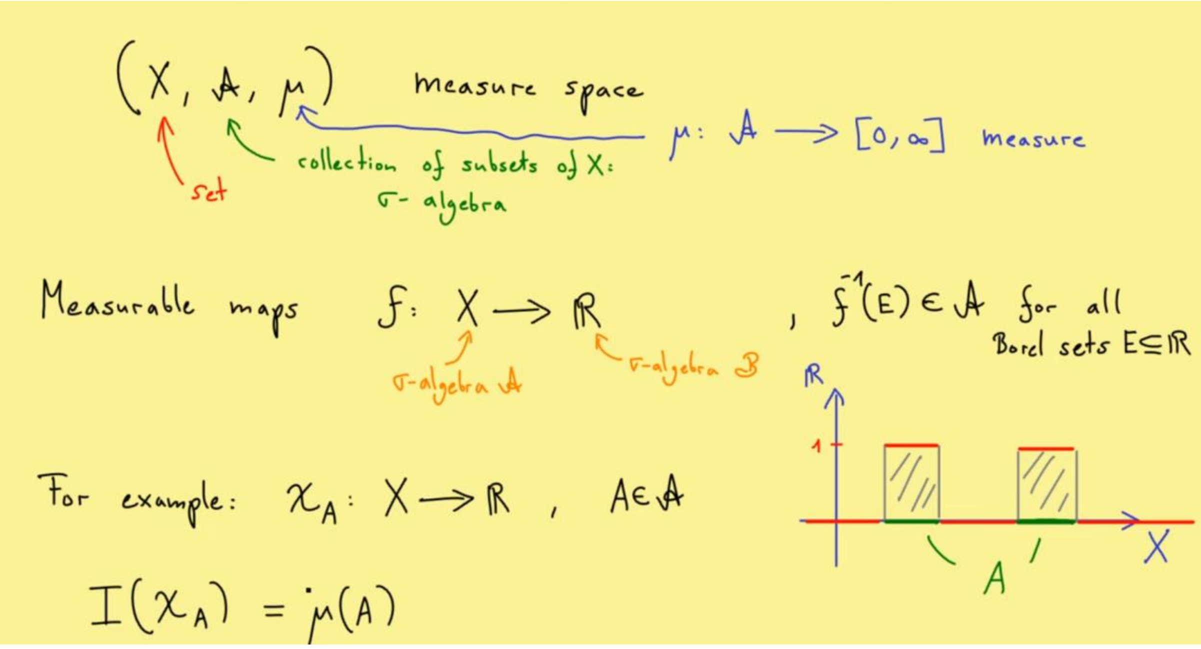
Lebesgue Integral

When we move on to Stochastic Calculus in Finance, finding Expectations and moments plays an important role. When we try to do this over arbitrary measurable spaces, an abstraction of the integral functional is required and this is where Lebesgue Integral comes into the picture.

Let X be a set and f be a measurable map from X to R. Then we wish to find the integral of this map. X has the σAlgebra $\mathcal{A}$ and R has the Borel σ Algebra. As a starting example let us look at maps that we know are measurable (see we are already using the examples we saw earlier to reduce more complex problems into known ones). Let us look at the characteristic function χ_A.



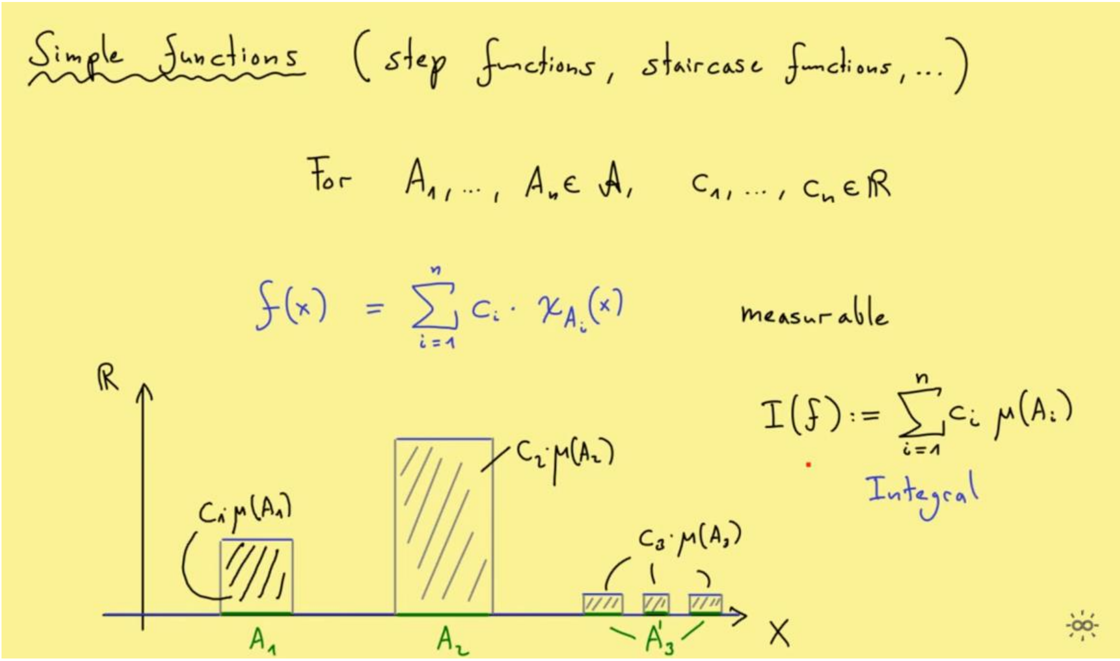
The integral of this characteristic function is the area in the shaded rectangles, which turn out to be some kind of measure of the set A. So intuitively, the integral turns to be the measure of some set in the domain.

