time = time + 1;
  G[s].ftime = time;
  return false;
}
```

MD5: `3cef44fd916e1aecfb0e3eacc355e2e3` | $\mathcal{O}(|V| + |E|)$

3.19 Strongly Connected Components

```java
public static void fDFS(Vertex u, LinkedList<Integer> sorting) {
  //compare with TS
  u.vis = true;
  for(Vertex v : u.out)
    if(!v.vis)
      fDFS(v, sorting);
  sorting.addFirst(u.id);
  return sorting;
}

public static void sDFS(Vertex u, int cnt) {
  //basic DFS, all visited vertices get cnt
  u.vis = true;
  u.comp = cnt;
  for(Vertex v : u.in)
    if(!v.vis)
      sDFS(v, cnt);
}

public static void doubleDFS(Vertex[] G) {
  //first calc a topological sort by first DFS
  LinkedList<Integer> sorting = new LinkedList<Integer>();
  for(int i = 0; i < G.length; i++)
    if(!G[i].vis)
      fDFS(G[i], sorting);
  for(int i = 0; i < G.length; i++)
    G[i].vis = false;
  //then go through the sort and do another DFS on G^T
  //each tree is a component and gets a unique number
  int cnt = 0;
  for(int i : sorting)
    if(!G[i].vis)
      sDFS(G[i], cnt++);
}
```

MD5: `1e023258a9249a1bc0d6898b670139ea` | $\mathcal{O}(|V| + |E|)$

3.20 Suurballe

Finds the min cost of two edge disjoint paths in a graph. If vertex disjoint needed, split vertices.

Input: Graph G, Source s, Target t

Output: Min cost as int

```java
public static int suurballe(Vertex[] G, int s, int t){
  //this uses the usual dijkstra implementation with stored predecessors
  dijkstra(G, s);
  //Modifying weights
  for(int i = 0; i < G.length; i++)
    for(Edge e : G[i].adj)
      e.dist = e.dist - e.t.dist + G[i].dist;
  //reversing path and storing used edges
  int old = t;
  int pre = G[t].pre;
  HashMap<Integer, Integer> hm = new HashMap<Integer, Integer>();
  while(pre != -1) {
    for(int i = 0; i < G[pre].adj.size(); i++) {
      if(G[pre].adj.get(i).t.id == old) {
        hm.put(pre * G.length + old, G[pre].adj.get(i).tdist);
        G[pre].adj.remove(i);
        break;
      }
    }
    boolean found = false;
    for(int i = 0; i < G[old].adj.size(); i++) {
      if(G[old].adj.get(i).t.id == pre) {
```

```java
        G[old].adj.get(i).dist = 0;
        found = true;
        break;
      }
    }
    if(!found)
      G[old].adj.add(new Edge(G[pre], 0));
    old = pre;
    pre = G[pre].pre;
  }
  //reset graph
  for(int i = 0; i < G.length; i++) {
    G[i].pre = -1;
    G[i].dist = Integer.MAX_VALUE;
    G[i].vis = false;
  }

  dijkstra(G, s);
  //store edges of second path
  old = t;
  pre = G[t].pre;
  while(pre != -1) {
    //store edges and remove if reverse
    for(int i = 0; i < G[pre].adj.size(); i++) {
      if(G[pre].adj.get(i).t.id == old) {
        if(!hm.containsKey(pre + old * G.length))
          hm.put(pre * G.length + old, G[pre].adj.get(i).tdist);
        else
          hm.remove(pre + old * G.length);
        break;
      }
    }
    old = pre;
    pre = G[pre].pre;
  }
  //sum up weights
  int sum = 0;
  for(int i : hm.keySet())
    sum += hm.get(i);
  return sum;
}
```

MD5: 222dac2a859273efbbdd0ec0d6285dd7 | $\mathcal{O}(VlogV + E)$

3.21 Kahns Algorithm for TS

Gives the specific TS where Vertices first in G are first in the sorting

```java
public static LinkedList<Integer> TS(Vertex[] G) {
  LinkedList<Integer> sorting = new LinkedList<Integer>();
  PriorityQueue<Vertex> p = new PriorityQueue<Vertex>();
  //inc counts the number of incoming edges, if they are zero put the vertex in the queue
  for(int i = 0; i < G.length; i++) {
    if(G[i].inc == 0) {
      p.add(G[i]);
      G[i].vis = true;
    }
  }
  while(!p.isEmpty()) {
    Vertex u = p.poll();
    sorting.add(u.id);
    //update inc
    for(Vertex v : u.out) {
      if(v.vis) continue;
      v.inc--;
      if(v.inc == 0) {
        p.add(v);
        v.vis = true;
      }
    }
  }
  return sorting;
}
```

MD5: e53d13c7467873d1c5d210681f4450d8 | $\mathcal{O}(V + E)$

3.22 Topological Sort

