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Scalable Data Science

Lecture 13a: Multi Probe LSH

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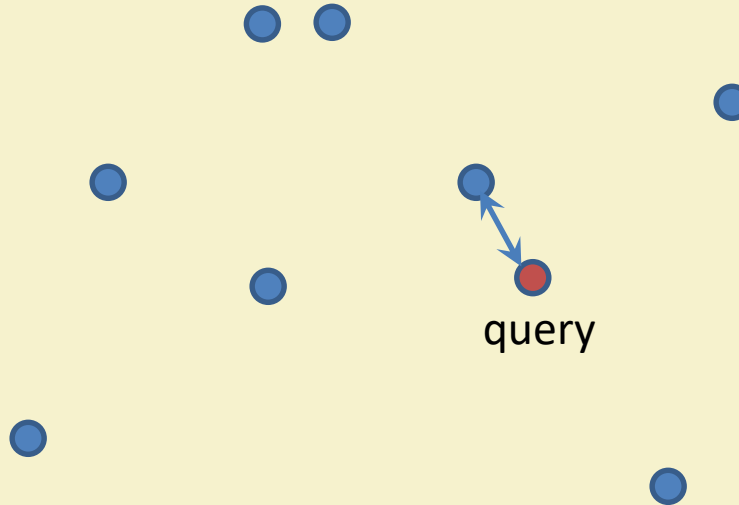
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Finding Near Neighbors

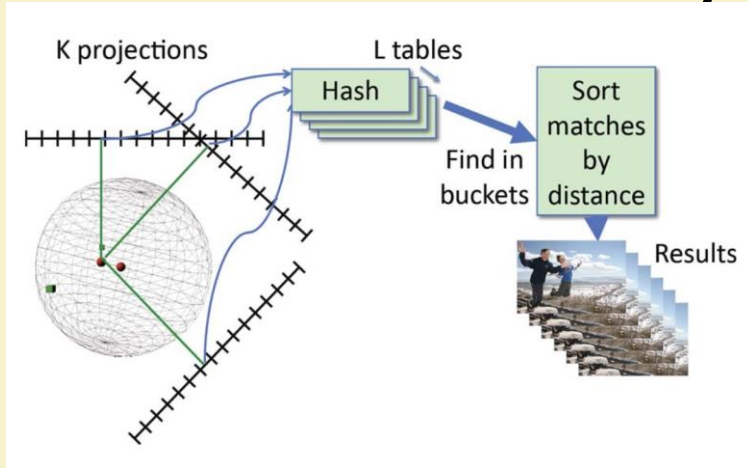


Given a set of data points
and a query

Can we find what is the nearest
datapoint to the query?

- K-nearest neighbors
- $d(p, \text{query}) < r$

Locality Sensitive Hashing

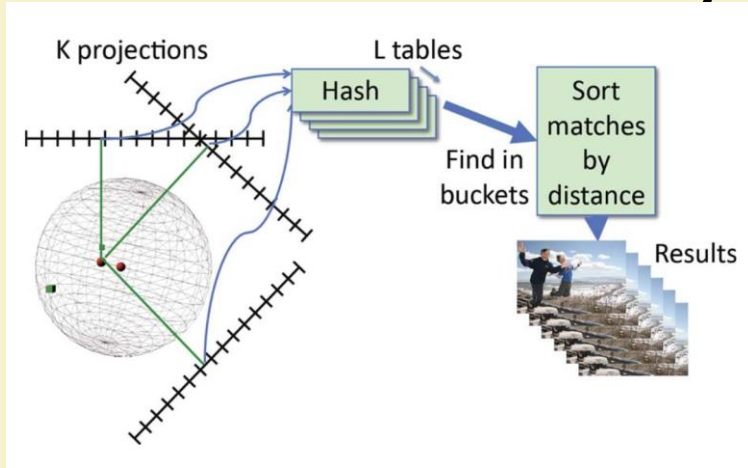


Given input data, radius r , approx factor c and confident δ

Output: if there is any point at distance $\leq r$ then w.p. $1 - \delta$ return one at distance $\leq cr$

Picture courtesy Slaney et al.

