SINGAPORE MANAGEMENT UNIVERSITY

QF605: FIXED INCOME SECURITIES

Bootstrapping Swap Curves, Swaption Calibration, Convexity Correction, and Decompounded Options

Group 5

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Part 1: Bootstrapping Swap Curves

1.1: Bootstrap the OIS discount factor $D_O(0, T)$ and plot the discount curve for $T \in [0, 30]$

In the process of bootstrapping the OIS discount factor, the given constraints are given below.

1) The Day count convention is 30/360.
2) The overnight leg frequency is daily based.
3) The fixed leg frequency is annual based.

Firstly, with the Tenor of 6 months to 30 years has been set. Let $f_{0.5}$ defines as the daily compounded overnight rate for [0,6m]. With the 6m OIS discount factor, we could obtain:

$$OIS_{6m} = \left(1 + \frac{f_0}{360}\right)^{180} - 1$$

$$D_0(0,6m) = \frac{1}{\left(1 + \frac{f_0}{360}\right)^{180}}$$

Using the 6m OIS, the present values for fixed and floated legs will be used for solving the value of the compounded overnight rate f_0. After calculating the $f_{0.5}$, now we could calculate the OIS discount factor for the 6m tenor with the formula mentioned above.

For the following years, we could calculate the discount factor with the relationship equation as listed below.

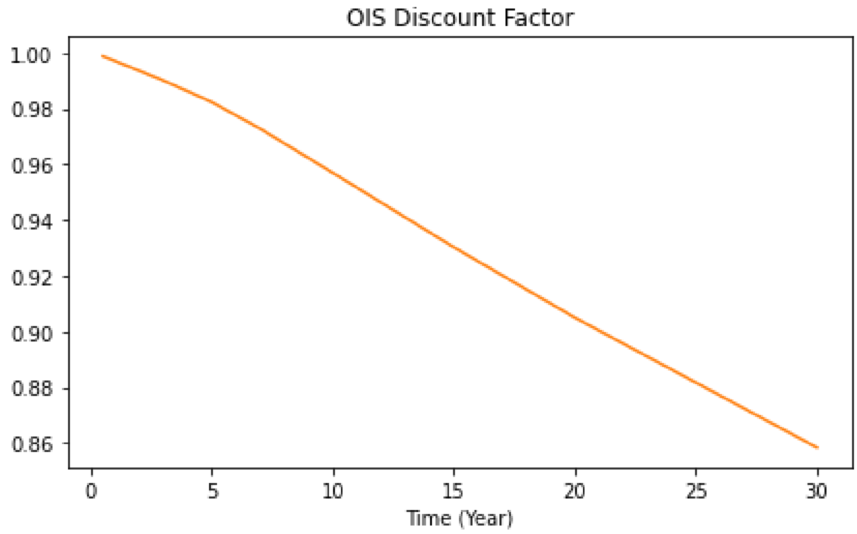
$$D_0(0, T_i) = D_0(0, T_{i-1}) \times \frac{1}{\left(1 + \frac{f_i}{360}\right)^{180}}$$

$$IRS\ rate \times \sum_{i=0.5}^{n} D_0(0, T_i) = \sum_{i=0.5}^{n} D_0(0, T_{i-1}) \times \frac{1}{\left(1 + \frac{f_i}{360}\right)^{180}}$$

With the mentioned equations, we can substitute the value of the discount factor to find the compounded overnight rate f_i and use linear interpolation for the semiannual data until the 30th year. The result of the OIS discount factor calculation has been shown in the table below.

