Testing of convergence of infinite series

Dr. P. Debnath

1. A) A necessary condition for the convergence of an infinite series $\sum u_n$ is

$$\lim_{n\to\infty} u_n = 0$$

B) Abel's Test: A necessary condition for the convergence of an infinite series $\sum u_n$ is

$$\lim_{n\to\infty} nu_n = 0$$

Example 1 : Test the convergence of the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Solution: The given series $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

Now, $\lim_{n\to\infty} nu_n = \lim_{n\to\infty} n\frac{1}{n} = 1 \neq 0.$

Therefore, the given series is not convergent. Now, a positive term series can never oscillate.

Thus, the given series is divergent.

NOTE: Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$, So, we cannot draw any conclusion from this test.

2. *Comparison Test*

A) Let $\sum u_n$ and $\sum v_n$ be two series of positive terms.

i) Suppose, $\sum u_n \leq k \sum v_n, \forall n \geq N$ and the series $\sum v_n$ is a convergent series (we know the property of $\sum v_n$). Then the series $\sum u_n$ is also convergent.

ii) Suppose, $\sum u_n \leq k \sum v_n, \forall n \geq N$ and the series $\sum u_n$ is a divergent series (we know the property of $\sum u_n$). Then the series $\sum v_n$ is also divergent.

B) Let $\sum u_n$ and $\sum v_n$ be two series of positive terms and

Dr. P. Debnath

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l,\quad l\ is\ a\ nonzero\ finite\ number$$

Then the series $\sum u_n$ and $\sum v_n$ has same property (converges or diverges together).

Example 2 : Test the convergence of the infinite series

$$1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots$$

Solution: $\sum u_n = 1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots$

Therefore, $u_n = \frac{2^{n-1}}{(n-1)!} = \frac{2.2.2....2((n-1)times)}{1.2.3......(n-1)}$

$$=\frac{2}{1}.\frac{2}{2}.\frac{2}{3}.\frac{2}{4}\cdots\frac{2}{n-1}(product\ of\ (n-1)terms).$$

$$=2[.\frac{2}{3}.\frac{2}{4}\cdots\frac{2}{n-1}(product\ of\ (n-3)terms)].$$

Now, $\frac{2}{3}\le\frac{2}{3},\quad \frac{2}{4}\le\frac{2}{3},\quad \cdots\frac{2}{n-1}.\le\frac{2}{3}$

Therefore, $u_n = 2\left[.\frac{2}{3}.\frac{2}{4}\cdots\frac{2}{n-1}(product\ of\ (n-3)terms)\right].$

$$\le 2.\left[\frac{2}{3}\frac{2}{3}\frac{2}{3}\cdots\frac{2}{3}(n-3\ times)\right]\ \forall n\ge 3$$

$$=2.\left(\frac{2}{3}\right)^{n-3}$$

Now, let $\sum v_n = \sum\left(\frac{2}{3}\right)^{n-3}$

Thus, $u_n \le 2v_n \quad \forall n\ge 3$

Now, the series $\sum v_n$ is a convergent series being a G.P. series with common ratio $\frac{2}{3}$.

So, by comparison test the series $\sum u_n$ is also convergent.

Example 3 : Test the convergence of the infinite series

Dr. P. Debnath

$$1+\frac{1}{1.3}+\frac{1}{1.3.5}+\cdots$$

Solution: General term of the series is $u_n=\frac{1}{1.3.5.\cdots(2n-1)}$

Now, $\frac{1}{1.3.5.\cdots(2n-1)}<\frac{1}{1.\{2.2.2\cdots2(n-1\ times)\}}=\frac{1}{2^{n-1}}$

Now, let $v_n=\frac{1}{2^{n-1}}$. So, $u_n<v_n\ \ for\ n=1,2,\ldots$

The series $\sum v_n=\sum\frac{1}{2^{n-1}}$ is a convergent series.

Hence by comparison test the given series is convergent.

