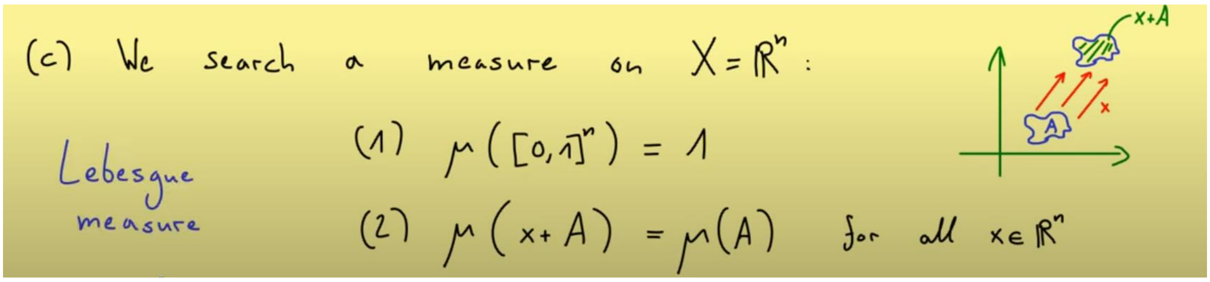
Motivation for Lebesgue Measure

When we try to define a measure on subsets of R^n we want that measure function to satisfy some properties which we may intuitively visualize in a physical sense. For example, we know the length of a unit line segment $[0,1]$ is 1. The area of a unit square $[0,1]X[0,1]$ is 1. Similarly, we can extend it for a unit cube and so on.

We want this property for our generalized measure too, the measure for our Unit Object should be 1.

Similarly, one other physical property is that of translation, the length of $[0,1]$, $[2,3]$, $[4,5]$ and so on is always 1. The area of $[0,1]X[0,1]$, $[4,5]X[4,5]$, $[100,101]X[100,101]$ is always 1 square units. So intuitively our generalized measure should also be translation invariant.

This gives us two properties we desire:



Lebesgue Measure

We find that there exists no such measure μ which can be defined on the whole powerset of R which satisfies these two properties we want:

$$\mu([a,b]) = b - a$$

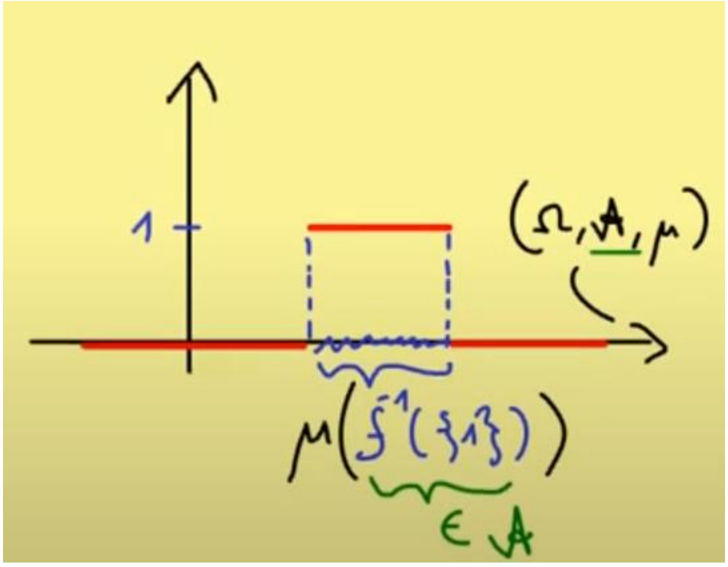
$$\mu(x + A) = \mu(A)$$

You can find a really good proof for this in any Measure Theory book or for a more detailed lecture I insist you to try this lecture by The Bright Side of Mathematics.

Measurable Maps

After Measurable Sets and Measure Functions, we discuss Measurable Maps. Our motivation for this section is that since we have already measured sets, we will now try to measure maps between these sets. This is where we start to dive deep into the essence of measure theory. But why do we need measurable maps? This is something I always ask when I am studying some new concept, why do we need this concept?

Well measurable maps will pave the way for us to interlink two measurable spaces. Consider the function $f: R \to R$ with the following map. It takes the value 1 in a given interval and 0 everywhere else. Now we wish to measure the interval where it takes the value 1. If (X axis) R has σ Algebra $\mathcal{A}$ and measure function μ. Then we want to measure the interval $f^{-1}(\{1\})$. For which we need this to be measurable. That is the motivation for the definition of measurable maps.



Definition: If $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. And f is a function $f: \Omega_1 \to \Omega_2$, then f is measurable if $\boldsymbol{f^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2}$.

Now let us look at an important property of these measurable maps, remember this will be used extensively later to simplify bigger problems

Properties

1. **Composition of measurable maps is measurable**

If $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2), (\Omega_3, \mathcal{A}_3)$ be three measurable spaces and $f: \Omega_1 \to \Omega_2$ and $g: \Omega_2 \to \Omega_3$, and $g \circ f: \Omega_1 \to \Omega_3$. Then if f and g are measurable then $g \circ f$ is measurable.

Proof: $\boldsymbol{(g \circ f)^{-1}(A_3) = f^{-1}\left((g^{-1}A_3)\right)}$

Now $g^{-1}(A_3) \in \mathcal{A}_2$ and $g^{-1}(A_2) \in \mathcal{A}_1$ so $(g \circ f)$ is measurable by definition.

2. **Addition, Subtraction, Multiplication and Absolute of measurable functions from a set to the Real Line with Borel σ Algebra are measurable**

If $(\Omega, \mathcal{A})$ is a measurable space and $f, g: \to (\mathrm{R}, \mathrm{B}(\mathrm{R}))$ are measurable then $(f+g), (f-g), (f \cdot g), |f|$ are measurable.