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# Scalable Data Science

## Lecture 3: Background on Linear Algebra

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# In this review

- Recall concepts we'll need in this class
- Geometric intuition for linear algebra
- Outline:
  - Matrices as linear transformations.
  - Linear systems & vector spaces.
  - Solving linear systems.
  - Eigenvalues & eigenvectors.


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# Basic concepts

- *Vector* in $R^n$ is an ordered set of $n$ real numbers.
  - e.g. $v = (1,6,3,4)$ is in $R^4$
  - $(1,6,3,4)$ is a column vector: $\begin{pmatrix} 1 \\ 6 \\ 3 \\ 4 \end{pmatrix}$
  - as opposed to a row vector: $\begin{pmatrix} 1 & 6 & 3 & 4 \end{pmatrix}$
- $m - by - n$ *matrix* is an object with m rows and n columns, each entry fill with a real number: $\begin{pmatrix} 1 & 2 & 8 \\ 4 & 78 & 6 \\ 9 & 3 & 2 \end{pmatrix}$


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# Basic concepts

- Transpose: reflect vector/matrix on line: $\left[\ \diagdown\ \right]$

$$\begin{pmatrix} a \\ b \end{pmatrix}^T = \begin{pmatrix} a & b \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

  - Note: $(Ax)^T = x^T A^T$
- Vector norms:
  - $L_p$ norm of $v = (v_1, \ldots, v_k)$ is $(\Sigma_i\, |v_i|^p)^{1/p}$
  - Common norms: $L_1, L_2$
  - $L_{infinity} = maxi\ |vi|$
- Length of a vector $v$ is $L_2(v)$


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# Basic concepts

- Vector dot product:

$$u \bullet v = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \bullet \begin{pmatrix} v_1 & v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

  - Note dot product of u with itself is the square of the length of u.

- Matrix product (multiplication):

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$


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# Basic concepts

- Vector products in matrix multiplication notation:
  - Dot product:

$$u \bullet v = u^T v = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

  - Outer product:

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

# Special matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$ diagonal

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$ upper-triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ I (identity matrix)

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$ lower-triangular


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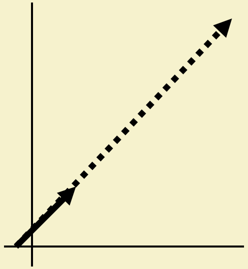
# Matrices as linear transformations

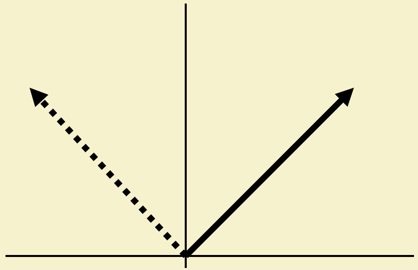
Multiplication with $m \times n$ matrices transform vectors in $R^n$ into vectors in $R^m$

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



Scaling: scalar product of identity matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Rotation: Orthogonal matrices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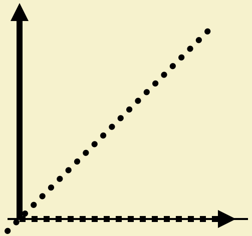
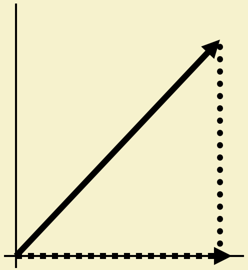
# Matrices as linear transformations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Projection onto axis

# Vector spaces

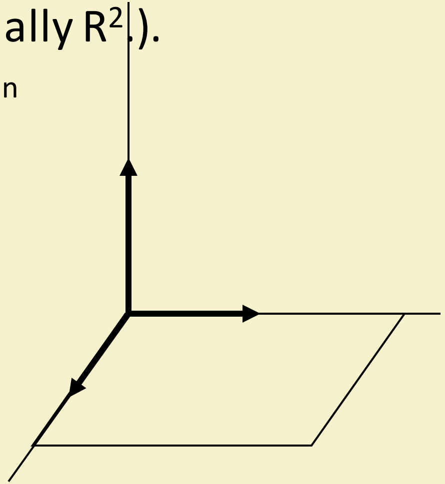
- Formally, a *vector space* is a set of vectors which is closed under addition and multiplication by real numbers (also called linear combination).