```java
public static LinkedList<Integer> TS(Vertex[] G) {
  LinkedList<Integer> sorting = new LinkedList<Integer>();
  for(int i = 0; i < G.length; i++)
    if(!G[i].vis)
      recTS(G[i], sorting);
    //check sorting for a -1 if the graph is not necessarily dag
    //maybe checking if there are too many values in sorting is easier?!
    return sorting;
}

public static LinkedList<Integer> recTS(Vertex u, LinkedList<Integer> sorting) {
  u.vis = true;
  for(Vertex v : u.adj)
    if(v.vis)
      //the -1 indicates that it will not be possible to find an TS
      //there might be a much faster and elegant way (flag?!)
      sorting.addFirst(-1);
    else
      recTS(v, sorting);
  sorting.addFirst(u.id);
  return sorting;
}
```

MD5: f6459575bf0d53344ddd9e5daf1dfbb8 | $\mathcal{O}(|V| + |E|)$

3.23 Tuple

Simple tuple class used for priority queue in Dijkstra and Prim

```java
class Tuple implements Comparable<Tuple> {

  int id;
  int dist;

  public Tuple(int id, int dist) {
    this.id = id;
    this.dist = dist;
  }

  public int compareTo(Tuple other) {
    return Integer.compare(this.dist, other.dist);
  }
}
```

MD5: `fb1aa32dc32b9a2bac6f44a84e7f82c7` | $\mathcal{O}(1)$

3.24 Reference for Vertex classes

Used in many graph algorithms, implements a vertex with its edges. Needs testing.

```
class Vertex {

  int id;
  boolean vis = false;
  int pre = -1;

  //for dijkstra and prim
  int dist = Integer.MAX_VALUE;

  //for SCC store number indicating the dedicated
      component
  int comp = -1;

  //for DFS we could store the start and finishing
      times
  int dtime = -1;
  int ftime = -1;

  //use an ArrayList of Edges if those information are
       needed
  ArrayList<Edge> adj = new ArrayList<Edge>();
  //use an ArrayList of Vertices else
  ArrayList<Vertex> adj = new ArrayList<Vertex>();
  //use two ArrayLists for SCC
  ArrayList<Vertex> in = new ArrayList<Vertex>();
  ArrayList<Vertex> out = new ArrayList<Vertex>();

  //for EdmondsKarp we need a HashMap to store Edges,
      Integer is target
  HashMap<Integer, Edge> adj = new HashMap<Integer,
      Edge>();

  //for bipartite graph check
  int color = -1;

  //we store as key the target
  public Vertex(int id) {
    this.id = id;
  }
}
```

MD5: `90e8120ce9f665b07d4388e30395dd36` | $\mathcal{O}(1)$

4 Math

4.1 Binomial Coefficient

Gives binomial coefficient (n choose k)

```
public static long bin(int n, int k) {
  if (k == 0)
    return 1;
  else if (k > n/2)
    return bin(n, n-k);
  else
    return n*bin(n-1, k-1)/k;
}
```

MD5: `32414ba5a444038b9184103d28fa1756` | $\mathcal{O}(k)$

4.2 Binomial Matrix

Gives binomial coefficients for all K <= N.

```
public static long[][] binomial_matrix(int N, int K) {
  long[][] B = new long[N+1][K+1];
  for (int k = 1; k <= K; k++)
    B[0][k] = 0;
  for (int m = 0; m <= N; m++)
    B[m][0] = 1;
  for (int m = 1; m <= N; m++)
    for (int k = 1; k <= K; k++)
      B[m][k] = B[m-1][k-1] + B[m-1][k];
  return B;
}
```

MD5: `e6f103bd9852173c02a1ec64264f4448` | $\mathcal{O}(N \cdot K)$

4.3 Divisability

Calculates (alternating) k-digitSum for integer number given by M.

```
public static long digit_sum(String M, int k, boolean
     alt) {
  long dig_sum = 0;
  int vz = 1;
  while (M.length() > k) {
    if (alt) vz *= -1;
    dig_sum += vz*Integer.parseInt(M.substring(M.
        length()-k));
    M = M.substring(0, M.length()-k);
  }
  if (alt)
    vz *= -1;
  dig_sum += vz*Integer.parseInt(M);
  return dig_sum;
}

// example: divisibility of M by 13
public static boolean divisible13(String M) {
  return digit_sum(M, 3, true)%13 == 0;
}
```

MD5: `33b3094ebf431e1e71cd8e8db3c9cdd6` | $\mathcal{O}(|M|)$

4.4 Graham Scan

Multiple unresolved issues: multiple points as well as collinearity. N denotes the number of points

```
public static Point[] grahamScan(Point[] points) {
  //find leftmost point with lowest y-coordinate
  int xmin = Integer.MAX_VALUE;
  int ymin = Integer.MAX_VALUE;
  int index = -1;
  for(int i = 0; i < points.length; i++) {
    if(points[i].y < ymin || (points[i].y == ymin &&
        points[i].x < xmin)) {
      xmin = points[i].x;
      ymin = points[i].y;
```

