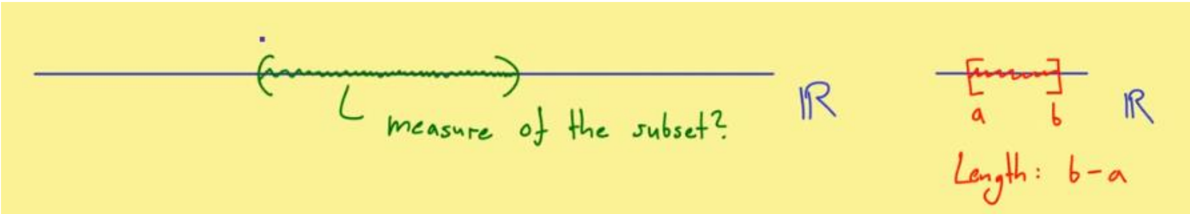
σ-Algebra

Let us start with measuring the real line:



For any interval $[a, b]$, we know its **"Length"** is $(b - a)$. Similarly, for a square of length a, we know its **"Area"** is a^2, we have a similar notion for three dimensions. All of these **"notions"** can be abstractly generalised as the notion of **"Measure".** We now wish to extend this notion of Measure to arbitrary sets or at least as many kinds of sets possible (see where I am going). We will call these special kind of sets as measurable sets, sets which we can ***measure.***

Now let us dive deeper into this idea of ***special kind of sets.*** Since we are trying to generalize, as with most other cases let us bring in some conditions and a name for this class of measurable sets.

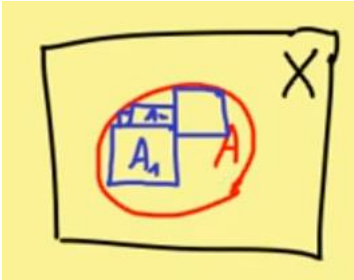
Let X be a set, and let

$$\mathcal{A} \subseteq P(X)$$

Now we define a few conditions for this to be called a σ algebra. The motivation for these conditions is also presented (to try to explain the origins of these assumptions as logical Deductive Reasoning). We want the subsets of $\mathcal{A}$ to be measurable sets, and this will be our guiding star for the following conditions:

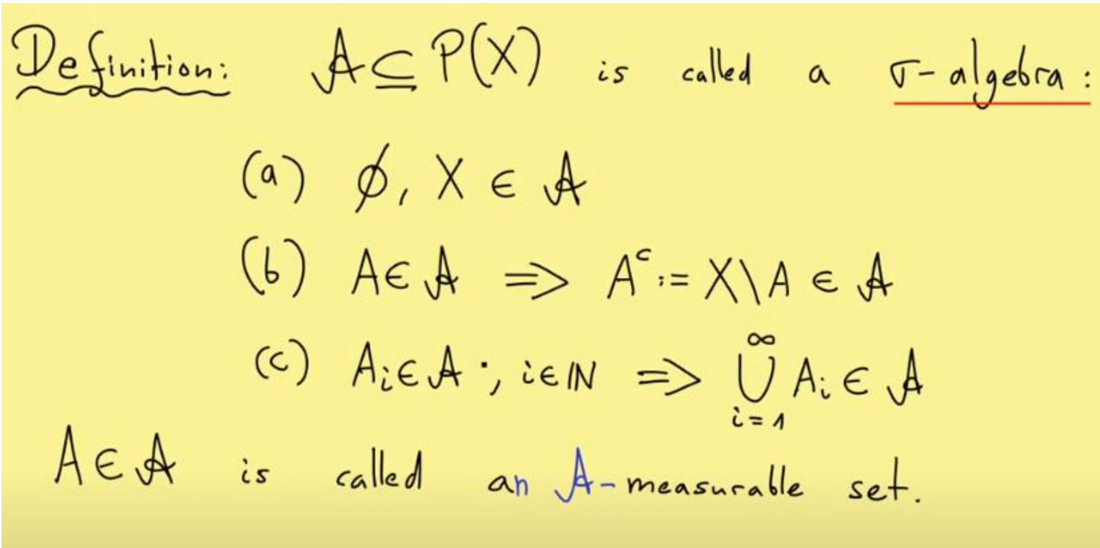
1. We want the empty set and full set to be measurable.
$$\Phi, X \in \mathcal{A}$$
2. If a set A is measurable, we want its complement to be measurable.
$$A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$$

3. If we have a countably many subsets $A_i \in \mathcal{A}$ which are measurable. Then we have the countable union of these subsets to be measurable. $\cup_{i \in N} A_i \in \mathcal{A}$.
So, if we have a set A, and subsets A_i which kind of sum up to (hence the σ) to the set A. Then if each of these subsets is measurable, we want the limiting case of the sum of their measures as the measure of A, for which we need A to be measurable.



All such systems of subsets ($\mathcal{A}$) which satisfy these three rules are known as σ algebras. And the elements of these subsets are called as $\mathcal{A}$-measurable sets. So essentially a σ-Algebra $\mathcal{A}$on the set X is a collection of subsets of the powerset of X, and each of the elements (subsets of powerset) are $\mathcal{A}$-measurable sets.

Summarizing:



Properties

Like any other field of Mathematics, when we introduce a new object, we wish to look at the properties of this object, we deduce these properties so that we can take advantage of them to break down bigger problems as use these properties as tools to solve the bigger problem.

Arbitrary Intersection of σ- Algebra's on a set X is a σ- Algebra on the set X

Proof:

Property 1)

We have, $\Phi, X \in \mathcal{A}_i \forall i \in I$.
So $\Phi, X \in \cap_i \mathcal{A}_i$

Property 2)

If $A \in \cap_i \mathcal{A}_i$
then $A \in \mathcal{A}_i \forall i$
Since $\mathcal{A}_i$ is a σ- Algebra, we have, $A^C \in \mathcal{A}_i \forall i$
So $A^C \in \cap_i \mathcal{A}_i$

Property 3)
If $A_j \in \cap_i \mathcal{A}_i$
then $A_j \in \mathcal{A}_i \forall i$
Since $\mathcal{A}_i$ is a σ- Algebra, we have, $\cup_j A_j \in \mathcal{A}_i \forall i$
So $\cup_j A_j \in \cap_i \mathcal{A}_i$

So, Arbitrary Intersection of σ- Algebra's on a set X is a σ- Algebra on the set X.

σ- Algebra Generated by a collection

Often, Abstract Mathematics tries to find the **most efficient** object once we have defined and explored that object for some time. With Linear Algebra, after explaining a Vector Space, and a generating set, we try to find the basis of a given vector space. Or, given a set we try to find its spanning set. We have come across similar exercises in Group Theory, Ring Theory, Field Theory, Galois Theory, and every other field. These **most efficient** objects help us immensely in our pursuit of solving problems that we wish to solve. With this motivation, we look at a σ Algebra generated by a subcollection, and then we will look at a Borel σ Algebra.

Definition: Given a subcollection μ ⊆ $P(X)$, the smallest σ Algebra which contains μ is known as the σ Algebra generated by μ. It is denoted by σ(μ). Mathematically,

$$\sigma(\mu) = \cap_{\mu \subseteq \mathcal{A}} \mathcal{A} \text{ where } \mathcal{A} \text{ is a } \sigma \text{ Algebra}$$

Borel σ Algebra

For the purpose of this section, the set X which is the most important aspect, but we were not paying much heed to, has to be a topological space. For help of understanding and simplicity we can just think of it to be a Metric Space or just any Real Space (R^2, R^3, ...). We wish to look at the σ Algebra generated by the open sets in X. That is μ = {Open Sets in X}.