Tenor	**OIS discount factor**	**Tenor**	**OIS discount factor**	**Tenor**	**OIS discount factor**	**Tenor**	**OIS discount factor**	**Tenor**	**OIS discount factor**	**Tenor**	**OIS discount factor**
0.5	0.998751561	5.5	0.979929365	10.5	0.954267275	15.5	0.92770704	20.5	0.902646962	25.5	0.879342823
1	0.997008973	6	0.977540459	11	0.951596751	16	0.925181522	21	0.900316548	26	0.877012409
1.5	0.995275475	6.5	0.975151552	11.5	0.948926227	16.5	0.922656003	21.5	0.897986134	26.5	0.874681996
2	0.993541977	7	0.972762646	12	0.946255703	17	0.920130485	22	0.89565572	27	0.872351582
2.5	0.991795987	7.5	0.970125171	12.5	0.943585179	17.5	0.917604967	22.5	0.893325306	27.5	0.870021168
3	0.990049998	8	0.967487697	13	0.940914655	18	0.915079449	23	0.890994893	28	0.867690754
3.5	0.988121646	8.5	0.964850222	13.5	0.93824413	18.5	0.91255393	23.5	0.888664479	28.5	0.86536034
4	0.986193294	9	0.962212748	14	0.935573606	19	0.910028412	24	0.886334065	29	0.863029926
4.5	0.984255783	9.5	0.959575274	14.5	0.932903082	19.5	0.907502894	24.5	0.884003651	29.5	0.860699513
5	0.982318271	10	0.956937799	15	0.930232558	20	0.904977376	25	0.881673237	30	0.858369099

The OIS discount factor has been plotted as shown below.



1.2: Bootstrap the LIBOR discount factor D_O (0, T) and plot the discount curve for T ∈ [0, 30]

Considering the same constraints as the OIS discount factor, the previous OIS approach will be adapted for the derivation of the LIBOR discount factor together with the forward LIBOR rate. The derivation is as shown below.

$$PV_{fix} = PV_{float}$$

$$0.5 \times IRS_{1y} \times [D_0(0,6m) + D_0(0,1y)] = 0.5 \times [D_0(0,6m) \times L(0,6m) + D_0(0,1y) \times L(6m,1y)]$$
$$0.5 \times IRS_{2y} \times [D_0(0,6m) + D_0(0,1y) + \cdots + D_0(0,2y)] = 0.5 \times [D_0(0,6m) \times L(0,6m) + \cdots + D_0(0,2y) \times L(1.5y,2y)]$$
.
.
.
$$0.5 \times IRS_{30y} \times [D_0(0,6m) + D_0(0,1y) + \cdots + D_0(0,30y)] = 0.5 \times [D_0(0,6m) \times L(0,6m) + \cdots + D_0(0,30y) \times L(29.5y,30y)].$$

With the derivation, we could calculate the first tenor of (0,6m) by the equation mentioned below.

$$D_L(0,6m) = \frac{1}{1 + 0.5 \times L(0,6m)}$$

Considering the following year and tenor, we could calculate by the formula as mentioned:

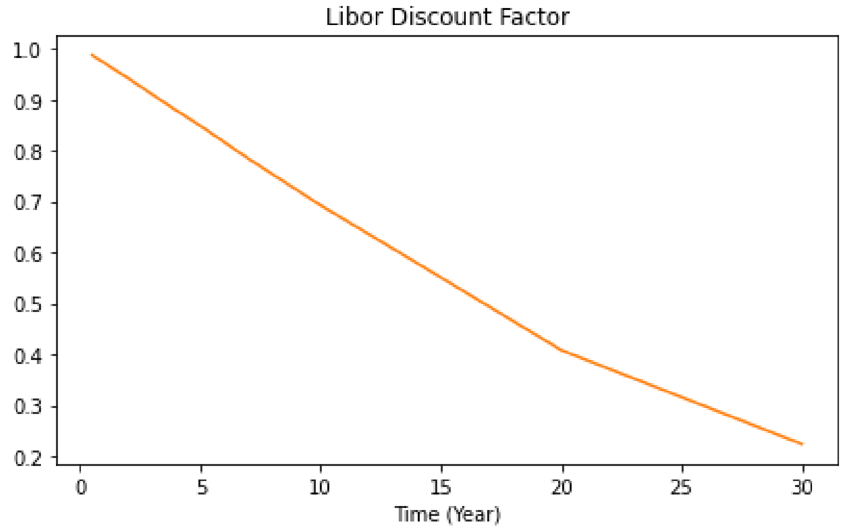
$$L(T_{i-1}, T_i) = \frac{D_L(0,T_{i-1}) - D_L(0,T_i)}{\Delta \times D_L(0,T_i)}$$

Now we obtain the value of the LIBOR rate, which we will substitute in the derivation of the present value between floating leg and fixed leg as stated previously, which has the unknown of $D_L(0, T_i)$. The linear interpolation has been utilized for the semiannual year until the 30th year as well.