Locality Sensitive Hashing



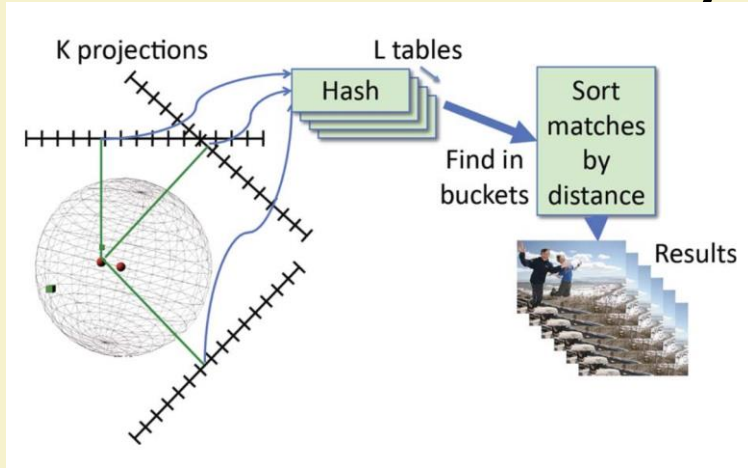
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Algo: Choose (k, L) .

Picture courtesy Slaney et al.

Locality Sensitive Hashing



Given input data, radius r , approx factor c and confident δ

Output: if there is any point at distance $\leq r$ then w.p. $1 - \delta$ return one at distance $\leq cr$

Algo: Choose (k, L) .

do L times

iid hash functions : $\{h_{i1} \dots h_{ik}\}$

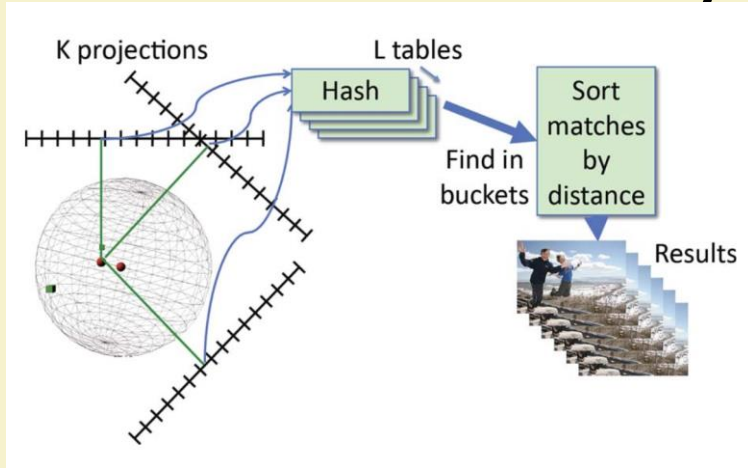
Create hash table H_i by putting each x in bucket

$$H_i(x) = (h_{i1}(x), \dots, h_{ik}(x))$$

Store non-empty buckets in normal hash table

Picture courtesy Slaney et al.

Locality Sensitive Hashing



Given input data, radius r , approx factor c and confident δ

Output: if there is any point at distance $\leq r$ then w.p. $1 - \delta$ return one at distance $\leq cr$

Query: Find out all points in buckets $H_1(q) \dots H_L(q)$ and return ones that are $\leq cr$

Picture courtesy Slaney et al.

Drawbacks

- Trading space with time, strongly super-linear space
 - Even in practice, typically 5-20 times more memory than dataset itself
- Space-time tradeoff mostly practical effective for medium-high dimensions, dense vectors
 - recent advances in ML about dense embeddings

Probing multiple times

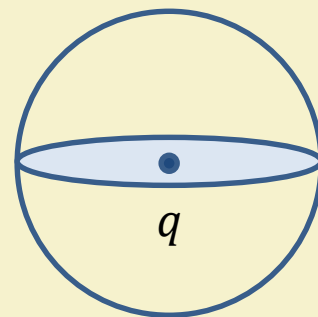
- Idea: Can we reduce space while not affecting query time by too much?
 - need to hit buckets that have high probability of the containing the nearest neighbour