Other functions like the characteristic function which satisfy this desirable property: "the integral turns to be the measure of some set in the domain." are known as simple functions.

Simple Functions

A function $f: X \rightarrow R$ is known as a simple function if it can be written as the summation of such characteristic functions. That is $\exists A_i \in \mathcal{A}$ and $c_i \in R$ such that for $x \in X$, we have

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$



However, there is a problem that may arise. What is we have subsets with measure ∞, and we get $I(f) = \cdots . +2\infty - 3\ \infty$, how do we find this? To solve this, we may have two solutions, one where we only restrict ourselves to subsets of finite measure or where c_i's are all positive.

So, we form another class of such functions, with some additional restrictions: We can see that we can have infinite representations for the summation (we can break A_2 into two parts and get a new representation). So, we only take functions which are positive (so that $c_i \geq 0$) and only has finitely many values.

$\mathcal{S}^+$ = { $f: X$ \righarrow R | f is simple measurable function which takes finitely many values and $f \ \geq 0$ }

For functions in this class, we can define the Lebesgue Integral.

The Lebesgue integral of $f \in \mathcal{S}^+$, $f = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$ is defined as:

$$\mathrm{I}(f) = \sum_{i=1}^{n} c_i \ \cdot \ \mu(A_i)$$

This integral is well defined and $I(f) \in [0, \infty]$

This is denoted by $\int_X f(x) d\,\mu = I(f)$

Definition: $S^+ := \{ f : X \to \mathbb{R} \mid f \text{ simple function}, \ f \geq 0 \}$

· measurable
· only finitely many values

$f \in S^+$ choose representation $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, $c_i \geq 0$

The (Lebesgue) integral f w.r.t. μ:

$$\int_X f(x)\, d\mu(x) = \int_X f \, d\mu = I(f) = \sum_{i=1}^{n} c_i \cdot \mu(A_i) \in [0, \infty]$$

well-defined

Yet again, we look at some of the important properties of this Lebesgue Integral which will be used later to simplify more complex problems.

Properties

1. **Linearity**

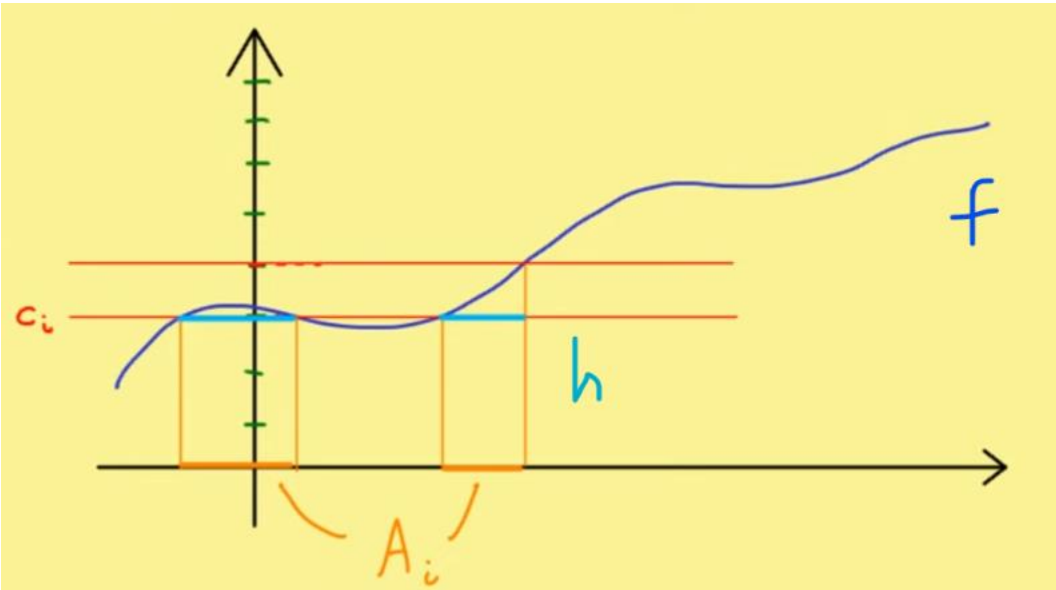
$$\boldsymbol{I(\alpha f + \beta g) = \alpha I(f) + \beta I(g), \forall \alpha, \beta \geq 0}$$

2. **Monotonicity**

   If $f \leq g, I(f) \leq I(g)$

We use this property of monotonicity of simple functions (also know as step functions) to define the Generalized Lebesgue Integral of any measurable function (YES FINALLY!!!):

If $f: X \to [0, \infty)$ is a measurable function. We find simple functions h such that at every point the value of $h \leq f$. We already know how to find the integral of these simple functions. For a given function, and a subset A_i, we define h in such a way that it obtains the value c_i in A_i and c_i is infimum value of f in A_i.



In doing so, as we create finer and finer partitions (A_i's), we get closer and closer to f and the integral of f gets closer and closer to the integral of h. And so, we get:

$$\int_X \boldsymbol{f}\, \boldsymbol{d\mu} = \boldsymbol{sup}\,\{\, \boldsymbol{I(h)} \mid \boldsymbol{h} \in \boldsymbol{\mathcal{S}^+}, \boldsymbol{h} \le \boldsymbol{f}\,\}$$

This is the Generalized Lebesgue Integral of the measurable function.