The result of the LIBOR discount factor calculation has been shown below in the table.

Tenor	LIBOR discount factor	Tenor	LIBOR discount factor	Tenor	LIBOR discount factor	Tenor	LIBOR discount factor	Tenor	LIBOR discount factor	Tenor	LIBOR discount factor
0.5	0.987654321	5.5	0.832796872	10.5	0.678559127	15.5	0.536844556	20.5	0.399142093	25.5	0.306946661
1	0.97257683	6	0.816604071	11	0.664399272	16	0.522568676	21	0.38992255	26	0.297727118
1.5	0.957377926	6.5	0.800411269	11.5	0.650239417	16.5	0.508292796	21.5	0.380703007	26.5	0.288507574
2	0.942179022	7	0.784218468	12	0.636079563	17	0.494016916	22	0.371483463	27	0.279288031
2.5	0.926330319	7.5	0.768968554	12.5	0.621919708	17.5	0.479741036	22.5	0.36226392	27.5	0.270068488
3	0.910481617	8	0.753718639	13	0.607759854	18	0.465465156	23	0.353044377	28	0.260848945
3.5	0.894731261	8.5	0.738468725	13.5	0.593599999	18.5	0.451189276	23.5	0.343824834	28.5	0.251629401
4	0.878980906	9	0.72321881	14	0.579440145	19	0.436913396	24	0.33460529	29	0.242409858
4.5	0.86398529	9.5	0.707968896	14.5	0.56528029	19.5	0.422637516	24.5	0.325385747	29.5	0.233190315
5	0.848989673	10	0.692718981	15	0.551120436	20	0.408361636	25	0.316166204	30	0.223970772

The LIBOR discount factor has been plotted as shown below.



1.3: Forward Swap Rate calculations

The forward swap rate could be calculated with the equation as mentioned below.

$$S(T_i, T_{i+m}) = \frac{0.5 \times \sum_{n=i+1}^{i+m} D_0(0, T_n) \times L(T_{n-0.5}, T_n)}{0.5 \times \sum_{n=i+1}^{i+m} D_0(0, T_n)}$$

After the calculation, we could obtain the result of the forward swap rates as shown below.

Tenor	1y	2y	3y	4y	5y	10y
1y	3.20%	3.33%	3.40%	3.43%	3.53%	3.84%
5y	3.93%	4.01%	4.01%	4.05%	4.11%	4.36%
10y	4.22%	4.31%	4.41%	4.51%	4.62%	5.34%

Part 2: Swaption Calibration

2.1: Calibrating DDM (Displaced-Diffusion Model)

Under the Swaption tab of IR Data.xlsm, swaption implied volatilities are provided. We can use the par forward swap rate calculated in Part I to derive corresponding strike prices for different implied volatilities. Based on strikes and implied volatility, the payer swaption and receiver swaption price can then be calculated using Black 76 Formula.
The Displaced-diffusion model (DD Model) is essentially the same as the Black 76 model, except that it models the movements of (F + Shift) as the underlying asset, instead of F. To better fit the market, the model can be thought of as a "weighted average" between a normal and a lognormal model, with β weight given to the normal model. Then the Displaced-diffusion Model states the price for a Swap is given by:

$$C^{DD} = C^{B76}\left(\frac{F}{\beta}, K + \frac{1-\beta}{\beta}F, \sigma\beta, T\right) = PVBP[F'N(d1) - KN(d2)]$$

$$P^{DD} = P^{B76}\left(\frac{F}{\beta}, K + \frac{1-\beta}{\beta}F, \sigma\beta, T\right) = PVBP[KN(d1) - F'N(d2)]$$

Displaced-Diffusion Model is a model that combine the normal distribution and lognormal distribution, where beta is the weight of lognormal behavior and (1-beta) is the weight of normal behavior. The think we need to do here is to use Displaced-Diffusion Model to fit the market data.