Entropy based LSH

- Assume that we know $R(p, q)$ = distance from query q to nearest neighbour p
 - Buckets are a random partition of the data
 - The success probability of a bucket (i.e. of containing p) depends only on $R(p, q)$
 - Ideally, we can sort the buckets by this probability

Entropy based LSH [Panigrahy' 06]

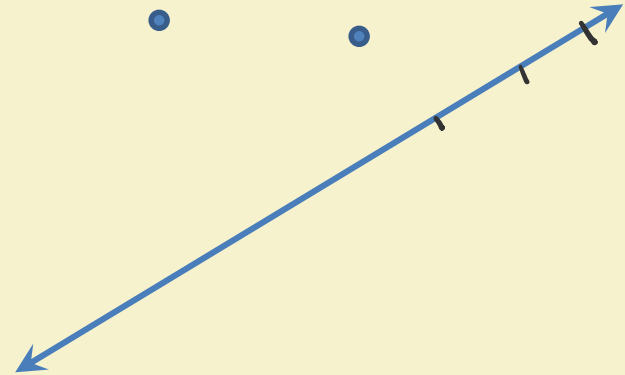
- Elegant way to sample from the success probability distribution
 - Perturb the query point repeatedly and probe
 - Buckets that have high probability should come up often
 - Theoretical guarantee



Multi-probe LSH

- Look at neighbouring buckets!
- Consider LSH for L2

$$h_{v,b}(q) = \left\lfloor \frac{q \cdot v + b}{w} \right\rfloor$$



Multi-probe LSH

- Suppose $k = 3$
- $H_1(q) = (5, 8, 3)$
- We consider buckets that differ in one position, two positions, ...



Formalizing

- $\Delta \in \{-1, 0, +1\}^k$ be a “perturbation” vector
 - E.g. $\Delta = (-1, 0, +1, +1, 0 \dots -1)$
 - We get a new hash bucket by doing $H(q) + \Delta$
 - Say Δ has at most S nonzeros
 - Number of possible Δ is:
- Is there a natural way to order these buckets for searching?



Success Probability Estimation

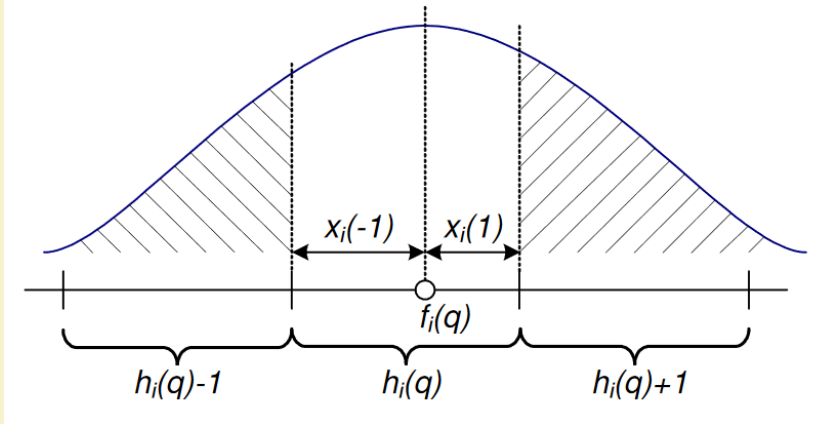


Image from Lv et al.