```java
      index = i;
    }
  }
  //get that point to the start of the array
  Point tmp = new Point(points[index].x, points[index].y);
  points[index] = points[0];
  points[0] = tmp;
  for(int i = 1; i < points.length; i++)
    points[i].src = points[0];
  Arrays.sort(points, 1, points.length);
  //for collinear points eliminate all but the farthest
  boolean[] isElem = new boolean[points.length];
  for(int i = 1; i < points.length-1; i++) {
    Point a = new Point(points[i].x - points[i].src.x, points[i].y - points[i].src.y);
    Point b = new Point(points[i+1].x - points[i+1].src.x, points[i+1].y - points[i+1].src.y);
    if(Calc.crossProd(a, b) == 0)
      isElem[i] = true;
  }
  //works only if there are more than three non-collinear points
  Stack<Point> s = new Stack<Point>();
  int i = 0;
  for(; i < 3; i++) {
    while(isElem[i++]);
    s.push(points[i]);
  }
  for(; i < points.length; i++) {
    if(isElem[i]) continue;
    while(true) {
      Point first = s.pop();
      Point second = s.pop();
      s.push(second);
      Point a = new Point(first.x - second.x, first.y - second.y);
      Point b = new Point(points[i].x - second.x, points[i].y - second.y);
      //use >= if straight angles are needed
      if(Calc.crossProd(a, b) > 0) {
        s.push(first);
        s.push(points[i]);
        break;
      }
    }
  }
  Point[] convexHull = new Point[s.size()];
  for(int j = s.size()-1; j >= 0; j--)
    convexHull[j] = s.pop();
  return convexHull;
  /*Sometimes it might be necessary to also add points to the convex hull that form a straight angle. The following lines of code achieve this. Only at the first and last diagonal we have to add those. Of course the previous return-statement has to be deleted as well as allowing straight angles in the above implementation. */
}
class Point implements Comparable<Point> {
  Point src; //set seperately in GrahamScan method
  int x;
  int y;

  public Point(int x, int y) {
    this.x = x;
    this.y = y;
  }

  //might crash if one point equals src
  //major issues with multiple points on same location!
  public int compareTo(Point cmp) {
  Point a = new Point(this.x - src.x, this.y - src.y);
  Point b = new Point(cmp.x - src.x, cmp.y - src.y);
  //checks if points are identical
  if(a.x == b.x && a.y == b.y) return 0;
  //if same angle, sort by dist
  if(Calc.crossProd(a, b) == 0 && Calc.dotProd(a, b) > 0)
    return Integer.compare(Calc.dotProd(a, a), Calc.dotProd(b, b));
  //angle of a is 0, thus b>a
  if(a.y == 0 && a.x > 0) return -1;
  //angle of b is 0, thus a>b
  if(b.y == 0 && b.x > 0) return 1;
  //a ist between 0 and 180, b between 180 and 360
  if(a.y > 0 && b.y < 0) return -1;
  if(a.y < 0 && b.y > 0) return 1;
  //return negative value if cp larger than zero
  return Integer.compare(0, Calc.crossProd(a, b));
  }
}

class Calc {
  public static int crossProd(Point p1, Point p2) {
    return p1.x * p2.y  - p2.x * p1.y;
  }
  public static int dotProd(Point p1, Point p2) {
    return p1.x * p2.x + p1.y * p2.y;
  }
}
```

MD5: `2555d858fadcfe8cb404a9c52420545d` | $\mathcal{O}(N \log N)$

4.5 Iterative EEA

Berechnet den ggT zweier Zahlen a und b und deren modulare Inverse $x = a^{-1} \mod b$ und $y = b^{-1} \mod a$.

```java
// Extended Euclidean Algorithm - iterativ
public static long[] eea(long a, long b) {
  if (b > a) {
    long tmp = a;
    a = b;
    b = tmp;
  }
  long x = 0, y = 1, u = 1, v = 0;
  while (a != 0) {
    long q = b / a, r = b % a;
    long m = x - u * q, n = y - v * q;
    b = a; a = r; x = u; y = v; u = m; v = n;
  }
  long gcd = b;
  // x = a^-1 % b, y = b^-1 % a
  // ax + by = gcd
  long[] erg = { gcd, x, y };
  return erg;
}
```

MD5: `81fe8cd4adab21329dcbe1ce0499ee75` | $\mathcal{O}(\log a + \log b)$

4.6 Polynomial Interpolation

```java
public class interpol {

  // divided differences for points given by vectors x and y
  public static rat[] divDiff(rat[] x, rat[] y) {
    rat[] temp = y.clone();
    int n = x.length;
    rat[] res = new rat[n];
    res[0] = temp[0];
    for (int i=1; i < n; i++) {
      for (int j = 0; j < n-i; j++) {
        temp[j] = (temp[j+1].sub(temp[j])).div(x[j+i].sub(x[j]));
      }
      res[i] = temp[0];
    }
    return res;
  }

  // evaluates interpolating polynomial p at t for given
  // x-coordinates and divided differences
  public static rat p(rat t, rat[] x, rat[] dD) {
    int n = x.length;
    rat p = new rat(0);
    for (int i = n-1; i > 0; i--) {
      p = (p.add(dD[i])).mult(t.sub(x[i-1]));
    }
    p = p.add(dD[0]);
    return p;
  }
}