We choose to assign more weight on ATM swaption to make the optimal solution to be stable instead of giving us only the local optimal solution. By doing so, we can obtain the calibration Displaced-diffusion model parameters. The calibrated parameters are showing as follow:

Sigma – Displaced Diffusion Model Calibration

	Tenor				
Expiry	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.2250	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.2660	0.2451
10Y	0.2854	0.2928	0.2940	0.2674	0.2437

Beta– Displaced Diffusion Model Calibration

	Tenor				
Expiry	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.000001	0.000001	0.000001	0.000001	0.000006
5Y	0.000001	0.000001	0.000002	0.000026	0.137303
10Y	0.000001	0.000003	0.000033	0.000258	0.100000

2.2: Calibrating SABR Model

Like what we did in Displaced Diffusion Model, we first use the given SABR function to get σ_{SABR}, then use SABR calibration function to find the error term between the Market data and σ_{SABR}, After getting the calibration function, we then search best fit parameters using least squares.

Alpha – Displaced Diffusion Model Calibration

	Tenor				
Expiry	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.139	0.185	0.197	0.178	0.170
5Y	0.167	0.200	0.210	0.190	0.175
10Y	0.180	0.196	0.211	0.205	0.183

Rho – Displaced Diffusion Model Calibration

	Tenor				
Expiry	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	-0.633	-0.525	-0.483	-0.408	-0.234
5Y	-0.582	-0.545	-0.550	-0.504	-0.414
10Y	-0.545	-0.548	-0.551	-0.566	-0.514

Nu – Displaced Diffusion Model Calibration

	Tenor				
Expiry	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	2.049	1.678	1.438	1.028	0.715
5Y	1.311	1.040	0.937	0.655	0.466
10Y	0.987	0.938	0.836	0.670	0.467

2.3: Price the following swaptions using the calibrated displaced-diffusion and SABR model:

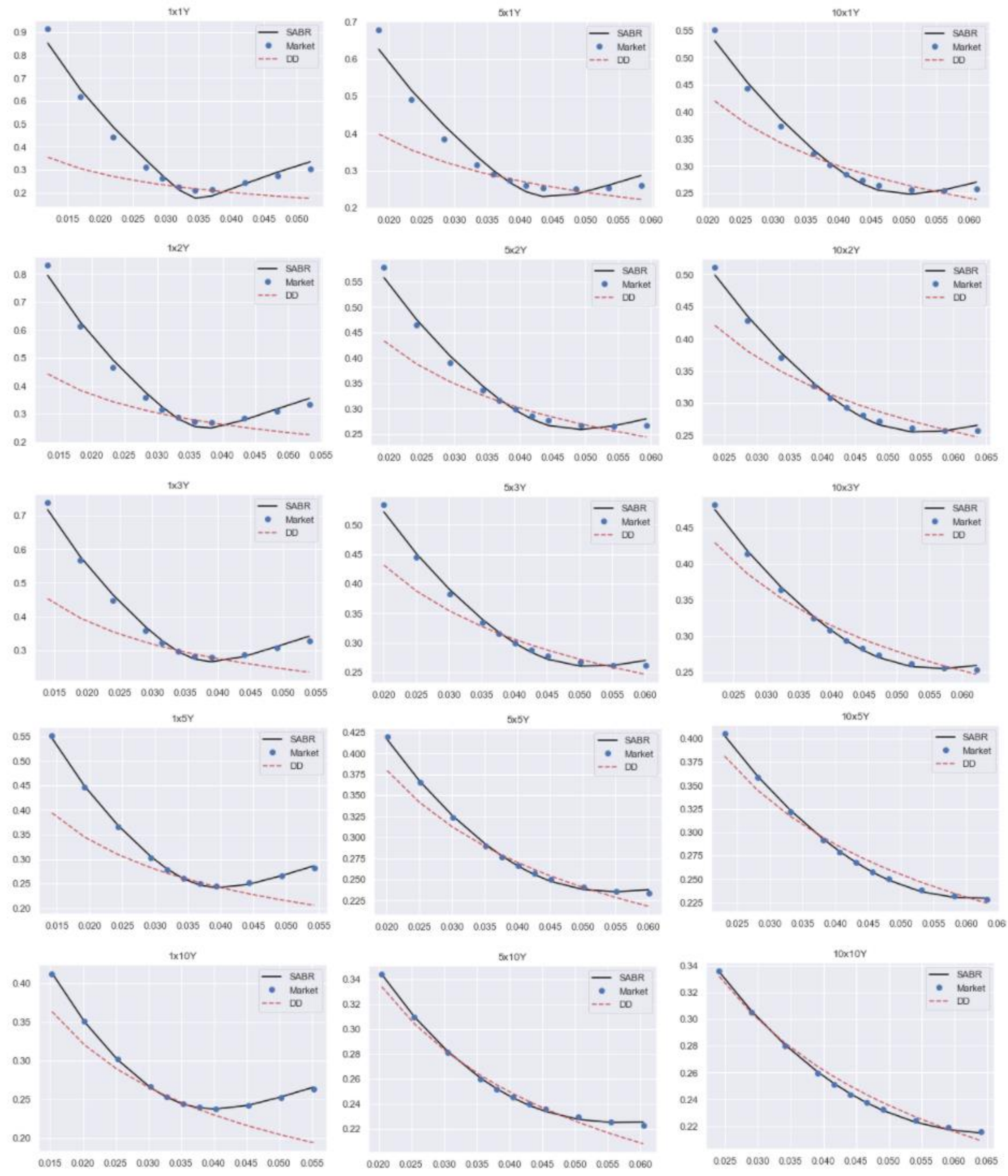
- payer: 2y X 10y K = 1%, 2%, 3%, 4%, 5%, 6%, 7%, 8%
- receiver: 8y X 10y K=1%,2%,3%,4%,5%,6%,7%,8%

According to the Displaced Diffusion model and SABR model, we need to know the following parameters F, K, T, sigma, alpha, beta, rho, and nu. Based on parameters' data we calculated in earlier, we set the interpolation function using cubic spline method to acquire calibrated sigma, alpha, beta, rho and nu for 2y x 10y and 8y x10y respectively.

The value of forward swap rate (F) is calculated using the same equation from Question 3 of part I. After we put the calibrated parameters into our Displaced Diffusion model and SABR model, we obtain the following price of the swaptions:

	1%	2%	3%	4%	5%	6%	7%	8%
Payer 2y x 10y DD	0.288	0.195	0.112	0.051	0.018	0.004	0.001	0.000
Payer 2y x 10y SABR	0.289	0.197	0.114	0.051	0.021	0.010	0.006	0.004
Receiver 8y x 10y DD	0.016	0.031	0.055	0.088	0.133	0.188	0.254	0.328
Receiver 8y x 10y SABR	0.013	0.031	0.054	0.085	0.128	0.185	0.255	0.335

The following charts shows the volatility smile of displaced diffusion and SABR models calibrated using swaptions of all tenors and expiries:



It is clear from the above that Displaced diffusion model does not have enough parameters to incorporate the skew and curvature of the market implied volatilities. The SABR model, on the other hand, fits the market surface almost perfectly, given we have additional parameters.