$f_i(q) = q \cdot v_i + b_i$ be the projection of q

$x_i(+1)$ and $x_i(-1)$ be the distance of the projection to the two boundaries

$f_i(q) - f_i(p) \sim N(0, C|p - q|)$ by property of normal distribution

Success Probability Estimation

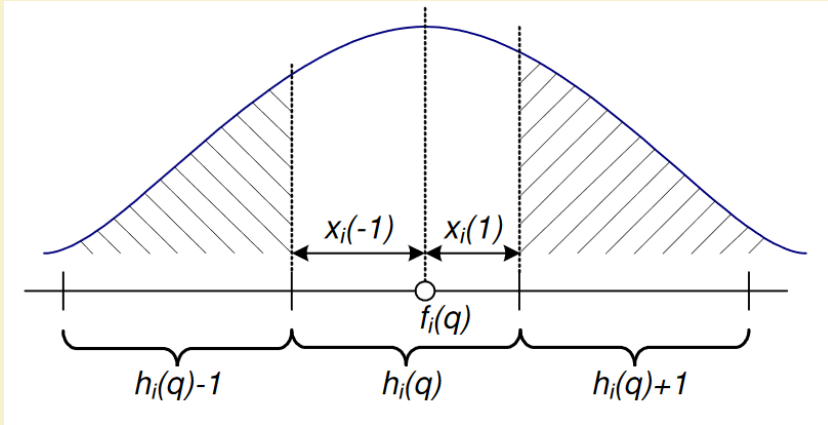


Image from Lv et al.

$x_i(+1)$ and $x_i(-1)$ be the distance of the projection to the two boundaries

$f_i(q) - f_i(p) \sim N(0, C|p - q|)$ by property of normal distribution

$$\Pr[h_i(p) = h_i(q) + 1] \approx \exp(-Cx_i(+1)^2)$$

Ordering buckets

- If $\Delta = (\delta_1 \dots \delta_k)$ then

$$\Pr[H(p) = H(q) + \Delta] = \Pr \prod [h_i(q) = h_i(q) + \delta_i]$$

$$\approx \prod \exp(-C x_i (\delta_i)^2) = \exp\left(-C \sum x_i (\delta_i)^2\right)$$

Ex: $\Delta = (+1, 0, -1)$,

Ordering buckets

- Define $score(\Delta) = \sum x_i(\delta_i)^2$
- Lower the score, higher the probability of p being in the bucket

Ordering buckets

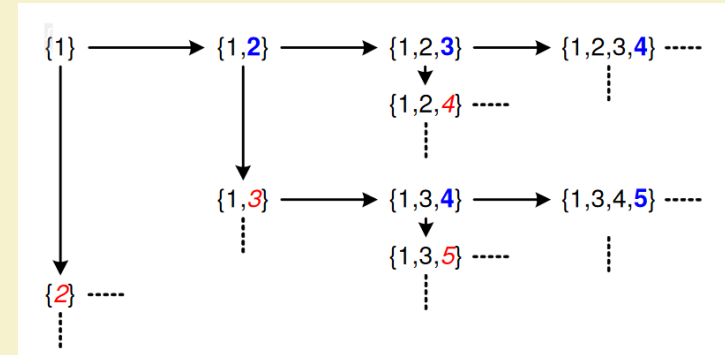
- Define $score(\Delta) = \sum x_i(\delta_i)^2$
- Lower the score, higher the probability of p being in the bucket
- Order the buckets by the score and search them in this order

Query directed ordering

- When a query q arrives
 - Calculate $H(q)$
 - Calculate $\{x_i(+1)^2, x_i(-1)^2, i = 1 \dots k\}$
 - Sort

Query directed ordering

- When a query q arrives
 - Calculate $H(q)$
 - Calculate $\{x_i(+1)^2, x_i(-1)^2, i = 1 \dots k\}$
 - Sort (call these as $z_1 \leq z_2 \dots \leq z_{2k}$)
- Start with $A = \{1\}$
- Repeatedly do either *shift* or *expand*
 - *shift* replace $\max(A)$ by $1+\max(A)$
 - *expand* adds $1+\max(A)$ to A

