Monotone Convergence Theorem

We now look into some important theorems about the Lebesgue Integral, with the help of these theorems we start to see the importance of Lebesgue Integral and will use Lebesgue Integral to solve more complex problems with the help of such theorems. We start off by defining a new notion and a few properties which will help us to look at the Monotone Convergence Theorem. Then we will look at an immediate application of the theorem to appreciate just how important this theorem is.

We start off by introducing the notion of almost everywhere. This will help us relax our assumptions which will greatly increase the use cases where Lebesgue Integral and its varied desirable properties may be used.

μ almost everywhere

A property P is said to be true μ almost everywhere, it the μ measure of the set where it is not true is 0.
For example, $f = g,\ \mu$ a.e. if $\mu(\{x \in X | f(x) \neq g(x)\}) = 0$.

Next, we investigate a few properties of the Lebesgue Measure which will serve as assumptions for the Monotone Convergence Theorem.

1. If $f = g$ a.e. then $\int_X f\, d\mu = \int_X g\, d\mu$
2. If $f \leq g$ a.e. then $\int_X f\, d\mu \leq \int_X g\, d\mu$
3. If $f = 0$ a.e. then $\int_X f\, d\mu = 0$

The proofs for this may be found at this link courtesy of The Bright Side Of Mathematics - YouTube.

Finally, we look at the **Monotone Convergence Theorem**:

If $(X, \mathcal{A}, \mu)$ is a measurable space, and f_n's are a series of measurable functions

$$f_n: X \to [0, \infty)\ \forall n \in N \text{ with } f_1 \leq f_2 \leq f_3 \leq \cdots \mu \text{ a.e.}$$

$$\text{And } \lim_{n\to\infty} f_n(x) = f(x)\ \mu \text{ a.e. } \forall x \in X.$$

$$\text{Then, } \boldsymbol{\lim_{n\to\infty} \int_X f_n d\mu = \int_X f\, d\mu}.$$

So ideally if we have a series monotonically increasing functions whose limit is a given function, then we can interchange Limit and Lebesgue Integral.

Application

We now look at an application of the Monotone Convergence Theorem we just saw.

If we have a series of functions $g_n: X \to [0, \infty]$, then we can interchange Summation and Lebesgue Integral. See the assumptions, almost none. This is the power of the Monotone Convergence Theorem.

$$\boldsymbol{\sum_{n=1}^{\infty} \int_X g_n d\mu = \int_X \sum_{n=1}^{\infty} g_n d\mu}$$

To prove this, we take the monotonic f_n's (in accordance with MCT) to be the partial sums of g_n.

That is, $f_m = \sum_{n=1}^{m} g_n$. Then these f_n's satisfies the two properties for the MCT and hence using the MCT we can get this property.