// implementation of rational numbers
class rat {

  public long c;
  public long d;

  public rat (long c, long d) {
    this.c = c;
    this.d = d;
    this.shorten();
  }

  public rat (long c) {
    this.c = c;
    this.d = 1;
  }

  public static long ggT(long a, long b) {
    while (b != 0) {
      long h = a%b;
      a = b;
      b = h;
    }
    return a;
  }

  public static long kgV(long a, long b) {
    return a*b/ggT(a,b);
  }

  public static rat[] commonDenominator(rat[] c) {
    long kgV = 1;
    for (int i = 0; i < c.length; i++) {
      kgV = kgV(kgV, c[i].d);
    }
    for (int i = 0; i < c.length; i++) {
      c[i].c *= kgV/c[i].d;
      c[i].d *= kgV/c[i].d;
    }
    return c;
  }

  public void shorten() {
    long ggT = ggT(this.c, this.d);
    this.c = this.c / ggT;
    this.d = this.d / ggT;
    if (d < 0) {
      this.d *= -1;
      this.c *= -1;
    }
  }

  public String toString() {
    if (this.d == 1) return ""+c;
    return ""+c+"/"+d;
  }

  public rat mult(rat b) {
    return new rat(this.c*b.c, this.d*b.d);
  }

  public rat div(rat b) {
    return new rat(this.c*b.d, this.d*b.c);
  }

  public rat add(rat b) {
    long new_d = kgV(this.d, b.d);
    long new_c = this.c*(new_d/this.d) + b.c*(new_d/b.d);
    return new rat(new_c, new_d);
  }

  public rat sub(rat b) {
    return this.add(new rat(-b.c, b.d));
  }
}
```

MD5: e7b408030f7e051e93a8c55056ba930b | $\mathcal{O}(?)$

4.7 Root of permutation

Calculates the K'th root of permutation of size N. Number at place i indicates where this dancer ended. needs commenting

```java
public static int[] rop(int[] perm, int N, int K) {
  boolean[] incyc = new boolean[N];
  int[] cntcyc = new int[N+1];
  int[] g = new int[N+1];
  int[] needed = new int[N+1];
  for(int i = 1; i < N+1; i++) {
    int j = i;
    int k = K;
    int div;
    while(k > 1 && (div = gcd(k, i)) > 1) {
      k /= div;
      j *= div;
    }
    needed[i] = j;
    g[i] = gcd(K, j);
  }
```

```java

HashMap<Integer, ArrayList<Integer>> hm = new HashMap<Integer, ArrayList<Integer>>();
for(int i = 0; i < N; i++) {
  if(incyc[i]) continue;
  ArrayList<Integer> cyc = new ArrayList<Integer>();
  cyc.add(i);
  incyc[i] = true;
  int newelem = perm[i];
  while(newelem != i) {
    cyc.add(newelem);
    incyc[newelem] = true;
    newelem = perm[newelem];
  }
  int len = cyc.size();
  cntcyc[len]++;
  if(hm.containsKey(len)) {
    hm.get(len).addAll(cyc);
  } else {
    hm.put(len, cyc);
  }
}
boolean end = false;
for(int i = 1; i < N+1; i++) {
  if(cntcyc[i] % g[i] != 0) end = true;
}
if(end) {
  //not possible
  return null;
} else {
  int[] out = new int[N];
  for(int length = 0; length < N; length++) {
    if(!hm.containsKey(length)) continue;
    ArrayList<Integer> p = hm.get(length);
    int totalsize = p.size();
    int diffcyc = totalsize / needed[length];
    for(int i = 0; i < diffcyc; i++) {
      int[] c = new int[needed[length]];
      for(int it = 0; it < needed[length]; it++) {
        c[it] = p.get(it + i * needed[length]);
      }
      int move = K / (needed[length]/length);
      int[] rewind = new int[needed[length]];
      for(int set = 0; set < needed[length]/length; set++) {
        int pos = set * length;
        for(int it = 0; it < length; it++) {
          rewind[pos] = c[it + set * length];
          pos = ((pos - set * length + move) % length)+ set * length;
        }
      }
      int[] merge = new int[needed[length]];
      for(int it = 0; it < needed[length]/length; it++) {
        for(int set = 0; set < length; set++) {
          merge[set * needed[length] / length + it] = rewind[it * length + set];
        }
      }
      for(int it = 0; it < needed[length]; it++) {
        out[merge[it]] = merge[(it+1) % needed[length]];
      }
    }
  }
  return out;
}
}
```

MD5: `b446a7c21eddf7d14dbdc71174e8d498` | $\mathcal{O}(?)$

4.8 Sieve of Eratosthenes

Calculates Sieve of Eratosthenes.
Input: A integer N indicating the size of the sieve.
Output: A boolean array, which is true at an index i iff i is prime.