Part 3: Convexity Correction

2.1: Value the following constant maturity swaps (CMS) products:

- **PV of a leg receiving CMS10y semi-annually over the next 5 years**
- **PV of a leg receiving CMS2y quarterly over the next 10 years**

A CMS leg is a collection of CMS rates paid over a period and the PV is the sum of the discounted values of the CMS rates, multiplied by the day count fraction. To calculate the PV, we will use the following formula:

$$CMS\ leg\ PV = \sum_{i=0.5}^{N} D_0(0,T_i) * Delta * CMS\left(S_{n,N}(T_i)\right)$$

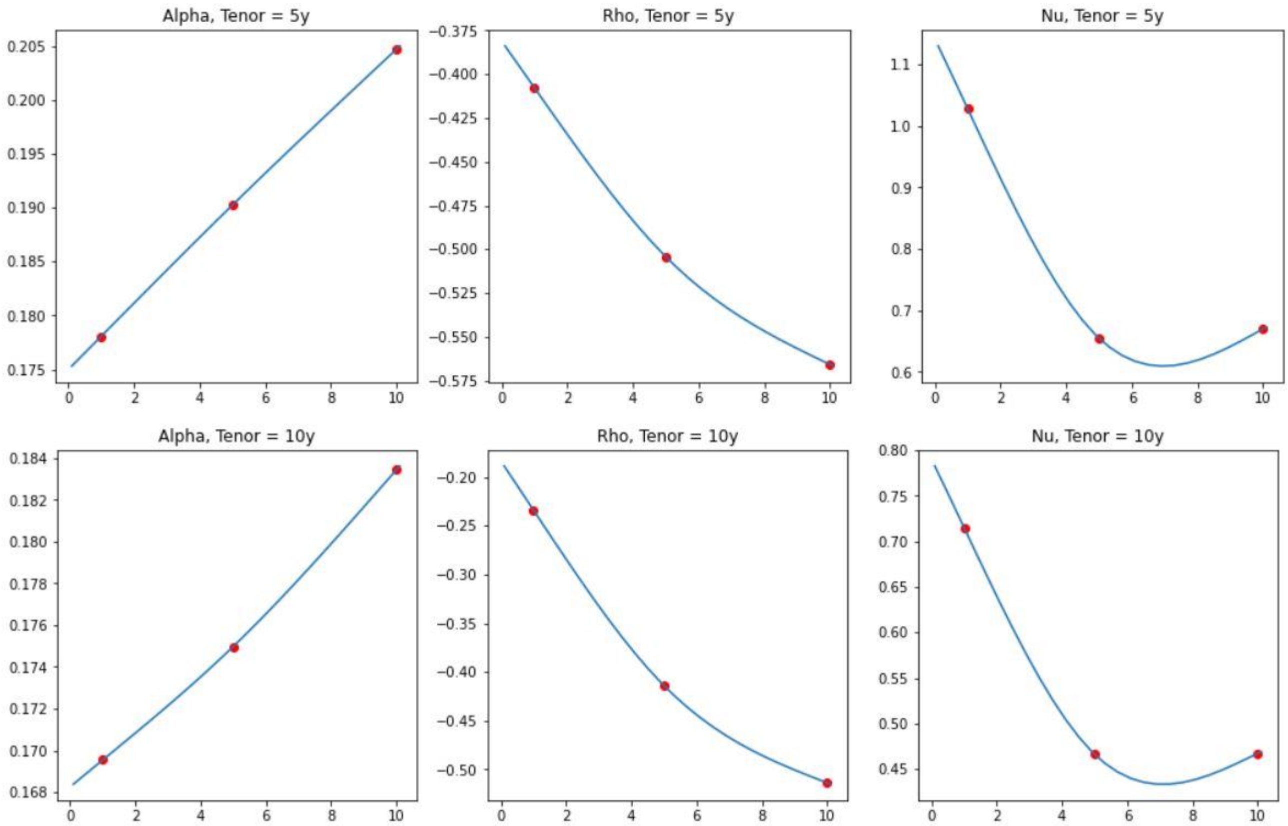
And $CMS(S_{n,N}(T_i))$ is given by:

$$E^T\left[S_{n,N}(T)\right] = g(F) + \frac{1}{D_0(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]$$