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

- A *subspace* is a subset of a vector space which is a vector space itself, e.g. the plane z=0 is a subspace of $R^3$ (It is essentially $R^2$.).
- We'll be looking at $R^n$ and subspaces of $R^n$

Our notion of planes in $R^3$ may be extended to *hyperplanes* in $R^n$ (of dimension n-1)

Note: subspaces must include the origin (zero vector).



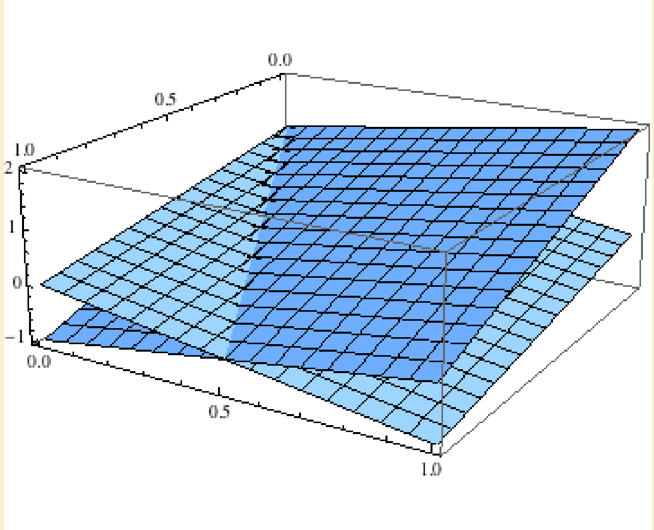
# Matrices as sets of constraints

Matrix equations (linear system of equations) can encode a set of linear constraints

$$x + y + z = 1$$

$$2x - y + z = 2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$





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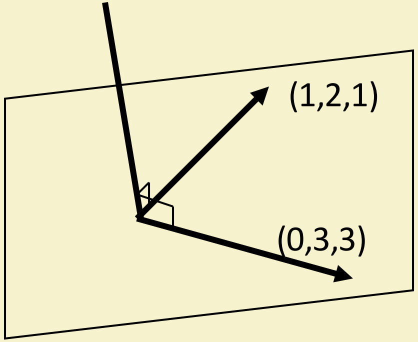
# Linear system & subspaces

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$u \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- $Ax = b$ is solvable iff b may be written as a linear combination of the columns of A
- The set of all possible vectors b forms a subspace called the *column space* of A







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# Linear system & subspaces

The set of solutions to $Ax = 0$ forms a subspace called the *null space* of A.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

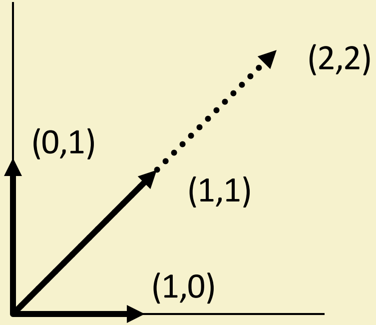
→ Null space: {(0,0)}

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

→ Null space: {(c,c,-c)}

# Linear independence and basis

- Vectors $v_1, \dots, v_k$ are linearly independent if $c_1 v_1 + \cdots + c_k v_k = 0$ implies $c_1 = \cdots = c_k = 0$




i.e. the nullspace is the origin

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Recall nullspace contained only $(u, v) = (0,0)$.
i.e. the columns are linearly independent.