```java
public static boolean[] sieveOfEratosthenes(int N) {
  boolean[] isPrime = new boolean[N+1];
  for (int i=2; i<=N; i++) isPrime[i] = true;
  for (int i = 2; i*i <= N; i++)
    if (isPrime[i])
      for (int j = i*i; j <= N; j+=i)
        isPrime[j] = false;
  return isPrime;
}
```

MD5: `95704ae7c1fe03e91adeb8d695b2f5bb` | $\mathcal{O}(n)$

4.9 Greatest Common Devisor

Calculates the gcd of two numbers a and b or of an array of numbers $input$.
Input: Numbers a and b or array of numbers $input$
Output: Greatest common devisor of the input

```java
private static long gcd(long a, long b) {
    while (b > 0) {
        long temp = b;
        b = a % b; // % is remainder
        a = temp;
    }
    return a;
}

private static long gcd(long[] input) {
    long result = input[0];
    for(int i = 1; i < input.length; i++)
    result = gcd(result, input[i]);
    return result;
}
```

MD5: `48058e358a971c3ed33621e3118818c2` | $\mathcal{O}(\log a + \log b)$

4.10 Least Common Multiple

Calculates the lcm of two numbers a and b or of an array of numbers $input$.
Input: Numbers a and b or array of numbers $input$
Output: Least common multiple of the input

```java
private static long lcm(long a, long b) {
    return a * (b / gcd(a, b));
}

private static long lcm(long[] input) {
  long result = input[0];
  for(int i = 1; i < input.length; i++)
    result = lcm(result, input[i]);
  return result;
```

```
}
```

MD5: `3cfaab4559ea05c8434d6cf364a24546` | $\mathcal{O}(\log a + \log b)$

5 Misc

5.1 Binary Search

Binary searchs for an element in a sorted array.
Input: sorted $array$ to search in, amount N of elements in $array$, element to search for a
Output: returns the index of a in $array$ or -1 if $array$ does not contain a

```
public static int BinarySearch(int[] array,
                                int N, int a) {
  int lo = 0;
  int hi = N-1;
  // a might be in interval [lo,hi] while lo <= hi
  while(lo <= hi) {
    int mid = (lo + hi) / 2;
    // if a > elem in mid of interval,
    // search the right subinterval
    if(array[mid] < a)
      lo = mid+1;
    // else if a < elem in mid of interval,
    // search the left subinterval
    else if(array[mid] > a)
      hi = mid-1;
    // else a is found
    else
      return mid;
  }
  // array does not contain a
  return -1;
}
```

MD5: `203da61f7a381564ce3515f674fa82a4` | $\mathcal{O}(\log n)$

5.2 Next number with n bits set

From x the smallest number greater than x with the same amount of bits set is computed. Little changes have to be made, if the calculated number has to have length less than 32 bits.
Input: number x with n bits set $(x = (1 << n) - 1)$
Output: the smallest number greater than x with n bits set

```
public static int nextNumber(int x) {
  //break when larger than limit here
  if(x == 0) return 0;
  int smallest = x & -x;
  int ripple = x + smallest;
  int new_smallest = ripple & -ripple;
  int ones  = ((new_smallest/smallest) >> 1) - 1;
  return ripple | ones;
}
```

MD5: `2d8a79cb551648e67fc3f2f611a4f63c` | $\mathcal{O}(1)$

5.3 Next Permutation

Returns true if there is another permutation. Can also be used to compute the nextPermutation of an array.

Input: String a as char array
Output: `true`, if there is a next permutation of a, `false` otherwise

```
public static boolean nextPermutation(char[] a) {
  int i = a.length - 1;
  while(i > 0 && a[i-1] >= a[i])
    i--;
  if(i <= 0)
    return false;
  int j = a.length - 1;
  while (a[j] <=  a[i-1])
    j--;
  char tmp = a[i - 1];
  a[i - 1] = a[j];
  a[j] = tmp;

  j = a.length - 1;
  while(i < j) {
    tmp = a[i];
    a[i] = a[j];
    a[j] = tmp;
    i++;
    j--;
  }
  return true;
}
```

MD5: `7d1fe65d3e77616dd2986ce6f2af089b` | $\mathcal{O}(n)$

6 String

6.1 Knuth-Morris-Pratt

Input: String s to be searched, String w to search for.
Output: Array with all starting positions of matches

```
public static ArrayList<Integer> kmp(String s, String w) {
  ArrayList<Integer> ret = new ArrayList<>();
  //Build prefix table
  int[] N = new int[w.length()+1];
  int i=0; int j =-1; N[0]=-1;
  while (i<w.length()) {
    while (j>=0 && w.charAt(j) != w.charAt(i))
      j = N[j];
    i++; j++; N[i]=j;
  }
  //Search string
  i=0; j=0;
  while (i<s.length()) {
    while (j>=0 && s.charAt(i) != w.charAt(j))
      j = N[j];
      i++; j++;
      if (j==w.length()) { //match found
      ret.add(i-w.length()); //add its start index
      j = N[j];
    }
  }
  return ret;
}
```

MD5: `3cb03964744db3b14b9bff265751c84b` | $\mathcal{O}(n + m)$

6.2 Levenshtein Distance

Calculates the Levenshtein distance for two strings (minimum number of insertions, deletions, or substitutions).
Input: A string a and a string b.
Output: An integer holding the distance.

