Radon–Nikodym theorem

The Radon-Nikodym Theorem helps us in understanding the change of measure. We can measure length (physical real word length) in centimetres as well as inches. How do these two relate to each other? In abstract Measure Theory Radon-Nikodym Theorem answers this question.

But before we go on to the Theorem statement, we need to get some definitions in place.

Lebesgue Measure

The Lebesgue Measure of a set in R is the measure which takes each interval to its length. It is denoted by λ.

$$\lambda([a, b]) = b - a$$

Absolutely Continuous

A measure μ is known as absolutely continuous with respect to the Lebesgue Measure, if μ is not finer than the Lebesgue Measure. Mathematically this can be written as

If $(R, \mathcal{B}(R), \lambda)$ is the Real Space with Lebesgue Measure and $(R, \mathcal{B}(R), \mu)$ is the Real Space with measure μ, then μ is absolutely continuous with respect to the Lebesgue Measure if,

$$\lambda(A) = 0 \Rightarrow \mu(A) = 0 \forall A \in \mathcal{B}(R)$$

This is denoted by $\mu \ll \lambda$. Two elementary examples would be The Lebesgue Measure itself is Absolutely Continuous with respect to the Lebesgue Measure, and the Zero Measure (assigns zero measure to every set) is absolutely continuous with respect to the Lebesgue Measure.

Singular Measure

This suggests that μ and λ in some sense are disjoint. Mathematically this can be written as:

μ is singular with respect to the Lebesgue Measure if there exists a set $N \in \mathcal{B}(R)$ such that

$$\lambda(N) = 0 \text{ and } \mu(N^C) = 0$$

This is denoted by $\mu \perp \lambda$

As an intuitive example, we can take the Dirac Measure (Again examples from recent history helping us right away). Let $N = \{0\}$

$$\mu(N) = 0 \text{ and } \delta_0(N^C) = 0$$

σ-finite Measure

A positive measure μ defined on a σ-algebra of a set X is called a finite measure if μ(X) is a finite real number (rather than ∞). The measure μ is called σ-finite if X is the countable union of measurable sets with finite measure.
As an example the Lebesgue Measure is σ-finite. This is because the Real Space can be expressed as

$$R = \cup_{k \in Z} [k, k+1)$$

And each of these intervals have a finite measure ($= 1$).

Now we finally come to **Radon Nikodym Theorem,** the theorem states that any Absolutely Continuous σ-finite measure can be expressed as the Lebesgue Integral of a Lebesgue Measure.

So, it μ is σ-finite absolutely continuous with respect to the Lebesgue Measure then we will get a measurable map $h: R \to [0, \infty]$, such that

$$\boldsymbol{\mu}(\boldsymbol{A}) = \int_{\boldsymbol{A}} \boldsymbol{h}\, \boldsymbol{d\lambda} \quad \forall A \in \mathcal{B}(R)$$

So, in other terms, it becomes easier for us to deal with this abstract measure because now we only must deal with a measurable function.