

Trinomial Trees

MScFE 620 Group Project Submission 2

Avhad, Prajakta
prajakta.s.avhad@gmail.com

Hugo, Pepar Thomas Jay
peparhugo@gmail.com

Lin, Hong
plantvsbird@gmail.com

Ramsay, Ben
ramsay.ben@gmail.com

Wang, Jiao Yu
joseph_wang@live.ie

December 14, 2020

1 Part A - Construction of Trinomial Tree

Pepar's code goes here.

2 Part B - Arbitrage-free conditions and market completeness

For this section of the task, we will apply the arbitrage-free conditions to the numerical trinomial tree construction above and also to a completely general trinomial tree. In the later case, we will also derive conditions under which the market is complete. We begin by defining a market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with $T = 2$ and $d = 1$. $X = (1, X_t)$ is comprised of one risk-free asset (normalised to 1) and one risky asset which we will denote as X_t . Since we are trying to describe the trinomial model, we know that the sample space will consist of the three possibilities for asset evolution at any given time, namely up u , down d and nothing n . Assuming the completely general case where the asset price can move a different amount at each time step, we have $ud \neq du$ (commutativity property does not hold) and so $\Omega = \{uu, un, ud, nu, nn, nd, du, dn, dd\}$. For \mathcal{F} , we use the natural filtration which prevents look ahead bias. Specifically, $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ where

$$\begin{aligned}\mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_1 &= \sigma(\{\{uu, un, ud\}, \{nu, nn, nd\}, \{du, dn, dd\}\}) \\ \mathcal{F}_2 &= 2^\Omega\end{aligned}$$

We can define the evolution of the risky asset as:

$$\begin{aligned}X_{t+1} &= X_t Z_{t+1}, \quad t : 0 < t \leq 2 \\ X_0 &= c \in \mathbb{R}^+ \neq 0\end{aligned}$$

with Z_t , $t : 1 \leq t \leq 2$ a sequence of random variable taking on three values:

$$Z_t(w) = \begin{cases} U_t, & \omega = u \\ N_t, & \omega = n \\ D_t, & \omega = d \end{cases}$$

In this case, U_t denotes the magnitude of the stock price change at time t and the corresponding definitions hold for N_t and D_t . Note that the condition of order is imposed on the values of Z_t : $0 < D_t < N_t < U_t$ for all t . Finally, the probability measure is defined below with the following constraints enforced (these arise from the definition of a probability measure): $0 < p_u, p_d < 1$ and $0 < p_u + p_d < 1$

$$\mathbb{P} = p_u \delta_u + p_d \delta_d + (1 - p_u - p_d) \delta_n$$

For the purpose of this assignment, we will assume that the coefficients of \mathbb{P} do not vary with time. Below is an enumeration of all possible values of X_t from 0:

ω	$Z_1(w)$	$Z_2(w)$	$X_0(w)$	$X_1(w)$	$X_2(w)$
uu	U_1	U_2	c	cU_1	cU_1U_2
un	U_1	N_2	c	cU_1	cU_1N_2
ud	U_1	D_2	c	cU_1	cU_1D_2
nu	N_1	U_2	c	cN_1	cN_1U_2
nn	N_1	N_2	c	cN_1	cN_1N_2
nd	N_1	D_2	c	cN_1	cN_1D_2
du	D_1	U_2	c	cD_1	cD_1U_2
dn	D_1	N_2	c	cD_1	cD_1N_2
dd	D_1	D_2	c	cD_1	cD_1D_2

At $t = 1$, we can write the Martingale condition as follows:

$$p_u \times cU_1 + p_d \times cD_1 + (1 - p_u - p_d) \times cN_1 = c \quad (1)$$

$$\implies p_u U_1 + p_d D_1 + (1 - p_u - p_d) N_1 = 1 \quad (2)$$

assuming that $c \neq 0$. At $t = 2$, we can write three equations representing the Martingale requirement:

$$p_u \times cU_1U_2 + p_d \times cU_1D_2 + (1 - p_u - p_d) \times cU_1N_2 = cU_1 \quad (3)$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d) N_2 = 1 \quad (4)$$

$$p_u \times cN_1U_2 + p_d \times cN_1D_2 + (1 - p_u - p_d) \times cN_1N_2 = cN_1 \quad (5)$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d) N_2 = 1$$

$$p_u \times cD_1U_2 + p_d \times cD_1D_2 + (1 - p_u - p_d) \times cD_1N_2 = cD_1 \quad (6)$$

$$\implies p_u U_2 + p_d D_2 + (1 - p_u - p_d) N_2 = 1$$

assuming that $U_1, N_1, D_1 \neq 0$. Note that equations 3, 5 and 6 all simplify to 4 and therefore equations 2, 4 and $\{c, U_t, N_t, D_t \neq 0\}$ are the constraints for our trinomial tree.

In order to determine the conditions for which this market is arbitrage-free / complete, we use the Fundamental Theorem of Asset Pricing (FTAP) 1 and 2. Therefore, we must determine an equivalent Martingale measure and to do this, we construct it arbitrarily as follows:

$$\mathbb{P}^* = \alpha\delta_u + \beta\delta_d + \gamma\delta_n \quad (7)$$

For \mathbb{P}^* to be an EMM, it must satisfy two conditions: 1) it must be a probability measure and equivalent to \mathbb{P} and 2) X_t must be an $(\mathbb{F}, \mathbb{P}^*)$ -Martingale. The first condition is confirmed by recognising that both \mathbb{P} and \mathbb{P}^* have the same null set (the empty set \emptyset) as well as enforcing the following condition:

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha, \beta, \gamma &> 0 \end{aligned} \quad (8)$$

The second condition is satisfied when the following constraints are observed (see 2 and 4 except $\{p_u, p_d, 1 - p_u - p_d\}$ have been replaced by $\{\alpha, \beta, \gamma\}$):

$$\alpha U_1 + \beta D_1 + \gamma N_1 = 1 \quad (9)$$

$$\alpha U_2 + \beta D_2 + \gamma N_2 = 1 \quad (10)$$

Then equations 8, 9 and 10 form a system of linear equations.

$$\begin{aligned} Ax &= b \\ \begin{bmatrix} U_1 & D_1 & N_1 \\ U_2 & D_2 & N_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \quad (11)$$

Recall that the FTAPI dictates that there are no arbitrage opportunities if and only if $|\mathcal{P}| \neq 0$ and FTAPII states that the market is complete if and only if $|\mathcal{P}| = 1$. This is equivalent to determining under which conditions 11 has at least one solution and what conditions does it have only one solution. If Z_t is time invariant i.e. $U_s = U_t$ for all s, t (similar requirement for D_t and N_t), we automatically have that the market is arbitrage free since the matrix A contains two equivalent rows which results in an infinite number of solutions. This set of solutions will depend on a parameter (let's denote it p) and therefore to enforce market completeness, all that is needed is to select one value of p within its allowable domain. Since symbolic calculations involving 11 could become tedious, we have instead taken the numerical examples in part A to show an example of this. Recall that X_t evolves according to the table below:

ω	$X_0(w)$	$X_1(w)$	$X_2(w)$
uu	1	2	6
uu	1	2	2
uu	1	2	-6
uu	1	1	4
uu	1	1	1
uu	1	1	-4
uu	1	-2	-10
uu	1	-2	-2
uu	1	-2	10

and so equation 11 becomes:

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (12)$$

assuming that:

$$Z_t(w) = \begin{cases} 2, & \omega = u \\ 1, & \omega = n \\ -2, & \omega = d \end{cases}$$