```java
public static int levenshteinDistance(String a, String
    b) {
  a = a.toLowerCase();
  b = b.toLowerCase();
  int[] costs = new int[b.length() + 1];

  for (int j = 0; j < costs.length; j++)
    costs[j] = j;

  for (int i = 1; i <= a.length(); i++) {
    costs[0] = i;
    int nw = i - 1;
    for (int j = 1; j <= b.length(); j++) {
      int cj = Math.min(1 + Math.min(costs[j], costs[j
          - 1]),
        a.charAt(i - 1) == b.charAt(j - 1) ? nw : nw +
          1);
      nw = costs[j];
      costs[j] = cj;
    }
  }
  return costs[b.length()];
}
```

MD5: `79186003b792bc7fd5c1ffbbcfc2b1c6` | $\mathcal{O}(|a| \cdot |b|)$

6.3 Longest Common Subsequence

Finds the longest common subsequence of two strings.
Input: Two strings $string1$ and $string2$.
Output: The LCS as a string.

```java
public static String longestCommonSubsequence(String
    string1, String string2) {
  char[] s1 = string1.toCharArray();
  char[] s2 = string2.toCharArray();
  int[][] num = new int[s1.length + 1][s2.length + 1];
  // Actual algorithm
  for (int i = 1; i <= s1.length; i++)
    for (int j = 1; j <= s2.length; j++)
      if (s1[i - 1] == s2[j - 1])
        num[i][j] = 1 + num[i - 1][j - 1];
      else
        num[i][j] = Math.max(num[i - 1][j], num[i][j -
          1]);
  // System.out.println("length of LCS = " + num[s1.
      length][s2.length]);
  int s1position = s1.length, s2position = s2.length;
  List<Character> result = new LinkedList<Character>()
      ;
  while (s1position != 0 && s2position != 0) {
    if (s1[s1position - 1] == s2[s2position - 1]) {
      result.add(s1[s1position - 1]);
      s1position--;
      s2position--;
    } else if (num[s1position][s2position - 1] >= num[
        s1position][s2position])
      s2position--;
    else
      s1position--;
  }
  Collections.reverse(result);
  char[] resultString = new char[result.size()];
  int i = 0;
  for (Character c : result) {
    resultString[i] = c;
    i++;
  }
  return new String(resultString);
}
```

MD5: `4dc4ee3af14306bea5724ba8a859d5d4` | $\mathcal{O}(n \cdot m)$

6.4 Longest common substring

gets two String and finds all LCSs and returns them in a set

```java
public static TreeSet<String> LCS(String a, String b)
    {
  int[][] t = new int[a.length()+1][b.length()+1];
  for(int i = 0; i <= b.length(); i++)
    t[0][i] = 0;

  for(int i = 0; i <= a.length(); i++)
    t[i][0] = 0;