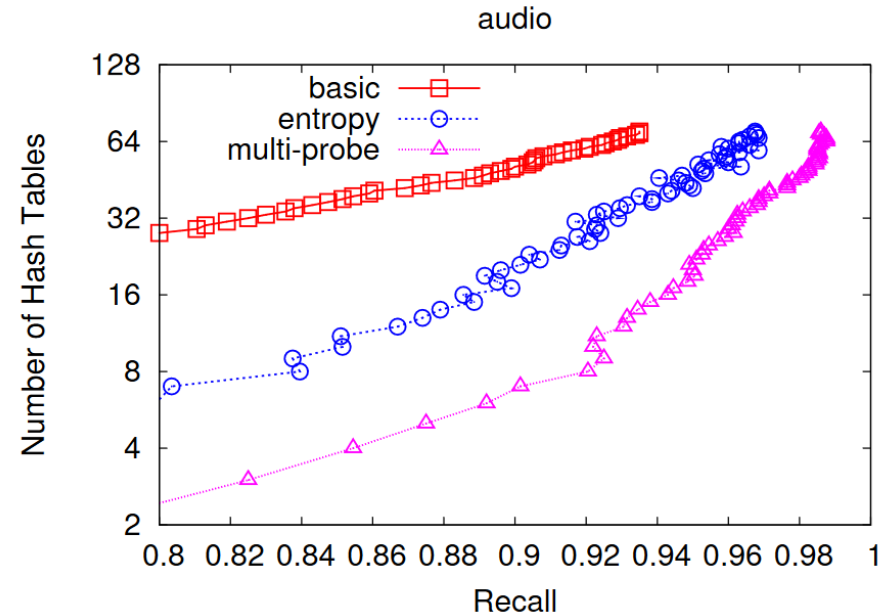
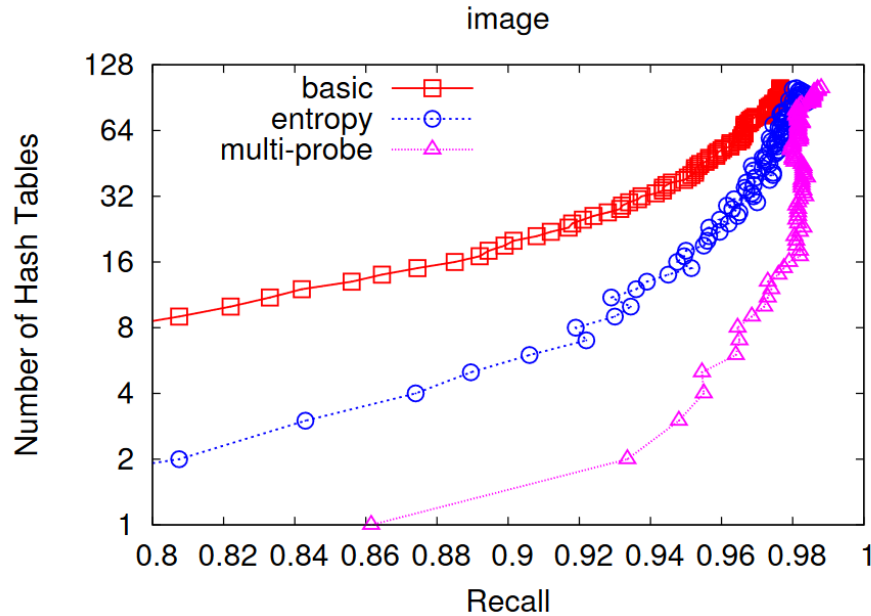


Multiprobe LSH

- Using a min-heap at query time we can use the shift and expand operations to explore all buckets in order
 - Can optimize further
- In practice, will stop after a budget



Experiments



Summary

- While LSH is a powerful technique, there are few areas of concern, memory usage among them
- Entropy and Multi-probe LSH are elegant solutions that are useful in practice
 - Shown to be useful in practice, reduce space usage by a factor
 - also form part of the state-of-art LSH system
- Intuition based on idea of probing multiple buckets in a query-dependent manner



References:

- Primary references for this lecture
 - Multi-Probe LSH: Efficient Indexing for High Dimensional Similarity Search. By Qin Lv, William Josephson, Zhe Wang, Moses Charikar, Kai Li, VLDB 2007
 - R. Panigrahy. Entropy based nearest neighbor search in high dimensions. In Proc. of ACM-SIAM Symposium on Discrete Algorithms(SODA), 2006.

Thank You!!



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