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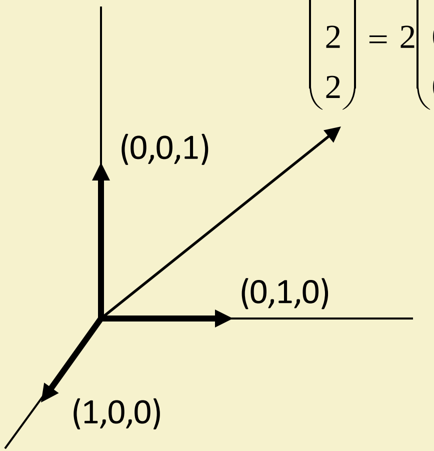
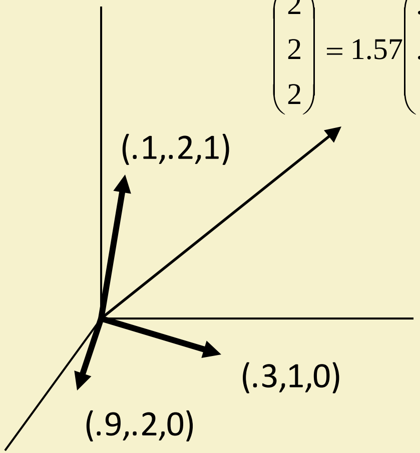
# Linear independence and basis

- If all vectors in a vector space may be expressed as linear combinations of $v_1, \dots, v_k$, then $v_1, \dots, v_k$ *span* the space.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$



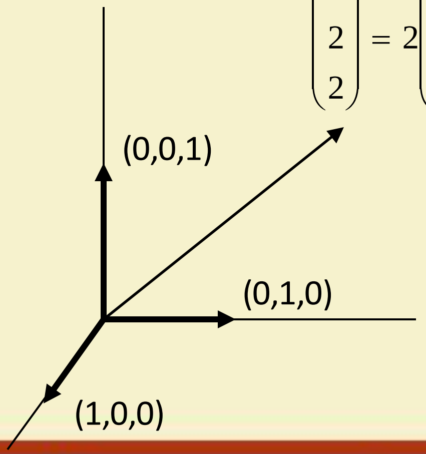

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



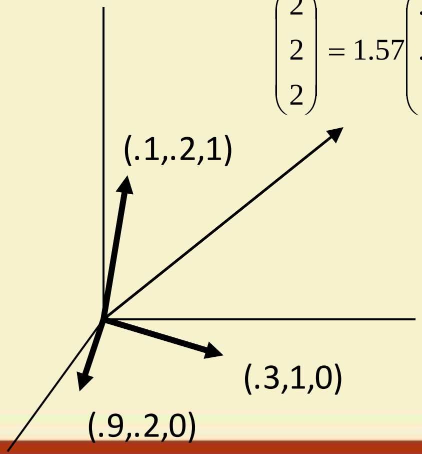
# Linear independence and basis

- A *basis* is a set of linearly independent vectors which span the space.
- The *dimension* of a space is the # of “degrees of freedom” of the space; it is the number of vectors in any basis for the space.
- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$




$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



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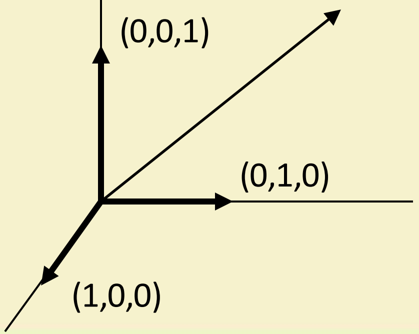
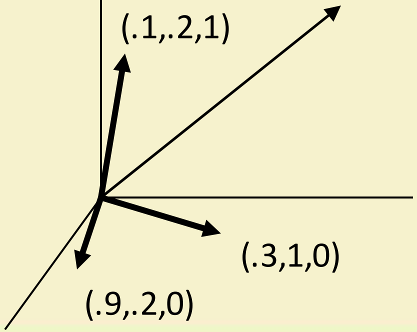
# Linear independence and basis

- Two vectors are *orthogonal* if their dot product is 0.
- An *orthogonal basis* consists of orthogonal vectors.
- An *orthonormal basis* consists of orthogonal vectors of unit length.