  for(int i = 1; i <= a.length(); i++)
    for(int j = 1; j <= b.length(); j++)
      if(a.charAt(i-1) == b.charAt(j-1))
        t[i][j] = t[i-1][j-1] + 1;
      else
        t[i][j] = 0;
  int max = -1;
  for(int i = 0; i <= a.length(); i++)
    for(int j = 0; j <= b.length(); j++)
      if(max < t[i][j])
        max = t[i][j];
  if(max == 0 || max == -1)
    return new TreeSet<String>();
  TreeSet<String> res = new TreeSet<String>();
  for(int i = 0; i <= a.length(); i++)
    for(int j = 0; j <= b.length(); j++)
      if(max == t[i][j])
        res.add(a.substring(i-max, i));
  return res;
}
```

MD5: `9de393461e1faebe99af3ff8db380bde` | $\mathcal{O}(|a| * |b|)$

7 Math Roland

7.1 Divisability Explanation

$D \mid M \Leftrightarrow D \mid$ `digit_sum(M, k, alt)`, refer to table for values of D, k, alt.

7.2 Combinatorics

- Variations (ordered): k out of n objects (permutations for $k = n$)
  - without repetition:
    $M = \{(x_1, \ldots, x_k) : 1 \leq x_i \leq n,\ x_i \neq x_j \text{ if } i \neq j\}$,
    $|M| = \frac{n!}{(n-k)!}$

  - with repetition: $M = \{(x_1, \ldots, x_k) : 1 \le x_i \le n\}, |M| = n^k$
- Combinations (unordered): k out of n objects
  - without repetition: $M = \{(x_1, \ldots, x_n) : x_i \in \{0, 1\},\ x_1 + \ldots + x_n = k\}, |M| = \binom{n}{k}$
  - with repetition: $M = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, k\},\ x_1 + \ldots + x_n = k\}, |M| = \binom{n+k-1}{k}$
- Ordered partition of numbers: $x_1 + \ldots + x_k = n$ (i.e. 1+3 = 3+1 = 4 are counted as 2 solutions)
  - #Solutions for $x_i \in \mathbb{N}_0$: $\binom{n+k-1}{k-1}$
  - #Solutions for $x_i \in \mathbb{N}$: $\binom{n-1}{k-1}$
- Unordered partition of numbers: $x_1 + \ldots + x_k = n$ (i.e. 1+3 = 3+1 = 4 are counted as 1 solution)
  - #Solutions for $x_i \in \mathbb{N}$: $P_{n,k} = P_{n-k,k} + P_{n-1,k-1}$ where $P_{n,1} = P_{n,n} = 1$
- Derangements (permutations without fixed points): $!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$

7.3 Polynomial Interpolation

7.3.1 Theory

Problem: for $\{(x_0, y_0), \ldots, (x_n, y_n)\}$ find $p \in \Pi_n$ with $p(x_i) = y_i$ for all $i = 0, \ldots, n$.
Solution: $p(x) = \sum\limits_{i=0}^{n} \gamma_{0,i} \prod\limits_{j=0}^{i-1} (x - x_i)$ where $\gamma_{j,k} = y_j$ for $k = 0$ and $\gamma_{j,k} = \frac{\gamma_{j+1,k-1} - \gamma_{j,k-1}}{x_{j+k} - x_j}$ otherwise.
Efficient evaluation of $p(x)$: $b_n = \gamma_{0,n}$, $b_i = b_{i+1}(x - x_i) + \gamma_{0,i}$ for $i = n - 1, \ldots, 0$ with $b_0 = p(x)$.

7.4 Fibonacci Sequence

7.4.1 Binet's formula

$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow f_n = \frac{1}{\sqrt{5}}(\phi^n - \tilde{\phi}^n)$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\tilde{\phi} = \frac{1-\sqrt{5}}{2}$.

7.4.2 Generalization

$g_n = \frac{1}{\sqrt{5}}(g_0(\phi^{n-1} - \tilde{\phi}^{n-1}) + g_1(\phi^n - \tilde{\phi}^n)) = g_0 f_{n-1} + g_1 f_n$ for all $g_0, g_1 \in \mathbb{N}_0$

7.4.3 Pisano Period

Both $(f_n \bmod k)_{n\in\mathbb{N}_0}$ and $(g_n \bmod k)_{n\in\mathbb{N}_0}$ are periodic.

8 Java Knowhow

8.1 System.out.printf() und String.format()

Syntax: `%[flags][width][.precision][conv]`
flags:

`-`	left-justify (default: right)
`+`	always output number sign
`0`	zero-pad numbers
(space)	space instead of minus for pos. numbers
,	group triplets of digits with ,

width specifies output width
precision is for floating point precision
conv:

d	byte, short, int, long
f	float, double
c	char (use C for uppercase)
s	String (use S for all uppercase)

8.2 Modulo: Avoiding negative Integers

```java
int mod = (((nums[j] % D) + D) % D);
```

8.3 Speed up IO

Use

```java
BufferedReader br = new BufferedReader(new
InputStreamReader(System.in));
```

Use