Example 4: Test the convergence of $\frac{6}{1.3.5}+\frac{8}{3.5.7}+\frac{10}{5.7.9}+\cdots$

Solution: General term of the series is $u_n=\frac{2n+4}{(2n-1)(2n+1)(2n+3)}$

Note that the degree of numerator is 1 but the degree of denominator is 3, thus there difference is $(3-1=)2$.

So, let us compare it with the $p-series$ where $p=2\ \ i.e. with\ \sum v_n=\sum\frac{1}{n^2}$

So,

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{2n+4}{(2n-1)(2n+1)(2n+3)}}{\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{(2n+4)n^2}{(2n-1)(2n+1)(2n+3)}$$

$$=2\lim_{n\to\infty}\frac{(1+\frac{2}{n})}{(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})}=\frac{2}{2.2.2}=\frac{1}{4}$$

Therefore by comparison test both the series converges or diverges together but the series $\sum\frac{1}{n^2}$ is a convergent series. Therefore the given series is also convergent.

An important limit: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

Testing of convergence of infinite series

Dr. P. Debnath

Problem: Check the convergence of the infinite series $\sum_{n=2}^{\infty}\frac{\log n}{\sqrt{n+1}}$

Solution:

Let us apply Abel's test.

$$\lim_{n\to\infty} nu_n = \lim_{n\to\infty}\frac{n\, logn}{\sqrt{n+1}}$$

$$= \lim_{n\to\infty}\frac{\sqrt{n}\log n}{\sqrt{1+\frac{1}{n}}} = \infty$$

Now, this is a series of positive terms and by Abel's test the series is not convergent.

Thus the given series is divergent.

Problem: Check the convergence of the infinite series

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$$

Solution: Omitting the first term of the series, remaining series is

$$\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$$

General term of this series $u_n = \frac{n^n}{(n+1)^{(n+1)}}$

The difference of degree of numerator and denominator is 1. So, let us compare this series with the p-series where $p = 1\ i.e. \sum\frac{1}{n}$.

Now, $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{n^n}{(n+1)^{(n+1)}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^{n+1}}{(n+1)^{(n+1)}} = \lim_{n\to\infty}\frac{1}{\left(\frac{1+n}{n}\right)^{n+1}}$

$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)} = \frac{1}{e.1} = \frac{1}{e}$ [as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$]

So, we see that $\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{e} = a\ non\ zero\ finite\ number.$

So, the series $\sum u_n$ and $\sum v_n$ converges or diverges together but the series $\sum \frac{1}{n}$ is a divergent series. So, by comparison test the given series is divergent.

C) ***D' Alembert's Ratio Test***

Let $\sum u_n$ be a series of positive terms and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$. The series is convergent if $l < 1$ and divergent for $l > 1$. For $l = 1$, the test fails.

Problem: Discuss the convergence of the series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots$

Solution: Omitting the first term, the general term of the series is

$$u_n = \frac{x^n}{n^2+1}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

Now, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)^2+1}}{\frac{x^n}{n^2+1}} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n}$

$$\lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} x = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} x = x$$

So, by D' Alembert's ratio test the series is convergent for $x < 1$ and divergent for $x > 1$.

For $x = 1$, the series becomes $\sum \frac{1}{n^2+1}$ and we have $\frac{1}{n^2+1} < \frac{1}{n^2}$ $\forall n \geq 1$.

Now the series $\sum \frac{1}{n^2}$ is convergent. So, by comparison test the series $\sum \frac{1}{n^2+1}$ is also convergent.

Thus the given series is convergent for $x \leq 1$ and divergent for $x > 1$.

Problem: Check the convergence of the infinite series $\sum_{n=1}^{\infty} \frac{n! 2^n}{n^n}$.

Solution: $\boldsymbol{u_n} = \frac{n!2^n}{n^n}$

To check the convergence let us apply the D'Alembert's ratio test.

$$u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$$

So, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}}{\frac{n!2^n}{n^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!2^n}$

$= \lim_{n\to\infty} \frac{(n+1).2}{(n+1)(n+1)^n} . n^n$

$= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = 2\lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$

$= 2\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$= \frac{2}{e}$

Now, $2 < e < 3$, therefore, $\frac{2}{e} < 1$.