$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$




$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 1.57\begin{pmatrix}.9\\.2\\0\end{pmatrix} + 1.29\begin{pmatrix}.3\\1\\0\end{pmatrix} + 2\begin{pmatrix}.1\\.2\\1\end{pmatrix}$$



# About subspaces

- The *rank* of A is the dimension of the column space of A.
- It also equals the dimension of the *row space* of A (the subspace of vectors which may be written as linear combinations of the rows of A).

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}$$

(1,3) = (2,3) – (1,0)

Only 2 linearly independent rows, so rank = 2.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# About subspaces

<u>Fundamental Theorem of Linear Algebra</u>:

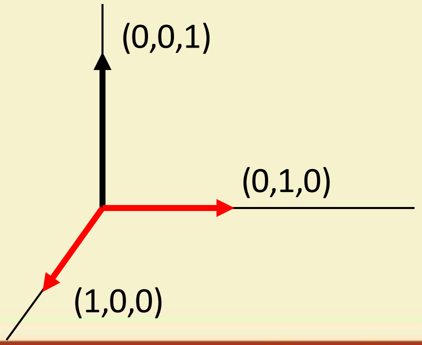
If A is $m\ x\ n$ with rank r,

Column space(A) has dimension $r$

Nullspace(A) and Nullspace($A^T$) has dimension $n - r$ (= *nullity* of A)

Row space(A) = Column space($A^T$) has dimension $r$

<u>Rank-Nullity Theorem</u>: rank + nullity = n




$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

m = 3
n = 2
r = 2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# Matrix inversion

- To solve $Ax = b$, we can write a closed-form solution if we can find a matrix $A^{-1}$ s.t. $AA^{-1} = A^{-1}A = I$ (identity matrix)

- Then $Ax = b$ iff $x = A^{-1}b$:

$$x = Ix = A^{-1}Ax = A^{-1}b$$

- A is *non-singular* iff $A^{-1}$ exists iff $Ax = b$ has a unique solution.

- Note: If $A^{-1}$,$B^{-1}$ exist, then $(AB)^{-1} = B^{-1}A^{-1}$, and $(A^T)^{-1} = (A^{-1})^T$


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# Special matrices

- Matrix A is *symmetric* if $A = A^T$
- A is *positive definite* if $x^T A x > 0$ for all non-zero x (*positive semi-definite* if inequality is not strict).

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 + b^2 + c^2 \qquad \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 - b^2 + c^2$$

- Useful fact: Any matrix of form $A^TA$ is positive semi-definite.

  To see this, $x^T(ATA)x = (x^T A^T)(Ax) = (Ax)^T(Ax) \geq 0$



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# Determinants

- If det(A) = 0, then A is singular, also called rank deficient
- If det(A) ≠ 0, then A is invertible.
- To compute:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

# Eigenvalues & eigenvectors

- How can we characterize matrices?
- The solutions to $Ax = \lambda x$ in the form of eigenpairs (λ,x) = (eigenvalue,eigenvector) where x is non-zero.
- To solve this, $(A - \lambda I)x = 0$
- λ is an eigenvalue iff $\det(A - \lambda I) = 0$



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# Eigenvalues & eigenvectors

$(A - \lambda I)x = 0$

$\lambda$ is an eigenvalue iff $\det(A - \lambda I) = 0$

Example:

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 3/4 & 6 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 3/4-\lambda & 6 \\ 0 & 0 & 1/2-\lambda \end{pmatrix} = (1-\lambda)(3/4-\lambda)(1/2-\lambda)$$

$$\lambda = 1, \lambda = 3/4, \lambda = 1/2$$


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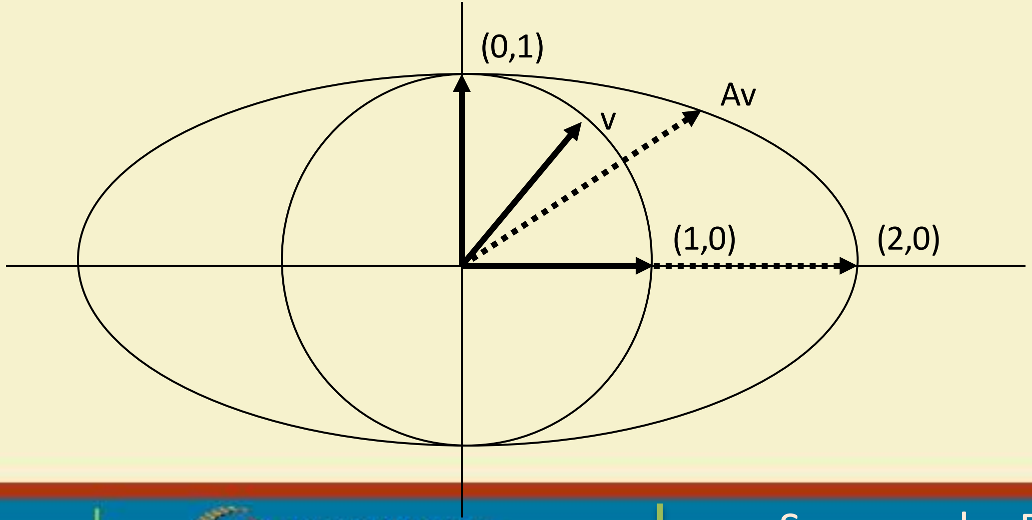
# Eigenvalues & eigenvectors

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues $\lambda = 2, 1$ with eigenvectors (1,0), (0,1)

Eigenvectors of a linear transformation A are not rotated (but will be scaled by the corresponding eigenvalue) when A is applied.





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# Properties of Eigenvalues and Eigenvectors

– If $\lambda_1, \ldots, \lambda_n$ are *distinct* eigenvalues of a matrix, then the corresponding eigenvectors $e_1, \ldots, e_n$ are linearly independent.

– If $e_1$ is an eigenvector of a matrix with corresponding eigenvalue $\lambda_1$, then any nonzero scalar multiple of $e_1$ is also an eigenvector with eigenvalue $\lambda_1$.

– A real, symmetric square matrix has real eigenvalues, with orthogonal eigenvectors (can be chosen to be orthonormal).

# SVD: Singular Value Decomposition

- Any matrix $A$ $(m \quad n)$ can be written as the product of three matrices:
  $A = UDV^T$
  - $U$ is an $m \times m$ orthonormal matrix
  - $D$ is an $m \times n$ diagonal matrix. Its diagonal elements, $\sigma_1, \sigma_2, \ldots$, are called the **singular values** of A, and satisfy $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$.
  - $V$ is an $n \times n$ orthonormal matrix
- Example: if $m > n$

$$
\underset{A}{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}} = \underset{U}{\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ u_1 & u_2 & u_3 & \mathrm{L} & u_m \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}} \underset{D}{\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{V^{\mathrm{T}}}{\begin{bmatrix} \leftarrow & v_1^T & \rightarrow \\ \mathrm{M} & \mathrm{M} & \mathrm{M} \\ \leftarrow & v_n^T & \rightarrow \end{bmatrix}}
$$


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# Some Properties of SVD

- The rank of matrix $A$ is equal to the number of nonzero singular values $\sigma_i$ .
- A square $(n \times n)$ matrix $A$ is singular

  if and only if

  at least one of its singular values $\sigma_1, \ldots, \sigma_n$ is zero.



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# References:


- Strang, Gilbert. **Introduction to Linear Algebra.** 4th ed. *Wellesley-Cambridge Press*, 2009.

- Wikipedia.

# Thank You!!