Trinomial Trees
MScFE 620 Group Project Submission 2

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1 Part A - Construction of Trinomial Tree

Pepar's code goes here. This is the other extreme where β, ρ, v are 0.

$$\alpha + \gamma + \theta + \eta + \psi + \lambda = 1 \tag{1}$$
$$2\alpha + 2\gamma + \eta - 2\psi - 2\lambda = 1 \tag{2}$$
$$6\alpha - 6\gamma + 4\theta - 4\eta - 10\psi + 10\lambda = 1 \tag{3}$$
$$\alpha = 2\gamma \tag{4}$$
$$\theta = \frac{5}{3}\eta \tag{5}$$
$$\psi = \frac{4}{3}\lambda \tag{6}$$

Substituting 4, 5, 6 into 1, 3 and 2 gives (Pepar, please check)

$$3\gamma + \frac{8}{3}\eta + \frac{7}{3}\lambda = 1$$
$$6\gamma + \frac{8}{3}\eta - \frac{14}{3}\lambda = 1$$
$$\implies -3\gamma + 7\lambda = 0$$
$$\implies \lambda = \frac{3}{7}\gamma$$
$$\implies \psi = \frac{4}{7}\gamma$$

Then

$$\begin{aligned} 3\gamma + \frac{3}{8}\eta + \frac{3}{7}\gamma + \frac{4}{7}\gamma &= 1 \\ \frac{8}{3}\eta &= 1 - 4\gamma \\ \implies \eta &= \frac{3(1-4)}{8}\gamma \\ \implies \theta &= \frac{5-20\gamma}{8} \end{aligned}$$

The set of equivalent Martingale measures is therefore:

$$\mathcal{P} = \left\{ \left(2\gamma, 0, \gamma, \frac{5-20\gamma}{8}, 0, \frac{3-12\gamma}{8}, \frac{4}{7}\gamma, 0, \frac{3}{7}\gamma \right) : 0 \geq \gamma \geq \frac{1}{4} \right\}$$

I copied the condition for γ from the notes but this is unphysical. Could you please double check this?

2 Part B - Arbitrage-free conditions and market completness

For this section of the task, we will apply the arbitrage-free conditions to the numerical trinomial tree construction above and also to a completely general trinomial tree. In the later case, we will also derive conditions under which the market is complete. We begin by definining a market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with $T = 2$ and $d = 1$. $X = (1, X_t)$ is comprised of one risk-free asset (normalised to 1) and one risky asset which we will denote as X_t. Since we are trying to describe the trinomial model, we know that the sample space will consist of the three possibilities for asset evolution at any given time, namely up u, down d and nothing n. Assuming the completely general case where the asset price can move a different amount at each time step, we have $ud \neq du$ (commutativity property does not hold) and so $\Omega = \{uu, un, ud, nu, nn, nd, du, dn, dd\}$. For $\mathcal{F}$, we use the natural filtration which prevents look ahead bias. Specifically, $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ where

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_1 &= \sigma(\{\{uu, un, ud\}, \{nu, nn, nd\}, \{du, dn, dd\}\}) \\ \mathcal{F}_2 &= 2^{\Omega} \end{aligned}$$

We can define the evolution of the risky asset as:

$$\begin{aligned} X_{t+1} &= X_t Z_{t+1},\ t : 0 < t \leq 2 \\ X_0 &= c \in \mathbb{R}^+ \neq 0 \end{aligned}$$

with $Z_t,\ t : 1 \leq t \leq 2$ a sequence of random variable taking on three values:

$$Z_t(w) = \begin{cases} U_t,\ \omega = u \\ N_t,\ \omega = n \\ D_t,\ \omega = d \end{cases}$$

In this case, U_t denotes the magnitude of the stock price change at time t and the corresponding definitions hold for N_t and D_t. Note that the condition of order is imposed on the values of Z_t: $0 < D_t < N_t < U_t$ for all t. Finally, the probability measure is defined below with the following constraints enforced (these arise from the definition of a probability measure): $0 < p_u, p_d < 1$ and $0 < p_u + p_d < 1$

$$\mathbb{P} = p_u \delta_u + p_d \delta_d + (1 - p_u - p_d)\delta_n$$

For the purpose of this assignment, we will assume that the coefficients of $\mathbb{P}$ do not vary with time. Below is an enumeration of all possible values of X_t from 0:

ω	$Z_1(w)$	$Z_2(w)$	$X_0(w)$	$X_1(w)$	$X_2(w)$
uu	U_1	U_2	c	cU_1	cU_1U_2
un	U_1	N_2	c	cU_1	cU_1N_2
ud	U_1	D_2	c	cU_1	cU_1D_2
nu	N_1	U_2	c	cN_1	cN_1U_2
nn	N_1	N_2	c	cN_1	cN_1N_2
nd	N_1	D_2	c	cN_1	cN_1D_2
du	D_1	U_2	c	cD_1	cD_1U_2
dn	D_1	N_2	c	cD_1	cD_1N_2
dd	D_1	D_2	c	cD_1	cD_1D_2

At $t = 1$, we can write the Martingale condition as follows:

$$p_u \times cU_1 + p_d \times cD_1 + (1 - p_u - p_d) \times cN_1 = c \tag{7}$$

$$\implies p_u U_1 + p_d D_1 + (1 - p_u - p_d)N_1 = 1 \tag{8}$$

assuming that $c \neq 0$. At $t = 2$, we can write three equations representing the Martingale requirement:

$$p_u \times cU_1U_2 + p_d \times cU_1D_2 + (1 - p_u - p_d) \times cU_1N_2 = cU_1 \tag{9}$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d)N_2 = 1 \tag{10}$$

$$p_u \times cN_1U_2 + p_d \times cN_1D_2 + (1 - p_u - p_d) \times cN_1N_2 = cN_1 \tag{11}$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d)N_2 = 1$$

$$p_u \times cD_1U_2 + p_d \times cD_1D_2 + (1 - p_u - p_d) \times cD_1N_2 = cN_1 \tag{12}$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d)N_2 = 1$$

assuming that $U_1, N_1, D_1 \neq 0$. Note that equations 9, 11 and 12 all simplify to 10 and therefore equations 8, 10 and $\{c, U_t, N_t, D_t \neq 0\}$ are the constraints for our trinomial tree.

In order to determine the conditions for which this market is arbitrage-free / complete, we use the Fundamental Theorem of Asset Pricing (FTAP) 1 and 2. Therefore, we must determine an equivalent Martingale measure and to do this, we construct it arbitrarily as follows:

$$\mathbb{P}^* = \alpha\delta_u + \beta\delta_d + \gamma\delta_n \tag{13}$$

For 13 to be an EMM, it must satisfy two conditions: 1) it must be a probability measure and equivalent to $\mathbb{P}$ and 2) X_t must be an $(\mathbb{F}, \mathbb{P}^*)$-Martingale. The first condition is confirmed by recognising that both $\mathbb{P}$ and $\mathbb{P}^*$ have the same null set (the empty set $\emptyset$) as well as enforcing the following condition:

$$\begin{aligned} &\alpha + \beta + \gamma = 1 \\ &\alpha, \beta, \gamma > 0 \end{aligned} \tag{14}$$

The second condition is satisfied when the following constraints are observed (see 8 and 10 except $\{p_u, p_d, 1 - p_u - p_d\}$ have been replaced by $\{\alpha, \beta, \gamma\}$):

$$\alpha U_1 + \beta D_1 + \gamma N_1 = 1 \tag{15}$$

$$\alpha U_2 + \beta D_2 + \gamma N_2 = 1 \tag{16}$$

Then equations 14, 15 and 16 form a system of linear equations.

$$Ax = b$$

$$\begin{bmatrix} U_1 & D_1 & N_1 \\ U_2 & D_2 & N_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{17}$$

Recall that the FTAPI dictates that there are no arbitrage opportunities if and only if $|\mathcal{P}| \neq 0$ and FTAPII states that the market is complete if and only if $|\mathcal{P}| = 1$. This is equivalent to determining under which conditions 17 has at least one solution and what conditions does it have only one solution. If Z_t is time invariant i.e. $U_s = U_t$ for all s, t (similar requirement for D_t and N_t), we automatically have that the market is arbitrage free since the matrix A contains two equivalent rows which results in an infinite number of solutions. This set of solutions will depend on a parameter (let's denote it p) and therefore to enforce market completeness, all that is needed is to select one value of p within its allowable domain. Since symbolic calculations involving 17 could become tedious, we have instead taken the numerical examples in part A to show an example of this. Recall that X_t evolves according to the table below:

ω	$X_0(w)$	$X_1(w)$	$X_2(w)$
uu	1	2	6
uu	1	2	2
uu	1	2	-6
uu	1	1	4
uu	1	1	1
uu	1	1	-4
uu	1	-2	-10
uu	1	-2	-2
uu	1	-2	10

and so equation 17 becomes:

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{18}$$

assuming that:

$$Z_t(w) = \begin{cases} 2, \; \omega = u \\ 1, \; \omega = n \\ -2, \; \omega = d \end{cases}$$

the probabilities become:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix} \tag{19}$$