```java
Double.parseDouble(Scanner.next());
```

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Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in$ such that $b \le s$, $\forall s \in S$.
$\liminf_{n\to\infty} a_n$	$\lim_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \}$.
$\limsup_{n\to\infty} a_n$	$\lim_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \quad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\binom{n}{k} = \binom{n}{n-k}$,

4. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\left[{n \atop 1}\right] = (n-1)!$, **15.** $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\left[{n \atop n}\right] = 1$, **17.** $\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\left\langle{0 \atop k}\right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0$ for $n \ne 0$,

34. $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[{k \atop m}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\left[{k \atop m}\right]$, **39.** $\left[{x \atop x-n}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$,

40. $\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k}$, **41.** $\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k}$,

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k\left\{{n+k \atop k}\right\}$, **43.** $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} k(n+k)\left[{n+k \atop k}\right]$,

44. $\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k}$, **45.** $(n-m)!\binom{n}{m} = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}$, for $n \geq m$,

46. $\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right]$, **47.** $\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\}$,

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k}$, **49.** $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_k \left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}$.

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:
$$\sum_{i=1}^{n} 2^{-d_i} \leq 1,$$
and equality holds only if every internal node has 2 sons.

Recurrences

Master method:
$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then
$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then
$$T(n) = \Theta(n^{\log_b a}\log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then
$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence
$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have
$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get
$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find
$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence
$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side
$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \quad \vdots \quad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get
$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have
$$n \sum_{i=0}^{m-1} c^i = n\left(\frac{c^m - 1}{c - 1}\right)$$
$$= 2n(c^{\log_2 n} - 1)$$
$$= 2n(c^{(k-1)\log_c n} - 1)$$
$$= 2n^k - 2n,$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider
$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that
$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find
$$T_{i+1} - T_i = 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j$$
$$= T_i.$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:
$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:
$$\sum_{i\geq 0} g_{i+1}x^i = \sum_{i\geq 0} 2g_i x^i + \sum_{i\geq 0} x^i.$$

We choose $G(x) = \sum_{i\geq 0} x^i g_i$. Rewrite in terms of $G(x)$:
$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\geq 0} x^i.$$

Simplify:
$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:
$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:
$$G(x) = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$$
$$= x\left(2\sum_{i\geq 0} 2^i x^i - \sum_{i\geq 0} x^i\right)$$
$$= \sum_{i\geq 0} (2^{i+1} - 1)x^{i+1}.$$

So $g_i = 2^i - 1$.

Theoretical Computer Science Cheat Sheet

$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520}, \ldots$$

$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x)\Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$
$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i<\cdots<i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

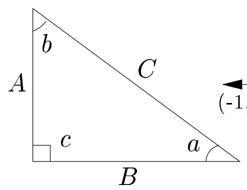
$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

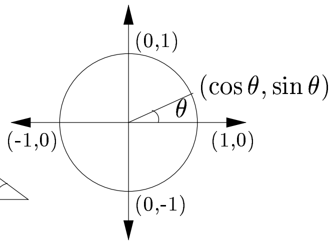
Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

Theoretical Computer Science Cheat Sheet

Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2\sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{array}{c} aei + bfg + cdh \\ - ceg - fha - ibd. \end{array}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

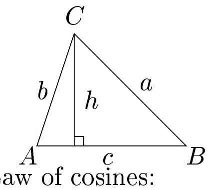
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2\sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

. . . in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$\begin{aligned} C &\equiv r_1 \bmod m_1 \\ \vdots\ &\ \vdots\ \ \vdots \\ C &\equiv r_n \bmod m_n \end{aligned}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d \mid x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d \mid a} F(d),$$

then

$$F(a) = \sum_{d \mid a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embedded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k,\ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

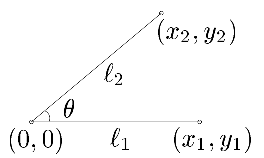
$$\left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p},$$

$$\lim_{p \to \infty} \left[\lvert x_1 - x_0\rvert^p + \lvert y_1 - y_0\rvert^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:
$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \, \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \, \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1-u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \, \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \, \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$,

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\, dx = \tan x,$

20. $\int \csc^2 x\, dx = -\cot x,$

21. $\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx,$

22. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx,$

23. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \quad n \neq 1,$

24. $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \quad n \neq 1,$

25. $\int \sec^n x\, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx, \quad n \neq 1,$

26. $\int \csc^n x\, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x\, dx, \quad n \neq 1,$

27. $\int \sinh x\, dx = \cosh x,$

28. $\int \cosh x\, dx = \sinh x,$

29. $\int \tanh x\, dx = \ln|\cosh x|,$

30. $\int \coth x\, dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x\, dx = \ln \left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x\, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln|a^2 - x^2|,$

38. $$\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$$

39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

41. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left|\frac{a + x}{a - x}\right|,$

45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln\left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left|\frac{x}{a + bx}\right|,$

49. $\int x\sqrt{a + bx}\, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

51. $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln \left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

53. $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \frac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

57. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a,} \quad a > 0,$

58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

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Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66.
$$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

67.
$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

68.
$$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69.
$$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70.
$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax) + \tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax) - \tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$x^1 =$	$x^{\underline{1}}$	$=$	$x^{\overline{1}}$	
$x^2 =$	$x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{2}}-x^{\overline{1}}$	
$x^3 =$	$x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}$	
$x^4 =$	$x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}$	
$x^5 =$	$x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}$	

$x^{\overline{1}} =$	x^1	$x^{\underline{1}} =$	x^1
$x^{\overline{2}} =$	x^2+x^1	$x^{\underline{2}} =$	x^2-x^1
$x^{\overline{3}} =$	$x^3+3x^2+2x^1$	$x^{\underline{3}} =$	$x^3-3x^2+2x^1$
$x^{\overline{4}} =$	$x^4+6x^3+11x^2+6x^1$	$x^{\underline{4}} =$	$x^4-6x^3+11x^2-6x^1$
$x^{\overline{5}} =$	$x^5+10x^4+35x^3+50x^2+24x^1$	$x^{\underline{5}} =$	$x^5-10x^4+35x^3-50x^2+24x^1$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x) + C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u + \Delta v,$

$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right]x^k.$$

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Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i, \qquad \left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i, \qquad (e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!}, \qquad x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!}, \qquad \zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x}, \qquad \frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

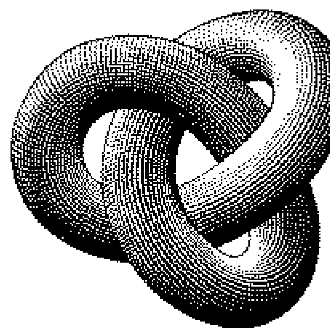
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^i i!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fibonacci Numbers

$1,1,2,3,5,8,13,21,34,55,89,\ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$
$$F_{-i} = (-1)^{i-1}F_i,$$
$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$
$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)