Measure

Now we finally come to the most important part of **Measure** Theory, the Measure. Remember, how we visualized our object $\mathcal{A}$ as the collection of measurable sets, and intuitively we thought this is just the collection of objects we can measure. Well turns out our intuition was correct.

Given any set X, and a σ Algebra $\mathcal{A}$. The tuple $(X, \mathcal{A})$ is known as a measurable space.

Given a measurable space, we define a function μ,

$$\boldsymbol{\mu}: \boldsymbol{\mathcal{A}} \rightarrow [\mathbf{0}, \infty) \cup \{\infty\}$$

Note: We wish to visualize this function as a function which gives a measure (see as generalized length or generalized volume) to elements of $\mathcal{A}$, so it makes perfect sense to restrict the codomain to only positive part of the real line. Also note how ∞ is a part of the codomain, this is our way of saying that certain elements may also have infinite measure (or length/ volume) like what is the length of the whole real line?

This map μ is known as a measure if it satisfies the following conditions (here we go again):

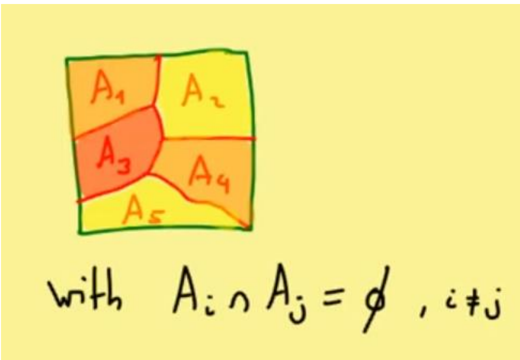
1. What is the length of nothing on a real line, or what is the volume of nothing in R^3

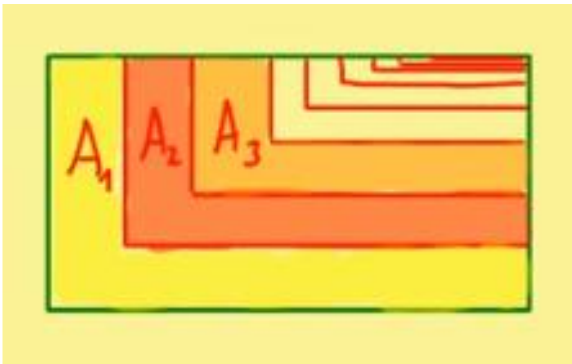
$$\boldsymbol{\mu(\Phi) = 0}$$

2. What is the total length of pairwise disjoint intervals of the real line like $[0,1] \cup [2,3]$?

$$\boldsymbol{\mu(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} A_i}$$

   This is known as σ Additivity.





$(X, \mathcal{A}, \mu)$ is known as the measure space.

Bonus: If this measure μ satisfies $\mu(X) = 1$. Then μis a probability measure. $(X, \mathcal{A}, \mu)$ is called a probability space.

Examples

I do not know if this happens with you, but something that happens a lot with me, is I skim through examples real fast because well these are case-specific right? They would not be of much use later when things get complex and we need generalizations and abstractness! And then it turns out they are defining these simple examples everywhere to make the problem smaller and I must go back again and again to understand those simple examples. The examples I use here are again those ubiquitous omnipotent examples.

Counting Measure

Examples: X, $\mathcal{A} = \mathcal{P}(X)$

(a) Counting measure: $(A \in \mathcal{A})$

number of elements in A

$$\mu(A) := \begin{cases} \# A & , \text{ A has finitely many elements} \\ \infty & , \text{ else} \end{cases}$$

Simply put, like the name suggests, we count the number of elements in a set A. How many elements are there in $\mathrm{A} = \{\, \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \,\}$, $\mu(A) = 4$. If X is the real line, and $\mathrm{A} = [0{,}5]$. How many points are there in A? $\mu(A) = \infty$

Dirac Measure

(b) Dirac measure for $p \in X$

$$\delta_p(A) := \begin{cases} 1 & , \; p \in A \\ 0 & , \; \text{else} \end{cases}$$

The Dirac Measure like we can see from the definition, serves as tool to give weightage (measure) to sets which contain a particular point we are interested about. The Dirac Measure will be used extensively in Financial Engineering Stochastic Calculus.