IRR-settled option pricer (V^{rec} *or* V^{pay}) is given by:

$$V_{n,N}(0) = D_0(0,T_n) * IRR\left(S_{n,N}(0)\right) * Black\ 76\left(S_{n,N}(0), K, \sigma_{sabr}, T\right)$$

Hence, to calculate the PV of a leg receiving CMS, we need to find SABR parameters at different expiries to price each CMS rate. We use natural cubic spline interpolation with second derivative at boundary = 0 between α , ν, and ρ of 1y × 10y, 5y × 10y and 10y × 10y SABR models we have calibrated. We have shown our interpolated curves for 5y and 10y tenors below.



In addition to SABR parameters interpolation, due to quarterly arrangement, more discrete OIS discount rates and Libor discount rates are linearly interpolated as calculated in Part 1.

Based on our calculation:

- The PV of a leg receiving CMS10y semi-annually over the next 5 years is **0.20205**
- The PV of a leg receiving CMS2y quarterly over the next 10 years is **0.38207.**

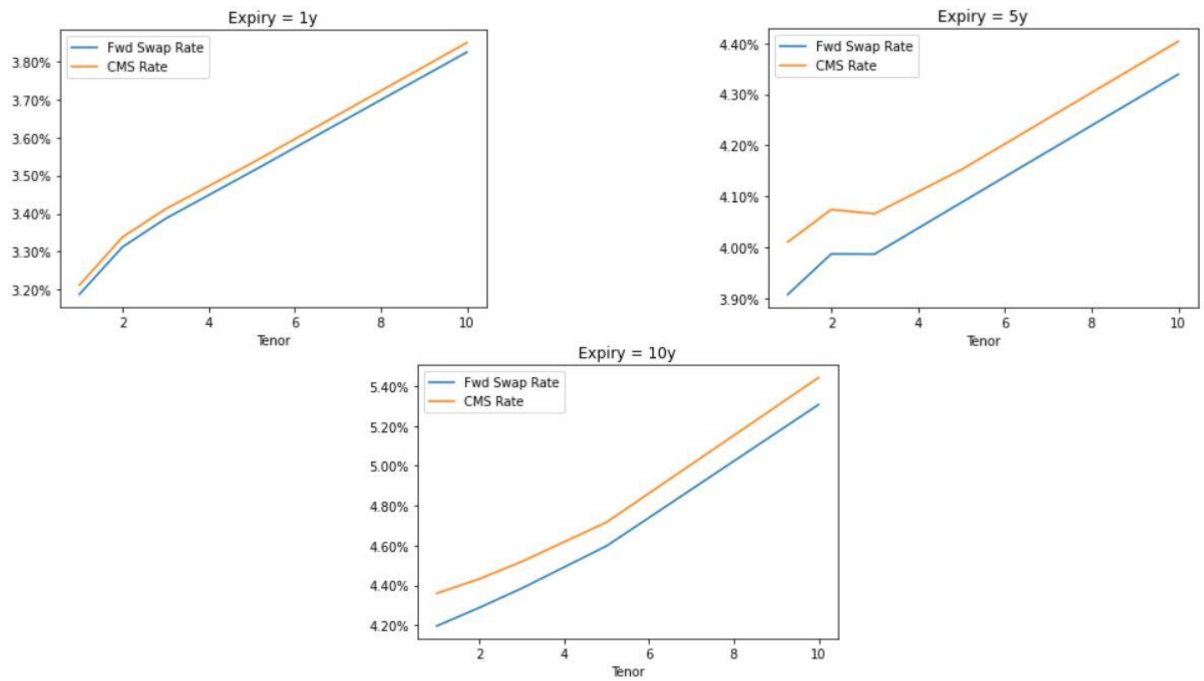
2.2: Compare the forward swap rates with the CMS rate:

- 1y × 1y, 1y × 2y, 1y × 3y, 1y × 5y, 1y × 10y
- 5y × 1y, 5y × 2y, 5y × 3y, 5y × 5y, 5y × 10y
- 10y × 1y, 10y × 2y, 10y × 3y, 10y × 5y, 10y × 10y