Thus, by D'Alembert's ratio test the given series is convergent.

D) ***Cauchy Root test:*** Let $\sum u_n$ be a series of positive terms and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$. The series is convergent if $l < 1$ and divergent for $l > 1$. For $l = 1$, the test fails.

Problem: Examine the nature of the given series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

Dr. P. Debnath

Solution:

$$u_n = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}$$

Now, applying Cauchy root test

$$\lim_{n\to\infty} (\,u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-1}$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)^{n}(n+1)}{n^{n}.n} - \frac{n+1}{n}\right)^{-1}$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right)^{-1}$$

$$= (e-1)^{-1}$$

Now, Now, 2$< e < 3$, therefore, $e-1>1$, therefore, $(e-1)^{-1}<1$.

So, by Cauchy root test, given series is convergent.

Home work: Examine the nature of the given series $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Note: According to limit theorem of differential calculus, If both of , $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$ and $\lim_{n\to\infty}(\,u_n)^{\frac{1}{n}}$ exists then they are equal. So, if D' Alembert's ratio test fails, we will not try for Cauchy root test. We will directly apply ***Raabe's Test.***

Raabe's Test: Let $\sum u_n$ be a series of positive terms and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = l$, The series is convergent if $l>1$ and divergent for $l<1$.

Problem: Test the convergence of

Testing of convergence of infinite series

Dr. P. Debnath

$$1+\alpha+\frac{\alpha(\alpha+1)}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3}+\cdots \quad (0<\alpha<1)$$

Solution: Omitting the first term, general term of the remaining infinite series is $u_n=\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+(n-1))}{1.2.3\ldots n}$

At first, let us apply D' Alembert's ratio test

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+(n-1))(\alpha+n)}{1.2.3\ldots n(n+1)}}{\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+(n-1))}{1.2.3\ldots n}}$$

$$=\lim_{n\to\infty}\frac{\alpha+n}{n+1}=\lim_{n\to\infty}\frac{\frac{\alpha}{n}+1}{1+\frac{1}{n}}=1$$

So, D' Alembert's ratio test fails. Let us now apply Raabe's test

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)$$

$$=\lim_{n\to\infty}n\left(\frac{n+1}{\alpha+n}-1\right)$$

$$=\lim_{n\to\infty}n\frac{1-\alpha}{\alpha+n}$$

$$\lim_{n\to\infty}\frac{1-\alpha}{\frac{\alpha}{n}+1}=1-\alpha$$

Now, $0<\alpha<1$, therefore, $0<1-\alpha<1$. Hence the given series is divergent.

Home work: Check the convergence of $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\cdots$

1. i) The infinite series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is

   a) Convergent b) divergent c) oscillatory d) convergent omitting the first term

   ii) The p-$series$ is convergent if

   a) $p\le 1$ b) $p>1$ c) $p\ge 1$ d) $p<1$

   iii) The sequence $\{\frac{1}{1+2n}\}$ is

a) Increasing and bounded b) Increasing and unbounded

c) Decreasing and unbounded d) Decreasing and bounded

$iv)\ \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}$ is equal to

a) 1 b) e c) 0 d) $^1/_e$

v) The infinite series $\sum r^{n-1}$ is

a) convergent for $|r| > 1$, oscillatory for $r = 1$, otherwise divergent

b) divergent for $|r| > 1$, oscillatory for $r = -1$, otherwise divergent

c) convergent for $|r| > 1$, oscillatory for $r = -1$, otherwise divergent

d) convergent for $|r| < 1$, oscillatory for $r = -1$, otherwise divergent

2. Check the convergence of the series $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots$

3. Test the convergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots\cdot$

4. Discuss the convergence of the series $\sum\frac{n!}{n^n}$

5. $1+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{3^2}\right)+\left(\frac{1}{4}-\frac{1}{4^2}\right)+\cdots$