Comparison Table

Expiry	Tenor	Fwd Swap Rate	CMS Rate
1Y	1Y	3.19%	3.21%
1Y	2Y	3.31%	3.34%
1Y	3Y	3.39%	3.41%
1Y	5Y	3.51%	3.53%
1Y	10Y	3.82%	3.85%
5Y	1Y	3.91%	4.01%
5Y	2Y	3.99%	4.07%
5Y	3Y	3.99%	4.07%
5Y	5Y	4.09%	4.15%
5Y	10Y	4.34%	4.40%
10Y	1Y	4.20%	4.36%
10Y	2Y	4.29%	4.43%
10Y	3Y	4.38%	4.52%
10Y	5Y	4.60%	4.72%
10Y	10Y	5.31%	5.44%

We can see that the CMS Rate is consistently higher than the Par forward swap rate. To further illustrate the difference, we plot the rates by expiry.



We can observe from the above that the CMS rates and the forward swap rates increase as the expiration period increases. This indicates that as the expiration increases the magnitude of convexity correction also increases. In contrast, we observe that Tenor has little effect on convexity correction.

Part 4: Decompounded Options

4.1: Static replication to value the PV of the decompounded option payoff CMS 10y$^{1/4}$-0.04$^{1/2}$ at T=5y

According to the Leibniz's rule, the payer IRR swaption and the receiver IRR swaption has been derived as the equations shown below:

Payer IRR swaption	Receiver IRR swaption
$V^{pay}(K) = D(0,T)\int_K^{\infty} IRR(S) \times (S-K) \times f(S)dS$	$V^{rec}(K) = D(0,T)\int_0^{F} IRR(S) \times (K-S) \times f(S)dS$
$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(0,T) \times IRR(K) \times f(K)$	$\frac{\partial^2 V^{rec}(K)}{\partial K^2} = D(0,T) \times IRR(K) \times f(K)$

Considering the calculation of the PV of the payoff function CMS 10y$^{1/4}$-0.04$^{1/2}$, we could derive the generic contract valuation with the integration by parts, then the yield will be:

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^{\infty} h''(K)\,V^{pay}(K)dK$$

With the consideration of:

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$$

$$g(K) = K^{1/4} - 0.2$$
$$g'(K) = \frac{1}{4}K^{-3/4}$$
$$g''(K) = -\frac{3}{16}K^{-7/4}$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2 \times IRR'(K)g'(K)}{IRR(K)^2} + \frac{2 \times IRR'(K)^2 g(K)}{IRR(K)^3}$$

After calculating with the static replication and the equations derived above, considered with the T = 5y and 5y x 10y forward swap rate with the OIS discount factor $D_0(0,5)$, we could obtain the value of the PV of this payoff which is 0.2419.

4.2: Static replication to value the PV of the payoff (CMS 10y$^{1/p}$-0.04$^{1/q}$)$^+$

According to the given information, the payoff is (CMS 10y$^{1/p}$-0.04$^{1/q}$)$^+$, which must be positive.
Consequently, we can now consider:

$$g(S_T) = S_T^{1/4} - 0.2 > 0$$
$$S_T > 0.2^4$$
$$S_T > 0.0016 = L$$

Accordingly, we could use the CMS caplet strike (V^+) at L = 0.0016. The following process will be:

$$V_0^+ = D(0,T)\int_L^{\infty} g(K)f(K)dK$$
$$V_0^+ = \int_L^{\infty} \frac{\partial^2 V^{pay}(K)}{\partial K^2}dK$$
$$V_0^+ = h'(L)V^{pay}(L) + \int_L^{\infty} h''(K)V^{pay}(K)dK$$

After calculating with the static replication and the equations derived above, considered with the T = 5y and other parameters using previously in the question 4.1, we could obtain the value of the PV of this payoff (V_0^+), which is **0.0294.**